Spaces of Valuations

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Abstract

Valuations are measure-like functions mapping the open sets of a topological space X into positive real numbers. They can be classified into finite, point continuous, and Scott continuous valuations. We define corresponding spaces of valuations $\mathsf{V}_{\mathrm{f}} X \subseteq \mathsf{V}_{\mathrm{p}} X \subseteq \mathsf{V}X$. The main results of the paper are that $\mathsf{V}_{\mathrm{p}} X$ is the soberification of $\mathsf{V}_{\mathrm{f}} X$, and that $\mathsf{V}_{\mathrm{p}} X$ is the free sober locally convex topological cone over X. From this universal property, the notion of the integral of a real-valued function over a Scott continuous valuation can be easily derived. The integral is used to characterize the spaces $\mathsf{V}_{\mathrm{p}} X$ and $\mathsf{V}X$ as dual spaces of certain spaces of real-valued functions on X.

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1 Introduction

For a topological space X, a *valuation* on X is a function ν which maps the open sets of X to real numbers in the range from zero to infinity (inclusively) with the following properties [11, 8, 9, 10, 13]:

(1) The empty set is mapped to zero: $\nu\emptyset = 0$ (strictness).

(2) The values assigned to binary union and intersection are related by the following equation:

$$\nu(U \cup V) + \nu(U \cap V) = \nu U + \nu V \qquad \text{for all opens } U \text{ and } V \text{ (modularity).}$$

(3) Bigger sets are mapped to bigger numbers: if $U \subseteq V$, then $\nu U \leq \nu V$ (monotonicity).

Most often, we consider *Scott continuous valuations* which enjoy the additional property $\nu(\bigcup_{i\in I} V_i) = \bigsqcup_{i\in I} \nu V_i$ for every directed family $(V_i)_{i\in I}$ of opens V_i of X.

Some authors [8, 9] write *evaluations* instead of valuations, and some authors immediately require Scott continuity.

The concept of valuations has some similarity with the concept of *measures*. Borel measures are defined for all Borel sets of a space, and every open set is a Borel set. Hence, every measure can be restricted to a valuation. This valuation is not Scott continuous in general, since measures have to satisfy a weaker property where only countable directed

families $(V_i)_{i \in I}$ are considered. On the other hand, a valuation cannot always be extended to a measure.

Previously, valuations were already used in the following contexts:

- The *probabilistic power domain* of Jones and Plotkin [8, 9] over some dcpo X is the dcpo of Scott continuous valuations on X which are bounded by 1. It is used to model the semantics of probabilistic programs. If the semantics of a program p is ν, this means that for every open set U, the number νU is the probability that the result of running p is in U. The difference $1 - \nu(X)$ is the probability that running p yields no result at all, i.e., does not terminate.
- In [6], the author presented a *lower bag domain* as an analogue to the lower power domain, but without idempotence of addition. It can be used to specify a bag semantics for non-deterministic programs which takes multiplicities of results into account. One possible description of the lower bag domain consists of Scott continuous integer-valued valuations.
- In [3], Edalat used Scott continuous valuations to obtain a domain-theoretic treatment of measures and Riemann-like integrals. In [1], he connected dynamical systems and fractals with domain theory. The probabilistic power domain, i.e., the collection of Scott continuous valuations bounded by 1, plays a major role in this connection which leads to better algorithms for fractal image generation [2].

Because of these applications, we feel that the concept of valuations deserves further interest.

In this paper, we investigate valuations in a topological setting: the set $\mathsf{V}X$ of Scott continuous valuations on a space X is made into a topological space, whose structure and properties are studied. Thus, we present some background information about dcpo's and topological spaces in Section 2.

In Section 3, valuations are defined formally. In addition to Scott continuity, we introduce *point continuity* as a possible property of valuations which is stronger than Scott continuity. Some special valuations are defined, e.g., the *point valuations* $\widehat{x}$ which map open sets containing x to 1 and all other open sets to 0. We also consider some operations on valuations such as addition, multiplication by a constant from $\overline{\mathbf{R}}_+$, and restriction to an open set.

In Section 4, we study the relationship between point continuous and Scott continuous valuations. In particular, we prove that for continuous dcpo's, all Scott continuous valuations are point continuous. We also introduce *finite valuations* which are finite linear combinations of point valuations. Their representation as linear combination is unique if X is a $\mathcal{T}_0$-space.

In Section 5, we define the space $\mathsf{V}X$ of Scott continuous valuations on X, and the subspaces of point continuous $(\mathsf{V}_{\mathrm{p}}\, X)$ and finite valuations $(\mathsf{V}_{\mathrm{f}}\, X)$. We also prove one of the main results of this paper: the space $\mathsf{V}_{\mathrm{p}}\, X$ is the soberification of $\mathsf{V}_{\mathrm{f}}\, X$.

The other main results are shown in Section 6. For a $\mathcal{T}_0$-space X, the space $\mathsf{V}_{\mathrm{f}}\, X$ of finite valuations is the free locally convex $\mathcal{T}_0$-cone over X, and the space $\mathsf{V}_{\mathrm{p}}\, X$ of point continuous valuations is the free locally convex sober cone over X. Here, a $\mathcal{T}_0$-cone is an $\mathbf{R}_+$-module with a $\mathcal{T}_0$-topology such that addition and multiplication are continuous, and a sober cone is a $\mathcal{T}_0$-cone with sober topology. We have not found a universal property for V.

Nevertheless, the universal property of V_p suffices to find a novel definition of integration $\int_X : [X \to \overline{\mathbf{R}}_+] \times \mathsf{V}X \to \overline{\mathbf{R}}_+$ of a real-valued function w.r.t. a Scott continuous valuation (see Section 7). This definition allows for elegant proofs of the properties of integration.

In Section 8, integration is used to prove that the spaces of valuations are isomorphic to the dual spaces of certain function spaces, namely

$$\mathsf{V}X \cong [[X \to \overline{\mathbf{R}}_+]_\mathrm{i} \xrightarrow{lin} \overline{\mathbf{R}}_+]_\mathrm{p} \quad \text{and} \quad \mathsf{V}_\mathrm{p}\, X \cong [[X \to \overline{\mathbf{R}}_+]_\mathrm{p} \xrightarrow{lin} \overline{\mathbf{R}}_+]_\mathrm{p},$$

where the index '$[\,]_\mathrm{p}$' means a function space with pointwise topology, and '$[\,]_\mathrm{i}$' a function space with Isbell topology.

2 Topological Background

The main purpose of this section is to introduce some notion and notation which may not be familiar to everybody since it does not occur in usual topological textbooks. In particular, we introduce dcpo's, d-spaces, sober spaces, spaces of opens, and function spaces. Then we present a characterization of the soberification of a space which seems to be novel. Finally, we briefly look at the space $\overline{\mathbf{R}}_+$ of positive real numbers.

2.1 Some Notation

Some Set-Theoretic Notation

We only mention some slightly non-standard notation. If A is a subset of a fixed set X, then the complement of A in X is written $\neg A$.

If $f : X \to Y$ is a function, then we denote the image of a set $A \subseteq X$ by $f^+A = \{fx \mid x \in A\}$, and the inverse image of a set $B \subseteq Y$ by $f^-B = \{x \in X \mid fx \in B\}$.

We use *λ-notation* for the definition of functions, e.g., $\lambda x.\, x + y$ is the function which maps the argument x into the result $x + y$, where y is a fixed constant.

Topological Spaces

We write the set of opens of a topological space X as ΩX, and denote the topological closure of a subset A of X by $\mathsf{cl}\, A$. For a subset A of a space X, let $\mathcal{O}(A) = \{O \in \Omega X \mid A \subseteq O\}$. We abbreviate $\mathcal{O}(\{x\})$ to $\mathcal{O}(x)$.

Preorders and Posets

A *preorder* $(X, \sqsubseteq)$ is a set X together with a reflexive and transitive relation '$\sqsubseteq$'. For a subset A of a preorder X, we define $\downarrow A = \{x \in X \mid \exists a \in A : x \sqsubseteq a\}$ and $\uparrow A = \{x \in X \mid \exists a \in A : a \sqsubseteq x\}$. We shall often abbreviate $\downarrow\{a\}$ by $\downarrow a$ and $\uparrow\{a\}$ by $\uparrow a$. The set A is *lower* if $A = \downarrow A$, and *upper* if $A = \uparrow A$.

A *poset* is a preorder with anti-symmetric relation '$\sqsubseteq$'. In a poset, the *least upper bound* or *join* of a subset A is denoted by $\bigsqcup A$ (if it exists).

Directed Sets and Dcpo's

A subset D of a poset is *directed* if it is non-empty, and for all x, y in D, there is z in D with $x, y \sqsubseteq z$. A *dcpo* is a poset where every directed set has a least upper bound. A function

$f : \mathbf{X} \to \mathbf{Y}$ between dcpo's $\mathbf{X}$ and $\mathbf{Y}$ is *Scott continuous* if for all directed subsets D of $\mathbf{X}$, $f(\bigsqcup D) = \bigsqcup f^+ D$ holds.

A subset O of a dcpo $\mathbf{X}$ is *Scott open* if it is upper, and for all directed sets D, $\bigsqcup D \in O$ implies $D \cap O \neq \emptyset$. With this definition, every dcpo becomes a topological space. A function $f : \mathbf{X} \to \mathbf{Y}$ between dcpo's is Scott continuous in the sense defined above iff it is topologically continuous [7, II.1.10]. *Continuous dcpo's* can be topologically characterized as follows: if x is in an open set U, then there are an open V and a point y such that $x \in V \subseteq \uparrow y \subseteq U$.

The Specialization Preorder

Every topological space can be preordered by defining $x \sqsubseteq x'$ iff every open set which contains x also contains x'. This is called the *specialization preorder* of the space. Every continuous function between two spaces is monotonic in their specialization preorders.

A space is a $\mathcal{T}_0$*-space* iff its specialization preorder is a partial order, i.e., anti-symmetric. It is a $\mathcal{T}_1$*-space* iff the preorder coincides with equality. In this paper, most spaces will be $\mathcal{T}_0$, but not $\mathcal{T}_1$. The specialization preorder of a dcpo with its Scott topology is the original order of the dcpo. Hence, every dcpo is a $\mathcal{T}_0$-space.

If we use order notions such as lower, upper, $\downarrow A$, and $\uparrow A$ in a topological space, this always refers to the specialization preorder. All open sets are upper sets, and all closed sets are lower sets. For finite F, $\mathsf{cl}\, F = \downarrow F$ holds.

Joint Continuity versus Separate Continuity

For topological spaces X, Y, and Z, a function $f : X \times Y \to Z$ is *jointly continuous* if it is continuous in the product topology of $X \times Y$. It is *separately continuous* if all the functions $f_x : Y \to Z$ with $f_x y = f(x, y)$ and $f^y : X \to Z$ with $f^y x = f(x, y)$ are continuous. Every jointly continuous function is separately continuous; the opposite implication does not hold in general. We consider joint continuity as the default case, i.e., when we say '$f : X \times Y \to Z$ is continuous', we mean jointly continuous.

2.2 Spaces of Opens and Function Spaces

Spaces of Open Sets

The set ΩX of open sets of a space X can be topologized in several different ways.

First, $(\Omega X, \subseteq)$ is a dcpo, whence it can be endowed with the *Scott topology*. We call the resulting space $\Omega_{\mathrm{s}} X$.

Second, ΩX can be given the *point topology* with subbase $\{\mathcal{O}(x) \mid x \in X\}$. A set $\mathcal{O}$ of opens is open in the point topology (*point open*) iff for every O in $\mathcal{O}$ there is a finite set F such that $O \in \mathcal{O}(F) \subseteq \mathcal{O}$. We call the resulting space $\Omega_{\mathrm{p}} X$.

