Spaces of Valuations

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Abstract

Valuations are measure-like functions mapping the open sets of a topological space into positive real numbers. They can be classified according to some additional properties. Some topological spaces are defined whose elements are valuations from various classes. The relationships among these spaces are studied, and universal properties are shown for some of them.

1 Introduction

For a topological space X, a *valuation* on X is a function ν which maps the open sets of X to real numbers in the range from zero to infinity (inclusively) with the following properties:

(1) The empty set is mapped to zero: $\nu\emptyset = 0$ (strictness).

(2) The values assigned to binary union and intersection are related by the following equation:

$$\nu(U \cup V) + \nu(U \cap V) = \nu U + \nu V \qquad \text{for all opens } U \text{ and } V \text{ (modularity).}$$

(3) Bigger sets are mapped to bigger numbers: if $U \subseteq V$, then $\nu U \leq \nu V$ (monotonicity).

Most often, we consider *Scott continuous valuations* which enjoy the additional property $\nu(\bigcup_{i\in I} V_i) = \bigsqcup_{i\in I} \nu V_i$ for every directed family $(V_i)_{i\in I}$ of opens V_i of X.

Some authors write *evaluations* instead of valuations, and some authors immediately require Scott continuity.

The concept of valuations has some similarity with the concept of *measures*. Borel measures are defined for all Borel sets of a space, and every open set is a Borel set. Hence, every measure can be restricted to a valuation. This valuation is not Scott continuous in general, since measures have to satisfy a weaker property where only countable directed families $(V_i)_{i\in I}$ are considered. On the other hand, a valuation cannot always be extended to a measure.

Previously, valuations were already used in the following contexts:

- The *probabilistic power domain* of Jones and Plotkin [6, 7] over some dcpo X is the dcpo of Scott continuous valuations on X which are bounded by 1. It is used to model the semantics of probabilistic programs. If the semantics of a program p is ν, this means that for every open set U, the number νU is the probability that the result of running p is in U. The difference $1 - \nu(X)$ is the probability that running p yields no result at all, i.e., does not terminate.
- In [5], the author presented a *lower bag domain* as an analogue to the lower power domain, but without idempotence of addition. It can be used to specify a bag semantics for non-deterministic programs which takes multiplicities of results into account. One possible description of the lower bag domain consists of Scott continuous integer-valued valuations. In fact, some of the results about integer valuations which are presented in Section 12 were already contained in [5], but with different proofs.
- In [3], Edalat used Scott continuous valuations to obtain a domain-theoretic treatment of measures and Riemann-like integrals. In [1], he connected dynamical systems and fractals with domain theory. The probabilistic power domain, i.e., the collection of Scott continuous valuations bounded by 1, plays a major role in this connection which leads to better algorithms for fractal image generation [2].

Because of these applications, we feel that the concept of valuations deserves further interest.

In this paper, we investigate valuations in a topological setting: the set $\mathsf{V}X$ of Scott continuous valuations on a space X is made a topological space, whose structure and properties are studied. Thus, we present some background information about dcpo's and topological spaces in Section 2.

In Section 3, valuations are defined formally. In addition to Scott continuity, we introduce *point continuity* as a possible property of valuations which is stronger than Scott continuity. Some special valuations are defined, e.g., the *point valuations* $\widehat{x}$ which map open sets containing x to 1 and all other open sets to 0. We also consider some operations on valuations such as addition, multiplication by a constant from $\overline{\mathbf{R}}_+$, and restriction to an open set.

In Section 4, we define some classes of valuations. In particular, *finite valuations* are finite linear combinations of point valuations. The connections among these classes are studied. In particular, we prove that for continuous dcpo's, all Scott continuous valuations are point continuous.

In Section 5, finite valuations are studied. It is shown that they can be represented by assigning finite weights to a finite number of points. We compare finite valuations in terms of these representations, and prove that the representation is unique.

In Section 6, we define the space $\mathsf{V}X$ of Scott continuous valuations on X, and the subspaces of point continuous ($\mathsf{V}_\mathrm{p}X$) and finite valuations ($\mathsf{V}_\mathrm{f}X$). We also prove one of the main results of this paper: the space $\mathsf{V}_\mathrm{p}X$ is the sobrification of $\mathsf{V}_\mathrm{f}X$.

The other main results are shown in Section 7. The space $\mathsf{V}_\mathrm{f}X$ of finite valuations is the free locally convex $\mathcal{T}_0$-cone over X, and the space $\mathsf{V}_\mathrm{p}X$ of point continuous valuations is the free locally convex sober cone over X. Here, a $\mathcal{T}_0$-cone is an $\mathbf{R}_+$-module with a $\mathcal{T}_0$-topology such that addition and multiplication are continuous, and a sober cone is a $\mathcal{T}_0$-cone with sober topology. We did not find a universal property for V.

Nevertheless, the universal property of V_p suffices to find a novel definition of integration $\int_X : [X \to \overline{\mathbf{R}}_+] \times \mathsf{V}X \to \overline{\mathbf{R}}_+$ of a real-valued function w.r.t. a Scott continuous valuation (see Section 8). This definition allows for elegant proofs of the properties of integration.

In Section 9, integration is used to prove that the spaces of valuations are isomorphic to certain second order function spaces, namely

$$\mathsf{V}X \cong [[X \to \overline{\mathbf{R}}_+]_\mathrm{i} \to \overline{\mathbf{R}}_+]_\mathrm{p} \quad \text{and} \quad \mathsf{V}_\mathrm{p} X \cong [[X \to \overline{\mathbf{R}}_+]_\mathrm{p} \to \overline{\mathbf{R}}_+]_\mathrm{p},$$

where the index '$[\,]_\mathrm{p}$' means a function space with pointwise topology, and '$[\,]_\mathrm{i}$' a function space with Isbell topology.

The final Section 12 is devoted to the special case of *integer valuations* which map all opens to numbers from $\overline{\mathbf{N}}_0$. In the case of integer valuations, the notions of point continuity and Scott continuity coincide. The space $\mathsf{V}^\mathrm{N}X$ of continuous integer valuations is the free sober $\mathbf{N}_0$-module over X.

2 Some Topology

After fixing some set-theoretical notation (Subsection 2.1), we present a brief overview of the topological notions needed in this paper. In particular, we introduce dcpo's, topological spaces, product and function spaces, sobriety and sobrification, and the spaces $\overline{\mathbf{R}}_+$ of positive real numbers and $\overline{\mathbf{N}}_0$ of positive integers.

2.1 Some Set-Theoretic Notation

We only mention some slightly non-standard notation. If A is a subset of a fixed set X, then the complement of A in X is written $\neg A$.

If $f : X \to Y$ is a function, then we denote the image of a set $A \subseteq X$ by $f^+A = \{fx \mid x \in A\}$, and the inverse image of a set $B \subseteq Y$ by $f^-B = \{x \in X \mid fx \in B\}$. The set f^+X, which is the image of the function, is also denoted by $\Im f$.

For $A \cap B \neq \emptyset$, we briefly write $A \between B$. Note that $f^+A \subseteq B$ iff $A \subseteq f^-B$, and $f^+A \between B$ iff $A \between f^-B$.

2.2 Dcpo's

We use the standard definitions: a poset $(X, \sqsubseteq)$ is a set X together with a reflexive, anti-symmetric, and transitive relation '$\sqsubseteq$'. For a subset A of a poset X, we define $\downarrow A = \{x \in X \mid \exists a \in A : x \sqsubseteq a\}$ and $\uparrow A = \{x \in X \mid \exists a \in A : a \sqsubseteq x\}$. We shall often abbreviate $\downarrow\{a\}$ by $\downarrow a$ and $\uparrow\{a\}$ by $\uparrow a$. The set A is *lower* if $A = \downarrow A$, and *upper* if $A = \uparrow A$. The *least upper bound* or *join* of A is denoted by $\bigsqcup A$ (if it exists).

A subset D of a poset is *directed* if it is non-empty, and for all x, y in D, there is z in D with $x, y \sqsubseteq z$. A *dcpo* is a poset where every directed set has a least upper bound. A function $f : \mathbf{X} \to \mathbf{Y}$ between dcpo's $\mathbf{X}$ and $\mathbf{Y}$ is *Scott continuous* if for all directed subsets D of $\mathbf{X}$, $f(\bigsqcup D) = \bigsqcup f^+D$ holds. The category of dcpo's and continuous functions is called $\mathcal{DCPO}$. This category is small complete and cartesian closed. The function space $[\mathbf{X} \to \mathbf{Y}]$ for instance consists of the continuous functions from $\mathbf{X}$ to $\mathbf{Y}$ ordered by $f \sqsubseteq g$ iff $fx \sqsubseteq gx$ for all x in $\mathbf{X}$.

2.3 Topological Spaces

A *topological space* is a set X together with a set ΩX of subsets of X which is closed under finite intersections and arbitrary unions. The sets in ΩX are called *open*, and their complements are called *closed*. A function $f : X \to Y$ between two topological spaces X and Y is *continuous* iff $f^- V$ is open in X for every open set V of Y. Equivalently, $f^- C$ is closed for every closed set C of Y.

A *subbase* of a space X is a collection $\mathcal{S}$ of opens of X such that every open set of X is a union of finite intersections of members of $\mathcal{S}$. Often, the set of opens of a space to be constructed is specified by defining a subbase.

For a subset A of a space X, let $\mathcal{O}(A) = \{O \in \Omega X \mid A \subseteq O\}$. We abbreviate $\mathcal{O}(\{x\})$ to $\mathcal{O}(x)$. For every subset A of a space X, the *closure* $\mathsf{cl}\, A$ is the least closed superset of A. A point x is in $\mathsf{cl}\, A$ iff every O in $\mathcal{O}(x)$ meets A.

A subset O of a dcpo $\mathbf{X}$ is *Scott open* if it is upper, and for all directed sets D, $\bigsqcup D \in O$ implies $D \between O$. With this definition, every dcpo becomes a topological space. A function $f : \mathbf{X} \to \mathbf{Y}$ between dcpo's is Scott continuous iff it is topologically continuous. Thus, $\mathcal{DCPO}$ can be considered as a full subcategory of the category of topological spaces. (Beware: this inclusion functor does not preserve products, not even binary products.)

Every topological space can be preordered by defining $x \sqsubseteq x'$ iff every open set which contains x also contains x'. This is called the *specialization preorder* of the space. A space is a $\mathcal{T}_0$*-space* iff this preorder is a partial order, i.e., anti-symmetric. The specialization preorder of a dcpo with its Scott topology is the original order of the dcpo. Hence, every dcpo is a $\mathcal{T}_0$-space.

If we use order-notions such as lower, upper, $\downarrow A$, and $\uparrow A$ in a topological space, this always refers to the specialization preorder. All open sets are upper sets, and all closed sets are lower sets. Hence, $A \subseteq \downarrow A \subseteq \mathsf{cl}\, A$ holds for all subsets A of a space. For finite F, even $\mathsf{cl}\, F = \downarrow F$ holds. For every subset A of a topological space, $\uparrow A$ is the intersection of all open supersets of A.

2.4 D-Spaces

A space X is a *d-space* if the induced preorder is a dcpo and all open sets of X are Scott open w.r.t. this dcpo. A continuous function $f : X \to Y$ between two d-spaces is Scott continuous. Since the order in a dcpo is anti-symmetric, all d-spaces are $\mathcal{T}_0$-spaces. Every dcpo with its Scott topology is a d-space, and every $\mathcal{T}_1$-space is a d-space.

2.5 Embeddings and Subspaces

A function $e : X \to Y$ between two topological spaces is a (topological) *embedding* iff it is continuous and injective and $e^- : \Omega Y \to \Omega X$ is surjective. Every topological embedding is an order embedding as well, i.e., $ex \sqsubseteq ey$ iff $x \sqsubseteq y$. If Y is a $\mathcal{T}_0$-space, the condition of injectivity is redundant.

If $e : X \to Y$ is an embedding, then a function $f : Z \to X$ is continuous iff $e \circ f : Z \to Y$ is continuous.

Let Y be a topological space, and S a subset thereof. We make S into a *subspace* of Y by defining a subset U of S as open iff $U = S \cap V$ for some open set V of Y. The $\mathcal{T}_0$ property is preserved by subspace formation. The d-space property is not preserved in general, since some directed joins may be omitted.

If S is a subspace of Y, then the subset inclusion $e : S \to Y$ is a topological embedding. Conversely, if $e : X \to Y$ is an embedding, then X is isomorphic to the subspace e^+X of Y.

Equalizers are a special kind of subspace. The equalizer of two continuous functions $f, g : X \to Y$ is the subspace $\{x \in X \mid fx = gx\}$ of X.

2.6 The Product of Topological Spaces

For a family $(X_i)_{i \in I}$ of topological spaces, we define the *product space* $\prod_{i \in I} X_i$ with points $(x_i)_{i \in I}$ and subbasis $\{\langle i, O\rangle \mid i \in I,\ O \in \Omega X_i\}$ where $\langle j, O\rangle = \{(x_i)_{i \in I} \mid x_j \in O\}$. The preorder of $\prod_{i \in I} X_i$ is $(x_i)_{i \in I} \sqsubseteq (y_i)_{i \in I}$ iff $x_i \sqsubseteq y_i$ for all i in I.

The projections π_j with $\pi_j((x_i)_{i \in I}) = x_j$ are continuous for every j in I, and moreover, a function $f : Y \to \prod_{i \in I} X_i$ is continuous iff the functions $\pi_j \circ f$ are continuous for every j in I. If all the spaces X_i are $\mathcal{T}_0$ / d-spaces, then so is $\prod_{i \in I} X_i$.

A special case is the binary product $X \times Y$ of two spaces X and Y. An alternative subbase of $X \times Y$ is $\{U \times V \mid U \in \Omega X,\ V \in \Omega Y\}$.

2.7 The Tensor Product

The tensor product or cross product $X \otimes Y$ of two spaces X and Y has the same carrier set as the product space $X \times Y$. A set W is open in $X \otimes Y$ if for every (x, y) in W, there are open sets U of X and V of Y such that $(x, y) \in \{x\} \times V \subseteq W$ and $(x, y) \in U \times \{y\} \subseteq W$. To compare, W is open in $X \times Y$ if for every (x, y) in W, there are open sets U of X and V of Y such that $(x, y) \in U \times V \subseteq W$.

The spaces $X \otimes Y$ and $X \times Y$ share the same specialization preorder. The topology of $X \otimes Y$ is a superset of the topology of $X \times Y$, whence a function $f : X \times Y \to Z$ is also continuous as a function from $X \otimes Y$ to Z.

A function $f : X \otimes Y \to Z$ is continuous if and only if all the functions $f_x : Y \to Z$ with $f_x y = f(x,\ y)$ and $f^y : X \to Z$ with $f^y x = f(x,\ y)$ are continuous as well. Thus, a continuous function $f : X \otimes Y \to Z$ is often called *continuous in the two arguments separately*, whereas continuous functions $f : X \times Y \to Z$ are sometimes called *jointly continuous*.

If X and Y are $\mathcal{T}_0$ / d-spaces, then so is $X \otimes Y$. If X and Y are dcpo's, then the Scott topology on the product set $X \times Y$ is identical with the tensor product topology.

2.8 Spaces of Open Sets

The set ΩX of open sets of a space X can be topologized in several different ways.

First, $(\Omega X, \subseteq)$ is a dcpo, whence it can be endowed with the *Scott topology*. We call the resulting space $\Omega_s X$.

Second, ΩX can be given the *point topology* with subbase $\{\mathcal{O}(x) \mid x \in X\}$. A set $\mathcal{O}$ of opens is open in the point topology (*point open*) iff for every O in $\mathcal{O}$ there is a finite set F such that $O \in \mathcal{O}(F) \subseteq \mathcal{O}$. We call the resulting space $\Omega_{\mathrm{p}} X$.

Since every set $\mathcal{O}(x)$ is Scott open, the topology of $\Omega_{\mathrm{p}} X$ is contained in that of $\Omega_{\mathrm{s}} X$. The two spaces $\Omega_{\mathrm{s}} X$ and $\Omega_{\mathrm{p}} X$ have the same preorder, namely subset inclusion. Both are d-spaces.

2.9 The Pointwise Function Space

For two spaces X and Y, the *pointwise function space* $[X \to Y]_{\mathrm{p}}$ consists of all continuous functions $f : X \to Y$ with subbase $\{\langle x \to V \rangle \mid x \in X, V \in \Omega Y\}$ where $\langle x \to V \rangle = \{f : X \to Y \mid fx \in V\}$. It is a subspace of the product $\prod_{x \in X} Y$ of copies of Y. The preorder on $[X \to Y]_{\mathrm{p}}$ is given 'pointwise': $f \sqsubseteq g$ iff $fx \sqsubseteq gx$ for all x in X.

The properties $\mathcal{T}_0$ and d-space carry over from Y to $[X \to Y]_{\mathrm{p}}$, no matter which properties X has.

A function $f : X \otimes Y \to Z$ is continuous iff its curried variant $g : X \to [Y \to Z]_{\mathrm{p}}$ with $gxy = f(x, y)$ is well-typed and continuous.

If $+ : Y \times Y \to Y$ is continuous, then so is $+ : [X \to Y]_{\mathrm{p}} \times [X \to Y]_{\mathrm{p}} \to [X \to Y]_{\mathrm{p}}$ with $(f + g)x = fx + gx$. (It is mathematical custom to reuse the name of the simple function for that of the function defined for functions.)

Composition $\circ : [Y \to Z]_{\mathrm{p}} \otimes [X \to Y]_{\mathrm{p}} \to [X \to Z]_{\mathrm{p}}$ with $(g \circ f)x = g(fx)$ is continuous (in the two arguments separately).

For every two spaces X and Y, the function $\Omega_{\mathrm{p}} : [X \to Y]_{\mathrm{p}} \to [\Omega_{\mathrm{p}} Y \to \Omega_{\mathrm{p}} X]_{\mathrm{p}}$ with $\Omega_{\mathrm{p}} f(V) = f^{-} V$ is well defined and continuous. For, $\Omega_{\mathrm{p}} f$ maps opens to opens by continuity of f, and is continuous since $\Omega_{\mathrm{p}} f(V) \in \mathcal{O}(x)$ iff $fx \in V$ iff $V \in \mathcal{O}(fx)$. The function Ω_{p} itself is continuous since $\Omega_{\mathrm{p}} f \in \langle V \to \mathcal{O}(x) \rangle$ iff $f^{-} V \in \mathcal{O}(x)$ iff $fx \in V$ iff $f \in \langle x \to V \rangle$.

2.10 The Isbell Function Space

For two spaces X and Y, the *Isbell function space* $[X \to Y]_{\mathrm{i}}$ consists of all continuous functions $f : X \to Y$ with subbase $\{\langle \mathcal{U} \leftarrow V \rangle \mid \mathcal{U} \in \Omega(\Omega_{\mathrm{s}} X), V \in \Omega Y\}$ where $\langle \mathcal{U} \leftarrow V \rangle = \{f : X \to Y \mid \mathcal{U} \ni f^{-} V\}$. Since $\langle x \to V \rangle = \langle \mathcal{O}(x) \leftarrow V \rangle$, the topology of $[X \to Y]_{\mathrm{i}}$ includes that of $[X \to Y]_{\mathrm{p}}$. Both function spaces have the same preorder, namely $f \sqsubseteq g$ iff $fx \sqsubseteq gx$ for all x in X.

If Y is $\mathcal{T}_0$ / a d-space, then so is $[X \to Y]_{\mathrm{i}}$, no matter which properties X has.

If $f : X \times Y \to Z$ is (jointly) continuous, then its curried variant $g : X \to [Y \to Z]_{\mathrm{i}}$ is well-defined and continuous. *Composition* $\circ : [Y \to Z]_{\mathrm{i}} \otimes [X \to Y]_{\mathrm{i}} \to [X \to Z]_{\mathrm{i}}$ with $(g \circ f)x = g(fx)$ is continuous (in the two arguments separately).

If X and Y are $\mathcal{T}_0$-spaces, the function $\Omega_{\mathrm{s}} : [X \to Y] \to [\Omega_{\mathrm{s}} Y \to \Omega_{\mathrm{s}} X]$ with $\Omega_{\mathrm{s}} f = f^{-}$ is injective, no matter which topologies are chosen for the two function spaces. If $[\Omega_{\mathrm{s}} Y \to \Omega_{\mathrm{s}} X]$ is equipped with the pointwise topology, it is just the Isbell topology on $[X \to Y]$ which makes Ω_{s} into an embedding. For, $\Omega_{\mathrm{s}} f \in \langle V \to \mathcal{U} \rangle$ iff $f^{-} V \in \mathcal{U}$ iff $f \in \langle \mathcal{U} \leftarrow V \rangle$. Hence, we obtain a continuous function $\Omega_\mathrm{s} : [X \to Y]_\mathrm{i} \to [\Omega_\mathrm{s} Y \to \Omega_\mathrm{s} X]_\mathrm{p}$, whose type differs from that of the continuous function $\Omega_\mathrm{p} : [X \to Y]_\mathrm{p} \to [\Omega_\mathrm{p} Y \to \Omega_\mathrm{p} X]_\mathrm{p}$ of the previous section.

The spaces of open sets $\Omega_\mathrm{s} X$ and $\Omega_\mathrm{p} X$ can also be seen as special instances of function spaces. The *Sierpinski space* **2** has points 0 and 1 with subbase $\{\{1\}\}$. Equivalently, **2** is the dcpo $\{0, 1\}$ with $0 \sqsubset 1$. The opens of a space X are in one-to-one correspondence with the continuous functions from X to **2** by $U \mapsto \chi_U$ and $f \mapsto f^-\{1\}$. By this correspondence, we get the isomorphisms $\Omega_\mathrm{p} X \cong [X \to \mathbf{2}]_\mathrm{p}$ and $\Omega_\mathrm{s} X \cong [X \to \mathbf{2}]_\mathrm{i}$.

2.11 Sobriety

A subset A of a space X is *irreducible* if whenever $A \subseteq \bigcup_{i \in I} C_i$ for some finite family $(C_i)_{i \in I}$ of closed sets, then $A \subseteq C_i$ for some i in I. Continuous images of irreducible sets are irreducible. A set A is irreducible iff $\mathsf{cl}\, A$ is so, and singleton closures $\mathsf{cl}\,\{x\} = \downarrow x$ are irreducible.

Definition 2.1 A space X is sober if for every irreducible set A, there is exactly one point x such that $\mathsf{cl}\, A = \mathsf{cl}\,\{x\}$.

Every Hausdorff space is sober, and every sober space is $\mathcal{T}_0$. Every finite $\mathcal{T}_0$-space is sober. Every sober space is a d-space, and every continuous function between sober spaces is Scott continuous. (To prove these facts, note that directed sets are irreducible.)

Another equivalent definition of sobriety involves sets of open sets. A set $\mathcal{O}$ of open sets of X is a *prime filter* iff it is upper, closed under finite intersections, and inaccessible by unions. Equivalently, $\mathcal{O}$ is a prime filter iff it is Scott open, contains the whole space, does not contain $\emptyset$, is closed under binary intersection, and has the property that $U \cup V \in \mathcal{O}$ implies $U \in \mathcal{O}$ or $V \in \mathcal{O}$. Every set $\mathcal{O}(x) = \{O \in \Omega X \mid x \in O\}$ is a prime filter.

Theorem 2.2 A space X is sober iff for every prime filter $\mathcal{O}$, there is a unique point x such that $\mathcal{O} = \mathcal{O}(x)$.

For the proof, note that if A is irreducible then $\{O \in \Omega X \mid O \between A\}$ is a prime filter, and conversely, if $\mathcal{O}$ is a prime filter, then the complement of $\bigcup\{O \in \Omega X \mid O \notin \mathcal{O}\}$ is an irreducible closed set.