Since every set $\mathcal{O}(x)$ is Scott open, the topology of $\Omega_{\mathrm{p}} X$ is contained in that of $\Omega_{\mathrm{s}} X$. The two spaces $\Omega_{\mathrm{s}} X$ and $\Omega_{\mathrm{p}} X$ have the same specialization preorder, namely subset inclusion.

The following two properties are easily proved:

Proposition 2.1 If $f : X \to Y$ is continuous, then $f^- : \Omega Y \to \Omega X$ is continuous from $\Omega_{\mathrm{s}} Y$ to $\Omega_{\mathrm{s}} X$ as well as from $\Omega_{\mathrm{p}} Y$ to $\Omega_{\mathrm{p}} X$.

Proposition 2.2 Binary intersection $\cap : \Omega X \times \Omega X \to \Omega X$ is separately continuous in $\Omega_{\mathrm{s}} X$, and jointly continuous in $\Omega_{\mathrm{p}} X$.

The Pointwise Function Space

For two spaces X and Y, the *pointwise function space* $[X \to Y]_{\mathrm{p}}$ consists of all continuous functions $f : X \to Y$ with subbase $\{\langle x \to V\rangle \mid x \in X, V \in \Omega Y\}$ where $\langle x \to V\rangle = \{f : X \to Y \mid fx \in V\}$. It is a subspace of the product $\prod_{x \in X} Y$ of copies of Y. The function $\Omega_{\mathrm{p}} : [X \to Y]_{\mathrm{p}} \to [\Omega_{\mathrm{p}} Y \to \Omega_{\mathrm{p}} X]_{\mathrm{p}}$ with $\Omega_{\mathrm{p}} f = f^-$ is continuous.

The Isbell Function Space

For two spaces X and Y, the *Isbell function space* [12] $[X \to Y]_{\mathrm{i}}$ consists of all continuous functions $f : X \to Y$ with subbase $\{\langle \mathcal{U} \leftarrow V\rangle \mid \mathcal{U} \in \Omega(\Omega_{\mathrm{s}} X), V \in \Omega Y\}$ where $\langle \mathcal{U} \leftarrow V\rangle = \{f : X \to Y \mid \mathcal{U} \ni f^- V\}$. Since $\langle x \to V\rangle = \langle \mathcal{O}(x) \leftarrow V\rangle$, the topology of $[X \to Y]_{\mathrm{i}}$ includes that of $[X \to Y]_{\mathrm{p}}$. Both function spaces have the same specialization preorder, namely $f \sqsubseteq g$ iff $fx \sqsubseteq gx$ for all x in X.

If X and Y are $\mathcal{T}_0$-spaces, the function $\Omega_{\mathrm{s}} : [X \to Y] \to [\Omega_{\mathrm{s}} Y \to \Omega_{\mathrm{s}} X]$ with $\Omega_{\mathrm{s}} f = f^-$ is injective, no matter which topologies are chosen for the two function spaces. If $[\Omega_{\mathrm{s}} Y \to \Omega_{\mathrm{s}} X]$ is equipped with the pointwise topology, the Isbell topology on $[X \to Y]$ is characterized as the topology which makes Ω_{s} into an embedding. For, $\Omega_{\mathrm{s}} f \in \langle V \to \mathcal{U}\rangle$ iff $f^- V \in \mathcal{U}$ iff $f \in \langle \mathcal{U} \leftarrow V\rangle$. Hence, we obtain a continuous embedding $\Omega_{\mathrm{s}} : [X \to Y]_{\mathrm{i}} \hookrightarrow [\Omega_{\mathrm{s}} Y \to \Omega_{\mathrm{s}} X]_{\mathrm{p}}$.

2.3 D-Spaces and Sober Spaces

A space X is a *d-space* if its specialization preorder is a dcpo and all open sets of X are Scott open w.r.t. this dcpo. (These spaces are called *monotone convergence spaces* in [5, Def. II.3.9]). A continuous function $f : X \to Y$ between two d-spaces is Scott continuous. Every dcpo with its Scott topology is a d-space, and every $\mathcal{T}_1$-space is a d-space. All d-spaces are $\mathcal{T}_0$-spaces.

A closed subset C of a space X is *irreducible* if whenever $C \subseteq \bigcup_{i \in I} C_i$ for some finite family $(C_i)_{i \in I}$ of closed sets, then $C \subseteq C_i$ for some i in I. A space X is *sober* if all irreducible closed subsets of X are the closure $\mathsf{cl}\{x\}$ of a unique point x of X [5, page 79]. Every Hausdorff space is sober. Every sober space is a d-space [5, II.3.17]. (The proof relies on the fact that closures of directed sets are irreducible.)

The properties sober and d-space are preserved by products and $\mathcal{T}_0$-equalizer subspaces. A $\mathcal{T}_0$-equalizer subspace of a space X is a subspace $\{x \in X \mid fx = gx\}$ where $f, g : X \to Y$ are continuous and Y is $\mathcal{T}_0$. If Y is sober or a d-space, then so is $[X \to Y]_{\mathrm{p}}$ (no matter what X is). If Y is a d-space, then so is $[X \to Y]_{\mathrm{i}}$. The proofs of the statements about sobriety are contained in Chapter 5 of [13], following ideas of the author. The proofs of the d-space properties can be performed in a similar fashion.

2.4 Soberification

Every space X has a *soberification* $\overline{X}$ which is characterized by the following property:

Theorem 2.3 X can be embedded into $\overline{X}$, and every continuous $f : X \to Z$ to a sober space Z has a unique continuous extension $\overline{f} : \overline{X} \to Z$.

There are several known ways to construct the soberification, which we do not consider here [5, V.4.7,V.4.9]. Instead, we are interested in a criterion which tells us whether a given sober space Y is (homeomorphic to) the soberification of a subspace X. Since this criterion seems to be novel, we include its derivation.

If X is a subspace of a space Y, the function $\lambda V.X \cap V$ is a surjective frame homomorphism from ΩY to ΩX. A sober space Y is the soberification of X iff this function is a frame isomorphism. This is the starting point for the following criterion:

Theorem 2.4 For a subspace X of a sober space Y, the following are equivalent:

(1) Y is the soberification of X.

(2) For all opens U and V of Y, $X \cap U \subseteq V$ implies $U \subseteq V$.

(3) For every y in Y and O in ΩY with $y \in O$, there is some x in X with $x \in O$ and $x \sqsubseteq y$.

Proof: By the discussion above, (1) holds if $\lambda O.\, X \cap O$ is a frame isomorphism from ΩY to ΩX. Since it is a surjective homomorphism anyway, this is equivalent to $\lambda O.\, X \cap O$ being an order embedding, i.e., $X \cap U \subseteq X \cap V \Rightarrow U \subseteq V$. This implication is equivalent to (2).

Assume (3) and $X \cap U \subseteq V$. For u in U, there is x in $X \cap U \subseteq V$ with $x \sqsubseteq u$. Since V is upper, u in V follows.

Assume (2) and let $y \in O$. If $x \not\sqsubseteq y$ held for all x in $X \cap O$, then $X \cap O \subseteq \neg{\downarrow}y$, whence $O \subseteq \neg{\downarrow}y$ by (2). Then $y \in O$ would imply $y \in \neg{\downarrow}y$ — a contradiction. Thus there is x in $X \cap O$ with $x \sqsubseteq y$. □

In the sequel, we present some consequences of the theorem.

Proposition 2.5 Let Y be the soberification of its subspace X, and let Z be a $\mathcal{T}_0$-space. Two continuous functions $f, g : Y \to Z$ with $fx = gx$ for all x in X are equal.

Proof: Let y in Y. We prove $fy \sqsubseteq gy$. If $fy \in V$ open, then $y \in f^{-}V$. By Theorem 2.4 (3), there is x in X such that $x \sqsubseteq y$ and $x \in f^{-}V$. Then $gy \sqsupseteq gx = fx \in V$. Similarly, $gy \sqsubseteq fy$ is shown, whence $fy = gy$ since Z is $\mathcal{T}_0$. □

Soberification commutes with binary products:

Proposition 2.6 If X is the soberification of $A \subseteq X$ and Y is the soberification of $B \subseteq Y$, then $X \times Y$ is the soberification of $A \times B$.

Proof: Let (x, y) be in an open set W of $X \times Y$. Then there are open sets U of X and V of Y such that $(x, y) \in U \times V \subseteq W$. By hypothesis, there is a in A with $a \sqsubseteq x$ and $a \in U$, and b in B with $b \sqsubseteq y$ and $b \in V$. Thus, $(a, b) \sqsubseteq (x, y)$ and $(a, b) \in U \times V \subseteq W$. □

2.5 Numbers

Let $\mathbf{R}_+$ be the set of positive real numbers including 0, but without ∞, and let $\overline{\mathbf{R}}_+$ be $\mathbf{R}_+$ together with ∞. Arithmetic is extended to $\overline{\mathbf{R}}_+$ by $x + \infty = \infty + x = \infty$ for all x, $x \cdot \infty = \infty \cdot x = \infty$ for all $x \neq 0$, and $0 \cdot \infty = \infty \cdot 0 = 0$. Subtraction $x - y$ is only defined if $x \geq y$ and $x \neq \infty$.

The set $\overline{\mathbf{R}}_+$ is ordered in the standard way, which yields a dcpo. It is given the Scott topology. Hence, the open sets of $\overline{\mathbf{R}}_+$ are $\emptyset$, $\overline{\mathbf{R}}_+$ itself, and all the sets $\{x \in \overline{\mathbf{R}}_+ \mid x > r\}$ for

fixed numbers $r < \infty$. This space is sober. Addition and multiplication as defined above are continuous.

Sometimes, we shall need the split lemma for real numbers.

Lemma 2.7 (Split Lemma)
Let $(r_i)_{i\in I}$ and $(s_j)_{j\in J}$ be two families of members of $\mathbf{R}_+$, where the index sets I and J are finite, and let $R \subseteq I \times J$ be a relation. For $T \subseteq I$, we write $R^+(T)$ for $\{j \in J \mid \exists i \in T : (i,j) \in R\}$.

If for all $T \subseteq I$, $\sum_{i\in T} r_i \leq \sum_{j\in R^+(T)} s_j$ holds,

then there are numbers t_{ij} in $\mathbf{R}_+$ for i in I and j in J with

(1) $\sum_{j\in J} t_{ij} = r_i$ for all i in I,

(2) $\sum_{i\in I} t_{ij} \leq s_j$ for all j in J,

(3) if $t_{ij} > 0$, then $(i,j) \in R$.

Proof: This is essentially the proof of the Splitting Lemma 4.10 of [8] or Lemma 9.2 of [9]. The Max-Flow Min-Cut Theorem 5.1 of [4] is applied to a graph with nodes $\bot$ (source), i in I, j in J, and $\top$ (sink); the index sets I and J are assumed to be disjoint. There are edges from $\bot$ to i with capacities r_i, from i to j with 'large' capacity C if $(i,j) \in R$ and 0 otherwise, and from j to $\top$ with capacities s_j, where C is a constant which is bigger than the sums of all occurring numbers. The remainder of the proof is in analogy to [8, 9] and thus omitted.□

In contrast to addition, subtraction is not continuous since it is not even monotonic. Nevertheless, the following lemma holds.

Lemma 2.8 Let Y be a topological space with a separately continuous binary meet operation $\sqcap : Y \times Y \to Y$ (w.r.t. the specialization preorder), and let $f, g : Y \to \mathbf{R}_+$ be two functions with $f \sqsupseteq g$. If f is continuous, and both g and $f - g$ are monotonic, then $f - g$ is continuous.

Proof: For r in $\mathbf{R}_+$, we have to show that $W = \{y \in Y \mid (f-g)(y) > r\}$ is open. Let y be in W, i.e., $fy > r + gy$. By continuity of f, $U = \{a \in Y \mid fa > r + gy\}$ is open. By hypothesis, $h : Y \to Y$ with $hb = b \sqcap y$ is continuous. Hence, $V = h^- U$ is open. Since $hy = y \in U$, y is in V. We show $V \subseteq W$. Let b be in V.

$$\begin{array}{lcll} (f-g)(b) & \geq & (f-g)(b \sqcap y) & (f-g \text{ is monotonic}) \\ & = & f(b \sqcap y) - g(b \sqcap y) & \\ & > & r + gy - g(b \sqcap y) & (b \sqcap y = hb \text{ is in } U) \\ & \geq & r & (g \text{ is monotonic}) \quad \square \end{array}$$

This result is a bit surprising since g need not even be continuous. Note that the lemma applies in particular to the spaces $Y = \Omega_s X$ and $Y = \Omega_p X$ because of Prop. 2.2.

3 Valuations

In this section, we define valuations and their potential continuity properties. Then, some operations on valuations are introduced, e.g., addition of two valuations, multiplication by a real number, restriction and corestriction to an open set.

3.1 Definition and Continuity Properties

A *valuation* on a topological space X is a function $\nu : \Omega X \to \overline{\mathbf{R}}_+$ with the following properties:

- $\nu\emptyset = 0$ (strictness).
- $\nu(U \cup V) + \nu(U \cap V) = \nu U + \nu V$ for all opens U and V (modularity).
- $\nu U \leq \nu V$ for all opens U and V with $U \subseteq V$ (monotonicity).

Valuations are partially ordered by defining $\nu \sqsubseteq \nu'$ iff $\nu O \leq \nu' O$ for all O in ΩX. A valuation ν is *bounded* if $\nu(X) < \infty$.

Mostly, we shall consider valuations with an additional continuity condition. There are several such conditions according to which topology is chosen for ΩX.

(1) A valuation ν is *Scott continuous* iff $\nu : \Omega_{\mathrm{s}} X \to \overline{\mathbf{R}}_+$ is continuous. Equivalently, for every directed family $(V_i)_{i \in I}$ of opens, $\nu(\bigcup_{i \in I} V_i) = \bigsqcup_{i \in I} \nu V_i$ holds.

Since every Scott continuous function is monotonic, the condition of monotonicity in the definition of valuations becomes redundant once we consider Scott continuous valuations.

(2) A valuation ν is *point continuous* iff $\nu : \Omega_{\mathrm{p}} X \to \overline{\mathbf{R}}_+$ is continuous. Equivalently, $\nu^{-}\{s \in \overline{\mathbf{R}}_+ \mid s > r\}$ is point open for every r in $\mathbf{R}_+$, or: for every open O and number r in $\mathbf{R}_+$ with $\nu O > r$, there is some finite $F \subseteq O$ such that $F \subseteq O'$ implies $\nu O' > r$.

Since the topology of $\Omega_{\mathrm{p}} X$ is a subset of that of $\Omega_{\mathrm{s}} X$, we obtain:

Proposition 3.1 Every point continuous valuation is Scott continuous.

While many authors [8, 10, 13] have considered Scott continuous valuations, the notion of point continuity seems to be new.

3.2 Operations on Valuations

In this subsection, we present some basic ways to obtain valuations from other valuations or from scratch.

(1) The *zero function* $\lambda U.\, 0$, which maps every open set to 0, is a bounded valuation. As a constant function, it has all continuity properties you like; thus, it is point continuous.

(2) For every point x of X, there is a bounded valuation $\widehat{x}$, where $\widehat{x}(U)$ is 1 if $x \in U$, and 0 otherwise. Valuations of the form $\widehat{x}$ are called *point valuations*. Since $x \in U$ and $x \sqsubseteq y$ implies $y \in U$, $x \sqsubseteq y$ implies $\widehat{x} \sqsubseteq \widehat{y}$.

Every point valuation is point continuous. For, if $\widehat{x}(U) > r$, then $r < 1$ and $x \in U$. Choose $\{x\}$ as the finite set F in the characterization of point continuity.

(3) If r is a constant from $\overline{\mathbf{R}}_+$ and ν is a valuation, then $r \cdot \nu$ is a valuation, where $(r \cdot \nu)(U) = r \cdot \nu U$. If $r < \infty$ and ν is bounded, then $r \cdot \nu$ is bounded.

Since $\lambda s.\, r \cdot s : \overline{\mathbf{R}}_+ \to \overline{\mathbf{R}}_+$ is continuous and compositions of continuous functions are continuous, we obtain:

- If ν is Scott continuous / point continuous, then so is $r \cdot \nu$.

(4) If ν_1 and ν_2 are two valuations on X, then so is $\nu_1 + \nu_2$, where $(\nu_1 + \nu_2)(U) = \nu_1 U + \nu_2 U$. If both ν_1 and ν_2 are bounded, then so is $\nu_1 + \nu_2$.

Since $+ : \overline{\mathbf{R}}_+ \times \overline{\mathbf{R}}_+ \to \overline{\mathbf{R}}_+$ is continuous and $\nu_1 + \nu_2 = (+) \circ (\nu_1 \times \nu_2)$, we obtain:

- If ν_1 and ν_2 are Scott continuous / point continuous, then so is $\nu_1 + \nu_2$.

(5) Every directed family $(\nu_i)_{i \in I}$ of valuations has a least upper bound, namely the valuation $\bigsqcup_{i \in I} \nu_i$, which is defined by $(\bigsqcup_{i \in I} \nu_i)(U) = \bigsqcup_{i \in I} \nu_i U$. Even if all ν_i are bounded, $\bigsqcup_{i \in I} \nu_i$ may be unbounded.

For all open sets W of $\overline{\mathbf{R}}_+$, $(\bigsqcup_{i \in I} \nu_i)(U)$ is in W iff $\nu_i(U)$ in W for some i in I. Hence, $(\bigsqcup_{i \in I} \nu_i)^- W = \bigcup_{i \in I} \nu_i^- W$. Thus we obtain:

- If all ν_i are Scott continuous / point continuous, then so is $\bigsqcup_{i \in I} \nu_i$.

(6) If $f : X \to Y$ is continuous, then every valuation ν on X induces a valuation $\nu \circ f^-$ on Y. This operation maps point valuations to point valuations since $\widehat{x} \circ f^- = \widehat{fx}$. The valuation $\nu \circ f^-$ is bounded iff ν is bounded. Using Prop. 2.1, we obtain:

- If ν is Scott continuous / point continuous, then so is $\nu \circ f^-$.

(7) Let ν_1 and ν_2 be valuations on X, where ν_1 is bounded and $\nu_1 \sqsupseteq \nu_2$ holds. Then $\nu_1 - \nu_2$ with $(\nu_1 - \nu_2)(U) = \nu_1 U - \nu_2 U$ is a bounded strict modular function from ΩX to $\overline{\mathbf{R}}_+$. We require that ν_1 is bounded to avoid differences involving ∞. The condition $\nu_1 \sqsupseteq \nu_2$ is needed to ensure that $\nu_1 - \nu_2$ yields values in $\overline{\mathbf{R}}_+$.

Even if ν_1 and ν_2 are monotonic, the difference $\nu_1 - \nu_2$ may not be monotonic. On the other hand, monotonicity of the difference is sufficient to derive stronger continuity results.

Proposition 3.2 Let ν_1 and ν_2 be valuations on X, where ν_1 is bounded, $\nu_1 \sqsupseteq \nu_2$ holds, and $\nu_1 - \nu_2$ is monotonic. Then $\nu_1 - \nu_2$ is a valuation. If ν_1 is Scott continuous / point continuous, then so is $\nu_1 - \nu_2$.

Proof: The continuity properties follow from Lemma 2.8 with Prop. 2.2. □

Interestingly, this holds without requiring the corresponding kind of continuity for ν_2.

3.3 Restriction and Corestriction of Valuations

Let ν be a valuation on X, and let W be an open set of X. The *restriction* $\nu|_W$ of ν to W is defined by $\nu|_W(U) = \nu(W \cap U)$ for every U in ΩX. This is again a valuation on X. It is bounded iff $\nu(W) < \infty$. This holds in particular if ν is bounded.

By Prop. 2.2, the function $\lambda U.\, W \cap U : \Omega X \to \Omega X$ is continuous in $\Omega_{\mathrm{s}} X$ as well as in $\Omega_{\mathrm{p}} X$. Thus, we obtain:

- If ν is Scott continuous / point continuous, then so is $\nu|_W$.

Let ν be a *bounded* valuation on X, and W an open set of X. The *corestriction* $\nu|^W$ of ν to W is defined by $\nu|^W = \nu - \nu|_W$. Hence, $\nu|^W(U) = \nu U - \nu(U \cap W)$. By modularity, this is equal to $\nu(U \cup W) - \nu W$.

We require ν to be bounded in order to avoid problems with undefined differences. Monotonicity of ν guarantees $\nu \sqsupseteq \nu|_W$. Because of $\nu|^W(U) = \nu(U \cup W) - \nu W$, monotonicity of $\nu|^W$ follows from monotonicity of ν. Using Prop. 2.8, we obtain:

- If ν is Scott continuous / point continuous, then so is $\nu|^W$.

It is not difficult to prove some more properties of restriction and corestriction:

Proposition 3.3

$W \cap U = W \cap V \Rightarrow \nu|_W U = \nu|_W V$	$\neg W \cap U = \neg W \cap V \Rightarrow \nu\|^W U = \nu\|^W V$
$(\nu|_U)|_V = (\nu|_V)|_U = \nu|_{U \cap V}$	$(\nu\|^U)\|^V = (\nu\|^V)\|^U = \nu\|^{U \cup V}$
$(\nu|_U)\|^V = (\nu\|^V)|_U$	$\nu = \nu|_W + \nu\|^W$

We shall use the abbreviation $\nu|_U^V$ for $\nu|_U\|^V = \nu\|^V|_U$.

4 Classes of Valuations

In this section, we study the relationship among some classes of valuations. We already know the classes of Scott continuous, point continuous, and bounded valuations ($\nu X < \infty$).

4.1 Valuations on Locally Finitary Spaces

There is a class of spaces where every Scott continuous valuation is point continuous.

A subset F of a space X is *finitary* iff $F = \uparrow E$ for some finite set E. The space X is *locally finitary* iff for every point x in X and open U of X with $x \in U$, there are a finitary set F and an open V such that $x \in V \subseteq F \subseteq U$.

Since finitary sets are compact, every locally finitary space is locally compact. Every continuous dcpo (with its Scott topology) is locally finitary. A locally finitary $\mathcal{T}_1$-space is discrete.

Theorem 4.1

Every Scott continuous valuation on a locally finitary space is point continuous.

Proof: Let ν be a Scott continuous valuation and assume $\nu U > r$ for some open U and $r < \infty$. Let $\mathcal{V}$ be the set of all open sets V such that there is a finitary set F with $V \subseteq F \subseteq U$. Since the union of two open / finitary sets is again so, $\mathcal{V}$ is directed. Because of local finitariness, the union of $\mathcal{V}$ is U. Since ν is Scott continuous, there is some V in $\mathcal{V}$ such that $\nu V > r$. Let E be a finite set with $V \subseteq \uparrow E \subseteq U$. If $E \subseteq O'$, then $V \subseteq O'$, whence $\nu O' > r$. □

In general, the notions of point continuity and Scott continuity differ. For instance, the length or Lebesgue measure on the unit interval of the reals with the standard Hausdorff topology induces a bounded Scott continuous valuation which is not point continuous.

4.2 Approximation by Bounded Valuations

Theorem 4.2 Every Scott continuous / point continuous valuation can be obtained as a directed join of bounded Scott continuous / point continuous valuations.