Some topological constructions preserve sobriety:

- Products of sober spaces are sober.
- If Y is sober, then $[X \to Y]_\mathrm{p}$ is sober (no matter what X is).
- If $f, g : X \to Y$ are continuous, X is sober, and Y is $\mathcal{T}_0$, then $\{x \in X \mid fx = gx\}$ is a sober subspace of X.

2.12 Sobrification

Let X be a sober space and S a subset of X so that for every x in X and U in ΩX with $x \in U$, there is some a in S with $a \in U$ and $a \sqsubseteq x$. In this situation, we say that X is the *sobrification* of the subspace S. We first show that continuous functions from X to some $\mathcal{T}_0$-space are uniquely determined by their values on S.

Proposition 2.3 Let X be the sobrification of its subspace S and let Y be a $\mathcal{T}_0$-space. Let $f, g : X \to Y$ be two continuous functions with $fa = ga$ for all a in S. Then $f = g$ follows.

Proof: Let x in X. We prove $fx \sqsubseteq gx$. If $fx \in V$ open, then $x \in f^- V$. By hypothesis, there is a in S such that $a \sqsubseteq x$ and $a \in f^- V$. Then $gx \sqsupseteq ga = fa \in V$. Similarly, $gx \sqsubseteq fx$ is shown, whence $fx = gx$ since Y is a $\mathcal{T}_0$-space. □

The sobrification has the following universal property:

Theorem 2.4 If X is the sobrification of its subspace S, then for every sober space Y and continuous $f : S \to Y$, there is a unique continuous $F : X \to Y$ which extends f.

Proof: Uniqueness follows from Prop. 2.3.

In X, $\mathsf{cl}(S \cap \downarrow x) = \downarrow x$ holds. Thus, $S \cap \downarrow x$ is irreducible in X, and hence in S. By continuity of f, $f^+(S \cap \downarrow x)$ is irreducible in Y. Since Y is sober, there is Fx in Y such that $\mathsf{cl}\, f^+(S \cap \downarrow x) = \downarrow Fx$.

For continuity of F, consider the inverse image of a closed set C. From the definition of F, $Fx \in C$ iff $S \cap \downarrow x \subseteq f^- C$. The set $f^- C$ is closed in S. Thus, there is a closed set C' of X with $f^- C = S \cap C'$. Finally, $S \cap \downarrow x \subseteq S \cap C'$ iff $\mathsf{cl}(S \cap \downarrow x) \subseteq C'$ iff $x \in C'$.

For the extension property, we have to show $Fa = fa$ for all a in S. To this end, we show $\mathsf{cl}\, f^+(S \cap \downarrow a) = \downarrow fa$. The inclusion '$\subseteq$' holds by monotonicity of f, and '$\supseteq$' holds since $a \in S \cap \downarrow a$. □

Extension of functions is continuous:

Theorem 2.5 If X is the sobrification of its subspace S, then for every sober space Y, the function $\mathsf{E} : [S \to Y]_{\mathrm{p}} \to [X \to Y]_{\mathrm{p}}$ given by Theorem 2.4 is continuous.

Proof: Let $f : S \to Y$ be continuous, x in X, and V in ΩY such that $\mathsf{E}f \in \langle x \to V \rangle$. Then $x \in (\mathsf{E}f)^- V$ which is an open set of X. Since X is the sobrification of S, there is some a in S with $a \sqsubseteq x$ and $a \in (\mathsf{E}f)^- V$. Then $fa = \mathsf{E}fa \in V$, whence $f \in \langle a \to V \rangle$.

If g is in $\langle a \to V \rangle$, then $\mathsf{E}ga = ga \in V$. Since $a \sqsubseteq x$, $\mathsf{E}gx$ is in V as well, whence $\mathsf{E}g$ in $\langle x \to V \rangle$. □

Corollary 2.6 If X is the sobrification of its subspace S, then $[S \to Y]_{\mathrm{p}} \cong [X \to Y]_{\mathrm{p}}$ holds for every sober space Y.

Proof: One isomorphism is the extension function E given by Theorem 2.4. The other is restriction of a function $F : X \to Y$ to S. □

Finally, we show that sobrification commutes with binary products:

Proposition 2.7 If X is the sobrification of $A \subseteq X$ and Y is the sobrification of $B \subseteq Y$, then $X \times Y$ is the sobrification of $A \times B$.

Proof: Let (x, y) be in an open set W of $X \times Y$. Then there are open sets U of X and V of Y such that $(x, y) \in U \times V \subseteq W$. By hypothesis, there is a in A with $a \sqsubseteq x$ and $a \in U$, and b in B with $b \sqsubseteq y$ and $b \in V$. Thus, $(a, b) \sqsubseteq (x, y)$ and $(a, b) \in U \times V \subseteq W$. □

2.13 Numbers

Let $\mathbf{R}_+$ be the set of positive real numbers including 0, but without ∞, and let $\overline{\mathbf{R}}_+$ be $\mathbf{R}_+$ together with ∞. Similarly, $\mathbf{N}_0$ is the set of natural numbers including 0, and $\overline{\mathbf{N}}_0$ is $\mathbf{N}_0$ together with ∞. Arithmetic is extended to $\overline{\mathbf{R}}_+$ and $\overline{\mathbf{N}}_0$ by $x + \infty = \infty + x = \infty$ for all x, $x \cdot \infty = \infty \cdot x = \infty$ for all $x \neq 0$, and $0 \cdot \infty = \infty \cdot 0 = 0$. Subtraction $x - y$ is only defined if $x \geq y$ and $x \neq \infty$.

The set $\overline{\mathbf{R}}_+$ is ordered in the standard way, which yields a dcpo. It is given the Scott topology. Hence, the open sets of $\overline{\mathbf{R}}_+$ are $\emptyset$, $\overline{\mathbf{R}}_+$ itself, and all the sets $\{x \in \overline{\mathbf{R}}_+ \mid x > r\}$ for fixed numbers $r < \infty$. This space is sober. Addition and multiplication as defined above are continuous.

The subsets $\mathbf{R}_+$, $\overline{\mathbf{N}}_0$, and $\mathbf{N}_0$ are considered as subspaces of $\overline{\mathbf{R}}_+$. The subspace $\overline{\mathbf{N}}_0$ is again a sober dcpo with its Scott topology. The subspaces $\mathbf{R}_+$ and $\mathbf{N}_0$ are neither sober nor dcpo's.

Sometimes, we shall need the split lemma for real numbers and integers.

Lemma 2.8 (Split Lemma)

Let $(r_i)_{i \in I}$ and $(s_j)_{j \in J}$ be two families of members of $\mathbf{R}_+$ [$\mathbf{N}_0$], where the index sets I and J are finite, and let $R \subseteq I \times J$ be a relation. For $T \subseteq I$, we write $R^+(T)$ for $\{j \in J \mid \exists i \in T : (i,j) \in R\}$.

If for all $T \subseteq I$, $\sum_{i \in T} r_i \leq \sum_{j \in R^+(T)} s_j$ holds,

then there are numbers t_{ij} in $\mathbf{R}_+$ [$\mathbf{N}_0$] for i in I and j in J with

(1) $\sum_{j \in J} t_{ij} = r_i$ for all i in I,

(2) $\sum_{i \in I} t_{ij} \leq s_j$ for all j in J,

(3) if $t_{ij} > 0$, then $(i,j) \in R$.

Proof: This is essentially the proof of the Splitting Lemma 4.10 of [6] or Lemma 9.2 of [7]. The Max-Flow Min-Cut Theorem 5.1 of [4] is applied to a graph with nodes $\perp$ (source), i in I, j in J, and $\top$ (sink); the index sets I and J are assumed to be disjoint. There are edges from $\perp$ to i with capacities r_i, from i to j with 'large' capacity C if $(i,j) \in R$ and 0 otherwise, and from j to $\top$ with capacities s_j, where C is a constant which is bigger than the sums of all occurring numbers. The remainder of the proof is in analogy to [6, 7] and thus omitted.

The $\mathbf{N}_0$-version follows from the integrity assertion of the Max-Flow Min-Cut Theorem: if all capacities are integers, then the maximal flow has integer values. □

An immediate consequence of the Split Lemma is *Hall's Theorem* [9, Theorem 1.1.3].

Theorem 2.9 (Hall's Theorem)

Let I and J be finite sets, and let $R \subseteq I \times J$ be a relation.

If for all $T \subseteq I$, $|T| \leq |R^+(T)|$ holds,

then there is an injective function $j : I \to J$ such that $(i, j(i)) \in R$ for all i in I.

Proof: Let $r_i = s_j = 1$ for all i and j. From the $\mathbf{N}_0$-version of the Split Lemma, there are numbers t_{ij} in $\mathbf{N}_0$. Let $j(i) = j$ iff $t_{ij} = 1$. □

3 Valuations

In this section, we define valuations and their potential continuity properties. Then, some operations on valuations are introduced, e.g., addition of two valuations, multiplication by a real number, restriction and corestriction to an open set.

3.1 Definition and Continuity Properties

A *valuation* on a topological space X is a function $\nu : \Omega X \to \overline{\mathbf{R}}_+$ with the following properties:

- $\nu\emptyset = 0$ (strictness).
- $\nu(U \cup V) + \nu(U \cap V) = \nu U + \nu V$ for all opens U and V (modularity).
- $\nu U \leq \nu V$ for all opens U and V with $U \subseteq V$ (monotonicity).

Valuations are partially ordered by defining $\nu \sqsubseteq \nu'$ iff $\nu O \leq \nu' O$ for all O in ΩX. A valuation ν is *bounded* if $\nu(X) < \infty$.

Mostly, we shall consider valuations with an additional continuity condition. There are several such conditions according to which topology is chosen for ΩX.

(1) A valuation ν is *Scott continuous* iff $\nu : \Omega_{\mathrm{s}} X \to \overline{\mathbf{R}}_+$ is continuous. Equivalently, for every directed family $(V_i)_{i \in I}$ of opens, $\nu(\bigcup_{i \in I} V_i) = \bigsqcup_{i \in I} \nu V_i$ holds.

Since every Scott continuous function is monotonic, the condition of monotonicity in the definition of valuations becomes redundant once we consider Scott continuous valuations.

(2) A valuation ν is *point continuous* iff $\nu : \Omega_{\mathrm{p}} X \to \overline{\mathbf{R}}_+$ is continuous. Equivalently, $\nu^{-}\{s \in \overline{\mathbf{R}}_+ \mid s > r\}$ is point open for every r in $\mathbf{R}_+$, or: for every open O and number r in $\mathbf{R}_+$ with $\nu O > r$, there is some finite $F \subseteq O$ such that $F \subseteq O'$ implies $\nu O' > r$.

Since the topology of $\Omega_{\mathrm{p}} X$ is a subset of that of $\Omega_{\mathrm{s}} X$, we obtain:

Proposition 3.1 Every point continuous valuation is Scott continuous.

Remark: There is a notion of continuity in between point and Scott continuity. It uses the topology on ΩX which is generated by the sets $\mathcal{O}(K)$ with compact K. A valuation ν is continuous in this sense if for $\nu O > r$, there is some compact $K \subseteq O$ such that $K \subseteq O'$ implies $\nu O' > r$. This notion of continuity will not be considered in this paper since we did not find any remarkable properties for it.

3.2 Special Properties of Scott Continuous Valuations

Scott continuous valuations have some special properties which are needed later.

Proposition 3.2 Let X be a topological space with a base $\mathcal{B}$ which is closed under binary intersection. Then every Scott continuous valuation on X is uniquely determined by its values on members of $\mathcal{B}$.

Proof: Let ν be a Scott continuous valuation on X. First we show that the values of ν on finite unions $\nu(B_1 \cup \cdots \cup B_n)$ of members of $\mathcal{B}$ are uniquely determined. This is done by induction on n.

Case $n = 0$: $\nu\emptyset$ must be 0 by strictness.
Case $n+1$: By modularity,
$\nu(B_1 \cup \cdots \cup B_n \cup B_{n+1}) + \nu((B_1 \cap B_{n+1}) \cup \cdots \cup (B_n \cap B_{n+1})) = \nu(B_1 \cup \cdots \cup B_n) + \nu B_{n+1}$ holds. If one of the two terms on the right hand side is ∞, then $\nu(B_1 \cup \cdots \cup B_n \cup B_{n+1})$ must be ∞ by monotonicity. If they are finite, then the two terms on the left hand side must be finite as well, and $\nu(B_1 \cup \cdots \cup B_n \cup B_{n+1})$ is uniquely determined by the other three terms which are uniquely determined by the induction hypothesis and the general hypothesis.

Arbitrary opens of X are directed unions of finite unions of members of $\mathcal{B}$. Hence, the values of ν on arbitrary opens are uniquely determined by Scott continuity. □

Lemma 3.3 For every Scott continuous valuation ν, there is a closed set C such that $\nu O > 0$ iff $O \between C$.

Proof: Consider the set $\mathcal{W}$ of all open sets O with $\nu O = 0$. By strictness and modularity, this set is directed. Let $W = \bigcup \mathcal{W}$. By Scott continuity, $\nu W = 0$ holds. Hence, $\nu O = 0$ iff $O \subseteq W$. Negating both sides, we obtain $\nu O > 0$ iff $O \between \neg W$. Let C be the closed set $\neg W$. □

3.3 Operations on Valuations

In this subsection, we present some basic ways to obtain valuations from other valuations or from scratch.

(1) The *zero function* $\lambda U.\, 0$, which maps every open set to 0, is a bounded valuation. As a constant function, it has all continuity properties you like; thus, it is point continuous.

(2) For every point x of X, there is a bounded valuation $\widehat{x}$, where $\widehat{x}(U)$ is 1 if $x \in U$, and 0 otherwise. Valuations of the form $\widehat{x}$ are called *point valuations*. Since $x \in U$ and $x \sqsubseteq y$ implies $y \in U$, $x \sqsubseteq y$ implies $\widehat{x} \sqsubseteq \widehat{y}$.

Every point valuation is point continuous. For, if $\widehat{x}(U) > r$, then $r < 1$ and $x \in U$. Choose $\{x\}$ as the finite set F in the characterization of point continuity.

(3) If r is a constant from $\overline{\mathbf{R}}_+$ and ν is a valuation, then $r \cdot \nu$ is a valuation, where $(r \cdot \nu)(U) = r \cdot \nu U$. If $r < \infty$ and ν is bounded, then $r \cdot \nu$ is bounded.

Since $\lambda s.\, r \cdot s : \overline{\mathbf{R}}_+ \to \overline{\mathbf{R}}_+$ is continuous and compositions of continuous functions are continuous, we obtain:

- If ν is Scott continuous / point continuous, then so is $r \cdot \nu$.

(4) If ν_1 and ν_2 are two valuations on X, then so is $\nu_1 + \nu_2$, where $(\nu_1 + \nu_2)(U) = \nu_1 U + \nu_2 U$. If both ν_1 and ν_2 are bounded, then so is $\nu_1 + \nu_2$.

Since $+ : \overline{\mathbf{R}}_+ \times \overline{\mathbf{R}}_+ \to \overline{\mathbf{R}}_+$ is continuous and $\nu_1 + \nu_2 = (+) \circ (\nu_1 \times \nu_2)$, we obtain:

- If ν_1 and ν_2 are Scott continuous / point continuous, then so is $\nu_1 + \nu_2$.

(5) Every directed family $(\nu_i)_{i \in I}$ of valuations has a least upper bound, namely the valuation $\bigsqcup_{i \in I} \nu_i$, which is defined by $(\bigsqcup_{i \in I} \nu_i)(U) = \bigsqcup_{i \in I} \nu_i U$. Even if all ν_i are bounded, $\bigsqcup_{i \in I} \nu_i$ may be unbounded.

For all open sets W of $\overline{\mathbf{R}}_+$, $(\bigsqcup_{i \in I} \nu_i)(U)$ is in W iff $\nu_i(U)$ in W for some i in I. Hence, $(\bigsqcup_{i \in I} \nu_i)^{-} W = \bigcup_{i \in I} \nu_i^{-} W$. Thus we obtain:

- If all ν_i are Scott continuous / point continuous, then so is $\bigsqcup_{i \in I} \nu_i$.

(6) If $f : X \to Y$ is continuous, then every valuation ν on X induces a valuation $\nu \circ f^-$ on Y. This operation maps point valuations to point valuations since $\widehat{x} \circ f^- = \widehat{fx}$. The valuation $\nu \circ f^-$ is bounded iff ν is bounded.

For the continuity properties, we must consider $f^- : \Omega Y \to \Omega X$. This function is Scott continuous. For every x in X and V in ΩY, $f^- V \in \mathcal{O}(x)$ iff $fx \in V$ iff $V \in \mathcal{O}(fx)$ holds, whence $f^- : \Omega_{\mathrm{p}} X \to \Omega_{\mathrm{p}} X$ is continuous as well. Thus, we obtain:

- If ν is Scott continuous / point continuous, then so is $\nu \circ f^-$.

(7) Let ν_1 and ν_2 be valuations on X, where ν_1 is bounded and $\nu_1 \sqsupseteq \nu_2$ holds. Then $\nu_1 - \nu_2$ with $(\nu_1 - \nu_2)(U) = \nu_1 U - \nu_2 U$ is a bounded strict modular function from ΩX to $\overline{\mathbf{R}}_+$. We require that ν_1 is bounded to avoid differences involving ∞. The condition $\nu_1 \sqsupseteq \nu_2$ is needed to ensure that $\nu_1 - \nu_2$ yields values in $\overline{\mathbf{R}}_+$.

Even if ν_1 and ν_2 are monotonic, the difference $\nu_1 - \nu_2$ may not be monotonic. On the other hand, monotonicity of the difference is sufficient to derive stronger continuity results.

Proposition 3.4 Let ν_1 be bounded and ν_2 be monotonic so that $\nu_1 \sqsupseteq \nu_2$, and $\nu_1 - \nu_2$ is monotonic. Then $\nu_1 - \nu_2$ is a valuation. If ν_1 is Scott continuous / point continuous, then so is $\nu_1 - \nu_2$.

Interestingly, this holds without requiring the corresponding kind of continuity for ν_2.

Proof: For Scott continuity, let $(O_i)_{i \in I}$ be a directed family of open sets. We have to show $(\nu_1 - \nu_2)(\bigcup_{i \in I} O_i) = \bigsqcup_{i \in I} (\nu_1 - \nu_2)(O_i)$. The relation '$\geq$' follows from monotonicity of $\nu_1 - \nu_2$ which is part of the hypothesis. For '$\leq$', we have to show

$$\nu_1(\bigcup_{i \in I} O_i) \leq \bigsqcup_{i \in I} (\nu_1 O_i - \nu_2 O_i) + \nu_2(\bigcup_{i \in I} O_i)$$

By Scott continuity of ν_1, the left hand side is $\bigsqcup_{i \in I} \nu_1 O_i$. Fix some i in I.

$$\nu_1 O_i \;=\; \nu_1 O_i - \nu_2 O_i + \nu_2 O_i \;\leq\; \bigsqcup_{i \in I} (\nu_1 O_i - \nu_2 O_i) + \nu_2(\bigcup_{i \in I} O_i)$$

holds using monotonicity of ν_2.

For point continuity, assume $\nu_1 O - \nu_2 O > r$. Then $\nu_1 O > r + \nu_2 O$. By point continuity of ν_1, there is a finite $F \subseteq O$ such that $F \subseteq O'$ implies $\nu_1 O' > r + \nu_2 O$. We claim that $F \subseteq O'$ also implies $(\nu_1 - \nu_2)(O') > r$.

$$\begin{array}{rcll} (\nu_1 - \nu_2)(O') & \geq & (\nu_1 - \nu_2)(O \cap O') & (\nu_1 - \nu_2 \text{ is monotonic}) \\ & > & r + \nu_2 O - \nu_2(O \cap O') & (F \subseteq O \cap O') \\ & \geq & r & (\nu_2 \text{ is monotonic}) \qquad \square \end{array}$$

3.4 Restriction and Corestriction of Valuations

Let ν be a valuation on X, and let W be an open set of X. The *restriction* $\nu|_W$ of ν to W is defined by $\nu|_W(U) = \nu(W \cap U)$ for every U in ΩX. This is again a valuation on X. It is bounded iff $\nu(W) < \infty$. This holds in particular if ν is bounded.

For the continuity conditions, we have to consider the function $\lambda U.\, W \cap U : \Omega X \to \Omega X$. Obviously, it is Scott continuous. For every x in W, $W \cap U$ is in $\mathcal{O}(x)$ iff U is in $\mathcal{O}(x)$. For every x not in W, $W \cap U$ is in $\mathcal{O}(x)$ iff U is in $\emptyset$. Hence, $\lambda U.\, W \cap U : \Omega_{\mathrm{p}} X \to \Omega_{\mathrm{p}} X$ is continuous. Thus, we obtain:

- If ν is Scott continuous / point continuous, then so is $\nu|_W$.

For iterated restrictions, the following equations are obvious: $(\nu|_U)|_V = (\nu|_V)|_U = \nu|_{U\cap V}$. Furthermore, $\nu|_X = \nu$ and $\nu|_\emptyset = 0$ holds. If $O \subseteq W$, then $\nu|_W O = \nu O$, and if $O \cap W = \emptyset$, then $\nu|_W O = 0$. If U and V are open sets with $W \cap U = W \cap V$, then $\nu|_W U = \nu|_W V$.

Let ν be a *bounded* valuation on X, and W an open set of X. The *corestriction* $\nu|^W$ of ν to W is defined by $\nu|^W = \nu - \nu|_W$. Hence, $\nu|^W(U) = \nu U - \nu(U \cap W)$. By modularity, this is equal to $\nu(U \cup W) - \nu W$.

We require ν to be bounded in order to avoid problems with undefined differences. Monotonicity of ν guarantees $\nu \sqsupseteq \nu|_W$. Because of $\nu|^W(U) = \nu(U \cup W) - \nu W$, monotonicity of $\nu|^W$ follows from monotonicity of ν. Using Prop. 3.4, we obtain:

- If ν is a bounded valuation, then so is $\nu|^W$.
- If ν is in addition Scott continuous / point continuous, then so is $\nu|^W$.

As additional properties, we have $\nu|^X = \nu - \nu|_X = \nu - \nu = 0$ and $\nu|^\emptyset = \nu - \nu|_\emptyset = \nu - 0 = \nu$. If $O \subseteq W$, then $\nu|^W O = \nu O - \nu(O \cap W) = 0$, and if $O \cap W = \emptyset$, then $\nu|^W O = \nu O$. For two opens U and V with $W \cup U = W \cup V$, $\nu|^W U = \nu|^W V$ holds. Note that $W \cup U = W \cup V$ iff $\neg W \cap U = \neg W \cap V$.

For iterated corestriction, we claim $(\nu|^U)|^V = \nu|^{U\cup V}$. For,

$$\nu|^U|^V(O) = \nu|^U(V \cup O) - \nu|^U(V) = \nu(U \cup V \cup O) - \nu U - \nu(U \cup V) + \nu U = \nu|^{U\cup V}(O).$$

By the definition of corestriction, restriction and corestriction are related by the equation $\nu = \nu|_W + \nu|^W$ for all open sets W. We call this a partition of ν along W.

Restrictions and corestrictions commute with each other: $(\nu|_U)|^V = (\nu|^V)|_U$. For,

$$\nu|_U|^V(O) = \nu|_U(O) - \nu|_U(V \cap O) = \nu(U \cap O) - \nu(U \cap V \cap O) = \nu|^V(U \cap O) = \nu|^V|_U(O).$$

We shall use the abbreviation $\nu|_U^V$ for $\nu|_U|^V = \nu|^V|_U$.