Proof: Let ν be a Scott continuous valuation. Let $\Phi(\nu)$ be the set of all opens O with $\nu O < \infty$, and let C be $\neg \bigcup \Phi(\nu)$. For V in $\Phi(\nu)$, $F \subseteq_{fin} C$, and $n \in \mathbf{N}_0$, let $\nu_{V,F,n} = \nu|_V + \sum_{x \in F} n \cdot \widehat{x}$. This is a bounded valuation. If ν is point continuous, then so is $\nu_{V,F,n}$. The set $D = \{\nu_{V,F,n} \mid V \in \Phi(\nu), F \subseteq_{fin} C, n \in \mathbf{N}_0\}$ is directed. Let $\nu' = \bigsqcup D$. We claim $\nu' = \nu$. If there is an x in $O \cap C$, then $\nu' O \geq (n \cdot \widehat{x})(O) = n$ for all n in $\mathbf{N}_0$, whence $\nu' O = \infty$. On the other hand, if νO were finite, then $O \subseteq \bigcup \Phi(\nu) = \neg C$ in contradiction to $O \cap C \neq \emptyset$.

If $O \subseteq \bigcup \Phi(\nu)$, then $O = \bigcup\{O \cap V \mid V \in \Phi(\nu)\}$. By modularity, $\Phi(\nu)$ is directed.

$$\begin{array}{lcll} \nu O & = & \bigsqcup_{V \in \Phi(\nu)} \nu(O \cap V) & (\nu \text{ is Scott continuous}) \\ & = & \bigsqcup_{V \in \Phi(\nu)} \nu|_V(O) & \\ & = & \bigsqcup_{\nu^* \in D} \nu^*(O) & (\nu_{V,F,n}(O) = \nu|_V(O)) \\ & = & \nu' O & \end{array}$$ □

4.3 Finite Valuations

A valuation φ on a space X is *finite* if it is a finite linear combination of point valuations, i.e., if $\varphi = \sum_{x \in F} r_x \cdot \widehat{x}$ where F is a finite subset of X and $0 < r_x < \infty$ for all x in F. This representation is unique provided that the underlying space X is $\mathcal{T}_0$.

Theorem 4.3 (Unique Representation Theorem)
Let X be a $\mathcal{T}_0$-space. If $\sum_{x \in F} r_x \cdot \widehat{x} = \sum_{x \in G} s_x \cdot \widehat{x}$ where F and G are finite subsets of X, and $0 < r_x < \infty$ for all x in F and $0 < s_x < \infty$ for all x in G, then $F = G$ and $r_x = s_x$ for all x in F.

Proof: This is Lemma 2.1 in [10]. We sketch Kirch's proof. Let a be maximal in $F \cup G$, and let U be an open set with $U \cap (F \cup G) = \{a\}$. If a is in G, then $\sum_{x \in F} r_x \cdot \widehat{x}(U) = s_a$, whence a is in F as well, and $r_a = s_a$ holds. With induction, the claim follows. □

The $\mathcal{T}_0$ property is really needed. Consider the space $X = \{a, b\}$ where $\emptyset$ and X are the only open sets. The finite valuations $1 \cdot \widehat{a}$ and $1 \cdot \widehat{b}$ both map $\emptyset$ to 0 and X to 1. Hence, they are identical despite their different representation.

The theorem also relies on the fact that finite valuations have finite coefficients. Consider for instance the Sierpinski space $X = \{\bot, \top\}$ where $\emptyset$, $\{\top\}$, and X are open. The valuations $\infty \cdot \widehat{\top}$, $1 \cdot \widehat{\bot} + \infty \cdot \widehat{\top}$, and $\infty \cdot \widehat{\bot} + \infty \cdot \widehat{\top}$ all map $\emptyset$ to 0, and $\{\top\}$ and X to ∞.

5 Spaces of Valuations

In this section, we define various topological spaces of valuations and study their relationship.

Let $\mathsf{V}X$ be the set of all *Scott continuous valuations* on X, $\mathsf{V}_{\mathrm{p}}X$ the set of all *point continuous* valuations, and $\mathsf{V}_{\mathrm{f}}X$ the set of *finite* valuations. We topologize these sets as subspaces of the pointwise function space $[\Omega_{\mathrm{s}}X \to \overline{\mathbf{R}}_+]_{\mathrm{p}}$. Thus, the topology of $\mathsf{V}X$ is generated by the subbasic opens $\langle U > r\rangle = \{\nu \in \mathsf{V}X \mid \nu U > r\}$ where U ranges over the opens of X and r ranges over $\overline{\mathbf{R}}_+$ with $0 < r < \infty$. The order defined by this topology is $\nu \sqsubseteq \nu'$ iff $\nu O \leq \nu' O$ for all opens O.

In general, continuous operations on a space Y can be lifted to continuous operations on pointwise function spaces $[X \to Y]_{\mathrm{p}}$. Hence, addition $+ : \mathsf{V}X \times \mathsf{V}X \to \mathsf{V}X$ and multiplication $\cdot : \overline{\mathbf{R}}_+ \times \mathsf{V}X \to \mathsf{V}X$ are continuous.

The function $\mathsf{s} : X \to \mathsf{V}X$ with $\mathsf{s}x = \widehat{x}$ is continuous. For, $\mathsf{s}^-\langle U > r\rangle = U$ if $r < 1$, and $= \emptyset$ otherwise. This also shows that s^- is surjective, whence s is a topological embedding for $\mathcal{T}_0$-spaces X.

Every continuous function $f : X \to Y$ induces a function $\mathsf{V}f : \mathsf{V}X \to \mathsf{V}Y$ where $\mathsf{V}f\,(\nu) = \nu \circ f^-$. The function $\mathsf{V}f$ is linear w.r.t. addition and scalar multiplication of $\mathsf{V}X$. It

is continuous since $\mathsf{V}f(\nu) \in \langle V > r\rangle$ iff $\nu \in \langle f^{-}V > r\rangle$. Thus, $(\mathsf{V}f)^{-}\langle V > r\rangle = \langle f^{-}V > r\rangle$. From this equation, we see that $(\mathsf{V}f)^{-}$ is surjective if f^{-} is surjective. Hence, $\mathsf{V}f$ is a topological embedding if f is an embedding. The operation V is functorial, i.e., $\mathsf{V}\,\mathsf{id} = \mathsf{id}$ and $\mathsf{V}(g \circ f) = \mathsf{V}g \circ \mathsf{V}f$, and $\mathsf{V}f \circ \mathsf{s} = \mathsf{s} \circ f$ holds.

Let us now consider the topological properties of the various spaces of valuations.

Proposition 5.1

For every space X, the spaces $\mathsf{V}X$ and $\mathsf{V}_{\mathrm{p}}X$ are sober, and $\mathsf{V}_{\mathrm{f}}X$ is a $\mathcal{T}_0$-space.

Proof: The space $\mathsf{V}X$ is a subspace of the pointwise function space $F = [\Omega_{\mathrm{s}}X \to \overline{\mathbf{R}}_+]_{\mathrm{p}}$ which is sober as $\overline{\mathbf{R}}_+$ is sober. Since the elements of $\mathsf{V}X$ are characterized by continuous equations with values in $\overline{\mathbf{R}}_+$ (strictness and modularity), $\mathsf{V}X$ is a $\mathcal{T}_0$-equalizer subspace of F and thus sober. For $\mathsf{V}_{\mathrm{p}}X$, we start with $F' = [\Omega_{\mathrm{p}}X \to \overline{\mathbf{R}}_+]_{\mathrm{p}}$. The space $\mathsf{V}_{\mathrm{f}}X$ is $\mathcal{T}_0$ as a subspace of $\mathsf{V}_{\mathrm{p}}X$. □

Now, we come to one of the main results of this paper: For every space X, $\mathsf{V}_{\mathrm{p}}X$ is the soberification of $\mathsf{V}_{\mathrm{f}}X$. By Theorem 2.4 (3), we have to prove: For every point continuous valuation ν and open set $\mathcal{O}$ of $\mathsf{V}_{\mathrm{p}}X$ with ν in $\mathcal{O}$, there is a finite valuation $\varphi \sqsubseteq \nu$ with φ in $\mathcal{O}$. The proof of this statement is structured into several parts. The results of these parts are presented as auxiliary lemmas.

The first lemma contains the step from arbitrary to bounded valuations.

Lemma 5.2 For every point continuous valuation ν and open set $\mathcal{O}$ of $\mathsf{V}_{\mathrm{p}}X$ with ν in $\mathcal{O}$, there is a bounded point continuous valuation $\nu' \sqsubseteq \nu$ with ν' in $\mathcal{O}$.

Proof: By Theorem 4.2, ν is a directed join of bounded point continuous valuations. In a sober space such as $\mathsf{V}_{\mathrm{p}}X$, every open set is Scott open. Hence, there is some bounded point continuous valuation $\nu' \sqsubseteq \nu$ with ν' in $\mathcal{O}$. □

The next lemma deals with the step from bounded to finite valuations in a special case.

Lemma 5.3 Let ν be a bounded point continuous valuation with $\nu W > r$ for some open set W and real number r. Then there is a finite valuation $\varphi \sqsubseteq \nu$ with $\varphi W > r$.

Proof: Choose a real number r' such that $\nu W > r' > r$. Since ν is point continuous, there is a finite set $F = \{x_1, \ldots, x_n\} \subseteq W$ such that $F \subseteq O$ implies $\nu O > r'$ for all open sets O. Since $\nu\emptyset = 0$, n cannot be 0. Let $\epsilon = \frac{r'-r}{n}$. For every point x_i, choose an open set U_i such that $x_i \in U_i$ and $\nu U_i < \epsilon + \sqcap_{O \ni x_i} \nu O$. We also need the unions $V_i = \bigcup_{j=1}^{i} U_i$ for $0 \le i \le n$; in particular, $V_0 = \emptyset$.

Using restriction and corestriction, for every i with $1 \le i \le n$, $\nu|^{V_{i-1}} = \nu|^{V_{i-1}}|_{U_i} + \nu|^{V_{i-1}}|^{U_i} = \nu|_{U_i}^{V_{i-1}} + \nu|^{V_i}$. Let $\nu_i = \nu|_{U_i}^{V_{i-1}}$. Starting from $\nu = \nu|^{V_0}$, we obtain by iteration $\nu = (\sum_{i=1}^{n} \nu_i) + \nu|^{V_n}$. On the other hand, $\nu = \nu|_{V_n} + \nu|^{V_n}$ holds, whence $\sum_{i=1}^{n} \nu_i = \nu|_{V_n}$.

Let $a_i = \nu_i(X)$ and $b_i = 0 \sqcup (a_i - \epsilon)$. With these numbers, let $\varphi = \sum_{i=1}^{n} b_i \cdot \widehat{x_i}$.

To prove $\varphi W > r$, we first compute $\sum_{i=1}^{n} a_i = \sum_{i=1}^{n} \nu_i(X) = \nu|_{V_n}(X) = \nu(V_n) > r'$ since $F \subseteq V_n$. Since all x_i are in W, we obtain $\varphi W = \sum_{i=1}^{n} b_i \ge \sum_{i=1}^{n}(a_i - \epsilon) = \sum_{i=1}^{n} a_i - n \cdot \epsilon > r' - (r' - r) = r$.

To prove $\varphi \sqsubseteq \nu$, note that $\nu O \ge \sum_{i=1}^{n} \nu_i O \ge \sum_{i:x_i \in O} \nu_i O$ and $\varphi O = \sum_{i:x_i \in O} b_i$. Hence, it suffices to show $\nu_i O \ge b_i$ for all i with $x_i \in O$. By definition, $\nu_i O = \nu(O \cap U_i) - \nu(O \cap U_i \cap V_{i-1})$

holds. Since x_i in $O \cap U_i$, we obtain $\nu(O \cap U_i) > \nu U_i - \epsilon$ by the choice of U_i. Thus, $\nu_i O > \nu U_i - \epsilon - \nu(U_i \cap V_{i-1}) = \nu_i(X) - \epsilon = a_i - \epsilon$. Since $\nu_i O \geq 0$ also holds, $\nu_i O \geq b_i$ follows. □

The next lemma generalizes Lemma 5.3 from one open set W to any finite number of open sets O_i.

Lemma 5.4 Let ν be a bounded point continuous valuation with $\nu O_i > r_i$ for some open sets $O_1, \ldots, O_n$ and real numbers $r_1, \ldots, r_n$. Then there is a finite valuation $\varphi \sqsubseteq \nu$ with $\varphi O_i > r_i$ for all i with $1 \leq i \leq n$.

Proof: The valuation ν can be partitioned along O_1 into $\nu = \nu|_{O_1} + \nu|^{O_1}$. Both parts can be partitioned along O_2 into $\nu|_{O_1} = \nu|_{O_1 \cap O_2} + \nu|^{O_2}_{O_1}$ and $\nu|^{O_1} = \nu|^{O_1}_{O_2} + \nu|^{O_1 \cup O_2}$. Iterating this process, we obtain $\nu = \sum_T \nu_T$ where T ranges over the subsets of $I = \{1, \ldots, n\}$ and $\nu_T = \nu|^{V_T}_{U_T}$ with $U_T = \bigcap_{i \in T} O_i$ and $V_T = \bigcup_{i \in I \setminus T} O_i$. By construction, $\nu_T U_T = \nu_T X$ holds.
If i in T, then $U_T \subseteq O_i$, whence $\nu_T O_i = \nu_T X$. If i not in T, then $O_i \subseteq V_T$, whence $\nu_T O_i = 0$. Thus, $r_i < \nu O_i = \sum_T \nu_T O_i = \sum_{T \ni i} \nu_T X$. Choose some real number ρ with $0 < \rho < 1$ such that $r_i < \rho \cdot (\sum_{T \ni i} \nu_T X)$ still holds for all i. Let $q_T = \rho \cdot \nu_T X$. Then $\sum_{T \ni i} q_T > r_i$ for all i.
If $\nu_T X \neq 0$, then $\nu_T U_T = \nu_T X > q_T$. By Lemma 5.3, there is a finite valuation $\varphi_T \sqsubseteq \nu_T$ such that $\varphi_T U_T > q_T$. If $\nu_T X = 0$, then $q_T = 0$, and we set $\varphi_T = 0$. In both cases, we obtain a finite $\varphi_T \sqsubseteq \nu_T$ such that $\varphi_T U_T \geq q_T$.
Let $\varphi = \sum_T \varphi_T$. This is a finite valuation below ν. For all i, $\varphi O_i = \sum_T \varphi_T O_i \geq \sum_{T \ni i} \varphi_T U_T \geq \sum_{T \ni i} q_T > r_i$ holds. □

With these lemmas, we can now prove:

Theorem 5.5 For every space X, $\mathsf{V}_{\mathrm{p}} X$ is the soberification of $\mathsf{V}_{\mathrm{f}} X$.

Proof: Let ν be in $\mathsf{V}_{\mathrm{p}} X$ and $\mathcal{O}$ in $\Omega(\mathsf{V}_{\mathrm{p}} X)$ with $\nu \in \mathcal{O}$. By Lemma 5.2, there is a bounded $\nu' \sqsubseteq \nu$ with $\nu' \in \mathcal{O}$.
Using the subbase of $\mathsf{V}_{\mathrm{p}} X$, we have $\nu' \in \bigcap_{i \in I} \langle O_i > r_i \rangle \subseteq \mathcal{O}$ for some finite I, open sets O_i of X, and r_i in $\mathbf{R}_+$ with $0 < r_i < \infty$. From Lemma 5.4, we obtain a finite $\varphi \sqsubseteq \nu'$ with $\varphi \in \bigcap_{i \in I} \langle O_i > r_i \rangle \subseteq \mathcal{O}$. □

6 Universal Properties

In the sequel, we look for universal properties of the valuation spaces. We shall prove that $\mathsf{V}_{\mathrm{f}} X$ is the free locally convex $\mathcal{T}_0$-cone, and $\mathsf{V}_{\mathrm{p}} X$ is the free locally convex sober cone. We have not found a universal property for $\mathsf{V} X$.

6.1 Cones

A *cone* or $\mathbf{R}_+$*-module* is an algebraic structure $(M, +, 0, \cdot)$ where $+ : M \times M \to M$ is a commutative associative operation with neutral element $0 \in M$, and $\cdot : \mathbf{R}_+ \times M \to M$ is an operation satisfying the module (or vector space) axioms:

$$\begin{array}{rclcrcl} r \cdot 0 & = & 0 & & r \cdot (m_1 + m_2) & = & r \cdot m_1 + r \cdot m_2 \\ 0 \cdot m & = & 0 & & (r + s) \cdot m & = & r \cdot m + s \cdot m \\ 1 \cdot m & = & m & & (r \cdot s) \cdot m & = & r \cdot (s \cdot m) \end{array}$$

A *topological cone* is a cone with a topology such that '$+$' and '$\cdot$' are continuous if $\mathbf{R}_+$ is given the Scott topology. Often, we shall omit the word 'topological' if there is already a topological notion around such as 'sober'.

Homomorphisms between cones are linear functions, where linearity means $f(m + m') = fm + fm'$ and $f(r \cdot m) = r \cdot fm$ as usual. Homomorphisms between topological cones are continuous linear functions.

If $(M_i)_{i \in I}$ is a family of (topological) cones, then the product $\prod_{i \in I} M_i$ is a (topological) cone with $(m_i)_{i \in I} + (m'_i)_{i \in I} = (m_i + m'_i)_{i \in I}$, $0 = (0_i)_{i \in I}$, and $r \cdot (m_i)_{i \in I} = (r \cdot m_i)_{i \in I}$.

In every topological cone, 0 is the least element since continuous functions are monotonic, whence $0 = 0 \cdot m \sqsubseteq 1 \cdot m = m$ holds for all m in M. Thus, non-trivial topological cones cannot be $\mathcal{T}_1$ spaces.

Standard examples of topological cones are given by powers of $\mathbf{R}_+$ or $\overline{\mathbf{R}}_+$, and linear subspaces thereof. On the other hand, there are quite strange cones which have nothing to do with real numbers. Let $(L, \vee, \wedge)$ be a distributive lattice with least element F and greatest element T. Define $a + b = a \vee b$ for a and b in L, $0 = \mathsf{F}$, and for r in $\mathbf{R}_+$ and a in L, define $r \cdot a = \mathsf{F}$ if $r = 0$, and $= a$ otherwise. With these operations, L becomes a cone. If L is endowed with a topology which makes '$\vee$' and '$\cdot$' continuous, then L is a topological cone. A suitable topology is the Scott topology if L is a continuous lattice.

6.2 Uniqueness Properties

The notions introduced so far are sufficient to state the uniqueness parts of the universal properties of V_f and V_p.

Theorem 6.1 Let X be a topological space.

(1) Every linear function from $\mathsf{V}_\mathrm{f} X$ to some cone is uniquely determined by its values on point valuations.

(2) Every continuous linear function from $\mathsf{V}_\mathrm{p} X$ to a $\mathcal{T}_0$-cone is uniquely determined by its values on point valuations.

Proof: Part (1) is obvious since every finite valuation is a finite linear combination of point valuations. Part (2) follows from part (1) and Prop. 2.5, using the fact that $\mathsf{V}_\mathrm{p} X$ is the soberification of $\mathsf{V}_\mathrm{f} X$ (Theorem 5.5). □

Unfortunately, we do not know whether similar properties hold for $\mathsf{V}X$, the space of all Scott continuous valuations on X. We do not even know the answer for the special case that the target cone is $\overline{\mathbf{R}}_+$.

Problem 1 Are continuous linear functions from $\mathsf{V}X$ to $\overline{\mathbf{R}}_+$ uniquely determined by their values on point valuations?

If the answer to this problem is yes, then also the continuous linear functions from $\mathsf{V}X$ to $\mathsf{V}Y$ are uniquely determined by their values on point valuations. For, the functions $\lambda\nu.\, \nu V : \mathsf{V}Y \to \overline{\mathbf{R}}_+$ are continuous and linear for every open set V of Y.

For continuous dcpo's, the question in Problem 1 can be answered affirmatively. For, continuous dcpo's X are locally finitary, hence $\mathsf{V}_\mathrm{p} X$ and $\mathsf{V}X$ coincide by Theorem 4.1, and thus Theorem 6.1 (2) applies.

6.3 Convexity and Local Convexity

In order to formulate the universal properties for V_f and V_p, we need some more notions connected with cones.

A *convex combination* is a linear combination $r_1 \cdot x_1 + \cdots + r_n \cdot x_n$ whose coefficients sum up to 1. A subset S of a cone is *convex* if all convex combinations of points of S are back in S again. Intersections of convex sets are convex, hence every subset S of a cone has a least convex superset, the *convex hull* $\mathsf{con}\, S$. It is routine to show the following properties.

Proposition 6.2 Let $f : M \to M'$ be a linear map between two cones.

(1) If S is convex in M', then $f^- S$ is convex in M.

(2) $\mathsf{con}\, f^- S \subseteq f^-(\mathsf{con}\, S)$ holds for all $S \subseteq M$.

A topological cone is *locally convex* if whenever a point x is in an open set U, there is an open set V such that $x \in V \subseteq \mathsf{con}\, V \subseteq U$. It is easily verified that it suffices to consider open sets U from a subbase $\mathcal{S}$.

As long as only continuous dcpo's are considered, local convexity is not an issue.

Proposition 6.3
A topological cone whose underlying space is a continuous dcpo is locally convex.

Proof: Let x be in an open set U. By continuity, there are an open set V and a point y such that $x \in V \subseteq \uparrow y \subseteq U$. By monotonicity of addition and multiplication, $\uparrow y$ is convex. Hence, $x \in V \subseteq \mathsf{con}\, V \subseteq \uparrow y \subseteq U$. □

Thus, the topological cones $\mathbf{R}_+$ and $\overline{\mathbf{R}}_+$ are locally convex. Also, distributive lattices with the Scott topology are locally convex since convex combinations are finite non-empty joins, and open sets are upper sets. For the moment, we do not have any examples of cones which are not locally convex.

In the sequel, we present three properties of local convexity.

Proposition 6.4 (Products)
If all topological cones M_i are locally convex, then so is $\prod_{i \in I} M_i$.

Proof: The product $M = \prod_{i \in I} M_i$ has a subbase $\{\pi_i^- O \mid i \in I, O \in \Omega M_i\}$. If x in $\pi_i^- O$, then $\pi_i x \in O$. Since M_i is locally convex, there is an open V in M_i such that $\pi_i x \in V \subseteq \mathsf{con}\, V \subseteq O$, whence $x \in \pi_i^- V \subseteq \pi_i^-(\mathsf{con}\, V) \subseteq \pi_i^- O$. The projection π_i is linear, whence $\mathsf{con}\, \pi_i^- V \subseteq \pi_i^-(\mathsf{con}\, V)$ by Prop. 6.2. □

Proposition 6.5 (Linear subspaces) If M and M' are topological cones, $e : M \hookrightarrow M'$ is a linear topological embedding, and M' is locally convex, then so is M.

Proof: Let x in O for some open O of M. Since e is an embedding, $O = e^- U$ holds for some $U \in \Omega M'$. Thus ex is in U. Since M' is locally convex, there is an open V in M' such that $ex \in V \subseteq \mathsf{con}\, V \subseteq U$, whence $x \in e^- V \subseteq e^-(\mathsf{con}\, V) \subseteq e^- U$. Since e is linear, $\mathsf{con}\, e^- V \subseteq e^-(\mathsf{con}\, V)$ holds by Prop. 6.2. □

From the two propositions above, we may conclude that the spaces $\mathsf{V}X$, $\mathsf{V}_\mathrm{p} X$, and $\mathsf{V}_\mathrm{f} X$ are locally convex as linear subspaces of products of $\overline{\mathbf{R}}_+$.

Proposition 6.6 (Linear retracts)

If M and M' are topological cones, $e : M \to M'$ is continuous, and $r : M' \to M$ is linear and continuous with $r \circ e = \mathsf{id}$, then local convexity of M' implies local convexity of M.