The results of this subsection are summarized in the following theorem.

Theorem 3.5

	Restriction $\nu\|_W$	Corestriction $\nu\|^W$
defined by	$\nu\|_W(O) = \nu(W \cap O)$	$\nu\|^W(O) = \nu O - \nu(W \cap O)$
or		$\nu\|^W(O) = \nu(W \cup O) - \nu W$
when defined?	always	if ν is bounded
Dependencies:	$W \cap U = W \cap V \Rightarrow \nu\|_W U = \nu\|_W V$	$\neg W \cap U = \neg W \cap V \Rightarrow \nu\|^W U = \nu\|^W V$
$O \subseteq W \Rightarrow$	$\nu\|_W O = \nu O$	$\nu\|^W O = 0$
$O \cap W = \emptyset \Rightarrow$	$\nu\|_W O = 0$	$\nu\|^W O = \nu O$
Whole space:	$\nu\|_X = \nu$	$\nu\|^X = 0$
Empty set:	$\nu\|_\emptyset = 0$	$\nu\|^\emptyset = \nu$
Iteration:	$(\nu\|_U)\|_V = (\nu\|_V)\|_U = \nu\|_{U\cap V}$	$(\nu\|^U)\|^V = (\nu\|^V)\|^U = \nu\|^{U\cup V}$
Connections:	$(\nu\|_U)\|^V = (\nu\|^V)\|_U$	$\nu = \nu\|_W + \nu\|^W$

If ν is Scott continuous / point continuous, then so are $\nu|_W$ and $\nu|^W$.

4 A Taxonomy of Valuations

In this section, some classes of valuations are defined and their relationships are investigated.

$$
\begin{tikzcd}
 & \text{bounded} & & \\
\text{primitive} \arrow[r] & \text{simple} \arrow[u] \arrow[r] & \text{point continuous} \arrow[r] & \text{Scott continuous} \\
\text{point valuation} \arrow[u] \arrow[r] & \text{finite} \arrow[u] & &
\end{tikzcd}
$$

Figure 1: Classes of valuations

4.1 Some more Classes of Valuations

We already know the classes of Scott continuous, point continuous, and bounded valuations. Here, we define some more classes and consider their relationships.

- Valuations $\widehat{x}$ for some x in X are *point valuations.*
- Finite linear combinations of point valuations $r_1 \cdot \widehat{x_1} + \cdots + r_n \cdot \widehat{x_n}$ with $r_i < \infty$ are called *finite.*
- A Scott continuous valuation ν with $\Im\nu = \{0, 1\}$ is called *primitive.*
- A Scott continuous valuation ν whose image $\Im\nu$ is finite and does not contain ∞ is called *simple.*

The classes of finite and of simple valuations are closed under addition, multiplication by a finite scalar, restriction, and corestriction. The inclusions among the various classes are depicted in Figure 1. Most of the inclusions are obvious or were already handled. Point continuity of simple valuations is proved below. In Section 5, finite valuations are studied in greater detail.

4.2 Simple and Primitive Valuations

There are various further relationships among the valuation classes. We start with the left part of the middle line of Figure 1.

Theorem 4.1

A valuation is simple iff it is a finite linear combination of primitive valuations.

Proof: Finite linear combinations of primitive valuations are obviously simple. For the opposite direction, we use induction on the size of the finite set $\Im\nu$ for simple valuations ν. Because of strictness, $\Im\nu$ always contains 0.

If $\Im\nu = \{0\}$, then ν is the empty sum of primitive valuations. Otherwise, let r be the least non-zero element of $\Im\nu$, and let W be some open with $\nu W = r$. Since ν is bounded and monotonic, it can be restricted and corestricted to W, and we obtain $\nu = \nu|_W + \nu|^W$. Because of the minimality of r, the valuation $\nu|_W$ with $\nu|_W(U) = \nu(U \cap W)$ assumes the two values 0 and r only. Thus $\pi = 1/r \cdot \nu|_W$ is a well-defined primitive valuation, and $\nu = r \cdot \pi + \nu|^W$.

The proof is completed once we have shown that $\Im(\nu|^W)$ is strictly smaller than $\Im\nu$. The following argument for showing this is taken from a proof in [8].

Let $f : \Im(\nu|^W) \to \mathbf{R}_+$ be defined by $fs = r + s$. If $s = \nu|^W(U)$ for some U, then $s = \nu(U \cup W) - \nu W$. Since $\nu W = r$, $r + s = \nu(U \cup W) \in \Im\nu$ follows. Hence, we obtain $f : \Im(\nu|^W) \to \Im\nu$. This function is injective, but not surjective. For, 0 is in $\Im\nu$, but $fs \geq r > 0$ for all s in $\Im(\nu|^W)$. Thus, $\Im(\nu|^W)$ is strictly smaller than $\Im\nu$. □

Using this characterization of simple valuations, we can show:

Proposition 4.2 Every simple valuation is point continuous.

Proof: Since the class of point continuous valuations is closed under finite linear combinations, we only need to show that primitive valuations are point continuous.

Let π be primitive. By Lemma 3.3, there is a closed set C such that $\pi O > 0$ iff $O \circledast C$. Because of primitivity, $\pi O > 0$ is equivalent to $\pi O = 1$.

Let $\pi O > r$ for some open O and r in $\mathbf{R}_+$. Then $r < 1$ and $\pi O = 1$. Let x be a point of $O \cap C$. Then $\{x\}$ is a finite subset of O, and $\{x\} \subseteq O'$ implies $O' \circledast C$, whence $\pi O' = 1 > r$.□

4.3 Valuations on Sober Spaces

In case of sober spaces, some of the classes of Figure 1 coincide. We start with the following lemma about primitive valuations.

Lemma 4.3 There is a one-to-one correspondence between primitive valuations π on X and prime filters $\mathcal{O}$ of open sets. The correspondence is defined by $\pi O = 1$ iff $O \in \mathcal{O}$ and $\pi O = 0$ iff $O \notin \mathcal{O}$. Point valuations $\widehat{x}$ correspond to prime filters $\mathcal{O}(x)$.

Proof: Arbitrary functions $\pi : \Omega X \to \{0, 1\}$ are in one-to-one correspondence with subsets $\mathcal{O}$ of ΩX by $O \in \mathcal{O}$ iff $\pi O = 1$. The function π assumes the value 1 iff $\mathcal{O}$ contains X, is Scott continuous iff $\mathcal{O}$ is Scott open in $(\Omega X, \subseteq)$, strict iff $\mathcal{O}$ does not contain $\emptyset$, and modular iff $\mathcal{O}$ is closed under binary intersection and inaccessible by binary union. □

From this lemma, it is obvious that there are close connections to sobriety.

Theorem 4.4 For a $\mathcal{T}_0$-space X, the following statements are equivalent:

(1) X is sober.

(2) Every primitive valuation of X is a point valuation.

(3) Every simple valuation on X is finite.

(4) Every primitive valuation on X is finite.

Proof:

(1) ⇒ (2): Using Lemma 4.3; in a sober space, every prime filter $\mathcal{O}$ is the neighborhood filter $\mathcal{O}(x)$ of some point x.

(2) ⇒ (3): By Theorem 4.1 and part (2), every simple valuation is a finite linear combination of point valuations, i.e., finite.

(3) ⇒ (4): Every primitive valuation is simple.

(4) ⇒ (1): Let $\mathcal{O}$ be a prime filter. By Lemma 4.3, there is a primitive valuation π with $\pi O = 1$ iff $O \in \mathcal{O}$. By assumption, π is finite, whence $\pi = \sum_{i \in I} r_i \cdot \widehat{x}_i$ for some finite index set I. Since $X \in \mathcal{O}$, or $\pi(X) = 1$, there is some i in I with $r_i > 0$. We claim $\mathcal{O} = \mathcal{O}(x_i)$.

If x_i in O, then $\pi O \geq r_i > 0$, whence $\pi O = 1$, i.e., $O \in \mathcal{O}$. For the opposite direction, let $O \in \mathcal{O}$ and assume $x_i \notin O$. Then $1 = \pi(X) \geq r_i + \pi(O) = r_i + 1 > 1$, which is impossible. $\square$

4.4 Valuations on Locally Finitary Spaces

A subset F of a space X is *finitary* iff $F = \uparrow E$ for some finite set E. The space X is *locally finitary* iff for every point x in X and open U of X with $x \in U$, there are a finitary set F and an open V such that $x \in V \subseteq F \subseteq U$.

Since finitary sets are compact, every locally finitary space is locally compact. Every continuous dcpo (with its Scott topology) is locally finitary. For, if x in U, there is some $y \ll x$ with y in U, whence $x \in \Uparrow y \subseteq \uparrow y \subseteq U$. (We do not include the definitions of continuous dcpo's and of compactness and local compactness because they are not needed in this paper.) A locally finitary $\mathcal{T}_1$-space is discrete.

Theorem 4.5
Every Scott continuous valuation on a locally finitary space is point continuous.

Proof: Let ν be a Scott continuous valuation and assume $\nu U > r$ for some open U and $r < \infty$. Let $\mathcal{V}$ be the set of all open sets V such that there is a finitary set F with $V \subseteq F \subseteq U$. Since the union of two open / finitary sets is again so, $\mathcal{V}$ is directed. Because of local finitariness, the union of $\mathcal{V}$ is U. Since ν is Scott continuous, there is some V in $\mathcal{V}$ such that $\nu V > r$. Let E be a finite set with $V \subseteq \uparrow E \subseteq U$. If $E \subseteq O'$, then $V \subseteq O'$, whence $\nu O' > r$. $\square$

Corollary 4.6
On a continuous dcpo, every Scott continuous valuation is point continuous.

In general, the notions of point continuity and Scott continuity differ. For instance, the length or Lebesgue measure on the unit interval of the reals with the standard Hausdorff topology induces a bounded Scott continuous valuation which is not point continuous.

4.5 Approximation by Bounded Valuations

Every valuation can be approximated by bounded valuations.

Theorem 4.7 Every Scott continuous / point continuous valuation can be obtained as a directed join of bounded Scott continuous / point continuous valuations.

Proof: Let ν be a Scott continuous valuation. Let $\Phi(\nu)$ be the set of all opens O with $\nu O < \infty$, and let C be $\neg \bigcup \Phi(\nu)$, a closed set. For V in $\Phi(\nu)$, $F \subseteq_{fin} C$, and $n \in \mathbf{N}_0$, let $\nu_{V,F,n} = \nu|_V + \sum_{x \in F} n \cdot \widehat{x}$. Since $\nu_{V,F,n}(X) = \nu(V) + n \cdot |F|$, this is a bounded valuation. If ν is point continuous, then so is $\nu_{V,F,n}$, since this property is enjoyed by point valuations, and preserved by restriction to open sets, addition, and multiplication by finite numbers.

Let $D = \{\nu_{V,F,n} \mid V \in \Phi(\nu),\ F \subseteq_{fin} C,\ n \in \mathbf{N}_0\}$. This set is not empty since it contains $\nu_{\emptyset,\emptyset,0}$, and directed, since ν_{V_1,F_1,n_1} and ν_{V_2,F_2,n_2} are bounded by $\nu_{V_1 \cup V_2, F_1 \cup F_2, n_1 \sqcup n_2}$. Let $\nu' = \bigsqcup D$. We claim $\nu' = \nu$.

If $O \circledast C$, let x be in $O \cap C$. Then $\nu' O \geq (n \cdot \widehat{x})(O) = n$ for all n in $\mathbf{N}_0$, whence $\nu' O = \infty$. On the other hand, if νO were finite, then $O \subseteq \bigcup \Phi(\nu) = \neg C$ in contradiction to $O \circledast C$.

If $O \subseteq \bigcup \Phi(\nu)$, then $O = \bigcup \{O \cap V \mid V \in \Phi(\nu)\}$. By modularity, $\Phi(\nu)$ is directed.

$$\begin{aligned} \nu O &= \textstyle\bigsqcup_{V \in \Phi(\nu)} \nu(O \cap V) \quad (\nu \text{ is Scott continuous}) \\ &= \textstyle\bigsqcup_{V \in \Phi(\nu)} \nu|_V(O) \\ &= \textstyle\bigsqcup_{\nu^* \in D} \nu^*(O) \qquad (\nu_{V,F,n}(O) = \nu|_V(O)) \\ &= \nu' O \end{aligned}$$ □

5 Finite Valuations

In this section, we consider the finite valuations in greater detail. We introduce a standard representation by *finite point densities*, and characterize equality and order of the valuations in terms of the representing point densities.

5.1 Representing Finite Valuations by Point Densities

A valuation ν on a space X is finite if it is a finite linear combination of point valuations, i.e., if $\nu = r_1 \cdot \widehat{x_1} + \cdots + r_n \cdot \widehat{x_n}$ where $0 \leq r_i < \infty$ and x_i in X. This representation is not unique since summands may be permuted, summands with coefficient 0 may be omitted, and two summands with the same point may be combined into one. These three kinds of ambiguities can be avoided by writing $\nu = \sum_{x \in X} s_x \cdot \widehat{x}$ where $s_x \in \mathbf{R}_+$ with $s_x = 0$ for all but a finite number of x. Hence, ν can be represented by a function $x \mapsto s_x$. We call such functions point densities.

A *finite point density* or shortly *density* on a space X is a function $A : X \to \mathbf{R}_+$ whose *support* $\$A = \{x \in X \mid Ax > 0\}$ is finite. A density need not be continuous or even monotonic in any sense. We are interested in the finite valuation $A^* = \sum_{x \in X} Ax \cdot \widehat{x} = \sum_{x \in \$A} Ax \cdot \widehat{x}$ induced by the point density A, and in criteria for equality $A^* = B^*$ and order $A^* \sqsubseteq B^*$ on valuations stated in terms of the representing point densities A and B.

5.2 Finite Point Densities and their Action on Subsets

A density A is below a density B — $A \leq B$ — iff for all x in X, $Ax \leq Bx$ holds. Given a density A on X and an arbitrary subset S of X, we define $A[S] = \sum_{x \in S} Ax = \sum_{x \in S \cap \$A} Ax$. Since $\$A$ is finite and all Ax are finite, $A[X]$ and thus all $A[S]$ are finite. For the special case of an open set O, $A[O] = A^*(O)$ holds, where $A^* = \sum_{x \in X} Ax \cdot \widehat{x}$ is the finite valuation defined above.

The elementary properties of the notion $A[S]$ are as follows:

Proposition 5.1 For all finite point densities A on X and subsets S and T of X:

(1) $A[\emptyset] = 0$.

(2) If $S \cap T = \emptyset$, then $A[S \cup T] = A[S] + A[T]$.

(3) $A[S \cup T] + A[S \cap T] = A[S] + A[T]$.

(4) If $\mathcal{D}$ is a directed set of subsets of X, then $A[\bigcup \mathcal{D}] = \bigsqcup_{S \in \mathcal{D}} A[S]$.

(5) If $\mathcal{D}$ is $\supseteq$-directed, then $A[\bigcap \mathcal{D}] = \sqcap_{S \in \mathcal{D}} A[S]$.

Proof: From the definition of $A[S]$, it is not difficult to prove (1), (2), and (4).

Property (3) follows from (2) because of the following disjoint partitions: $S = (S \setminus T) \cup (S \cap T)$, $T = (T \setminus S) \cup (S \cap T)$, $S \cup T = (S \setminus T) \cup (T \setminus S) \cup (S \cap T)$.

Property (5) follows from (4) because of $A[S] = A[X] - A[X \setminus S]$ for all S in $\mathcal{D}$. □

Because of parts (1), (3), and (4) of the proposition, we see that every finite valuation $\nu = A^*$ can be extended to a strict modular Scott continuous function to $\mathbf{R}_+$ defined on *arbitrary* subsets of X. Because of part (5), this extended function is even a measure. Hence, finite valuations can be extended to measures which are defined not only on the Borel sets, but on all subsets of X. Note that this extension is not unique if $\nu = A^* = B^*$ for two different point densities A and B. In Subsection 5.4, we shall see that this is impossible in a $\mathcal{T}_0$-space.

5.3 Operations on Densities

The *zero density* 0 is the function $\lambda x.\, 0$. Obviously, $0[S] = 0$ holds for all $S \subseteq X$, whence 0^* is the zero valuation 0.

The *sum* $A + B$ of two densities A and B is defined pointwise: $(A + B)x = Ax + Bx$. Obviously, $(A+B)[S] = A[S]+B[S]$ holds for all subsets S of X, whence $(A+B)^* = A^*+B^*$.

The *product* $r \cdot A$ of a density A by a factor r in $\mathbf{R}_+$ is defined by $(r \cdot A)x = r \cdot Ax$. Obviously, $(r \cdot A)[S] = r \cdot A[S]$ holds for all subsets S of X, whence $(r \cdot A)^* = r \cdot A^*$.

For every point x, there is a density A_x with $A_x u = 1$ if $u = x$, and $= 0$ otherwise. Obviously, $A_x[S] = 1$ holds if $x \in S$, and $= 0$ if $x \notin S$. Thus, $(A_x)^*$ is the point valuation $\widehat{x}$.

The *restriction* $A|_W$ of a density A to an open set W is defined by $(A|_W)x = Ax$ if x in W, and $= 0$ otherwise. Obviously, $(A|_W)[S] = A[W \cap S]$ holds for all $S \subseteq X$, whence $(A|_W)^* = (A^*)|_W$.

The *corestriction* $A|^W$ of a density A to an open set W is defined by $(A|^W)x = Ax$ if x *not* in W, and $= 0$ otherwise. Obviously, $(A|^W)[S] = A[\neg W \cap S]$ holds for all $S \subseteq X$. Since S is the disjoint union of $W \cap S$ and $\neg W \cap S$, $(A|^W)[S] = A[S] - A[W \cap S]$ follows. Thus, $(A|^W)^* = (A^*)|^W$ holds.

As a consequence of these results, we see that the class of finite valuations on a space X is closed under addition, multiplication by scalars, restriction and corestriction.

If $f : X \to Y$ is continuous, and $\nu = \sum_{i \in I} r_i \cdot \widehat{x_i}$ is a finite valuation on X, then $\nu \circ f^-$ is a finite valuation on Y, namely $\nu \circ f^- = \sum_{i \in I} r_i \cdot \widehat{f x_i}$. In terms of point densities, $A^* \circ f^- = B^*$ holds where $By = \sum_{x \in f^-\{y\}} Ax$.

5.4 Uniqueness of Representation

In this subsection, we show that two different densities cannot represent the same valuation. We start with some auxiliary properties.

Proposition 5.2 If A is a density, and S an upper set in a space X, then $A[S] = \sqcap\{A^*(O) \mid O \text{ open } \supseteq \mathrm{S}\}$.

Proof: By Prop. 5.1 (5), since $S = \bigcap_{O \supseteq S} O$. $\square$

Lemma 5.3 Let X be a $\mathcal{T}_0$-space, A a density on X, and x a point of X. Then

$$Ax = \sqcap\{A^*(O) \mid x \in O \in \Omega X\} - \sqcap\{A^*(O) \mid O \in \Omega X, \uparrow x \subseteq O \cup \{x\}\}.$$

Proof: Since $\uparrow x$ is the disjoint union of $\{x\}$ and $\uparrow x \setminus \{x\}$, we obtain $A[\uparrow x] = A[\{x\}] + A[\uparrow x \setminus \{x\}]$ by Prop. 5.1 (2). Hence, $Ax = A[\uparrow x] - A[\uparrow x \setminus \{x\}]$.

The two sets $\uparrow x$ and $\uparrow x \setminus \{x\}$ are upper sets; the latter because of the $\mathcal{T}_0$ property. Thus, Prop. 5.2 can be applied. For an open set O, $\uparrow x \subseteq O$ holds iff $x \in O$, and $\uparrow x \setminus \{x\} \subseteq O$ iff $\uparrow x \subseteq O \cup \{x\}$. $\square$

Now we can prove the uniqueness of representation.

Theorem 5.4 For two densities A and B in a $\mathcal{T}_0$-space, $A^* = B^*$ implies $A = B$.

Proof: For every x in X, Ax and Bx can be expressed in terms of $A^* = B^*$ by the formula of Lemma 5.3. Thus, Ax and Bx are equal. $\square$

Corollary 5.5 For every finite valuation ν on a $\mathcal{T}_0$-space, there is a unique finite point density A such that $\nu = A^*$.

The $\mathcal{T}_0$ property is really needed. Consider the space $X = \{a, b\}$ where $\emptyset$ and X are the only open sets. The finite point densities A with $Aa = 1$ and $Ab = 0$, and B with $Ba = 0$ and $Bb = 1$ are different, but induce the same valuation ν with $\nu(\emptyset) = 0$ and $\nu(X) = 1$.

5.5 The Valuation Order in terms of Densities

Our goal in this subsection is to find a criterion for $A^* \sqsubseteq B^*$ in terms of the finite point densities A and B.

Theorem 5.6 For two finite point densities A and B on a space X, the following statements are equivalent:

(1) $A^* \sqsubseteq B^*$;

(2) $A[U] \leq B[U]$ for all opens sets U;

(3) $A[U] \leq B[U]$ for all upper sets U;

(4) $A[F] \leq B[\uparrow F]$ for all finite sets F;

(5) for all $T \subseteq \$A$: $\sum_{x \in T} Ax \leq \sum_{y \in \$B \cap \uparrow T} By$;

(6) there are numbers t_{xy} in $\mathbf{R}_+$ for x in $\$A$ and $y \in \$B$ with

(a) $\sum_{y \in \$B} t_{xy} = Ax$ for all x in $\$A$,

(b) $\sum_{x \in \$A} t_{xy} \leq By$ for all y in $\$B$,

(c) if $t_{xy} > 0$, then $x \sqsubseteq y$.

Proof:

$(1) \Rightarrow (2)$ by definition.

$(2) \Rightarrow (3)$: $B[U] = \sqcap_{O \in \mathcal{O}(U)} B[O]$ holds by Prop. 5.2. For every such O, $B[O] \geq A[O] \geq A[U]$ holds by (2), whence $B[U] \geq A[U]$.

$(3) \Rightarrow (4)$: $\uparrow F$ is an upper set. Hence, $A[F] \leq A[\uparrow F] \leq B[\uparrow F]$.

$(4) \Rightarrow (5)$: $\sum_{x \in T} Ax = A[T] \leq B[\uparrow T] = \sum_{y \in \$B \cap \uparrow T} By$.

(5) ⇒ (6): We apply the Split Lemma 2.8 with $I = \$A$, $J = \$B$, and $(x,y) \in R$ iff $x \sqsubseteq y$, whence $R^+(T) = \{y \in J \mid \exists x \in T : x \sqsubseteq y\} = \$B \cap \uparrow T$.

(6) ⇒ (1) :
$$\begin{array}{rcl} A^* & = & \sum_{x\in\$A} Ax \cdot \widehat{x} \\ & = & \sum_{x\in\$A} \sum_{y\in\$B} t_{xy} \cdot \widehat{x} \\ & \overset{!}{\leq} & \sum_{x\in\$A} \sum_{y\in\$B} t_{xy} \cdot \widehat{y} \\ & = & \sum_{y\in\$B} \sum_{x\in\$A} t_{xy} \cdot \widehat{y} \\ & \leq & \sum_{y\in\$B} By \cdot \widehat{y} \\ & = & B^* \end{array}$$

The relation '$\overset{!}{\leq}$' holds, since $t_{xy} > 0$ implies $x \sqsubseteq y$, whence $\widehat{x} \sqsubseteq \widehat{y}$. □

6 Spaces of Valuations

In this section, we define various topological spaces of valuations and study their relationship.