Proof: If $x = r(ex)$ in U where U in ΩM, then $ex \in r^{-}U$. Because M' is locally convex, there is an open V in $\Omega M'$ such that $ex \in V \subseteq \mathsf{con}\, V \subseteq r^{-}U$. Hence, $x \in e^{-}V$. We claim $\mathsf{con}\,(e^{-}V) \subseteq U$.

We show that every convex combination $\sum t_i \cdot x_i$ of points x_i from $e^{-}V$ is in U. Since ex_i is in V, $\sum t_i \cdot ex_i$ is in $\mathsf{con}\, V \subseteq r^{-}U$. Thus, $r(\sum t_i \cdot ex_i) = \sum t_i \cdot x_i$ is in U. Here, linearity of r is used. □

6.4 A Universal Property for the Space of Finite Valuations

In this subsection, we present a universal property for the space $\mathsf{V_f}\, X$ of finite valuations.

Theorem 6.7 $\mathsf{V_f}\, X$ is the free locally convex cone over X in $\mathcal{T}_0$, the category of $\mathcal{T}_0$-spaces and continuous maps.

This means: for a $\mathcal{T}_0$-space X, $\mathsf{V_f}\, X$ is itself a locally convex $\mathcal{T}_0$-cone, and for every continuous function $f : X \to M$ from X to a locally convex $\mathcal{T}_0$-cone M, there is a unique continuous linear function $\bar{f} : \mathsf{V_f}\, X \to M$ with $\bar{f} \circ \mathsf{s} = f$, i.e., $\bar{f}(\widehat{x}) = fx$ for all x in X.

Proof: We already know that $\mathsf{V_f}\, X$ is a locally convex $\mathcal{T}_0$-cone. The uniqueness statement is given by Theorem 6.1. We still have to show existence of $\bar{f}$.

Every finite valuation φ can be written as $\sum_{x \in F} r_x \cdot \widehat{x}$ for some finite set F and some numbers r_x with $0 < r_x < \infty$. By Theorem 4.3, this representation is unique. Hence, $\bar{f}(\varphi) = \sum_{x \in F} r_x \cdot fx$ is a well-defined element of M. The function $\bar{f} : \mathsf{V_f}\, X \to M$ defined in this manner is obviously linear and satisfies $\bar{f} \circ \mathsf{s} = f$. The only remaining task is to prove continuity of $\bar{f}$. This turns out to be quite complex.

Let U be an open set of M, and assume $\bar{f}(\varphi) \in U$. Then $\sum_{x \in F} r_x \cdot fx$ is in U. Since addition and multiplication are continuous in M, there are open sets R_x of $\mathbf{R}_+$ and V_x of M such that $r_x \in R_x$, $fx \in V_x$, and whenever $s_x \in R_x$ and $m_x \in V_x$, then $\sum_{x \in F} s_x \cdot m_x \in U$.

Choose numbers $r'_x > 0$ such that $r_x > r'_x \in R_x$, and applying local convexity of M, choose open sets W_x of M such that $fx \in W_x \subseteq \mathsf{con}\, W_x \subseteq V_x$, and let $O_x = f^{-}W_x$. By continuity of f, the sets O_x are open sets of X with x in O_x for all x in F. For every non-empty $T \subseteq F$, $\varphi(\bigcup_{x \in T} O_x) \geq \sum_{x \in T} r_x > \sum_{x \in T} r'_x$ holds. Hence, φ is in $\mathcal{O} = \bigcap_{\emptyset \neq T \subseteq F} \langle \bigcup_{x \in T} O_x > \sum_{x \in T} r'_x \rangle$, which is an open set of $\mathsf{V_f}\, X$. We have to show that for every ψ in $\mathcal{O}$, $\bar{f}(\psi)$ is in U.

Let $\psi = \sum_{y \in G} s_y \cdot \widehat{y}$ be in $\mathcal{O}$. Let $R \subseteq F \times G$ be the relation given by $(x, y) \in R$ iff $O_x \ni y$, whence $R^{+}(T) = \{y \in G \mid \exists x \in T : y \in O_x\}$ for subsets T of I. Since ψ is in $\mathcal{O}$,

$$\sum_{y \in R^{+}(T)} s_j = \psi(\bigcup_{x \in T} O_x) > \sum_{x \in T} r'_x$$

holds for all non-empty subsets T of I. For $T = \emptyset$, both sides are zero. Thus, '$\geq$' instead of '$>$' holds for all subsets T of I. Applying the Split Lemma 2.7, we obtain numbers $t_{xy} \in \mathbf{R}_+$ for x in F and y in G such that

(1) $\sum_{y \in G} t_{xy} = r'_x$ for all x in F,
(2) $\sum_{x \in F} t_{xy} \leq s_y$ for all y in G,
(3) if $t_{xy} > 0$, then $y \in O_x$.

Let $\psi' = \sum_{x \in F} \sum_{y \in G} t_{xy} \cdot \widehat{y}$. Then

$$\bar{f}(\psi') = \sum_{y \in G} (\sum_{x \in F} t_{xy}) \cdot fy \sqsubseteq \sum_{y \in G} s_y \cdot fy = \bar{f}(\psi)$$

using monotonicity of addition and multiplication in M. The valuation ψ' may alternatively be written as

$$\psi' = \sum_{x \in F} r'_x \cdot \psi_x \text{ where } \psi_x = \sum_{y \in G \cap O_x} (t_{xy}/r'_x) \cdot \widehat{y}$$

The coefficients of ψ_x sum up to 1. Thus, $\bar{f}(\psi_x)$ is a convex combination of the points fy where y in $G \cap O_x$. All these points are in W_x, whence $\bar{f}(\psi_x)$ in V_x by choice of W_x.

Since $\bar{f}(\psi') = \sum_{x \in F} r'_x \cdot \bar{f}(\psi_x)$ where r'_x in R_x and $\bar{f}(\psi_x)$ in V_x, it is in U. Since $\bar{f}(\psi)$ is above $\bar{f}(\psi')$, it is in U as well. □

In Theorem 6.7, local convexity cannot be dispensed with: if M is a topological cone with the property that identity $\mathsf{id} : M \to M$ has a continuous linear extension $\overline{\mathsf{id}} : \mathsf{V_f}\, M \to M$, then M is a linear retract of $\mathsf{V_f}\, M$, whence locally convex by Prop. 6.6. As a subspace of $\mathsf{V_f}\, M$, it is also $\mathcal{T}_0$.

6.5 A Universal Property for the Space of Point Continuous Valuations

Here, we present a universal property for the space $\mathsf{V_p}\, X$ of point continuous valuations.

Theorem 6.8 $\mathsf{V_p}\, X$ is the free locally convex sober cone over X in $\mathcal{T}_0$.

Proof: We already know that $\mathsf{V_p}\, X$ is a locally convex sober cone. Let M be an arbitrary locally convex sober cone, and let $f : X \to M$ be continuous. By Theorem 6.7, there is a unique continuous linear function $f^* : \mathsf{V_f}\, X \to M$ with $f^* \circ \mathsf{s} = f$. By Theorem 5.5, $\mathsf{V_p}\, X$ is the soberification of $\mathsf{V_f}\, X$. Hence, by Theorem 2.3, the continuous function $f^* : \mathsf{V_f}\, X \to M$ has a unique continuous extension $\bar{f} : \mathsf{V_p}\, X \to M$. Since $\bar{f}$ extends f^*, $\bar{f} \circ \mathsf{s} = f$ follows. The only thing which remains to be proved is linearity of $\bar{f}$.

Let r be a fixed element of $\mathbf{R}_+$. Consider the two functions $F, G : \mathsf{V_p}\, X \to \overline{\mathbf{R}}_+$ with $F(\nu) = \bar{f}(r \cdot \nu)$ and $G(\nu) = r \cdot \bar{f}(\nu)$. They are continuous and coincide on $\mathsf{V_f}\, X$ because of linearity of f^*. By Prop. 2.5, $F = G$ follows.

For addition, consider the two functions $F, G : \mathsf{V_p}\, X \times \mathsf{V_p}\, X \to \overline{\mathbf{R}}_+$ with $F(\nu, \nu') = \bar{f}(\nu + \nu')$ and $G(\nu, \nu') = \bar{f}(\nu) + \bar{f}(\nu')$. They are continuous and coincide on $\mathsf{V_f}\, X \times \mathsf{V_f}\, X$ because of linearity of f^*. By Prop. 2.6, $\mathsf{V_p}\, X \times \mathsf{V_p}\, X$ is the soberification of $\mathsf{V_f}\, X \times \mathsf{V_f}\, X$. By Prop. 2.5, $F = G$ follows. □

6.6 The Continuous Case

If X is a continuous dcpo, then $\mathsf{V}X$ equals $\mathsf{V_p}\, X$ by Theorem 4.1, and is sober by Prop. 5.1. Thus, its specialization preorder forms a dcpo, and every open set of $\mathsf{V}X$ is Scott open. It is however not immediately obvious that the topology of $\mathsf{V}X$ actually is identical with the Scott topology, and thus $\mathsf{V}X$ is the continuous dcpo of valuations considered by Jones [8].

Our universal property can be used to obtain a simple proof of the coincidence. For the purpose of this proof, let $\mathsf{V_S}\, X$ be the dcpo of Scott continuous valuations on X equipped with the Scott topology. As shown by Jones, continuity of X implies continuity of $\mathsf{V_S}\, X$, and

$\mathsf{s}_\mathrm{S} : X \to \mathsf{V}_\mathrm{S} X$ with $\mathsf{s}_\mathrm{S} x = \widehat{x}$ is Scott continuous. For continuous dcpo's, the Scott topology of the product is the product of the Scott topologies, whence $\mathsf{V}_\mathrm{S} X$ is a topological cone in our sense. By Prop. 6.3, it is locally convex. Thus, Theorem 6.8 applies, and we obtain a continuous linear function $F : \mathsf{V}X \to \mathsf{V}_\mathrm{S} X$ with $F \circ \mathsf{s} = \mathsf{s}_\mathrm{S}$. Let $G : \mathsf{V}_\mathrm{S} X \to \mathsf{V}X$ be the function which operates as identity on elements. It is obviously linear, and continuous since every open of $\mathsf{V}X$ is Scott open. Thus, $G \circ F : \mathsf{V}X \to \mathsf{V}X$ is continuous and linear. By Theorem 6.1 (2), it equals the identity function of $\mathsf{V}X$. Thus, $F : \mathsf{V}X \to \mathsf{V}_\mathrm{S} X$ must also operate as identity on elements. Its continuity implies that every open set of $\mathsf{V}_\mathrm{S} X$ is open in $\mathsf{V}X$. Therefore, the topologies of $\mathsf{V}X$ and $\mathsf{V}_\mathrm{S} X$ coincide.

Summarizing, we obtain:

Theorem 6.9 If X is a continuous dcpo with its Scott topology, then so is $\mathsf{V}X = \mathsf{V}_\mathrm{p} X$.

7 Integration

Several authors [8, 9, 10, 13] already defined integration of real-valued functions w.r.t. a valuation. Since they defined integration from scratch, the proofs of its properties are quite involved. Here, we present a novel definition of integration which is so simple that most proofs become trivial. (The complexity has not disappeared, though; it is now in the proofs of Theorems 2.3 and 6.7, which are needed to prove Theorem 6.8).

For every space X, integration will be a separately continuous function $\int_X : [X \to \overline{\mathbf{R}}_+]_\mathrm{i} \times \mathsf{V}X \to \overline{\mathbf{R}}_+$. Note that the function space $[X \to \overline{\mathbf{R}}_+]_\mathrm{i}$ is not topologized by the pointwise topology, but by the *Isbell topology* which has more open sets (see Subsection 2.2).