Let $\mathsf{V}X$ be the set of all *Scott continuous valuations* on X, $\mathsf{V}_{\mathrm{p}}X$ the set of all *point continuous* valuations, and $\mathsf{V}_{\mathrm{f}}X$ the set of *finite* valuations. We topologize these sets as subspaces of the pointwise function space $[\Omega_{\mathrm{s}}X \to \overline{\mathbf{R}}_+]_{\mathrm{p}}$. Thus, the topology of $\mathsf{V}X$ is generated by the subbasic opens $\langle U > r\rangle = \{\nu \in \mathsf{V}X \mid \nu U > r\}$ where U ranges over the opens of X and r ranges over $\overline{\mathbf{R}}_+$ with $0 < r < \infty$. The order defined by this topology is $\nu \sqsubseteq \nu'$ iff $\nu O \leq \nu' O$ for all opens O.

In general, continuous operations on a space Y can be lifted to continuous operations on pointwise function spaces $[X \to Y]_{\mathrm{p}}$. Hence, addition $+ : \mathsf{V}X \times \mathsf{V}X \to \mathsf{V}X$ and multiplication $\cdot : \overline{\mathbf{R}}_+ \times \mathsf{V}X \to \mathsf{V}X$ are continuous.

The function $\mathsf{s} : X \to \mathsf{V}X$ with $\mathsf{s}x = \widehat{x}$ is continuous. For, $\mathsf{s}^-\langle U > r\rangle = U$ if $r < 1$, and $= \emptyset$ otherwise. This also shows that s^- is surjective, whence s is a topological embedding for $\mathcal{T}_0$-spaces X.

Every continuous function $f : X \to Y$ induces a function $\mathsf{V}f : \mathsf{V}X \to \mathsf{V}Y$ where $\mathsf{V}f\,(\nu) = \nu \circ f^-$. The function $\mathsf{V}f$ is linear w.r.t. addition and scalar multiplication of $\mathsf{V}X$. It is continuous since $\mathsf{V}f(\nu) \in \langle V > r\rangle$ iff $\nu \in \langle f^- V > r\rangle$. Thus, $(\mathsf{V}f)^-\langle V > r\rangle = \langle f^- V > r\rangle$. From this equation, we see that $(\mathsf{V}f)^-$ is surjective if f^- is surjective. Hence, $\mathsf{V}f$ is a topological embedding if f is an embedding.

The operation V has functorial properties, i.e., $\mathsf{V}\,\mathsf{id} = \mathsf{id}$ and $\mathsf{V}(g \circ f) = \mathsf{V}g \circ \mathsf{V}f$. Because of $\mathsf{V}f\,(\widehat{x}) = \widehat{fx}$, we obtain $\mathsf{V}f \circ \mathsf{s} = \mathsf{s} \circ f$, i.e., the operation s is 'natural' w.r.t. the functor V. Let us now consider the topological properties of the various spaces.

Proposition 6.1

For every space X, the spaces $\mathsf{V}X$ and $\mathsf{V}_{\mathrm{p}}X$ are sober, and $\mathsf{V}_{\mathrm{f}}X$ is a $\mathcal{T}_0$-space.

Proof: The space $\mathsf{V}X$ is a subspace of the pointwise function space $F = [\Omega_{\mathrm{s}}X \to \overline{\mathbf{R}}_+]_{\mathrm{p}}$. We show that it can be described as an equalizer.

Let $J = \{0\} \cup \Omega X \times \Omega X$. For every j in J, we define functions f_j and $g_j : F \to \overline{\mathbf{R}}_+$ as follows:

$f_0(\nu) = \nu\emptyset$ $\qquad g_0(\nu) = 0$

$f_{U,V}(\nu) = \nu(U \cup V) + \nu(U \cap V)$ $\qquad g_{U,V}(\nu) = \nu U + \nu V$

All these functions are continuous, since for fixed U in ΩX, $\lambda\nu.\,\nu U : F \to \overline{\mathbf{R}}_+$ is continuous, and $+ : \overline{\mathbf{R}}_+ \times \overline{\mathbf{R}}_+ \to \overline{\mathbf{R}}_+$ is continuous. By tupling, we obtain continuous functions $f, g : F \to (\overline{\mathbf{R}}_+)^J$ such that $f(\nu) = g(\nu)$ iff ν is strict and modular, i.e., in $\mathsf{V}X$.

Since $\overline{\mathbf{R}}_+$ is sober, the pointwise function space F is sober, whence its equalizer subspace $\mathsf{V}X$ is sober. For $\mathsf{V}_\mathrm{p} X$, we start with $F' = [\Omega_\mathrm{p} X \to \overline{\mathbf{R}}_+]_\mathrm{p}$. The space $\mathsf{V}_\mathrm{f} X$ is $\mathcal{T}_0$ as a subspace of $\mathsf{V}_\mathrm{p} X$. □

Now, we come to one of the main results of this paper: For every space X, $\mathsf{V}_\mathrm{p} X$ is the sobrification of $\mathsf{V}_\mathrm{f} X$. As defined in Subsection 2.12, we have to prove: For every point continuous valuation ν and open set $\mathcal{O}$ of $\mathsf{V}_\mathrm{p} X$ with ν in $\mathcal{O}$, there is a finite valuation $\varphi \sqsubseteq \nu$ with φ in $\mathcal{O}$. The proof of this statement is structured into several parts. The results of these parts are presented as auxiliary lemmas.

The first lemma contains the step from arbitrary to bounded valuations.

Lemma 6.2 For every point continuous valuation ν and open set $\mathcal{O}$ of $\mathsf{V}_\mathrm{p} X$ with ν in $\mathcal{O}$, there is a bounded point continuous valuation $\nu' \sqsubseteq \nu$ with ν' in $\mathcal{O}$.

Proof: By Theorem 4.7, ν is a directed join of bounded point continuous valuations. In a sober space such as $\mathsf{V}_\mathrm{p} X$, every open set is Scott open. Hence, there is some bounded point continuous valuation $\nu' \sqsubseteq \nu$ with ν' in $\mathcal{O}$. □

The next lemma deals with the step from bounded to finite valuations in a quite special case.

Lemma 6.3 Let ν be a bounded point continuous valuation with $\nu W > r$ for some open set W and real number r. Then there is a finite valuation $\varphi \sqsubseteq \nu$ with $\varphi W > r$.

Proof: Choose a real number r' such that $\nu W > r' > r$. Since ν is point continuous, there is a finite set $F = \{x_1, \ldots, x_n\} \subseteq W$ such that $F \subseteq O$ implies $\nu O > r'$ for all open sets O. Since $\nu\emptyset = 0$, n cannot be 0. Let $\epsilon = \frac{r'-r}{n}$. For every point x_i, choose an open set U_i such that $x_i \in U_i$ and $\nu U_i < \epsilon + \sqcap_{O \ni x_i} \nu O$. We also need the unions $V_i = \bigcup_{j=1}^{i} U_i$ for $0 \le i \le n$; in particular, $V_0 = \emptyset$.

Using restriction and corestriction, for every i with $1 \le i \le n$, $\nu|^{V_{i-1}} = \nu|^{V_{i-1}}|_{U_i} + \nu|^{V_{i-1}}|^{U_i} = \nu|_{U_i}^{V_{i-1}} + \nu|^{V_i}$. Let $\nu_i = \nu|_{U_i}^{V_{i-1}}$. Starting from $\nu = \nu|^{V_0}$, we obtain by iteration $\nu = (\sum_{i=1}^{n} \nu_i) + \nu|^{V_n}$. On the other hand, $\nu = \nu|_{V_n} + \nu|^{V_n}$ holds, whence $\sum_{i=1}^{n} \nu_i = \nu|_{V_n}$.

Let $a_i = \nu_i(X)$ and $b_i = 0 \sqcup (a_i - \epsilon)$. With these numbers, let $\varphi = \sum_{i=1}^{n} b_i \cdot \widehat{x_i}$.

To prove $\varphi W > r$, we first compute $\sum_{i=1}^{n} a_i = \sum_{i=1}^{n} \nu_i(X) = \nu|_{V_n}(X) = \nu(V_n) > r'$ since $F \subseteq V_n$. Since all x_i are in W, we obtain $\varphi W = \sum_{i=1}^{n} b_i \ge \sum_{i=1}^{n}(a_i - \epsilon) = \sum_{i=1}^{n} a_i - n \cdot \epsilon > r' - (r' - r) = r$.

To prove $\varphi \sqsubseteq \nu$, note that $\nu O \ge \sum_{i=1}^{n} \nu_i O \ge \sum_{i: x_i \in O} \nu_i O$ and $\varphi O = \sum_{i: x_i \in O} b_i$. Hence, it suffices to show $\nu_i O \ge b_i$ for all i with $x_i \in O$. By definition, $\nu_i O = \nu(O \cap U_i) - \nu(O \cap U_i \cap V_{i-1})$ holds. Since x_i in $O \cap U_i$, we obtain $\nu(O \cap U_i) > \nu U_i - \epsilon$ by the choice of U_i. Thus, $\nu_i O > \nu U_i - \epsilon - \nu(U_i \cap V_{i-1}) = \nu_i(X) - \epsilon = a_i - \epsilon$. Since $\nu_i O \ge 0$ also holds, $\nu_i O \ge b_i$ follows. □

The next lemma generalizes Lemma 6.3 from one open set W to any finite number of open sets O_i.

Lemma 6.4 Let ν be a bounded point continuous valuation with $\nu O_i > r_i$ for some open sets $O_1, \ldots, O_n$ and real numbers $r_1, \ldots, r_n$. Then there is a finite valuation $\varphi \sqsubseteq \nu$ with $\varphi O_i > r_i$ for all i with $1 \leq i \leq n$.

Proof: The valuation ν can be partitioned along O_1 into $\nu = \nu|_{O_1} + \nu|^{O_1}$. Both parts can be partitioned along O_2 into $\nu|_{O_1} = \nu|_{O_1 \cap O_2} + \nu|_{O_1}^{O_2}$ and $\nu|^{O_1} = \nu|_{O_2}^{O_1} + \nu|^{O_1 \cup O_2}$. Iterating this process, we obtain $\nu = \sum_T \nu_T$ where T ranges over the subsets of $I = \{1, \ldots, n\}$ and $\nu_T = \nu|_{U_T}^{V_T}$ with $U_T = \bigcap_{i \in T} O_i$ and $V_T = \bigcup_{i \in I \setminus T} O_i$. By construction, $\nu_T U_T = \nu_T X$ holds.

If i in T, then $U_T \subseteq O_i$, whence $\nu_T O_i = \nu_T X$. If i not in T, then $O_i \subseteq V_T$, whence $\nu_T O_i = 0$. Thus, $r_i < \nu O_i = \sum_T \nu_T O_i = \sum_{T \ni i} \nu_T X$. Choose some real number ρ with $0 < \rho < 1$ such that $r_i < \rho \cdot (\sum_{T \ni i} \nu_T X)$ still holds for all i. Let $q_T = \rho \cdot \nu_T X$. Then $\sum_{T \ni i} q_T > r_i$ for all i.

If $\nu_T X \neq 0$, then $\nu_T U_T = \nu_T X > q_T$. By Lemma 6.3, there is a finite valuation $\varphi_T \sqsubseteq \nu_T$ such that $\varphi_T U_T > q_T$. If $\nu_T X = 0$, then $q_T = 0$, and we set $\varphi_T = 0$. In both cases, we obtain a finite $\varphi_T \sqsubseteq \nu_T$ such that $\varphi_T U_T \geq q_T$.

Let $\varphi = \sum_T \varphi_T$. This is a finite valuation below ν. For all i, $\varphi O_i = \sum_T \varphi_T O_i \geq \sum_{T \ni i} \varphi_T U_T \geq \sum_{T \ni i} q_T > r_i$ holds. □

With these lemmas, we can now prove:

Theorem 6.5 For every space X, $\mathsf{V}_\mathrm{p} X$ is the sobrification of $\mathsf{V}_\mathrm{f} X$.

Proof: Let ν be in $\mathsf{V}_\mathrm{p} X$ and $\mathcal{O}$ in $\Omega(\mathsf{V}_\mathrm{p} X)$ with $\nu \in \mathcal{O}$. By Lemma 6.2, there is a bounded $\nu' \sqsubseteq \nu$ with $\nu' \in \mathcal{O}$.

Using the subbase of $\mathsf{V}_\mathrm{p} X$, we have $\nu' \in \bigcap_{i \in I} \langle O_i > r_i \rangle \subseteq \mathcal{O}$ for some finite I, open sets O_i of X, and r_i in $\mathbf{R}_+$ with $0 < r_i < \infty$. From Lemma 6.4, we obtain a finite $\varphi \sqsubseteq \nu'$ with $\varphi \in \bigcap_{i \in I} \langle O_i > r_i \rangle \subseteq \mathcal{O}$. □

7 Universal Properties

In the sequel, we look for universal properties of the valuation spaces. We shall prove that $\mathsf{V}_\mathrm{f} X$ is the free locally convex $\mathcal{T}_0$-cone, and $\mathsf{V}_\mathrm{p} X$ is the free locally convex sober cone. We did not find a universal property for $\mathsf{V} X$.

7.1 Cones

A *cone* or $\mathbf{R}_+$-*module* is an algebraic structure $(M, +, 0, \cdot)$ where $+ : M \times M \to M$ is a commutative associative operation with neutral element $0 \in M$, and $\cdot : \mathbf{R}_+ \times M \to M$ is an operation satisfying the module (or vector space) axioms:

$$\begin{aligned} r \cdot 0 &= 0 & \qquad r \cdot (m_1 + m_2) &= r \cdot m_1 + r \cdot m_2 \\ 0 \cdot m &= 0 & (r + s) \cdot m &= r \cdot m + s \cdot m \\ 1 \cdot m &= m & (r \cdot s) \cdot m &= r \cdot (s \cdot m) \end{aligned}$$

A *topological cone* is a cone with a topology such that '+' and '·' are continuous if $\mathbf{R}_+$ is given the Scott topology. Often, we shall omit the word 'topological' if there is already a topological notion around such as 'sober'.

Homomorphisms between cones are linear functions, where linearity means $f(m+m') = fm + fm'$ and $f(r \cdot m) = r \cdot fm$ as usual. Homomorphisms between topological cones are continuous linear functions.

If $(M_i)_{i\in I}$ is a family of (topological) cones, then the product $\prod_{i\in I} M_i$ is a (topological) cone with $(m_i)_{i\in I} + (m'_i)_{i\in I} = (m_i + m'_i)_{i\in I}$, $0 = (0_i)_{i\in I}$, and $r \cdot (m_i)_{i\in I} = (r \cdot m_i)_{i\in I}$.

In every topological cone, 0 is the least element since continuous functions are monotonic, whence $0 = 0 \cdot m \sqsubseteq 1 \cdot m = m$ holds for all m in M. Thus, non-trivial topological cones cannot be $\mathcal{T}_1$ spaces.

Standard examples of topological cones are given by powers of $\mathbf{R}_+$ or $\overline{\mathbf{R}}_+$, and linear subspaces thereof. On the other hand, there are quite strange cones which have nothing to do with real numbers. Let $(L, \vee, \wedge)$ be a distributive lattice with least element F and greatest element T. Define $a + b = a \vee b$ for a and b in L, $0 = \mathsf{F}$, and for r in $\mathbf{R}_+$ and a in L, define $r \cdot a = \mathsf{F}$ if $r = 0$, and $= a$ otherwise. With these operations, L becomes a cone. If L is endowed with a topology which makes '$\vee$' and '$\cdot$' continuous, then L is a topological cone. A suitable topology is the Scott topology if L is a continuous lattice.

7.2 Uniqueness Properties

The notions introduced so far are sufficient to state the uniqueness parts of the universal properties of V_f and V_p.

Theorem 7.1 Let X and Y be topological spaces.

(1) Every linear function from $\mathsf{V}_\mathrm{f}\, X$ to some cone is uniquely determined by its values on point valuations.

(2) Every continuous linear function from $\mathsf{V}_\mathrm{p}\, X$ to a $\mathcal{T}_0$-cone is uniquely determined by its values on point valuations.

(3) Every continuous bilinear function from $\mathsf{V}_\mathrm{p}\, X \otimes \mathsf{V}_\mathrm{p}\, Y$ to a $\mathcal{T}_0$-cone is uniquely determined by its values on pairs $(\widehat{x}, \widehat{y})$ of point valuations. (Bilinear means linear in each argument if the other one is fixed.)

Proof: Part (1) is obvious since every finite valuation is a finite linear combination of point valuations. Part (2) follows from part (1) and Prop. 2.3, using the fact that $\mathsf{V}_\mathrm{p}\, X$ is the sobrification of $\mathsf{V}_\mathrm{f}\, X$ (Theorem 6.5).

Let f_1 and f_2 be continuous bilinear functions from $\mathsf{V}_\mathrm{p}\, X \otimes \mathsf{V}_\mathrm{p}\, Y$ to a $\mathcal{T}_0$-cone C which coincide on pairs $(\widehat{x}, \widehat{y})$. From the functions f_i, we derive functions $g_i : \mathsf{V}_\mathrm{p}\, X \to [Y \to C]_\mathrm{p}$ with $g_i(\alpha) = \lambda y.\, f_i(\alpha, \widehat{y})$. By raising the operations of C to functions, $[Y \to C]_\mathrm{p}$ becomes a $\mathcal{T}_0$-cone again. The functions g_i are continuous since they result from f_i by currying (Subsection 2.9) and composition with $\mathsf{s} = \lambda y.\, \widehat{y}$. They are also linear, and $g_1(\widehat{x}) = g_2(\widehat{x})$ holds for all x in X by hypothesis. By part (2), $g_1 = g_2$ follows, whence $f_1(\alpha, \widehat{y}) = f_2(\alpha, \widehat{y})$ holds for all α in $\mathsf{V}_\mathrm{p}\, X$ and y in Y. Currying f_i the other way round yields functions $h_i : \mathsf{V}_\mathrm{p}\, Y \to [\mathsf{V}_\mathrm{p}\, X \to C]_\mathrm{p}$ with $h_i(\beta) = \lambda\alpha.\, f_i(\alpha, \beta)$. Since $h_1(\widehat{y}) = h_2(\widehat{y})$ for all y in Y, we can apply part (2) again and obtain $h_1 = h_2$, i.e., $f_1 = f_2$. □

Unfortunately, we do not know whether similar properties hold for $\mathsf{V}X$, the space of all Scott continuous valuations on X. We do not even know the answer for the special case that the target cone is $\overline{\mathbf{R}}_+$.

Problem 1 Are continuous linear functions from $\mathsf{V}X$ to $\overline{\mathbf{R}}_+$ uniquely determined by their values on point valuations?

If the answer to this problem is yes, then also the continuous linear functions from $\mathsf{V}X$ to $\mathsf{V}Y$ are uniquely determined by their values on point valuations. For, the functions $\lambda\nu.\,\nu V : \mathsf{V}Y \to \overline{\mathbf{R}}_+$ are continuous and linear for every open set V of Y.

7.3 Convexity and Local Convexity

In order to formulate the universal properties for V_f and V_p, we need some more notions connected with cones.

In a cone, a *convex combination* is a linear combination $r_1 \cdot x_1 + \cdots + r_n \cdot x_n$ whose coefficients sum up to 1.

A subset S of a cone is *convex* iff all convex combinations of points of S are back in S again. This is equivalent to the condition that $r \cdot x + (1-r) \cdot y$ is in S for all x, y in S and r with $0 < r < 1$. Intersections of convex sets are convex, hence every subset S of a cone has a least convex superset, the *convex hull* $\mathsf{con}\, S$. The convex hull can be described as the set of all convex combinations of points of S.

Proposition 7.2 Let $f : M \to M'$ be a linear map between two cones.

(1) If S is convex in M', then $f^- S$ is convex in M.

(2) $\mathsf{con}\, f^- S \subseteq f^-(\mathsf{con}\, S)$ holds for all $S \subseteq M$.

Proof:

(1) Let x, y be in $f^- S$. Then $r \cdot x + (1-r) \cdot y$ is in $f^- S$ since $f(r \cdot x + (1-r) \cdot y) = r \cdot fx + (1-r) \cdot fy$ is in S.

(2) By (1), $f^-(\mathsf{con}\, S)$ is a convex superset of $f^- S$. □

A topological cone is *convex-based* if whenever a point x is in an open set U, there is a convex open set V such that $x \in V \subseteq U$ (or: there is an open set V such that $x \in V = \mathsf{con}\, V \subseteq U$). It is *locally convex* if whenever a point x is in an open set U, there is an open set V such that $x \in V \subseteq \mathsf{con}\, V \subseteq U$.

Clearly, every convex-based cone is locally convex. The two notions are quite similar and have similar properties. In our proofs, we shall concentrate on local convexity. The reader is invited to find the corresponding proofs for convex bases.

In both definitions, it suffices to consider open sets U from a subbase $\mathcal{S}$. For, if x is in an arbitrary open O, then there are opens $U_1, \ldots, U_n$ from $\mathcal{S}$ such that $x \in U_1 \cap \cdots \cap U_n \subseteq O$. Thus, x in U_i for all i, whence there are V_i such that $x \in V_i \subseteq \mathsf{con}\, V_i \subseteq U_i$. Since $\bigcap_i \mathsf{con}\, V_i$ is convex, $x \in \bigcap_i V_i \subseteq \mathsf{con}\,(\bigcap_i V_i) \subseteq \bigcap_i \mathsf{con}\, V_i \subseteq \bigcap_i U_i \subseteq O$ follows.

The topological cones $\mathbf{R}_+$ and $\overline{\mathbf{R}}_+$ are convex-based since all the opens $\{s \mid s > r\}$ are convex. Also, distributive lattices with the Scott topology are convex-based, since convex combinations are finite joins $a_1 \vee \cdots \vee a_n$ with $n > 0$, and open sets are upper sets. For the moment, we do not have any examples of cones which are not convex-based.

In the sequel, we present three properties of our notions.

Proposition 7.3 (Products)
If all topological cones M_i are convex-based / locally convex, then so is $\prod_{i \in I} M_i$.

Proof: The product $M = \prod_{i \in I} M_i$ has a subbase $\{\pi_i^- O \mid i \in I, O \in \Omega M_i\}$. If x in $\pi_i^- O$, then $\pi_i x \in O$. Since M_i is locally convex, there is an open V in M_i such that $\pi_i x \in V \subseteq \mathsf{con}\, V \subseteq O$, whence $x \in \pi_i^- V \subseteq \pi_i^-(\mathsf{con}\, V) \subseteq \pi_i^- O$. The projection π_i is linear, whence $\mathsf{con}\, \pi_i^- V \subseteq \pi_i^-(\mathsf{con}\, V)$ by Prop. 7.2. □

Proposition 7.4 (Linear subspaces) If M and M' are topological cones, $e : M \hookrightarrow M'$ is a linear topological embedding, and M' is convex-based / locally convex, then so is M.

Proof: Let x in O for some open O of M. Since e is an embedding, $O = e^- U$ holds for some $U \in \Omega M'$. Thus ex is in U. Since M' is locally convex, there is an open V in M' such that $ex \in V \subseteq \mathsf{con}\, V \subseteq U$, whence $x \in e^- V \subseteq e^-(\mathsf{con}\, V) \subseteq e^- U$. Since e is linear, $\mathsf{con}\, e^- V \subseteq e^-(\mathsf{con}\, V)$ holds by Prop. 7.2. □

From the two propositions above, we may conclude that the spaces $\mathsf{V}X$, $\mathsf{V}_{\mathrm{p}} X$, and $\mathsf{V}_{\mathrm{f}} X$ are convex-based as linear subspaces of products of $\overline{\mathbf{R}}_+$.

Proposition 7.5 (Linear retracts)
If M and M' are topological cones, $e : M \to M'$ is continuous, and $r : M' \to M$ is linear and continuous with $r \circ e = \mathsf{id}$, then local convexity of M' implies local convexity of M.

Proof: If $x = r(ex)$ in U where U in ΩM, then $ex \in r^- U$. Because M' is locally convex, there is an open V in $\Omega M'$ such that $ex \in V \subseteq \mathsf{con}\, V \subseteq r^- U$. Hence, $x \in e^- V$. We claim $\mathsf{con}\,(e^- V) \subseteq U$.

We show that every convex combination $\sum t_i \cdot x_i$ of points x_i from $e^- V$ is in U. Since ex_i is in V, $\sum t_i \cdot ex_i$ is in $\mathsf{con}\, V \subseteq r^- U$. Thus, $r(\sum t_i \cdot ex_i) = \sum t_i \cdot x_i$ is in U. Here, linearity of r is used. □

The corresponding property for convex-based cones is probably wrong, but we have no examples. Later, we shall see that the locally convex $\mathcal{T}_0$-cones are exactly the linear retracts of the convex-based $\mathcal{T}_0$-cones (Theorem 7.7).

7.4 A Universal Property for V_{f}

In this subsection, we present a universal property for the space $\mathsf{V}_{\mathrm{f}} X$ of finite valuations.

Theorem 7.6 $\mathsf{V}_{\mathrm{f}} X$ is the free locally convex $\mathcal{T}_0$-cone over X in $\mathcal{TOP}$, the category of topological spaces and continuous maps.

This means: $\mathsf{V}_{\mathrm{f}} X$ is itself a locally convex $\mathcal{T}_0$-cone, and for every continuous function $f : X \to M$ from a topological space X to a locally convex $\mathcal{T}_0$-cone M, there is a unique continuous linear function $\bar{f} : \mathsf{V}_{\mathrm{f}} X \to M$ with $\bar{f} \circ \mathsf{s} = f$, i.e., $\bar{f}(\widehat{x}) = fx$ for all x in X.

Proof: We already know that $\mathsf{V}_{\mathrm{f}} X$ is a convex-based $\mathcal{T}_0$-cone, hence locally convex. The uniqueness statement is given by Theorem 7.1. We still have to show existence of $\bar{f}$.

Every finite valuation φ can be written as $\sum_{x \in F} r_x \cdot \widehat{x}$ for some finite set F and some numbers r_x with $0 < r_x < \infty$. By Cor. 5.5, this representation is unique. Hence, $\bar{f}(\varphi) = \sum_{x \in F} r_x \cdot fx$ is a well-defined element of M. The function $\bar{f} : \mathsf{V}_{\mathrm{f}} X \to M$ defined in this manner is obviously linear and satisfies $\bar{f} \circ \mathsf{s} = f$. The only remaining task is to prove continuity of $\bar{f}$. This turns out to be quite complex.

Let $\varphi = \sum_{x \in F} r_x \cdot \widehat{x}$ be a member of $\mathsf{V_f}\, X$ where F is finite and $0 < r_x < \infty$. Let U be an open set of M, and assume $\bar{f}(\varphi) \in U$. Then $\sum_{x \in F} r_x \cdot fx$ is in U. Since addition and multiplication are continuous in M, there are open sets R_x of $\mathbf{R}_+$ and V_x of M such that $r_x \in R_x$, $fx \in V_x$, and whenever $s_x \in R_x$ and $m_x \in V_x$, then $\sum_{x \in F} s_x \cdot m_x \in U$.

Choose numbers $r'_x > 0$ such that $r_x > r'_x \in R_x$, and applying local convexity of M, choose open sets W_x of M such that $fx \in W_x \subseteq \mathsf{con}\, W_x \subseteq V_x$, and let $O_x = f^- W_x$. By continuity of f, the sets O_x are open sets of X with x in O_x for all x in F. For every non-empty $T \subseteq F$, $\varphi(\bigcup_{x \in T} O_x) \geq \sum_{x \in T} r_x > \sum_{x \in T} r'_x$ holds. Hence, φ is in $\mathcal{O} = \bigcap_{\emptyset \neq T \subseteq F} \langle \bigcup_{x \in T} O_x > \sum_{x \in T} r'_x \rangle$, which is an open set of $\mathsf{V_f}\, X$. We have to show that for every ψ in $\mathcal{O}$, $\bar{f}(\psi)$ is in U.

Let $\psi = \sum_{y \in G} s_y \cdot \widehat{y}$ be in $\mathcal{O}$. Let $R \subseteq F \times G$ be the relation given by $(x, y) \in R$ iff $O_x \ni y$, whence $R^+(T) = \{y \in G \mid \exists x \in T : y \in O_x\}$ for subsets T of I. Since ψ is in $\mathcal{O}$,

$$\sum_{y \in R^+(T)} s_j = \psi(\bigcup_{x \in T} O_x) > \sum_{x \in T} r'_x$$

holds for all non-empty subsets T of I. For $T = \emptyset$, both sides are zero. Thus, '$\geq$' instead of '$>$' holds for all subsets T of I. Applying the Split Lemma 2.8, we obtain numbers $t_{xy} \in \mathbf{R}_+$ for x in F and y in G such that

(1) $\sum_{y \in G} t_{xy} = r'_x$ for all x in F,
(2) $\sum_{x \in F} t_{xy} \leq s_y$ for all y in G,
(3) if $t_{xy} > 0$, then $y \in O_x$.

Let $\psi' = \sum_{x \in F} \sum_{y \in G} t_{xy} \cdot \widehat{y}$. Then

$$\bar{f}(\psi') = \sum_{y \in G} (\sum_{x \in F} t_{xy}) \cdot fy \sqsubseteq \sum_{y \in G} s_y \cdot fy = \bar{f}(\psi)$$

using monotonicity of addition and multiplication in M. The valuation ψ' may alternatively be written as

$$\psi' = \sum_{x \in F} r'_x \cdot \psi_x \text{ where } \psi_x = \sum_{y \in G \cap O_x} (t_{xy}/r'_x) \cdot \widehat{y}$$

The coefficients of ψ_x sum up to 1. Thus, $\bar{f}(\psi_x)$ is a convex combination of the points fy where y in $G \cap O_x$. All these points are in W_x, whence $\bar{f}(\psi_x)$ in V_x by choice of W_x.

Since $\bar{f}(\psi') = \sum_{x \in F} r'_x \cdot \bar{f}(\psi_x)$ where r'_x in R_x and $\bar{f}(\psi_x)$ in V_x, it is in U. Since $\bar{f}(\psi)$ is above $\bar{f}(\psi')$, it is in U as well. □

In Theorem 7.6, local convexity cannot be dispensed with: if M is a topological cone with the property that identity $\mathsf{id} : M \to M$ has a continuous linear extension $\overline{\mathsf{id}} : \mathsf{V_f}\, M \to M$, then M is a linear retract of $\mathsf{V_f}\, M$, whence locally convex by Prop. 7.5. As a subspace of $\mathsf{V_f}\, M$, it is also $\mathcal{T}_0$.

Theorem 7.6 also leads to a characterization of local convexity.

Theorem 7.7 A $\mathcal{T}_0$-cone is locally convex iff it is a linear retract of a convex-based cone.

Proof: Linear retracts of convex-based cones are locally convex by Prop. 7.5. Conversely, if M is a locally convex $\mathcal{T}_0$-cone, then identity $\mathsf{id} : M \to M$ has a continuous linear extension $\overline{\mathsf{id}} : \mathsf{V_f}\, M \to M$. Thus, M is a linear retract of the convex-based cone $\mathsf{V_f}\, M$. □

The extension function induced by Theorem 7.6 is continuous.

Theorem 7.8 For every space X and locally convex $\mathcal{T}_0$-cone M, the extension function $\mathsf{E} : [X \to M]_{\mathrm{p}} \to [\mathsf{V}_{\mathrm{f}} X \stackrel{lin}{\to} M]_{\mathrm{p}}$ given by Theorem 7.6 is continuous and linear.

Proof: For some $\varphi = \sum_{x \in F} r_x \cdot \widehat{x}$ in $\mathsf{V}_{\mathrm{f}} X$, continuous function $f : X \to M$, and open set U of M, assume $\mathsf{E} f \in \langle \varphi \to U \rangle$. Then $\sum_{x \in F} r_x \cdot fx$ is in U. As in the proof of Theorem 7.6, there are open sets R_x of $\mathbf{R}_+$ and V_x of M such that $r_x \in R_x$, $fx \in V_x$, and whenever $s_x \in R_x$ and $m_x \in V_x$, then $\sum_{x \in F} s_x \cdot m_x \in U$. From $fx \in V_x$, we obtain $f \in \bigcap_{x \in F} \langle x \to V_x \rangle$.

If g is in $\bigcap_{x \in F} \langle x \to V_x \rangle$, then $gx \in V_x$ for all x in F, whence $\sum_{x \in F} r_x \cdot gx$ is in U. Thus, $\mathsf{E} g$ is in $\langle \varphi \to U \rangle$. This proves continuity of E.

Linearity is meant to be w.r.t. the pointwise operations on $[X \to M]_{\mathrm{p}}$ and $[\mathsf{V}_{\mathrm{f}} X \stackrel{lin}{\to} M]_{\mathrm{p}}$. To prove the equality $\mathsf{E}(f + g) = \mathsf{E} f + \mathsf{E} g$, note that both functions are continuous and linear, and $\mathsf{E}(f + g) \circ \mathsf{s} = f + g = (\mathsf{E} f + \mathsf{E} g) \circ \mathsf{s}$ holds. The equality follows from the uniqueness statement of freeness. The second equality $\mathsf{E}(r \cdot f) = r \cdot \mathsf{E} f$ is shown by similar arguments. $\square$

Corollary 7.9 For every space X and locally convex $\mathcal{T}_0$-cone M, the function spaces $[X \to M]_{\mathrm{p}}$ and $[\mathsf{V}_{\mathrm{f}} X \stackrel{lin}{\to} M]_{\mathrm{p}}$ are isomorphic topological cones.

Proof: One isomorphism is given by the function E of Theorem 7.8. The opposite one is $F \mapsto F \circ \mathsf{s}$. $\square$

7.5 A Universal Property for V_{p}

Here, we present a universal property for the space $\mathsf{V}_{\mathrm{p}} X$ of point continuous valuations.

Theorem 7.10 $\mathsf{V}_{\mathrm{p}} X$ is the free locally convex sober cone over X in $\mathcal{TOP}$.

Proof: We already know that $\mathsf{V}_{\mathrm{p}} X$ is a convex-based, whence locally convex, sober cone. Let M be an arbitrary locally convex sober cone, and let $f : X \to M$ be continuous. By Theorem 7.6, there is a unique continuous linear function $f^* : \mathsf{V}_{\mathrm{f}} X \to M$ with $f^* \circ \mathsf{s} = f$. By Theorem 6.5, $\mathsf{V}_{\mathrm{p}} X$ is the sobrification of $\mathsf{V}_{\mathrm{f}} X$. Hence, by Theorem 2.4, the continuous function $f^* : \mathsf{V}_{\mathrm{f}} X \to M$ has a unique continuous extension $\bar{f} : \mathsf{V}_{\mathrm{p}} X \to M$. Since $\bar{f}$ extends f^*, $\bar{f} \circ \mathsf{s} = f$ follows. The only thing which remains to be proved is linearity of $\bar{f}$.

Let r be a fixed element of $\mathbf{R}_+$. Consider the two functions $F, G : \mathsf{V}_{\mathrm{p}} X \to \overline{\mathbf{R}}_+$ with $F(\nu) = \bar{f}(r \cdot \nu)$ and $G(\nu) = r \cdot \bar{f}(\nu)$. They are continuous and coincide on $\mathsf{V}_{\mathrm{f}} X$ because of linearity of f^*. By Prop. 2.3, $F = G$ follows.

For addition, consider the two functions $F, G : \mathsf{V}_{\mathrm{p}} X \times \mathsf{V}_{\mathrm{p}} X \to \overline{\mathbf{R}}_+$ with $F(\nu, \nu') = \bar{f}(\nu + \nu')$ and $G(\nu, \nu') = \bar{f}(\nu) + \bar{f}(\nu')$. They are continuous and coincide on $\mathsf{V}_{\mathrm{f}} X \times \mathsf{V}_{\mathrm{f}} X$ because of linearity of f^*. By Prop. 2.7, $\mathsf{V}_{\mathrm{p}} X \times \mathsf{V}_{\mathrm{p}} X$ is the sobrification of $\mathsf{V}_{\mathrm{f}} X \times \mathsf{V}_{\mathrm{f}} X$. By Prop. 2.3, $F = G$ follows. $\square$

Theorem 7.11 For every space X and locally convex sober cone M, the function $\mathsf{E} : [X \to M]_{\mathrm{p}} \to [\mathsf{V}_{\mathrm{p}} X \stackrel{lin}{\to} M]_{\mathrm{p}}$ induced by the freeness of $\mathsf{V}_{\mathrm{p}} X$ is continuous and linear.

Proof: It is continuous as the composition of the function $[X \to M]_{\mathrm{p}} \to [\mathsf{V}_{\mathrm{f}} X \stackrel{lin}{\to} M]_{\mathrm{p}}$ of Theorem 7.8 with a restriction of the function $[\mathsf{V}_{\mathrm{f}} X \to M]_{\mathrm{p}} \to [\mathsf{V}_{\mathrm{p}} X \to M]_{\mathrm{p}}$ of Theorem 2.5. Linearity follows from freeness as in the proof of Theorem 7.8. $\square$

Corollary 7.12 For every space X and locally convex sober cone M, the three function spaces $[X \to M]_{\mathrm{p}}$, $[\mathsf{V}_{\mathrm{f}} X \overset{lin}{\to} M]_{\mathrm{p}}$, and $[\mathsf{V}_{\mathrm{p}} X \overset{lin}{\to} M]_{\mathrm{p}}$ are isomorphic topological cones.

In particular, for every space X, the three topological cones $[X \to \overline{\mathbf{R}}_+]_{\mathrm{p}}$, $[\mathsf{V}_{\mathrm{f}} X \overset{lin}{\to} \overline{\mathbf{R}}_+]_{\mathrm{p}}$, and $[\mathsf{V}_{\mathrm{p}} X \overset{lin}{\to} \overline{\mathbf{R}}_+]_{\mathrm{p}}$ are isomorphic.

7.6 Universality for V_{p} and Tensor Products

In this subsection, we generalize Theorem 7.10 to functions with two arguments which are separately continuous, i.e., continuous functions $f : X \otimes Y \to M$.

Theorem 7.13 Let X and Y be two spaces and M a locally convex sober cone. For every continuous function $f : X \otimes Y \to M$, there is a unique continuous bilinear function $F : \mathsf{V}_{\mathrm{p}} X \otimes \mathsf{V}_{\mathrm{p}} Y \to M$ such that $F(\widehat{x}, \widehat{y}) = f(x, y)$ holds for all x in X and y in Y.

Proof: Starting from f, we obtain a continuous function $f' : X \to [Y \to M]_{\mathrm{p}}$ by currying (Subsection 2.9). As mentioned in Subsection 2.11, sobriety of M implies sobriety of $[Y \to M]_{\mathrm{p}}$. This space is also locally convex as a linear subspace (Prop. 7.4) of a power (Prop. 7.3) of M.

By Theorem 7.10, there is a unique continuous linear function $g : \mathsf{V}_{\mathrm{p}} X \to [Y \to M]_{\mathrm{p}}$ with $g(\widehat{x}) = f'x$ for all x in X. By Theorem 7.11, $\mathsf{E} : [Y \to M]_{\mathrm{p}} \to [\mathsf{V}_{\mathrm{p}} Y \overset{lin}{\to} M]_{\mathrm{p}}$ is continuous and linear. Composition of g and E produces a continuous linear function $h : \mathsf{V}_{\mathrm{p}} X \to [\mathsf{V}_{\mathrm{p}} Y \overset{lin}{\to} M]_{\mathrm{p}}$. Uncurrying h (Subsection 2.9) yields a continuous function $F : \mathsf{V}_{\mathrm{p}} X \otimes \mathsf{V}_{\mathrm{p}} Y \to M$ which is bilinear as required. The behavior on pairs on point valuations is as wanted:

$$F(\widehat{x}, \widehat{y}) \;=\; h\widehat{x}\widehat{y} \;=\; \mathsf{E}(g\widehat{x})\widehat{y} \;=\; g\widehat{x}y \;=\; f'xy \;=\; f(x, y).$$

Uniqueness of F follows from Theorem 7.1 (3). □

8 Integration

Several authors [6, 7, 8] already defined integration of real-valued functions w.r.t. a valuation. Since they defined integration from scratch, the proofs of its properties are quite involved. Here, we present a novel definition of integration which is so simple that most proofs become trivial. (The complexity has not disappeared, though; it is now in the proofs of Theorems 2.4 and 7.6, which are needed to prove Theorem 7.10).

For every space X, integration will be a function $\int_X : [X \to \overline{\mathbf{R}}_+]_{\mathrm{i}} \otimes \mathsf{V} X \to \overline{\mathbf{R}}_+$ which is continuous in the two arguments separately. (If one argument is fixed, then $\int_X$ is continuous in the other.)

Note that the function space $[X \to \overline{\mathbf{R}}_+]_{\mathrm{i}}$ is not topologized by the pointwise topology, but by the *Isbell topology* which has more open sets (see Subsection 2.10).

The function $\int_X$ is built from the following pieces:

1. The function $\Omega_{\mathrm{s}} : [X \to \overline{\mathbf{R}}_+]_{\mathrm{i}} \to [\Omega_{\mathrm{s}} \overline{\mathbf{R}}_+ \to \Omega_{\mathrm{s}} X]_{\mathrm{p}}$ with $\Omega_{\mathrm{s}} f = f^-$ is continuous. In fact, the Isbell topology was chosen to guarantee this.

2. Using Ω_s, we map from $[X \to \overline{\mathbf{R}}_+]_i \otimes \mathsf{V}X$ to $[\Omega_s \overline{\mathbf{R}}_+ \overset{\cup,\cap}{\to} \Omega_s X]_p \otimes [\Omega_s X \overset{mod}{\to} \overline{\mathbf{R}}_+]_p$, where the labels at the arrows indicate the properties of the resulting functions. Now, we can use function composition to reach $[\Omega_s \overline{\mathbf{R}}_+ \overset{mod}{\to} \overline{\mathbf{R}}_+]_p = \mathsf{V}\overline{\mathbf{R}}_+$. Function composition $\circ : [X \to Y]_p \otimes [Y \to Z]_p \to [X \to Z]_p$ is continuous in its two arguments separately.
3. Since $\overline{\mathbf{R}}_+$ is a continuous dcpo, it is locally finitary, whence $\mathsf{V}\overline{\mathbf{R}}_+ = \mathsf{V}_p \overline{\mathbf{R}}_+$ by Theorem 4.5.
4. $\overline{\mathbf{R}}_+$ is a locally convex (even convex-based) sober cone. By Theorem 7.10, identity $\mathsf{id} : \overline{\mathbf{R}}_+ \to \overline{\mathbf{R}}_+$ can be extended to a continuous linear function $\overline{\mathsf{id}} : \mathsf{V}_p \overline{\mathbf{R}}_+ \to \overline{\mathbf{R}}_+$ with the property $\overline{\mathsf{id}}\,(\widehat{r}) = r$ for all r in $\overline{\mathbf{R}}_+$.

Putting all pieces together, we yield a function $\int_X : [X \to \overline{\mathbf{R}}_+]_i \otimes \mathsf{V}X \to \overline{\mathbf{R}}_+$ with $\int_X(f, \nu) = \overline{\mathsf{id}}\,(\nu \circ f^-)$ which is continuous in its two arguments separately. Of course, this function can be restricted to a 'pointwise' function $[X \to \overline{\mathbf{R}}_+]_i \otimes \mathsf{V}_p X \to \overline{\mathbf{R}}_+$ which is also continuous in its two arguments separately. This continuity is not destroyed if the Isbell topology on the real-valued functions is replaced by the smaller pointwise topology. For, the function $\Omega_p : [X \to \overline{\mathbf{R}}_+]_p \to [\Omega_p \overline{\mathbf{R}}_+ \to \Omega_p X]_p$ with $\Omega_p f = f^-$ is continuous. Using Ω_p, we map from $[X \to \overline{\mathbf{R}}_+]_p \otimes \mathsf{V}_p X$ to $[\Omega_p \overline{\mathbf{R}}_+ \overset{\cup,\cap}{\to} \Omega_p X]_p \otimes [\Omega_p X \overset{mod}{\to} \overline{\mathbf{R}}_+]_p$, and composition can be used to reach $[\Omega_p \overline{\mathbf{R}}_+ \overset{mod}{\to} \overline{\mathbf{R}}_+]_p = \mathsf{V}_p \overline{\mathbf{R}}_+$.

Thus, we obtain two variants of integrations with the same definition $\int_X(f, \nu) = \overline{\mathsf{id}}(\nu \circ f^-)$, but different continuity properties; the Isbell variant $\int_X : [X \to \overline{\mathbf{R}}_+]_i \otimes \mathsf{V}X \to \overline{\mathbf{R}}_+$, and the pointwise variant $\int_X : [X \to \overline{\mathbf{R}}_+]_p \otimes \mathsf{V}_p X \to \overline{\mathbf{R}}_+$.

In the sequel, we derive the essential properties of integration. They hold for both variants because the defining equations are the same. They are collected in Theorem 8.1 at the end of this section.

From the construction of the two variants of $\int_X$, we know that they are continuous in the two arguments separately. Since $[X \to \overline{\mathbf{R}}_+]_i$, $[X \to \overline{\mathbf{R}}_+]_p$, $\mathsf{V}X$, $\mathsf{V}_p X$, and $\overline{\mathbf{R}}_+$ are d-spaces, they are also Scott continuous in both arguments. Integration is linear in the valuation argument, since $\int_X(f, \nu) = \overline{\mathsf{id}}\,(\nu \circ f^-)$, and $\overline{\mathsf{id}}$ is linear. The effect of integration on point valuations is as follows:

$$\textstyle\int_X(f, \widehat{x}) \;=\; \overline{\mathsf{id}}\,(\widehat{x} \circ f^-) \;=\; \overline{\mathsf{id}}\,(\widehat{fx}) \;=\; fx.$$

A kind of 'substitution theorem' is easily proved for continuous functions $h : X \to Y$, $f : Y \to \overline{\mathbf{R}}_+$, and valuations ν in $\mathsf{V}X$:

$$\textstyle\int_X(f \circ h, \nu) \;=\; \overline{\mathsf{id}}\,(\nu \circ (f \circ h)^-) \;=\; \overline{\mathsf{id}}\,(\nu \circ h^- \circ f^-) \;=\; \int_Y(f, \nu \circ h^-)$$

Our final goal is to show that integration is also linear in its functional argument. If we only considered point continuous valuations, the proof would be quite easy: Both sides of the equation $\int_X(f + g, \nu) = \int_X(f, \nu) + \int_X(g, \nu)$ are continuous and linear in $\nu \in \mathsf{V}_p X$. By Theorem 7.1, it suffices to consider the special case $\nu = \widehat{x}$. In this case, both sides are $fx + gx$. The equation $\int_X(r \cdot f, \nu) = r \cdot \int_X(f, \nu)$ would be handled similarly.

Unfortunately, this elegant proof is not possible in the general case since we do not have an analogous property for continuous linear functions defined on $\mathsf{V}X$ (cf. Problem 1). Fortunately, there is a way around the problem. Before we can present it, we have to consider the continuous functions $f : X \to \overline{\mathbf{R}}_+$ a bit closer.