The function $\int_X$ is built from the following pieces:

1. The function $\Omega_\mathrm{s} : [X \to \overline{\mathbf{R}}_+]_\mathrm{i} \to [\Omega_\mathrm{s}\overline{\mathbf{R}}_+ \to \Omega_\mathrm{s} X]_\mathrm{p}$ with $\Omega_\mathrm{s} f = f^-$ is continuous. In fact, the Isbell topology was chosen to guarantee this.
2. Using Ω_s, we map from $[X \to \overline{\mathbf{R}}_+]_\mathrm{i} \times \mathsf{V}X$ to $[\Omega_\mathrm{s}\overline{\mathbf{R}}_+ \stackrel{\cup,\cap}{\to} \Omega_\mathrm{s} X]_\mathrm{p} \times [\Omega_\mathrm{s} X \stackrel{mod}{\to} \overline{\mathbf{R}}_+]_\mathrm{p}$, where the labels at the arrows indicate the properties of the resulting functions. Now, we can use function composition to reach $[\Omega_\mathrm{s}\overline{\mathbf{R}}_+ \stackrel{mod}{\to} \overline{\mathbf{R}}_+]_\mathrm{p} = \mathsf{V}\overline{\mathbf{R}}_+$. Function composition $\circ : [X \to Y]_\mathrm{p} \times [Y \to Z]_\mathrm{p} \to [X \to Z]_\mathrm{p}$ is separately continuous.
3. Since $\overline{\mathbf{R}}_+$ is a continuous dcpo, it is locally finitary, whence $\mathsf{V}\overline{\mathbf{R}}_+ = \mathsf{V}_\mathrm{p}\overline{\mathbf{R}}_+$ by Theorem 4.1.
4. $\overline{\mathbf{R}}_+$ is a locally convex sober cone. By Theorem 6.8, identity $\mathsf{id} : \overline{\mathbf{R}}_+ \to \overline{\mathbf{R}}_+$ can be extended to a continuous linear function $\overline{\mathsf{id}} : \mathsf{V}_\mathrm{p}\overline{\mathbf{R}}_+ \to \overline{\mathbf{R}}_+$ with the property $\overline{\mathsf{id}}\,(\widehat{r}) = r$ for all r in $\overline{\mathbf{R}}_+$.

Putting all pieces together, we yield a separately continuous function $\int_X : [X \to \overline{\mathbf{R}}_+]_\mathrm{i} \times \mathsf{V}X \to \overline{\mathbf{R}}_+$ with $\int_X (f, \nu) = \overline{\mathsf{id}}\,(\nu \circ f^-)$. Of course, this function can be restricted to a 'pointwise' function $[X \to \overline{\mathbf{R}}_+]_\mathrm{i} \times \mathsf{V}_\mathrm{p} X \to \overline{\mathbf{R}}_+$ which is also separately continuous. This continuity is not destroyed if the Isbell topology on the real-valued functions is replaced by the smaller pointwise topology. For, the function $\Omega_\mathrm{p} : [X \to \overline{\mathbf{R}}_+]_\mathrm{p} \to [\Omega_\mathrm{p}\overline{\mathbf{R}}_+ \to \Omega_\mathrm{p} X]_\mathrm{p}$ with $\Omega_\mathrm{p} f = f^-$ is continuous. Using Ω_p, we map from $[X \to \overline{\mathbf{R}}_+]_\mathrm{p} \times \mathsf{V}_\mathrm{p} X$ to $[\Omega_\mathrm{p}\overline{\mathbf{R}}_+ \stackrel{\cup,\cap}{\to} \Omega_\mathrm{p} X]_\mathrm{p} \times [\Omega_\mathrm{p} X \stackrel{mod}{\to} \overline{\mathbf{R}}_+]_\mathrm{p}$, and composition can be used to reach $[\Omega_\mathrm{p}\overline{\mathbf{R}}_+ \stackrel{mod}{\to} \overline{\mathbf{R}}_+]_\mathrm{p} = \mathsf{V}_\mathrm{p}\overline{\mathbf{R}}_+$.

Thus, we obtain two variants of integration with the same definition $\int_X(f,\nu) = \overline{\mathsf{id}}(\nu \circ f^-)$, but different continuity properties; the Isbell variant $\int_X : [X \to \overline{\mathbf{R}}_+]_\mathrm{i} \times \mathsf{V}X \to \overline{\mathbf{R}}_+$, and the pointwise variant $\int_X : [X \to \overline{\mathbf{R}}_+]_\mathrm{p} \times \mathsf{V}_\mathrm{p}\, X \to \overline{\mathbf{R}}_+$.

In the sequel, we derive the essential properties of integration. They hold for both variants because the defining equations are the same. They are collected in Theorem 7.1 at the end of this section.

From the construction of the two variants of $\int_X$, we know that they are separately continuous. Since $[X \to \overline{\mathbf{R}}_+]_\mathrm{i}$, $[X \to \overline{\mathbf{R}}_+]_\mathrm{p}$, $\mathsf{V}X$, $\mathsf{V}_\mathrm{p}\, X$, and $\overline{\mathbf{R}}_+$ are d-spaces, they are also Scott continuous in both arguments. Integration is linear in the valuation argument, since $\int_X(f,\nu) = \overline{\mathsf{id}}\,(\nu \circ f^-)$, and $\overline{\mathsf{id}}$ is linear. The effect of integration on point valuations is as follows:

$$\int_X(f,\widehat{x}) \;=\; \overline{\mathsf{id}}\,(\widehat{x} \circ f^-) \;=\; \overline{\mathsf{id}}\,(\widehat{fx}) \;=\; fx.$$

A kind of 'substitution theorem' is easily proved for continuous functions $h : X \to Y$, $f : Y \to \overline{\mathbf{R}}_+$, and valuations ν in $\mathsf{V}X$:

$$\int_X(f \circ h, \nu) \;=\; \overline{\mathsf{id}}\,(\nu \circ (f \circ h)^-) \;=\; \overline{\mathsf{id}}\,(\nu \circ h^- \circ f^-) \;=\; \int_Y(f, \nu \circ h^-)$$

Our final goal is to show that integration is also linear in its functional argument. If we only considered point continuous valuations, the proof would be quite easy: Both sides of the equation $\int_X(f+g,\nu) = \int_X(f,\nu) + \int_X(g,\nu)$ are continuous and linear in $\nu \in \mathsf{V}_\mathrm{p}\, X$. By Theorem 6.1, it suffices to consider the special case $\nu = \widehat{x}$. In this case, both sides are $fx + gx$. The equation $\int_X(r \cdot f,\nu) = r \cdot \int_X(f,\nu)$ would be handled similarly.

Unfortunately, this elegant proof is not possible in the general case since we do not have an analogous property for continuous linear functions defined on $\mathsf{V}X$ (cf. Problem 1). Fortunately, there is a way around the problem. Before we can present it, we have to consider the continuous functions $f : X \to \overline{\mathbf{R}}_+$ a bit closer.

A continuous function $f : X \to \overline{\mathbf{R}}_+$ is *simple* if its image f^+X is finite and does not contain ∞. An arbitrary continuous function is the directed join of all the simple functions below it. Every finite linear combination $r_1 \cdot \widehat{U_1} + \cdots + r_n \cdot \widehat{U_n}$ of characteristic functions $\widehat{U_i}$ of open sets U_i is simple. Conversely, every simple function can be written as such a linear combination.

We need some auxiliary statements for our proof of the linearity of integration in the functional argument.

(1) For r in $\overline{\mathbf{R}}_+$: $\int_X(r \cdot f,\nu) = r \cdot \int_X(f,\nu)$.

Proof: Let $(r\cdot) : \overline{\mathbf{R}}_+ \to \overline{\mathbf{R}}_+$ be the function defined by $(r\cdot)(s) = r \cdot s$. Then $r \cdot f = (r\cdot) \circ f$. By the substitution property, we obtain $\int_X(r \cdot f,\nu) = \int_{\overline{\mathbf{R}}_+}((r\cdot), \nu \circ f^-)$, and by the definition of integration, $r \cdot \int_X(f,\nu) = r \cdot \overline{\mathsf{id}}(\nu \circ f^-)$ holds. We claim $\int_{\overline{\mathbf{R}}_+}((r\cdot),\sigma) = r \cdot \overline{\mathsf{id}}(\sigma)$ for all valuations σ in $\mathsf{V}\overline{\mathbf{R}}_+ = \mathsf{V}_\mathrm{p}\, \overline{\mathbf{R}}_+$. Since both sides of the equation are linear continuous functions in σ, it suffices by Theorem 6.1 to prove the equation for the special case of point valuations $\sigma = \widehat{s}$ where s in $\overline{\mathbf{R}}_+$. In this case, the left hand side is $\int_{\overline{\mathbf{R}}_+}((r\cdot),\widehat{s}) = (r\cdot)(s) = r \cdot s$, and the right hand side is $r \cdot \overline{\mathsf{id}}(\widehat{s}) = r \cdot s$. □

(2) $\int_X(\underline{0},\nu) = 0$ where $\underline{0}(x) = 0$ for all x in X.

Proof: By (1), using $\underline{0} = 0 \cdot \underline{0}$. □

(3) For r in $\overline{\mathbf{R}}_+$: $\int_X(r+f,\nu) = r\cdot\nu(X) + \int_X(f,\nu)$.

Proof: Apply the same idea as in the proof of (1). Since $X = f^-(\overline{\mathbf{R}}_+)$, equation (3) is equivalent to $\int_{\overline{\mathbf{R}}_+}((r+),\sigma) = r\cdot\sigma(\overline{\mathbf{R}}_+) + \overline{\mathsf{id}}(\sigma)$ where $\sigma = \nu\circ f^-$ in $\mathsf{V}\overline{\mathbf{R}}_+$. This equation holds for all σ in $\mathsf{V}\overline{\mathbf{R}}_+$, since both sides are continuous and linear in σ, and $\int_{\overline{\mathbf{R}}_+}((r+),\widehat{s}) = (r+)(s) = r+s$, and $r\cdot\widehat{s}(\overline{\mathbf{R}}_+) + \overline{\mathsf{id}}(\widehat{s}) = r\cdot 1 + s$. □

(4) If $fx = gx$ for all x in the open set W, then $\int_X(f,\nu|_W) = \int_X(g,\nu|_W)$.

Proof: By hypothesis, $W\cap f^-V = W\cap g^-V$ holds for all V in $\Omega\overline{\mathbf{R}}_+$. By Prop. 3.3, $\nu|_W(f^-V) = \nu|_W(g^-V)$ follows, whence $\nu|_W\circ f^- = \nu|_W\circ g^-$. □

(5) If $fx = gx$ for all x in $X\setminus W$ where W is open, and ν is bounded, then $\int_X(f,\nu|^W) = \int_X(g,\nu|^W)$.

Proof: The valuation ν must be bounded so that the corestriction $\nu|^W$ is well defined. By hypothesis, $\neg W\cap f^-V = \neg W\cap g^-V$ holds for all V in $\Omega\overline{\mathbf{R}}_+$. By Prop. 3.3, $\nu|^W\circ f^- = \nu|^W\circ g^-$ follows. □

(6) For r in $\overline{\mathbf{R}}_+$ and W in ΩX, $\int_X(r\cdot\widehat{W}+f,\nu) = r\cdot\nu(W) + \int_X(f,\nu)$.

Proof: First assume that ν is bounded. Then it can be partitioned along W into $\nu = \nu|_W + \nu|^W$. Since integration is linear in the valuation argument, $\int_X(r\cdot\widehat{W}+f,\nu) = \int_X(r\cdot\widehat{W}+f,\nu|_W) + \int_X(r\cdot\widehat{W}+f,\nu|^W)$ holds. Since $\widehat{W}(x) = 0$ for x in $X\setminus W$, the second summand equals $\int_X(f,\nu|^W)$ by (5). Since $\widehat{W}(x) = 1$ for x in W, the first summand equals $\int_X(r+f,\nu|_W)$ by (4). By (3), this is $r\cdot\nu|_W(X) + \int_X(f,\nu|_W)$. Hence, we obtain $r\cdot\nu(W) + \int_X(f,\nu|_W) + \int_X(f,\nu|^W) = r\cdot\nu(W) + \int_X(f,\nu)$ for the sum.