A continuous function $f : X \to \overline{\mathbf{R}}_+$ is *simple* if its image $\Im f$ is finite and does not contain ∞. An arbitrary continuous function is the directed join of all the simple functions below it. Every finite linear combination $r_1 \cdot \widehat{U_1} + \cdots + r_n \cdot \widehat{U_n}$ of characteristic functions $\widehat{U_i}$ of open sets U_i is simple. Conversely, every simple function can be written as such a linear combination.

We need some auxiliary statements for our proof of the linearity of integration in the functional argument.

(1) For r in $\overline{\mathbf{R}}_+$: $\int_X (r \cdot f, \nu) = r \cdot \int_X (f, \nu)$.

Proof: Let $(r\cdot) : \overline{\mathbf{R}}_+ \to \overline{\mathbf{R}}_+$ be the function defined by $(r\cdot)(s) = r \cdot s$. Then $r \cdot f = (r\cdot) \circ f$. By the substitution property, we obtain $\int_X (r \cdot f, \nu) = \int_{\overline{\mathbf{R}}_+} ((r\cdot), \nu \circ f^-)$, and by the definition of integration, $r \cdot \int_X (f, \nu) = r \cdot \overline{\mathsf{id}}(\nu \circ f^-)$ holds. We claim $\int_{\overline{\mathbf{R}}_+} ((r\cdot), \sigma) = r \cdot \overline{\mathsf{id}}(\sigma)$ for all valuations σ in $\mathsf{V}\overline{\mathbf{R}}_+ = \mathsf{V}_{\mathrm{p}}\overline{\mathbf{R}}_+$. Since both sides of the equation are linear continuous functions in σ, it suffices by Theorem 7.1 to prove the equation for the special case of point valuations $\sigma = \widehat{s}$ where s in $\overline{\mathbf{R}}_+$. In this case, the left hand side is $\int_{\overline{\mathbf{R}}_+} ((r\cdot), \widehat{s}) = (r\cdot)(s) = r \cdot s$, and the right hand side is $r \cdot \overline{\mathsf{id}}(\widehat{s}) = r \cdot s$. □

(2) $\int_X (\underline{0}, \nu) = 0$ where $\underline{0}(x) = 0$ for all x in X.

Proof: By (1), using $\underline{0} = 0 \cdot \underline{0}$. □

(3) For r in $\overline{\mathbf{R}}_+$: $\int_X (r + f, \nu) = r \cdot \nu(X) + \int_X (f, \nu)$.

Proof: Apply the same idea as in the proof of (1). Since $X = f^-(\overline{\mathbf{R}}_+)$, equation (3) is equivalent to $\int_{\overline{\mathbf{R}}_+} ((r+), \sigma) = r \cdot \sigma(\overline{\mathbf{R}}_+) + \overline{\mathsf{id}}(\sigma)$ where $\sigma = \nu \circ f^-$ in $\mathsf{V}\overline{\mathbf{R}}_+$. This equation holds for all σ in $\mathsf{V}\overline{\mathbf{R}}_+$, since both sides are continuous and linear in σ, and $\int_{\overline{\mathbf{R}}_+} ((r+), \widehat{s}) = (r+)(s) = r + s$, and $r \cdot \widehat{s}(\overline{\mathbf{R}}_+) + \overline{\mathsf{id}}(\widehat{s}) = r \cdot 1 + s$. □

(4) If $fx = gx$ for all x in the open set W, then $\int_X (f, \nu|_W) = \int_X (g, \nu|_W)$.

Proof: By hypothesis, $W \cap f^- V = W \cap g^- V$ holds for all V in $\Omega\overline{\mathbf{R}}_+$. By Theorem 3.5, $\nu|_W(f^- V) = \nu|_W(g^- V)$ follows, whence $\nu|_W \circ f^- = \nu|_W \circ g^-$. □

(5) If $fx = gx$ for all x in $X \setminus W$ where W is open, and ν is bounded, then $\int_X (f, \nu|^W) = \int_X (g, \nu|^W)$.

Proof: The valuation ν must be bounded so that the corestriction $\nu|^W$ is well defined. By hypothesis, $\neg W \cap f^- V = \neg W \cap g^- V$ holds for all V in $\Omega\overline{\mathbf{R}}_+$. By Theorem 3.5, $\nu|^W \circ f^- = \nu|^W \circ g^-$ follows. □

(6) For r in $\overline{\mathbf{R}}_+$ and W in ΩX, $\int_X (r \cdot \widehat{W} + f, \nu) = r \cdot \nu(W) + \int_X (f, \nu)$.

Proof: First assume that ν is bounded. Then it can be partitioned along W into $\nu = \nu|_W + \nu|^W$. Since integration is linear in the valuation argument, $\int_X (r \cdot \widehat{W} + f, \nu) = \int_X (r \cdot \widehat{W} + f, \nu|_W) + \int_X (r \cdot \widehat{W} + f, \nu|^W)$ holds. Since $\widehat{W}(x) = 0$ for x in $X \setminus W$, the second summand equals $\int_X (f, \nu|^W)$ by (5). Since $\widehat{W}(x) = 1$ for x in W, the first summand equals $\int_X (r + f, \nu|_W)$ by (4). By (3), this is $r \cdot \nu|_W(X) + \int_X (f, \nu|_W)$. Hence, we obtain $r \cdot \nu(W) + \int_X (f, \nu|_W) + \int_X (f, \nu|^W) \; = \; r \cdot \nu(W) + \int_X (f, \nu)$ for the sum.

The equation holds for every valuation ν in $\mathsf{V}X$, since integration is Scott continuous in the valuation argument, and ν is a directed join of bounded members of $\mathsf{V}X$ by Theorem 4.7. □

(7) $\int_X (r_1 \cdot \widehat{U_1} + \cdots + r_n \cdot \widehat{U_n}, \nu) = r_1 \cdot \nu(U_1) + \cdots + r_n \cdot \nu(U_n)$.
In particular, $\int_X (\widehat{U}, \nu) = \nu(U)$.

Proof: Apply (6) n times, then (2). $\square$

(8) $\int_X(f+g,\,\nu) = \int_X(f,\,\nu) + \int_X(g,\,\nu)$.

Proof: First assume that f and g are simple. Since every simple function is a finite linear combination of characteristic functions of opens, the statement follows from (7).

The equation for general f and g follows, since every continuous function is a directed join of simple functions, and integration is Scott continuous in the functional argument. $\square$

Summarizing our results, we obtain:

Theorem 8.1

(1) Integration $\int_X : [X \to \overline{\mathbf{R}}_+]_{\mathrm{i}} \otimes \mathsf{V}X \to \overline{\mathbf{R}}_+$ is continuous in its two arguments separately.

(2) The variant $\int_X : [X \to \overline{\mathbf{R}}_+]_{\mathrm{p}} \otimes \mathsf{V}_{\mathrm{p}}X \to \overline{\mathbf{R}}_+$ is also continuous in its two arguments separately.

(3) Integration (in both variants) is Scott continuous in its two arguments.

(4) Integration is linear in both arguments.

(5) For $f : X \to \overline{\mathbf{R}}_+$ and x in X, $\int_X(f, \widehat{x}) = fx$ holds.

(6) $\int_X(\widehat{U}, \nu) = \nu(U)$ where $\widehat{U}$ is the characteristic function of an open U of X.

(7) Let $h : X \to Y$ and $f : Y \to \overline{\mathbf{R}}_+$ be continuous, and let ν be in $\mathsf{V}X$. Then $\int_X(f \circ h,\, \nu) = \int_Y(f,\, \nu \circ h^-)$.

9 Isomorphic Descriptions

Integration may be used to derive isomorphic descriptions of $\mathsf{V}X$ and $\mathsf{V}_{\mathrm{p}}X$.

Theorem 9.1 For every space X, the topological cone of Scott continuous valuations on X, i.e., strict modular Scott continuous functions from ΩX to $\overline{\mathbf{R}}_+$, is isomorphic to the topological cone of linear continuous functions from $[X \to \overline{\mathbf{R}}_+]_{\mathrm{i}}$ to $\overline{\mathbf{R}}_+$ with the pointwise topology.

$$\mathsf{V}X \;=\; [\Omega_{\mathrm{s}}X \xrightarrow{mod} \overline{\mathbf{R}}_+]_{\mathrm{p}} \;\cong\; [[X \to \overline{\mathbf{R}}_+]_{\mathrm{i}} \xrightarrow{lin} \overline{\mathbf{R}}_+]_{\mathrm{p}}$$

Proof: For the proof, let $\mathcal{F} = [X \to \overline{\mathbf{R}}_+]_{\mathrm{i}}$.

One isomorphism is constructed from integration: Define $\alpha : \mathsf{V}X \to [\mathcal{F} \xrightarrow{lin} \overline{\mathbf{R}}_+]_{\mathrm{p}}$ by $\alpha(\nu) = \lambda f.\ \int_X(f, \nu)$. This function has the claimed type since integration is continuous and linear in its functional argument. The function α itself is linear since integration is linear in its valuation argument. It is continuous since integration is continuous in its two arguments separately (see Subsection 2.9).

The inverse isomorphism is defined using characteristic functions: For F in $[\mathcal{F} \xrightarrow{lin} \overline{\mathbf{R}}_+]_{\mathrm{p}}$, let $\beta(F) = F \circ \chi$ where $\chi : \Omega X \to \mathcal{F}$ with $\chi U = \widehat{U}$, the characteristic function of U. We first show that $\beta(F)$ is a Scott continuous valuation.

- $\beta(F)$ is strict since $F(\widehat{\emptyset}) = F(\lambda x.\, 0) = 0$ by linearity of F.
- $\beta(F)$ is modular since $F(f+g) = F(f) + F(g)$ and $\widehat{U \cup V} + \widehat{U \cap V} = \widehat{U} + \widehat{V}$.

- $\beta(F)$ is Scott continuous since both F and χ are Scott continuous. $F : \mathcal{F} \to \overline{\mathbf{R}}_+$ is Scott continuous since it is continuous and both $\mathcal{F}$ and $\overline{\mathbf{R}}_+$ are d-spaces. $\chi : \Omega X \to \mathcal{F}$ is Scott continuous since x in $\bigcup_{i \in I} O_i$ iff x in O_i for some i in I, whence $\chi(\bigcup_{i \in I} O_i)(x) = \bigsqcup_{i \in I} \chi(O_i)(x)$.

The function β itself is obviously linear. Since $[\mathcal{F} \stackrel{lin}{\to} \overline{\mathbf{R}}_+]_{\mathrm{p}}$ is a subspace of $\prod_{f \in \mathcal{F}} \overline{\mathbf{R}}_+$, the function $\beta_U : [\mathcal{F} \stackrel{lin}{\to} \overline{\mathbf{R}}_+]_{\mathrm{p}} \to \overline{\mathbf{R}}_+$ with $\beta_U(F) = F(\widehat{U})$ is continuous for every fixed U in ΩX. Since $[\Omega_{\mathrm{s}} X \stackrel{mod}{\to} \overline{\mathbf{R}}_+]_{\mathrm{p}}$ is a subspace of $\prod_{U \in \Omega X} \overline{\mathbf{R}}_+$, continuity of the function $\beta : [\mathcal{F} \stackrel{lin}{\to} \overline{\mathbf{R}}_+]_{\mathrm{p}} \to [\Omega_{\mathrm{s}} X \stackrel{mod}{\to} \overline{\mathbf{R}}_+]_{\mathrm{p}}$ follows.

At this point, we know that both α and β are continuous linear functions. We still have to show that they are inverse to each other.

For ν in $\mathsf{V}X$ and U in ΩX, $\beta(\alpha\nu)(U) = (\alpha\nu)(\widehat{U}) = \int_X(\widehat{U}, \nu) = \nu(U)$ holds using Theorem 8.1 (6), whence $\beta \circ \alpha = \mathsf{id}$.

For F in $[\mathcal{F} \stackrel{lin}{\to} \overline{\mathbf{R}}_+]_{\mathrm{p}}$, $\alpha(\beta F)(g) = \int_X(g, \beta F)$ holds for all g in $\mathcal{F}$. Hence, we have to show $\int_X(g, \beta F) = F(g)$ for all g in $\mathcal{F}$. Since both integration and the function F are Scott continuous, it suffices to show the equation for all simple functions g. Since both integration and the function F are linear, and every simple function is a finite linear combination of characteristic functions, it even suffices to show the equation for all functions $\widehat{U}$ with U in ΩX. By Prop. 8.1 (6), $\int_X(\widehat{U}, \beta F) = \beta F U = F \widehat{U}$ holds as required. □

An analogous theorem can be formulated for point continuous valuations.

Theorem 9.2 For every space X, the topological cone of point continuous valuations on X, i.e., strict modular continuous functions from $\Omega_{\mathrm{p}} X$ to $\overline{\mathbf{R}}_+$, is isomorphic to the topological cone of linear continuous functions from $[X \to \overline{\mathbf{R}}_+]_{\mathrm{p}}$ to $\overline{\mathbf{R}}_+$ with the pointwise topology.

$$\mathsf{V}_{\mathrm{p}} X \;=\; [\Omega_{\mathrm{p}} X \stackrel{mod}{\to} \overline{\mathbf{R}}_+]_{\mathrm{p}} \;\cong\; [[X \to \overline{\mathbf{R}}_+]_{\mathrm{p}} \stackrel{lin}{\to} \overline{\mathbf{R}}_+]_{\mathrm{p}}$$

Proof: Algebraically, the isomorphisms are the same as in the proof of Theorem 9.1. The difference is that the 'pointwise' version of integration is used. Hence, α defined by $\alpha(\nu) = \lambda f.\ \int_X(f, \nu)$ has type $\mathsf{V}_{\mathrm{p}} X \to [[X \to \overline{\mathbf{R}}_+]_{\mathrm{p}} \stackrel{lin}{\to} \overline{\mathbf{R}}_+]_{\mathrm{p}}$ in this case.

For F in $[[X \to \overline{\mathbf{R}}_+]_{\mathrm{p}} \stackrel{lin}{\to} \overline{\mathbf{R}}_+]_{\mathrm{p}}$, we have to show that $\beta(F) = F \circ \chi$ is point continuous, i.e., continuous with type $\Omega_{\mathrm{p}} X \to \overline{\mathbf{R}}_+$. We prove continuity of $\chi : \Omega_{\mathrm{p}} X \to [X \to \overline{\mathbf{R}}_+]_{\mathrm{p}}$. For x in X and V in $\Omega \overline{\mathbf{R}}_+$, $\chi^-\langle x \to V \rangle$ is $\Omega_{\mathrm{p}} X$ if $0 \in V$, $\mathcal{O}(x)$ if $0 \notin V$ and $1 \in V$, and $\emptyset$ if $1 \notin V$.

The remainder of the proof of Theorem 9.1 can be taken over unchanged. □

In general, the topology of a pointwise function space $[Y \to Z]_{\mathrm{p}}$ is the least such that for every y in Y, the function $\lambda f.\ fy : [Y \to Z]_{\mathrm{p}} \to Z$ is continuous. Applying this to $[[X \to \overline{\mathbf{R}}_+]_{\mathrm{i}} \stackrel{lin}{\to} \overline{\mathbf{R}}_+]_{\mathrm{p}}$, we see that its topology is the least such that for every continuous $f : X \to \overline{\mathbf{R}}_+$, the function $\lambda F.\ F(f)$ is continuous. Using the isomorphism α of the proof of Theorem 9.1, we conclude:

Theorem 9.3 The topology on $\mathsf{V}X$ is the least such that for every continuous function $f : X \to \overline{\mathbf{R}}_+$, the function $f^* : \mathsf{V}X \to \overline{\mathbf{R}}_+$ with $f^*(\nu) = \int_X(f, \nu)$ is continuous.

This characterizes the topology of $\mathsf{V}X$ as a 'weak topology'. Notice that in contrast to classical results which look similar, the space $\overline{\mathbf{R}}_+$ is equipped with the Scott topology instead of the usual Hausdorff topology.

10 V as a Kleisli Triple

In this section, we show that the construction V can be seen as the object map of a Kleisli triple in the category of topological spaces. The main work is already done; all what remains to do is to define the Kleisli operations and to show that they have the required properties.

For every space X, we need a function $\mathsf{s} : X \to \mathsf{V}X$. This function is given by $\mathsf{s}x = \widehat{x}$. It was already defined in Section 6. There, it was shown that s is continuous, and that it is a topological embedding of X into $\mathsf{V}X$ if X is a $\mathcal{T}_0$-space.

For every two spaces X and Y and every continuous function $f : X \to \mathsf{V}Y$, we need a continuous function $\bar{f} : \mathsf{V}X \to \mathsf{V}Y$. Using integration, $\bar{f}$ is defined as

$$\bar{f}(\nu) \;=\; \lambda V.\, \int_X (\lambda x.\, fx(V),\, \nu).$$

First, we have to show that $\bar{f}(\nu)$ is indeed in $\mathsf{V}Y$. It is strict since fx is strict for all x, and $\int_X(\lambda x.\, 0,\, \nu) = 0$. It is modular since fx is modular for all x, and integration is linear in its left argument. It is Scott continuous since fx, application, λ-abstraction, and integration are Scott continuous.

Second, we have to show continuity of $\bar{f}$. For fixed V in ΩY, the function $\nu \mapsto \bar{f}(\nu)(V)$ is continuous since integration is continuous in its right argument. Since $\mathsf{V}Y$ is a subspace of the power $\prod_{V \in \Omega Y} \overline{\mathbf{R}}_+$, continuity of $\bar{f}$ follows.

For s and extension $f \mapsto \bar{f}$ to be part of a Kleisli triple, we have to prove three properties.

(1) $\bar{f} \circ \mathsf{s} = f$:
For every point a of X, we compute: $\bar{f}(\mathsf{s}a) \;=\; \lambda V.\, \int_X(\lambda x. fx(V),\, \widehat{a}) \;=\; \lambda V. fa(V) \;=\; fa$.

(2) $\bar{\mathsf{s}} = \mathsf{id}$:
$\bar{\mathsf{s}}(\nu) \;=\; \lambda V.\, \int_X(\lambda x.\, \widehat{x}(V),\, \nu) \;=\; \lambda V.\, \int_X(\widehat{V},\, \nu) \;=\; \lambda V.\, \nu V \;=\; \nu.$

(3) For $f : X \to \mathsf{V}Y$ and $g : Y \to \mathsf{V}Z$, $\bar{g} \circ \bar{f} = \overline{\bar{g} \circ f}$:
For ν in $\mathsf{V}X$ and W in ΩX, we have to show

$$\int_Y (\lambda y.\, gyW, \lambda V.\, \int_X (\lambda x.\, fxV, \nu)) \;=\; \int_X (\lambda x.\, \int_Y (\lambda y.\, gyW, fx), \nu).$$

The function $\lambda y.\, gyW$ is continuous from Y to $\overline{\mathbf{R}}_+$. Generalizing a bit, we even prove

$$\int_Y (h, \lambda V.\, \int_X (\lambda x.\, fxV, \nu)) \;=\; \int_X (\lambda x.\, \int_Y (h, fx), \nu)$$

for all continuous $h : Y \to \overline{\mathbf{R}}_+$. Since both sides of the equation are continuous and linear in h, it suffices to prove the equation for the characteristic functions $h = \widehat{O}$ of opens O of Y. After substituting $\widehat{O}$ for h, both sides of the equation simplify to $\int_X(\lambda x.\, fxO, \nu)$.

This completes the proof that V is the object map of a Kleisli triple in the category of topological spaces.

Every Kleisli triple induces a functor which in our case is defined by $\mathsf{V}f \;=\; \overline{\mathsf{s} \circ f}$ for continuous $f : X \to Y$. We verify that this induced functor coincides with the functor defined in Section 6.

$$\overline{\mathsf{s} \circ f}(\nu)(V) \;=\; \int_X (\lambda x.\, \mathsf{s}(fx)(V), \nu) \;=\; \int_X (\widehat{f^- V}, \nu) \;=\; \nu(f^- V)$$

Every extended function $\bar{f}$ and extension $\mathsf{E} = \lambda f.\, \bar{f}$ itself are linear. This follows from the definition of $\bar{f}$ and linearity of integration in both arguments.

Next, we show that Kleisli extension $\mathsf{E} : [X \to \mathsf{V}Y]_{\mathrm{i}} \to [\mathsf{V}X \to \mathsf{V}Y]_{\mathrm{p}}$ is continuous. Since the topology on the right is pointwise, and the topology on $\mathsf{V}Y$ is pointwise as well, it suffices to show that for every fixed ν in $\mathsf{V}X$ and V in ΩY, the function $\lambda f. \mathsf{E} f \nu V = \lambda f. \int_X (\lambda x. f x V, \nu)$ is continuous from $[X \to \mathsf{V}Y]_{\mathrm{i}}$ to $\overline{\mathbf{R}}_+$. It is the composition of $G : [X \to \mathsf{V}Y]_{\mathrm{i}} \to [X \to \overline{\mathbf{R}}_+]_{\mathrm{i}}$ with $G(f) = \lambda x.\, f x V$ and $H : [X \to \overline{\mathbf{R}}_+]_{\mathrm{i}} \to \overline{\mathbf{R}}_+$ with $H(g) = \int_X (g, \nu)$. The latter is continuous because of the continuity properties of integration. The former is continuous since $G(f) = h \circ f$ holds where $h(\nu') = \nu'(V)$ is continuous, and composition is separately continuous w.r.t. the Isbell topologies of the function spaces.

The continuity property $\mathsf{E} : [X \to \mathsf{V}Y]_{\mathrm{i}} \to [\mathsf{V}X \to \mathsf{V}Y]_{\mathrm{p}}$ looks a bit awkward. At least, it is sufficient to conclude that E is Scott continuous since both function spaces are d-spaces.

From the continuity property of E, we can derive that the functorial mapping V is continuous from $[X \to Y]_{\mathrm{i}}$ to $[\mathsf{V}X \to \mathsf{V}Y]_{\mathrm{p}}$. For, $\mathsf{V} = \mathsf{E} \circ S$, where $S(f) = \mathsf{s} \circ f$, and S is continuous from $[X \to Y]_{\mathrm{i}}$ to $[X \to \mathsf{V}Y]_{\mathrm{i}}$ by separate continuity of composition in the Isbell case.

Finally, we show that the Kleisli triple V cuts down to a Kleisli triple V_{p}, i.e., we prove that for continuous $f : X \to \mathsf{V}_{\mathrm{p}} Y$, $\mathsf{E} f$ maps from $\mathsf{V}_{\mathrm{p}} X$ to $\mathsf{V}_{\mathrm{p}} Y$.

A priori, $\mathsf{E} f$ restricts to a continuous linear function $\mathsf{E}_{\mathrm{p}} f$ from $\mathsf{V}_{\mathrm{p}} X$ to $\mathsf{V}Y$ with $\mathsf{E}_{\mathrm{p}} f \circ \mathsf{s} = f$. Since $\mathsf{V}_{\mathrm{p}} Y$ is a locally convex sober cone, the universal property of $\mathsf{V}_{\mathrm{p}} X$ gives us a continuous linear function $\bar{f} : \mathsf{V}_{\mathrm{p}} X \to \mathsf{V}_{\mathrm{p}} Y \hookrightarrow \mathsf{V}Y$ with $\bar{f} \circ \mathsf{s} = f$. By the uniqueness part of the universal property, $\mathsf{E}_{\mathrm{p}} f$ and $\bar{f}$ coincide, whence $\mathsf{E}_{\mathrm{p}} f : \mathsf{V}_{\mathrm{p}} X \to \mathsf{V}_{\mathrm{p}} Y$. By universality again, E_{p} coincides with the function of Theorem 7.11. This shows that it is continuous from $[X \to \mathsf{V}_{\mathrm{p}} Y]_{\mathrm{p}}$ to $[\mathsf{V}_{\mathrm{p}} X \to \mathsf{V}_{\mathrm{p}} Y]_{\mathrm{p}}$. Thus, E_{p} satisfies a stronger continuity property than the Kleisli extension E of V. It follows that the functorial mapping V_{p} is continuous from $[X \to Y]_{\mathrm{p}}$ to $[\mathsf{V}_{\mathrm{p}} X \to \mathsf{V}_{\mathrm{p}} Y]_{\mathrm{p}}$. Summarizing, we obtain:

Theorem 10.1 Let $\mathsf{s} : X \to \mathsf{V}X$ with $\mathsf{s}x = \widehat{x}$ and $\mathsf{E} : [X \to \mathsf{V}Y]_{\mathrm{i}} \to [\mathsf{V}X \stackrel{lin}{\to} \mathsf{V}Y]_{\mathrm{p}}$ with

$$\mathsf{E} f(\nu)(V) \;=\; \int_X (\lambda x.\, f x(V),\, \nu).$$

Then s is a continuous embedding, E is continuous and linear, and $(\mathsf{V}, \mathsf{s}, \mathsf{E})$ forms a Kleisli triple. The induced functorial map is the continuous function $\mathsf{V} : [X \to Y]_{\mathrm{i}} \to [\mathsf{V}X \stackrel{lin}{\to} \mathsf{V}Y]_{\mathrm{p}}$ with $\mathsf{V} f(\nu) = \nu \circ f^{-}$.

In the point continuous case, E restricts to $\mathsf{E}_{\mathrm{p}} : [X \to \mathsf{V}_{\mathrm{p}} Y]_{\mathrm{p}} \to [\mathsf{V}_{\mathrm{p}} X \stackrel{lin}{\to} \mathsf{V}_{\mathrm{p}} Y]_{\mathrm{p}}$, and V to $\mathsf{V}_{\mathrm{p}} : [X \to Y]_{\mathrm{p}} \to [\mathsf{V}_{\mathrm{p}} X \stackrel{lin}{\to} \mathsf{V}_{\mathrm{p}} Y]_{\mathrm{p}}$.

11 Products of Valuations

In this section, we consider the problem to derive a product valuation on $X \times Y$ or $X \otimes Y$ from given valuations α on X and β on Y.

11.1 Topological Product

Our first problem is to derive a continuous product operation $\times : \mathsf{V}X \times \mathsf{V}Y \to \mathsf{V}(X \times Y)$ whose result is defined on the open sets of the topological product space.

As in [10], we start with a function $\mathsf{t} : X \times \mathsf{V}Y \to \mathsf{V}(X \times Y)$.

Proposition 11.1 Let $\mathsf{t} : X \times \mathsf{V}Y \to \mathsf{V}(X \times Y)$ be a function defined by $\mathsf{t}(x, \beta) = \lambda W.\, \beta(W_x)$ where $W_x = (\lambda y.\, (x, y))^{-}W = \{y \in Y \mid (x, y) \in W\}$. The function t is continuous, linear in the second argument, and satisfies $\mathsf{t}(x, \widehat{y}) = \widehat{(x, y)}$ and $\mathsf{t}(x, \beta)(U \times V) = \widehat{U}x \cdot \beta V$.

Proof: The result $\mathsf{t}(x, \beta) = \beta \circ (\lambda y.\, (x, y))^{-}$ is indeed a valuation on $X \times Y$. Linearity in the valuation argument is obvious.

By definition, $\mathsf{t}(x, \widehat{y})(W) = \widehat{y}(W_x)$ holds. Since $y \in W_x$ iff $(x, y) \in W$, this equals $\widehat{(x, y)}(W)$.

For the last equation, consider $\beta((U \times V)_x)$. If x in U, then $\widehat{U}x = 1$ and $(U \times V)_x = V$ holds, and if x is not in U, then $\widehat{U}x = 0$ and $(U \times V)_x$ is empty. In any case, $\beta((U \times V)_x)$ is $\widehat{U}x \cdot \beta V$.

For continuity of t, it suffices to show that all functions $\mathsf{t}_W : X \times \mathsf{V}Y \to \overline{\mathbf{R}}_+$ with $\mathsf{t}_W(x, \beta) = \beta W_x$ are continuous. Let $\mathsf{t}_W(x, \beta) > r$, i.e., $\beta W_x > r$. By Scott continuity of β, the set $\mathcal{V} = \{V \in \Omega Y \mid \beta V > r\}$ is Scott open. By assumption, W_x is in $\mathcal{V}$.

For every y in W_x, (x, y) is in W. By definition of the product topology, there are open sets U_y of X and V_y of Y such that $(x, y) \in U_y \times V_y \subseteq W$. From $y \in V_y$ for all y in W_x, $W_x \subseteq \bigcup_{y \in W_x} V_y$ follows. Since W_x is in $\mathcal{V}$, so is the union. By Scott continuity of $\mathcal{V}$, there is some finite subset G of W_x such that $V' = \bigcup_{y \in G} V_y$ is in $\mathcal{V}$. Let $U' = \bigcap_{y \in G} U_y$. Then $x \in U'$ and $\beta V' > r$, whence $(x, \beta) \in U' \times \langle V' > r \rangle$, an open set of $X \times \mathsf{V}Y$.

Let (x', β') be a member of $U' \times \langle V' > r \rangle$. From $U_y \times V_y \subseteq W$ for all y in $G \subseteq W_x$, $U' \times V' \subseteq W$ follows. Hence, $\{x'\} \times V' \subseteq W$, or $V' \subseteq W_x$. Thus, $\beta'(W_x) \geq \beta' V' > r$ holds. □

Of course, there is a dual function, i.e., a continuous function $\mathsf{t}' : \mathsf{V}X \times Y \to \mathsf{V}(X \times Y)$ with dual properties. A continuous product $\times : \mathsf{V}X \times \mathsf{V}Y \to \mathsf{V}(X \times Y)$ can then be obtained by composing an instance of t, namely $\mathsf{t} : \mathsf{V}X \times \mathsf{V}Y \to \mathsf{V}(\mathsf{V}X \times Y)$ with the Kleisli-extension $\overline{\mathsf{t}'}$ of t'. Hence

$$\begin{aligned}(\alpha \times \beta)(W) &= \overline{\mathsf{t}'}(\mathsf{t}(\alpha, \beta))(W) \\ &= \textstyle\int_{\mathsf{V}X \times Y}(\lambda(\nu, y).\, \mathsf{t}'(\nu, y)(W), \mathsf{t}(\alpha, \beta)) \\ &= \textstyle\int_{\mathsf{V}X \times Y}(\lambda(\nu, y).\, \mathsf{t}'(\nu, y)(W), \beta \circ (\lambda y.\, (\alpha, y))^{-}) \\ &= \textstyle\int_{Y}(\lambda y.\, \mathsf{t}'(\alpha, y)(W), \beta)\end{aligned}$$

where the substitution property Theorem 8.1 (7) was used for the last equality. Thus

$$\begin{aligned}(\alpha \times \beta)(U \times V) &= \textstyle\int_{Y}(\lambda y.\, \mathsf{t}'(\alpha, y)(U \times V), \beta) \\ &= \textstyle\int_{Y}(\lambda y.\, \alpha U \cdot \widehat{V}y, \beta) \\ &= \alpha U \cdot \textstyle\int_{Y}(\lambda y.\, \widehat{V}y, \beta) \\ &= \alpha U \cdot \beta V\end{aligned}$$

Since the rectangles $U \times V$ form a base of the topology of $X \times Y$, Prop. 3.2 implies that the product valuation $\alpha \times \beta$ is uniquely determined by the property $(\alpha \times \beta)(U \times V) = \alpha U \cdot \beta V$. This uniqueness can be used to derive some further properties. All facts are collected in the following theorem.

Theorem 11.2 For two spaces X and Y, there is a unique function $\times : \mathsf{V}X \times \mathsf{V}Y \to \mathsf{V}(X \times Y)$ with the property $(\alpha \times \beta)(U \times V) = \alpha U \cdot \beta V$ for all α in $\mathsf{V}X$, β in $\mathsf{V}Y$, U in ΩX, and V in ΩY. This function has the following properties:

(1) It is continuous.

(2) It is Scott continuous.

(3) It is linear in each argument.

(4) $\widehat{x} \times \widehat{y} = \widehat{(x,y)}$ holds for all x in X and y in Y.

(5) The product is symmetric: For all α in $\mathsf{V}X$ and β in $\mathsf{V}Y$, $\beta \times \alpha = \mathsf{V}g(\alpha \times \beta)$ holds where $g : X \times Y \to Y \times X$ is defined by $g(x,y) = (y,x)$.

(6) The product is associative: For all α in $\mathsf{V}X$, β in $\mathsf{V}Y$, and γ in $\mathsf{V}Z$, $\alpha \times (\beta \times \gamma) = \mathsf{V}h((\alpha \times \beta) \times \gamma)$ holds where $h : (X \times Y) \times Z \to X \times (Y \times Z)$ is defined by $h((x,y),z) = (x,(y,z))$.

The function '$\times$' can be restricted to a continuous function $\times : \mathsf{V}_{\mathrm{p}}X \times \mathsf{V}_{\mathrm{p}}Y \to \mathsf{V}_{\mathrm{p}}(X \times Y)$.

Proof: Existence, uniqueness, and continuity have already been shown. Scott continuity follows from continuity. One of the equalities belonging to linearity is the following:

$$(\alpha_1 + \alpha_2) \times \beta \;=\; (\alpha_1 \times \beta) + (\alpha_2 \times \beta)$$

Both sides of the equation are valuations on $X \times Y$. By Prop. 3.2, they are equal if they coincide on all open rectangles $U \times V$. The computation

$$((\alpha_1 + \alpha_2) \times \beta)(U \times V) = (\alpha_1 U + \alpha_2 U) \cdot \beta V$$
$$((\alpha_1 \times \beta) + (\alpha_2 \times \beta))(U \times V) = \alpha_1 U \cdot \beta V + \alpha_2 U \cdot \beta V$$

shows that the two sides are equal. All the remaining equalities in the theorem can be shown by similar arguments. The restriction to point continuous valuations works since t, t', and the Kleisli extension respect point continuity. □

11.2 Double Integral and Product Valuation

In this subsection, we prove that a continuous function $f : X \times Y \to \overline{\mathbf{R}}_+$ can be integrated in three different ways, yielding the same result. If α is a Scott continuous valuation on X and β a Scott continuous valuation on Y, then f can be integrated w.r.t. the product valuation: $\int_{X \times Y}(f, \alpha \times \beta)$. Alternatively, we may form the double integrals $\int_X(\lambda x.\ \int_Y(\lambda y.\, f(x,y), \beta), \alpha)$ and $\int_Y(\lambda y.\ \int_X(\lambda x.\, f(x,y), \alpha), \beta)$. We shall prove that the double integrals are well defined, and all three integrals yield the same value.

Consider the first double integral. For fixed x, the function $\lambda y.\, f(x,y)$ is continuous from Y to $\overline{\mathbf{R}}_+$. (For this, separate continuity of f, i.e., $f : X \otimes Y \to \overline{\mathbf{R}}_+$, would suffice.) Hence, the inner integral is well defined. We also have to show that $\lambda x.\ \int_Y(\lambda y.\, f(x,y), \beta)$ is continuous. This function can be written as composition of $F : X \to [Y \to \overline{\mathbf{R}}_+]_{\mathrm{i}}$ with $Fx = \lambda y. f(x,y)$ and $G : [Y \to \overline{\mathbf{R}}_+]_{\mathrm{i}} \to \overline{\mathbf{R}}_+$ with $G(h) = \int_Y(h, \beta)$. Function F is continuous as the currification of f (Subsection 2.10). Function G is continuous since $\int_Y : [Y \to \overline{\mathbf{R}}_+]_{\mathrm{i}} \otimes \mathsf{V}Y \to \overline{\mathbf{R}}_+$ is continuous. Thus, the outer integral in the first double integral is defined as well. The second double integral is handled analogously.

Our first step towards the main theorem is to derive a representation of the product valuation by double integrals.

Proposition 11.3 For two spaces X and Y, α in $\mathsf{V}X$, and β in $\mathsf{V}Y$:

$$\begin{aligned}\alpha \times \beta &= \lambda(W \in \Omega(X \times Y)).\ \textstyle\int_X(\lambda x.\ \int_Y(\lambda y.\, \widehat{W}(x,y), \beta), \alpha)\\ &= \lambda(W \in \Omega(X \times Y)).\ \textstyle\int_Y(\lambda y.\ \int_X(\lambda x.\, \widehat{W}(x,y), \alpha), \beta)\end{aligned}$$

Proof: The double integrals are well defined since for opens W of $X \times Y$, the characteristic function $\widehat{W} : X \times Y \to \overline{\mathbf{R}}_+$ is continuous. Because of linearity and Scott continuity of integration, all three terms in the proposition denote Scott continuous valuations on $X \times Y$. By Prop. 3.2, their equality can be shown by applying them to open rectangles $U \times V$. By Theorem 11.2, $(\alpha \times \beta)(U \times V)$ is $\alpha U \cdot \beta V$. We may compute:

$$\begin{aligned}
\textstyle\int_X(\lambda x.\ \int_Y(\lambda y.\ \widehat{U \times V}(x,y), \beta), \alpha) &= \textstyle\int_X(\lambda x.\ \int_Y(\lambda y.\ \widehat{U}x \cdot \widehat{V}y, \beta), \alpha) \\
&= \textstyle\int_X(\lambda x.\ \widehat{U}x \cdot \int_Y(\lambda y.\ \widehat{V}y, \beta), \alpha) \\
&= \textstyle\int_X(\lambda x.\ \widehat{U}x \cdot \beta V, \alpha) \\
&= \textstyle\int_X(\lambda x.\ \widehat{U}x, \alpha) \cdot \beta V \\
&= \alpha U \cdot \beta V
\end{aligned}$$

The third term yields the same result when applied to $U \times V$. Hence, all three terms are equal. □

The main theorem about double integrals is as follows:

Theorem 11.4 For spaces X and Y, α in $\mathsf{V}X$, β in $\mathsf{V}Y$, and continuous $f : X \times Y \to \overline{\mathbf{R}}_+$:

$$\begin{aligned}
\textstyle\int_{X \times Y}(f, \alpha \times \beta) &= \textstyle\int_X(\lambda x.\ \int_Y(\lambda y.\ f(x,y), \beta), \alpha) \\
&= \textstyle\int_Y(\lambda y.\ \int_X(\lambda x.\ f(x,y), \alpha), \beta)
\end{aligned}$$

Proof: By Scott continuity and linearity of integration, it suffices to consider the case where f is the characteristic function $\widehat{W}$ of an open W of $X \times Y$. Since $\int_{X \times Y}(\widehat{W}, \alpha \times \beta) = (\alpha \times \beta)(W)$, the theorem directly follows from Prop. 11.3. □

11.3 Tensor Product

Next, we define a tensor product operation $\otimes : \mathsf{V}_\mathrm{p} X \otimes \mathsf{V}_\mathrm{p} Y \to \mathsf{V}_\mathrm{p}(X \otimes Y)$ whose result is defined on the open sets of the tensor product space. We were not able to generalize the tensor product to Scott continuous valuations which are not point continuous.

Theorem 11.5 For two spaces X and Y, there is a unique continuous bilinear function $\otimes : \mathsf{V}_\mathrm{p} X \otimes \mathsf{V}_\mathrm{p} Y \to \mathsf{V}_\mathrm{p}(X \otimes Y)$ with the property $\widehat{x} \otimes \widehat{y} = \widehat{(x, y)}$ for all x in X and y in Y. This function has the following properties:

(1) It is Scott continuous.

(2) $(\alpha \otimes \beta)(U \times V) = \alpha U \cdot \beta V$ holds for all α in $\mathsf{V}_\mathrm{p} X$, β in $\mathsf{V}_\mathrm{p} Y$, U in ΩX, and V in ΩY.

(3) It is symmetric: For all α in $\mathsf{V}_\mathrm{p} X$ and β in $\mathsf{V}_\mathrm{p} Y$, $\beta \otimes \alpha = \mathsf{V}_\mathrm{p}\, g(\alpha \otimes \beta)$ holds where $g : X \otimes Y \to Y \otimes X$ is defined by $g(x, y) = (y, x)$.

(4) It is associative: For α in $\mathsf{V}_\mathrm{p} X$, β in $\mathsf{V}_\mathrm{p} Y$, and γ in $\mathsf{V}_\mathrm{p} Z$, $\alpha \otimes (\beta \otimes \gamma) = \mathsf{V}_\mathrm{p}\, h((\alpha \otimes \beta) \otimes \gamma)$ holds where $h : (X \otimes Y) \otimes Z \to X \otimes (Y \otimes Z)$ is defined by $h((x, y), z) = (x, (y, z))$.

For W in $\Omega(X \times Y)$, $(\alpha \otimes \beta)(W) = (\alpha \times \beta)(W)$ holds for all α in $\mathsf{V}_\mathrm{p} X$ and β in $\mathsf{V}_\mathrm{p} Y$, where '$\times$' is the product function of Theorem 11.2.

Proof: We apply Theorem 11.5 to the function $\mathsf{s} : X \otimes Y \to \mathsf{V}_\mathrm{p}(X \otimes Y)$ with $\mathsf{s}(x, y) = \widehat{(x, y)}$. This is possible since $\mathsf{V}_\mathrm{p}(X \otimes Y)$ is a locally convex sober cone. By the theorem, there is a unique continuous bilinear function $\otimes : \mathsf{V}_\mathrm{p} X \otimes \mathsf{V}_\mathrm{p} Y \to \mathsf{V}_\mathrm{p}(X \otimes Y)$ with $\widehat{x} \otimes \widehat{y} = \mathsf{s}(x, y) = \widehat{(x, y)}$. Scott continuity follows from continuity since all V_p-spaces are sober. For (2), fix U in ΩX and V in ΩY. The two functions $F, G : \mathsf{V}_\mathrm{p} X \otimes \mathsf{V}_\mathrm{p} Y \to \overline{\mathbf{R}}_+$ with $F(\alpha, \beta) = (\alpha \otimes \beta)(U \times V)$

and $G(\alpha, \beta) = \alpha U \cdot \beta V$ are continuous, bilinear, and coincide on pairs of point valuations since $\widehat{(x, y)}(U \times V) = \widehat{x}U \cdot \widehat{y}V$. By Theorem 7.1 (3), $F = G$ follows. The equalities (3) and (4) can be shown analogously.

The restriction of $\alpha \otimes \beta$ to $\Omega(X \times Y)$, which is a subset of $\Omega(X \otimes Y)$, equals $\alpha \times \beta$ since both valuations coincide on open rectangles $U \times V$. □

11.4 Double Integral and Tensor Product

In this subsection, we prove the analogon of Theorem 11.4 for tensor products and point continuous valuations. Let $f : X \otimes Y \to \overline{\mathbf{R}}_+$ be a (separately) continuous function, α a point continuous valuation on X, and β a point continuous valuation on Y. Then f can be integrated w.r.t. the tensor product valuation: $\int_{X \otimes Y}(f, \alpha \otimes \beta)$. Alternatively, we may form the double integrals $\int_X(\lambda x.\ \int_Y(\lambda y.\ f(x, y), \beta), \alpha)$ and $\int_Y(\lambda y.\ \int_X(\lambda x.\ f(x, y), \alpha), \beta)$. We shall prove that the double integrals are well defined, and all three integrals yield the same value.

Consider the first double integral. For fixed x, the function $\lambda y.\ f(x, y)$ is continuous from Y to $\overline{\mathbf{R}}_+$. Hence, the inner integral is well defined. We also have to show that $\lambda x.\ \int_Y(\lambda y.\ f(x, y), \beta)$ is continuous. This function can be written as composition of $F : X \to [Y \to \overline{\mathbf{R}}_+]_{\mathrm{p}}$ with $Fx = \lambda y. (x, y)$ and $G : [Y \to \overline{\mathbf{R}}_+]_{\mathrm{p}} \to \overline{\mathbf{R}}_+$ with $G(h) = \int_Y(h, \beta)$. Function F is continuous as the currification of f (Subsection 2.9). Function G is continuous since $\int_Y : [Y \to \overline{\mathbf{R}}_+]_{\mathrm{p}} \otimes \mathsf{V}_{\mathrm{p}} Y \to \overline{\mathbf{R}}_+$ is continuous. Thus, the outer integral in the first double integral is defined as well. The second double integral is handled analogously.

The main theorem about double integrals is as follows:

Theorem 11.6 For two spaces X and Y, α in $\mathsf{V}_{\mathrm{p}} X$, β in $\mathsf{V}_{\mathrm{p}} Y$, and (separately) continuous $f : X \otimes Y \to \overline{\mathbf{R}}_+$:

$$\begin{aligned} \textstyle\int_{X \otimes Y}(f, \alpha \otimes \beta) &= \textstyle\int_X(\lambda x.\ \int_Y(\lambda y.\ f(x, y), \beta), \alpha) \\ &= \textstyle\int_Y(\lambda y.\ \int_X(\lambda x.\ f(x, y), \alpha), \beta) \end{aligned}$$

Proof: For fixed f, the three terms of the theorem are separately continuous and linear in α and β. Hence, all three terms induce continuous bilinear functions from $\mathsf{V}_{\mathrm{p}} X \otimes \mathsf{V}_{\mathrm{p}} Y$ to $\overline{\mathbf{R}}_+$. By Theorem 7.1 (3), these functions are equal if they coincide on pairs $(\widehat{x}, \widehat{y})$ of point valuations.

$$\begin{aligned} \textstyle\int_{X \otimes Y}(f, \widehat{x} \otimes \widehat{y}) &= \textstyle\int_{X \otimes Y}(f, \widehat{(x, y)}) &&= f(x, y) \\ \textstyle\int_X(\lambda x.\ \int_Y(\lambda y.\ f(x, y), \widehat{y}), \widehat{x}) &= \textstyle\int_X(\lambda x.\ f(x, y), \widehat{x}) &&= f(x, y) \\ \textstyle\int_Y(\lambda y.\ \int_X(\lambda x.\ f(x, y), \widehat{x}), \widehat{y}) &= \textstyle\int_Y(\lambda y.\ f(x, y), \widehat{y}) &&= f(x, y) \end{aligned}$$

This proves the claimed equality. □

A property analogous to Prop. 11.3 easily follows.

Proposition 11.7 For two spaces X and Y, α in $\mathsf{V}_{\mathrm{p}} X$, and β in $\mathsf{V}_{\mathrm{p}} Y$:

$$\begin{aligned} \alpha \otimes \beta &= \lambda(W \in \Omega(X \otimes Y)).\ \textstyle\int_X(\lambda x.\ \int_Y(\lambda y. \widehat{W}(x, y), \beta), \alpha) \\ &= \lambda(W \in \Omega(X \otimes Y)).\ \textstyle\int_Y(\lambda y.\ \int_X(\lambda x. \widehat{W}(x, y), \alpha), \beta) \end{aligned}$$

Proof: This is a specialization of Theorem 11.6 to the case that $f : X \otimes Y \to \overline{\mathbf{R}}_+$ is the characteristic function $\widehat{W}$ of an open W of $X \otimes Y$. □

11.5 Comparison of the Product Operations

In the previous subsections, we defined a product operation $\times : \mathsf{V}X \times \mathsf{V}Y \to \mathsf{V}(X \times Y)$ and a tensor product operation $\otimes : \mathsf{V}_{\mathrm{p}} X \otimes \mathsf{V}_{\mathrm{p}} Y \to \mathsf{V}_{\mathrm{p}}(X \otimes Y)$. The two operations agree on arguments where both are defined, i.e., for α in $\mathsf{V}_{\mathrm{p}} X$, β in $\mathsf{V}_{\mathrm{p}} Y$, and W in $\Omega(X \times Y)$, $(\alpha \times \beta)(W) = (\alpha \otimes \beta)(W)$ holds. Apart from this, the two operations are of incomparable strength.