The equation holds for every valuation ν in $\mathsf{V}X$, since integration is Scott continuous in the valuation argument, and ν is a directed join of bounded members of $\mathsf{V}X$ by Theorem 4.2. □

(7) $\int_X(r_1\cdot\widehat{U_1}+\cdots+r_n\cdot\widehat{U_n},\nu) = r_1\cdot\nu(U_1)+\cdots+r_n\cdot\nu(U_n)$.
In particular, $\int_X(\widehat{U},\nu) = \nu(U)$.

Proof: Apply (6) n times, then (2). □

(8) $\int_X(f+g,\nu) = \int_X(f,\nu) + \int_X(g,\nu)$.

Proof: First assume that f and g are simple. Since every simple function is a finite linear combination of characteristic functions of opens, the statement follows from (7).

The equation for general f and g follows, since every continuous function is a directed join of simple functions, and integration is Scott continuous in the functional argument. □

Summarizing our results, we obtain:

Theorem 7.1

(1) Integration $\int_X : [X\to\overline{\mathbf{R}}_+]_{\mathrm{i}}\times\mathsf{V}X\to\overline{\mathbf{R}}_+$ is separately continuous.

(2) The variant $\int_X : [X\to\overline{\mathbf{R}}_+]_{\mathrm{p}}\times\mathsf{V}_{\mathrm{p}}X\to\overline{\mathbf{R}}_+$ is also separately continuous.

(3) Integration (in both variants) is Scott continuous in both arguments.

(4) Integration is linear in both arguments.

(5) For $f : X\to\overline{\mathbf{R}}_+$ and x in X, $\int_X(f,\widehat{x}) = fx$ holds.

(6) $\int_X(\widehat{U},\nu) = \nu(U)$ where $\widehat{U}$ is the characteristic function of an open U of X.

(7) Let $h : X \to Y$ and $f : Y \to \overline{\mathbf{R}}_+$ be continuous, and let ν be in $\mathsf{V}X$. Then $\int_X(f \circ h, \nu) = \int_Y(f, \nu \circ h^-)$.

Comparison with other Notions of Integral

Jones [8, Section 3.9], Kirch [10, Chapter 4], and Tix [13, Chapter 4] define integration on a space X w.r.t. a Scott continuous valuation. While Jones only deals with bounded continuous functions and bounded valuations, Kirch and Tix work with the extended non-negative reals $\overline{\mathbf{R}}_+$ as we do. Jones and Kirch define their integrals first for simple functions, using each their own standard representation. The integral notions are extended to all (bounded) continuous functions using the fact that these are directed joins of simple functions. The definition of Tix bears some similarity with ours: $\int_X(f,\nu)$ is defined as the Riemann integral $\int_0^\infty f_\nu(r)\,dr$ where $f_\nu(r) = \nu(f^-\{s \in \overline{\mathbf{R}}_+ \mid s > r\})$.

Jones's integral is Scott continuous and linear [8, Theorem 3.13 / 3.14] in its functional argument, and $\int_X(\widehat{U},\nu) = \nu U$ follows from its definition. Tix's integral has the same properties [13, Theorem 4.4]. Hence, these integrals coincide with ours since continuous functions from X to $\overline{\mathbf{R}}_+$ are directed joins of simple functions, which in turn are finite linear combinations of characteristic functions $\widehat{U}$ of opens.

Kirch's integral also coincides with ours because his definitions are instances of parts of our Theorem 7.1. He proves continuity and linearity of his integral for core-compact spaces only so that an argument as above is not possible.

Edalat [3] defines a Riemann-like integral on compact metric spaces X which differs from the integral notions mentioned so far in that the functions to be integrated are not necessarily continuous, and the integration is done w.r.t. a measure, not a valuation. There are some connections, though, as he uses the fact that every Borel measure on X can be approximated by finite valuations on a power space of X.

8 Isomorphic Descriptions

Integration may be used to derive isomorphic descriptions of $\mathsf{V}X$ and $\mathsf{V}_\mathrm{p}\,X$. Kirch [10, Section 8.1] and Tix [13, Section 4.3] prove similar descriptions; Kirch for dcpo's and Tix for the weak topology. In contrast, we do not start with the weak topology, but obtain it a posteriori (Theorem 8.3).

Theorem 8.1 For every space X, the topological cone of Scott continuous valuations on X, i.e., strict modular Scott continuous functions from ΩX to $\overline{\mathbf{R}}_+$, is isomorphic to the topological cone of linear continuous functions from $[X \to \overline{\mathbf{R}}_+]_\mathrm{i}$ to $\overline{\mathbf{R}}_+$ with the pointwise topology.

$$\mathsf{V}X \;=\; [\Omega_\mathrm{s}X \stackrel{mod}{\to} \overline{\mathbf{R}}_+]_\mathrm{p} \;\cong\; [[X \to \overline{\mathbf{R}}_+]_\mathrm{i} \stackrel{lin}{\to} \overline{\mathbf{R}}_+]_\mathrm{p}$$

Proof: For the proof, let $\mathcal{F} = [X \to \overline{\mathbf{R}}_+]_\mathrm{i}$.

One isomorphism is constructed from integration: Define $\alpha : \mathsf{V}X \to [\mathcal{F} \stackrel{lin}{\to} \overline{\mathbf{R}}_+]_\mathrm{p}$ by $\alpha(\nu) = \lambda f.\ \int_X(f,\nu)$. This function has the claimed type since integration is continuous and linear in its functional argument. The function α itself is linear since integration is linear in its valuation argument. It is continuous since integration is separately continuous.

The inverse isomorphism is defined using characteristic functions: For F in $[\mathcal{F} \stackrel{lin}{\to} \overline{\mathbf{R}}_+]_\mathrm{p}$, let $\beta(F) = F \circ \chi$ where $\chi : \Omega X \to \mathcal{F}$ with $\chi U = \widehat{U}$, the characteristic function of U. Strictness and modularity of $\beta(F)$ are easily verified. $\beta(F)$ is Scott continuous since both F and χ are Scott continuous. F is Scott continuous since it is continuous and both $\mathcal{F}$ and $\overline{\mathbf{R}}_+$ are d-spaces.

The function β itself is obviously linear. Since $[\mathcal{F} \stackrel{lin}{\to} \overline{\mathbf{R}}_+]_\mathrm{p}$ is a subspace of $\prod_{f \in \mathcal{F}} \overline{\mathbf{R}}_+$, the function $\beta_U : [\mathcal{F} \stackrel{lin}{\to} \overline{\mathbf{R}}_+]_\mathrm{p} \to \overline{\mathbf{R}}_+$ with $\beta_U(F) = F(\widehat{U})$ is continuous for every fixed U in ΩX. Since $[\Omega_\mathrm{s} X \stackrel{mod}{\to} \overline{\mathbf{R}}_+]_\mathrm{p}$ is a subspace of $\prod_{U \in \Omega X} \overline{\mathbf{R}}_+$, continuity of the function $\beta : [\mathcal{F} \stackrel{lin}{\to} \overline{\mathbf{R}}_+]_\mathrm{p} \to [\Omega_\mathrm{s} X \stackrel{mod}{\to} \overline{\mathbf{R}}_+]_\mathrm{p}$ follows.

At this point, we know that both α and β are continuous linear functions. We still have to show that they are inverse to each other.

For ν in $\mathsf{V}X$ and U in ΩX, $\beta(\alpha\nu)(U) = (\alpha\nu)(\widehat{U}) = \int_X (\widehat{U}, \nu) = \nu(U)$ holds using Theorem 7.1 (6), whence $\beta \circ \alpha = \mathsf{id}$.

For F in $[\mathcal{F} \stackrel{lin}{\to} \overline{\mathbf{R}}_+]_\mathrm{p}$, $\alpha(\beta F)(g) = \int_X (g, \beta F)$ holds for all g in $\mathcal{F}$. Hence, we have to show $\int_X (g, \beta F) = F(g)$ for all g in $\mathcal{F}$. Since both integration and the function F are Scott continuous, it suffices to show the equation for all simple functions g. Since both integration and the function F are linear, and every simple function is a finite linear combination of characteristic functions, it even suffices to show the equation for all functions $\widehat{U}$ with U in ΩX. By Prop. 7.1 (6), $\int_X (\widehat{U}, \beta F) = \beta F U = F\widehat{U}$ holds as required. □

An analogous theorem can be formulated for point continuous valuations.

Theorem 8.2 For every space X, the topological cone of point continuous valuations on X, i.e., strict modular continuous functions from $\Omega_\mathrm{p} X$ to $\overline{\mathbf{R}}_+$, is isomorphic to the topological cone of linear continuous functions from $[X \to \overline{\mathbf{R}}_+]_\mathrm{p}$ to $\overline{\mathbf{R}}_+$ with the pointwise topology.

$$\mathsf{V}_\mathrm{p} X \;=\; [\Omega_\mathrm{p} X \stackrel{mod}{\to} \overline{\mathbf{R}}_+]_\mathrm{p} \;\cong\; [[X \to \overline{\mathbf{R}}_+]_\mathrm{p} \stackrel{lin}{\to} \overline{\mathbf{R}}_+]_\mathrm{p}$$

Proof: Algebraically, the isomorphisms are the same as in the proof of Theorem 8.1. The difference is that the 'pointwise' version of integration is used. Hence, α defined by $\alpha(\nu) = \lambda f.\ \int_X (f, \nu)$ has type $\mathsf{V}_\mathrm{p} X \to [[X \to \overline{\mathbf{R}}_+]_\mathrm{p} \stackrel{lin}{\to} \overline{\mathbf{R}}_+]_\mathrm{p}$ in this case.

For F in $[[X \to \overline{\mathbf{R}}_+]_\mathrm{p} \stackrel{lin}{\to} \overline{\mathbf{R}}_+]_\mathrm{p}$, we have to show that $\beta(F) = F \circ \chi$ is point continuous, i.e., continuous with type $\Omega_\mathrm{p} X \to \overline{\mathbf{R}}_+$. We prove continuity of $\chi : \Omega_\mathrm{p} X \to [X \to \overline{\mathbf{R}}_+]_\mathrm{p}$. For x in X and V in $\Omega\overline{\mathbf{R}}_+$, $\chi^-\langle x \to V \rangle$ is $\Omega_\mathrm{p} X$ if $0 \in V$, $\mathcal{O}(x)$ if $0 \notin V$ and $1 \in V$, and $\emptyset$ if $1 \notin V$.

The remainder of the proof of Theorem 8.1 can be taken over unchanged. □

In general, the topology of a pointwise function space $[Y \to Z]_\mathrm{p}$ is the least such that for every y in Y, the function $\lambda f.\ fy : [Y \to Z]_\mathrm{p} \to Z$ is continuous. Applying this to $[[X \to \overline{\mathbf{R}}_+]_\mathrm{i} \stackrel{lin}{\to} \overline{\mathbf{R}}_+]_\mathrm{p}$, we see that its topology is the least such that for every continuous $f : X \to \overline{\mathbf{R}}_+$, the function $\lambda F.\ F(f)$ is continuous. Using the isomorphism α of the proof of Theorem 8.1, we conclude:

Theorem 8.3 The topology on $\mathsf{V}X$ is the least such that for every continuous function $f : X \to \overline{\mathbf{R}}_+$, the function $f^* : \mathsf{V}X \to \overline{\mathbf{R}}_+$ with $f^*(\nu) = \int_X (f, \nu)$ is continuous.

This characterizes the topology of $\mathsf{V}X$ as a 'weak topology'. Notice that in contrast to classical results which look similar, the space $\overline{\mathbf{R}}_+$ is equipped with the Scott topology instead of the usual Hausdorff topology.

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