For point continuous α and β, $\alpha \otimes \beta$ is more powerful than $\alpha \times \beta$, since the topology of $X \otimes Y$ is a superset of the topology of $X \times Y$. There are examples of spaces X and Y where this inclusion is strict, i.e., there are sets W such that $(\alpha \otimes \beta)(W)$ is defined, but $(\alpha \times \beta)(W)$ is not.

On the other hand, the operation '$\times$' is more powerful than '$\otimes$' since the former is defined for all Scott continuous valuations, whereas the latter is defined for point continuous valuations only. We do not know whether this restriction is a necessity.

Problem 2 Is it possible to extend $\otimes : \mathsf{V}_{\mathrm{p}} X \otimes \mathsf{V}_{\mathrm{p}} Y \to \mathsf{V}_{\mathrm{p}}(X \otimes Y)$ to a function $\otimes : \mathsf{V}X \otimes \mathsf{V}Y \to \mathsf{V}(X \otimes Y)$ with similar properties?

Another advantage of '$\times$' over '$\otimes$' is that '$\times$' is jointly continuous, whereas '$\otimes$' is continuous in its two arguments separately. This restriction is tight for $\mathcal{T}_0$-spaces: If there were a function $* : \mathsf{V}_{\mathrm{p}} X \times \mathsf{V}_{\mathrm{p}} Y \to \mathsf{V}_{\mathrm{p}}(X \otimes Y)$ with $\widehat{x} * \widehat{y} = \widehat{(x, y)}$, then there would be a continuous function $f : X \times Y \to \mathsf{V}_{\mathrm{p}}(X \otimes Y)$ defined by $f(x, y) = \widehat{x} * \widehat{y} = \widehat{(x, y)}$. Since the image of f would be a subset of the image of the embedding $\mathsf{s} : X \otimes Y \to \mathsf{V}_{\mathrm{p}}(X \otimes Y)$ with $\mathsf{s}z = \widehat{z}$, the identity-like function $g : X \times Y \to X \otimes Y$ with $g(x, y) = (x, y)$ would be continuous, whence $X \times Y = X \otimes Y$. However, there are $\mathcal{T}_0$-spaces X and Y where this equality does not hold.

12 Integer Valuations

An *integer valuation* ν on a space X is a valuation with the property that $\nu(O)$ is in $\overline{\mathbf{N}}_0$ for all O in ΩX. Almost all operations on valuations create integer valuations or map integer valuations to integer valuations. The only exception is, of course, multiplication by a non-integer constant.

Of course, integer valuations inherit all the properties of general real-valued valuations. In addition, they have some more properties because of their special nature. These additional properties are presented in this section.

12.1 The Taxonomy of Integer Valuations

Of course, integer valuations can be classified according to the same principles as general valuations (see Section 4). The difference is that in the integer case, some classes of valuations become identical which are different in the general case.

Theorem 12.1

(1) Every bounded Scott continuous integer valuation is simple.

(2) Every simple integer valuation is a finite sum of primitive valuations.

(3) Every Scott continuous integer valuation is a directed join of simple integer valuations.

(4) Every Scott continuous integer valuation is point continuous.

Proof:

(1) In $\overline{\mathbf{N}}_0$, every bounded set is finite.

(2) By Theorem 4.1, every simple valuation ν is a finite linear combination of primitive valuations. In the proof of 4.1, the coefficients of this linear combination are obtained as members of the images of corestrictions of ν. Hence, they are integers if ν is integer-valued. A finite linear combination with coefficients from $\mathbf{N}_0$ can be considered as a finite sum.

(3) In the proof of Theorem 4.7, a Scott continuous valuation ν is approximated by the bounded valuations $\nu_{V,F,n} = \nu|_V + \sum_{x \in F} n \cdot \widehat{x}$ where V is open, F is a finite set, and n is in $\mathbf{N}_0$. Clearly, this is an integer valuation if ν is integer-valued. By (1), bounded integer valuations are simple.

(4) This follows from (2), since simple valuations are point continuous by Prop. 4.2, and a directed join of point continuous valuations is again point continuous. $\square$

Thus, the notions of bounded and simple, and the notions of Scott continuous and point continuous coincide for integer valuations. This is not true for general valuations.

By Cor. 5.5, every finite valuation ν on a $\mathcal{T}_0$-space can be uniquely represented by a finite point density A. If ν is integer-valued, then so is A:

Proposition 12.2 Let A be a finite point density in a $\mathcal{T}_0$-space X. If A^* is an integer valuation, then A itself is integer-valued, i.e., Ax in $\overline{\mathbf{N}}_0$ for all x in X.

Proof: By Lemma 5.3,

$$Ax = \sqcap\{A^*(O) \mid x \in O \in \Omega X\} - \sqcap\{A^*(O) \mid O \in \Omega X, \uparrow x \subseteq O \cup \{x\}\}$$

holds. Hence Ax is in $\overline{\mathbf{N}}_0$ if all $A^*(O)$ are in $\overline{\mathbf{N}}_0$. $\square$

The $\mathcal{T}_0$ property is really needed. Consider the space $X = \{a, b\}$ where $\emptyset$ and X are the only open sets. The finite point density A with $Aa = Ab = 1/2$ induces the valuation ν with $\nu(\emptyset) = 0$ and $\nu(X) = 1$, which is an integer valuation.

12.2 Spaces of Integer Valuations

Starting from the space $\mathsf{V}X$ of Scott continuous valuations on X, the subspace of integer valuations is denoted by $\mathsf{V}^{\mathrm{N}}X$, and the subspace of finite integer valuations by $\mathsf{V}_{\mathrm{f}}^{\mathrm{N}}X$. We need not introduce a notation for the subspace of point continuous integer valuations since it is identical to $\mathsf{V}^{\mathrm{N}}X$ because of Theorem 12.1 (4).

The topology of $\mathsf{V}^{\mathrm{N}}X$ is generated by the subbasic opens $\langle U > n\rangle = \{\nu \in \mathsf{V}^{\mathrm{N}}X \mid \nu U > n\}$ where U ranges over the opens of X and n ranges over $\mathbf{N}_0$. Alternatively, we may use $\langle U \geq n\rangle = \{\nu \in \mathsf{V}^{\mathrm{N}}X \mid \nu U \geq n\}$.

In analogy with Prop. 6.1, we obtain:

Proposition 12.3 For every space X, the space $\mathsf{V}^{\mathrm{N}}X$ is sober, and $\mathsf{V}_{\mathrm{f}}^{\mathrm{N}}X$ is a $\mathcal{T}_0$-space.

In Section 6, we proved that $\mathsf{V_p}\,X$ is the sobrification of $\mathsf{V_f}\,X$ (Theorem 6.5). In the sequel, we want to show an analogous theorem for integer valuations. The proof for the general case used three auxiliary lemmas. This proof cannot be taken over because of some $\overline{\mathbf{R}}_+$-specific arguments in the proof of Lemma 6.3. Thus, we present a new proof for the integer case, which is simpler than the proof for $\mathsf{V_p}\,X$.

We want to show that for every space X, $\mathsf{V}^{\mathrm{N}}X$ is the sobrification of $\mathsf{V}_{\mathrm{f}}^{\mathrm{N}}X$. As defined in Subsection 2.12, we have to prove: For every continuous integer valuation ν and open set $\mathcal{O}$ of $\mathsf{V}^{\mathrm{N}}X$ with ν in $\mathcal{O}$, there is a finite integer valuation $\varphi \sqsubseteq \nu$ with φ in $\mathcal{O}$. We use two auxiliary lemmas.

The first lemma is analogous to Lemma 6.2.

Lemma 12.4 For every continuous integer valuation ν and open set $\mathcal{O}$ of $\mathsf{V}^{\mathrm{N}}X$ with ν in $\mathcal{O}$, there is a simple integer valuation $\nu' \sqsubseteq \nu$ with ν' in $\mathcal{O}$.

Proof: By Theorem 12.1 (3), ν is a directed join of simple integer valuations. In a sober space such as $\mathsf{V}^{\mathrm{N}}X$, every open set is Scott open. □

The next lemma deals with the step from primitive valuations to point valuations.

Lemma 12.5 For every primitive valuation π and open set $\mathcal{O}$ of $\mathsf{V}^{\mathrm{N}}X$ with π in $\mathcal{O}$, there is a point x of X with $\widehat{x} \sqsubseteq \pi$ and $\widehat{x}$ in $\mathcal{O}$.

Proof: By Lemma 3.3, there is a closed set C such that $\pi O > 0$ iff $O \between C$. Because of primitivity, $\pi O > 0$ is equivalent to $\pi O = 1$.

Using the subbase of $\mathsf{V}^{\mathrm{N}}X$, there are open sets $U_1, \ldots, U_n$, and numbers $k_1, \ldots, k_n$ in $\mathbf{N}_0$ such that $\pi \in \bigcap_{i=1}^{n}\langle U_i > k_i\rangle \subseteq \mathcal{O}$. By primitivity, i.e., $\Im(\pi) = \{0, 1\}$, $k_i = 0$ and $\pi U_i = 1$ follows for all i. Let $V = \bigcap_{i=1}^{n} U_i$. By modularity of π, $\pi V = 1$ holds. Thus, V meets C. Let x be in the intersection.

If $\widehat{x}O = 1$, then $x \in O$, whence $O \between C$, whence $\pi O = 1$. Thus, $\widehat{x} \sqsubseteq \pi$ holds. Since x is in V, $\widehat{x}U_i = 1$ holds for all i. Hence, $\widehat{x}$ is in $\bigcap_{i=1}^{n}\langle U_i > 0\rangle \subseteq \mathcal{O}$. □

With these lemmas, we can now prove:

Theorem 12.6 For every space X, $\mathsf{V}^{\mathrm{N}}X$ is the sobrification of $\mathsf{V}_{\mathrm{f}}^{\mathrm{N}}X$.

Proof: Let ν be in $\mathsf{V}^{\mathrm{N}}X$ and $\mathcal{O}$ in $\Omega(\mathsf{V}^{\mathrm{N}}X)$ with $\nu \in \mathcal{O}$. By Lemma 12.4, there is a simple integer valuation $\nu' \sqsubseteq \nu$ with $\nu' \in \mathcal{O}$. By Theorem 12.1 (2), ν' is a finite sum $\pi_1 + \cdots + \pi_n$ of primitive valuations. Since addition is continuous in $\mathsf{V}^{\mathrm{N}}X$, there are open sets $\mathcal{U}_1, \ldots, \mathcal{U}_n$ of $\mathsf{V}^{\mathrm{N}}X$ such that π_i in $\mathcal{U}_i$, and whenever ν_i in $\mathcal{U}_i$, then $\nu_1 + \cdots + \nu_n$ in $\mathcal{O}$.

By Lemma 12.5, there are points x_i in X such that $\widehat{x_i} \sqsubseteq \pi_i$ and $\widehat{x_i} \in \mathcal{U}_i$. Then $\varphi = \widehat{x_1} + \cdots + \widehat{x_n}$ is a finite integer valuation with $\varphi \sqsubseteq \nu' \sqsubseteq \nu$ and $\varphi \in \mathcal{O}$. □

12.3 Universal Properties

The spaces $\mathsf{V}_{\mathrm{f}}^{\mathrm{N}}X$ and $\mathsf{V}^{\mathrm{N}}X$ have universal properties analogous to those of $\mathsf{V_f}\,X$ and $\mathsf{V_p}\,X$.

A topological $\mathbf{N}_0$-*module* is defined analogously to a topological $\mathbf{R}_+$-module or cone. The only difference is that multiplication has type $\cdot : \mathbf{N}_0 \times M \to M$ instead of $\cdot : \mathbf{R}_+ \times M \to M$.

$\mathbf{N}_0$-modules can be equivalently characterized as commutative topological monoids $(M, +, 0)$ with the additional property that 0 is a least element in the specialization preorder.

Homomorphisms between topological $\mathbf{N}_0$-modules are continuous and linear functions. Here, linearity means $f(m + m') = fm + fm'$ and $f(n \cdot m) = n \cdot fm$ for n in $\mathbf{N}_0$, or equivalently $f(m + m') = fm + fm'$ and $f(0) = 0$.

Standard examples of topological $\mathbf{N}_0$-modules are given by powers of $\mathbf{N}_0$ or $\overline{\mathbf{N}}_0$, and linear subspaces thereof. In addition, every topological cone is an $\mathbf{N}_0$-module. Thus, the lattice examples of Subsection 7.1 are also $\mathbf{N}_0$-modules.

In convex combinations with integer coefficients, all coefficients are 0 except for one which is 1. Thus all sets are convex in the $\mathbf{N}_0$-sense. Hence, local convexity is not an issue in the following theorem.

Theorem 12.7 $\mathsf{V}_\mathrm{f}^\mathrm{N} X$ is the free $\mathcal{T}_0$-topological $\mathbf{N}_0$-module over X in $\mathcal{TOP}$.

Proof: Let X be a space, M be an $\mathbf{N}_0$-module with $\mathcal{T}_0$-topology, and $f : X \to M$ be continuous. As in the proof of Theorem 7.6, there is a unique linear function $\bar{f} : \mathsf{V}_\mathrm{f}^\mathrm{N} X \to M$ with $\bar{f} \circ \mathsf{s} = f$ which can easily be constructed explicitly. The only problem is to prove continuity of $\bar{f}$.

An element φ of $\mathsf{V}_\mathrm{f}^\mathrm{N} X$ can be written as a finite sum $\sum_{i \in I} \widehat{x_i}$ of point valuations. Let U be an open set of M, and assume $\bar{f}(\varphi) \in U$. Then $\sum_{i \in I} fx_i$ is in U. Since addition is continuous in M, there are open sets V_i of M such that $fx_i \in V_i$, and whenever $m_i \in V_i$, then $\sum_{i \in I} m_i \in U$.

Let $O_i = f^- V_i$. These are open sets of X with x_i in O_i for all i in I. For every $T \subseteq I$, $\varphi(\bigcup_{i \in T} O_i) \geq \sum_{i \in T} 1 = |T|$ holds. Hence, φ is in $\mathcal{O} = \bigcap_{T \subseteq F} \langle \bigcup_{i \in T} O_i \geq |T| \rangle$, which is an open set of $\mathsf{V}_\mathrm{f}^\mathrm{N} X$. We have to show that for every ψ in $\mathcal{O}$, $\bar{f}(\psi)$ is in U.

Let $\psi = \sum_{j \in J} \widehat{y_j}$ be in $\mathcal{O}$. Let $R \subseteq I \times J$ be the relation given by $(i, j) \in R$ iff $O_i \ni y_j$. Since ψ is in $\mathcal{O}$,

$$|R^+(T)| = \psi(\bigcup_{i \in T} O_i) \geq |T|$$

holds for all subsets T of I. Applying Hall's Theorem 2.9, we obtain an injective function $\iota : I \to J$ with $y_{\iota i} \in O_i$ for all i in I.

Since $fy_{\iota i} \in V_i$ for all i in I, $\sum_{i \in I} fy_{\iota i} \in U$ follows. Since ι is injective, this sum equals $\sum_{j \in \iota^+ I} fy_j$. In $\mathbf{N}_0$-modules, $m_1 \sqsubseteq m_1 + m_2$ holds. Hence, the sum over $\iota^+ I$ is below the full sum $\sum_{j \in J} fy_j = \bar{f}(\psi)$. Thus, $\bar{f}(\psi)$ is above some element of U, whence it is in U as well. □

The extension function induced by Theorem 12.7 is continuous.

Theorem 12.8 For every space X and $\mathcal{T}_0$-topological $\mathbf{N}_0$-module M, the function $\mathsf{E} : [X \to M]_\mathrm{p} \to [\mathsf{V}_\mathrm{f}^\mathrm{N} X \to M]_\mathrm{p}$ given by Theorem 12.7 is continuous and linear.

Proof: For some $\varphi = \sum_{i \in I} \widehat{x_i}$ in $\mathsf{V}_\mathrm{f}^\mathrm{N} X$, continuous function $f : X \to M$, and open set U of M, assume $\mathsf{E}f \in \langle \varphi \to U \rangle$. Then $\sum_{i \in I} fx_i$ is in U. By continuity of addition, there are open sets V_i of M such that $fx_i \in V_i$, and whenever $m_i \in V_i$, then $\sum_{i \in I} m_i \in U$. From $fx_i \in V_i$, we obtain $f \in \bigcap_{i \in I} \langle x_i \to V_i \rangle$. If g is in this set, then $\sum_{i \in I} gx_i$ is in U, whence $\mathsf{E}g$ is in $\langle \varphi \to U \rangle$.

Linearity of E follows from freeness as in the proof of Theorem 7.8. □

$$
\begin{tikzcd}
 & \mathsf{V}\overline{\mathbf{R}}_+ \arrow[rr, "\overline{\mathsf{id}}"] & & \overline{\mathbf{R}}_+ \\
\mathsf{V}\overline{\mathbf{N}}_0 \arrow[ur] & & \mathsf{V}^{\mathrm{N}}\overline{\mathbf{R}}_+ \arrow[ul] & \\
 & \mathsf{V}^{\mathrm{N}}\overline{\mathbf{N}}_0 \arrow[ul] \arrow[ur] \arrow[rr, "\overline{\mathsf{id}}"'] & & \overline{\mathbf{N}}_0 \arrow[uu]
\end{tikzcd}
$$

Figure 2: Two functions named $\overline{\mathsf{id}}$

Corollary 12.9 For every space X and $\mathcal{T}_0$-topological $\mathbf{N}_0$-module M, the function spaces $[X \to M]_{\mathrm{p}}$ and $[\mathsf{V}_{\mathrm{f}}^{\mathrm{N}} X \overset{lin}{\to} M]_{\mathrm{p}}$ are isomorphic.

Like in the general case, the universal property for $\mathsf{V}_{\mathrm{f}}^{\mathrm{N}} X$ can be used to derive a universal property for $\mathsf{V}^{\mathrm{N}} X$. The proofs of the following theorems are analogous to the corresponding proofs for the general case.

Theorem 12.10 $\mathsf{V}^{\mathrm{N}} X$ is the free sober $\mathbf{N}_0$-module over X in $\mathcal{TOP}$.

Theorem 12.11 For every space X and sober $\mathbf{N}_0$-module M, the extension function $\mathsf{E} : [X \to M]_{\mathrm{p}} \to [\mathsf{V}^{\mathrm{N}} X \to M]_{\mathrm{p}}$ induced by the freeness of $\mathsf{V}^{\mathrm{N}} X$ is continuous and linear.

Corollary 12.12 For every space X and sober $\mathbf{N}_0$-module M, the three function spaces $[X \to M]_{\mathrm{p}}$, $[\mathsf{V}_{\mathrm{f}}^{\mathrm{N}} X \overset{lin}{\to} M]_{\mathrm{p}}$, and $[\mathsf{V}^{\mathrm{N}} X \overset{lin}{\to} M]_{\mathrm{p}}$ are isomorphic $\mathbf{N}_0$-modules.

In particular, for every space X, the three $\mathbf{N}_0$-modules $[X \to \overline{\mathbf{N}}_0]_{\mathrm{p}}$, $[\mathsf{V}_{\mathrm{f}}^{\mathrm{N}} X \overset{lin}{\to} \overline{\mathbf{N}}_0]_{\mathrm{p}}$, and $[\mathsf{V}^{\mathrm{N}} X \overset{lin}{\to} \overline{\mathbf{N}}_0]_{\mathrm{p}}$ are isomorphic.

12.4 Integration over Integer Valuations

In Section 8, two variants of integration were derived. An analogous derivation yields two variants of integer integration:

- The Isbell variant $\int_X : [X \to \overline{\mathbf{N}}_0]_{\mathrm{i}} \otimes \mathsf{V}^{\mathrm{N}} X \to \overline{\mathbf{N}}_0$,
- and the pointwise variant $\int_X : [X \to \overline{\mathbf{N}}_0]_{\mathrm{p}} \otimes \mathsf{V}_{\mathrm{p}}^{\mathrm{N}} X \to \overline{\mathbf{N}}_0$.

In contrast to the general case, $\mathsf{V}^{\mathrm{N}} X$ and $\mathsf{V}_{\mathrm{p}}^{\mathrm{N}} X$ are actually identical. Thus, the two variants become comparable. The pointwise variant gives better information since the pointwise topology on the function space is included in the Isbell topology. Hence, both variants may be subsumed under $\int_X : [X \to \overline{\mathbf{N}}_0]_{\mathrm{p}} \otimes \mathsf{V}^{\mathrm{N}} X \to \overline{\mathbf{N}}_0$ where $\int_X$ is continuous in its two arguments separately.

General integration is defined by $\int_X(f, \nu) = \overline{\mathsf{id}}(\nu \circ f^-)$ where $\overline{\mathsf{id}} : \mathsf{V}\overline{\mathbf{R}}_+ \to \overline{\mathbf{R}}_+$ is the extension of $\mathsf{id} : \overline{\mathbf{R}}_+ \to \overline{\mathbf{R}}_+$. The definition of integer integration looks equally, but uses $\overline{\mathsf{id}} : \mathsf{V}^{\mathrm{N}}\overline{\mathbf{N}}_0 \to \overline{\mathbf{N}}_0$, the extension of $\mathsf{id} : \overline{\mathbf{N}}_0 \to \overline{\mathbf{N}}_0$. The relationship between the two functions named $\overline{\mathsf{id}}$ is shown in Figure 2. Because of the universal property of $\mathsf{V}^{\mathrm{N}}\overline{\mathbf{N}}_0$, the diagram in this figure commutes. Hence, $\overline{\mathsf{id}} : \mathsf{V}^{\mathrm{N}}\overline{\mathbf{N}}_0 \to \overline{\mathbf{N}}_0$ is a restriction of $\overline{\mathsf{id}} : \mathsf{V}\overline{\mathbf{R}}_+ \to \overline{\mathbf{R}}_+$. Thus, integer integration is merely a special instance of general integration, and satisfies all the properties listed in Theorem 8.1. For general integration, it follows that $\int_X(f, \nu)$ is an integer if both f and ν are integer valued.

Using integer integration, isomorphic descriptions of $\mathsf{V}^{\mathrm{N}}X = \mathsf{V}^{\mathrm{N}}_{\mathrm{p}}X$ can be derived which are analogous to Theorems 9.1 and 9.2.

Theorem 12.13

$$\begin{array}{rcl} \mathsf{V}^{\mathrm{N}}X \;=\; \mathsf{V}^{\mathrm{N}}_{\mathrm{p}}X & = & [\Omega_{\mathrm{s}}X \overset{mod}{\to} \overline{\mathbf{N}}_0]_{\mathrm{p}} \quad = \quad [\Omega_{\mathrm{p}}X \overset{mod}{\to} \overline{\mathbf{N}}_0]_{\mathrm{p}} \\ & \cong & [[X \to \overline{\mathbf{N}}_0]_{\mathrm{i}} \overset{lin}{\to} \overline{\mathbf{N}}_0]_{\mathrm{p}} \quad = \quad [[X \to \overline{\mathbf{N}}_0]_{\mathrm{p}} \overset{lin}{\to} \overline{\mathbf{N}}_0]_{\mathrm{p}} \end{array}$$

By Propositions 11.3 and 11.7, product $\alpha \times \beta$ and tensor product $\alpha \otimes \beta$ of two valuations can be obtained by a double integration involving α, β, and the characteristic functions of open sets which are integer valued. Hence, if α and β are integer valued, then so are $\alpha \times \beta$ and $\alpha \otimes \beta$. Thus, we obtain two continuous functions $\times : \mathsf{V}^{\mathrm{N}}X \times \mathsf{V}^{\mathrm{N}}Y \to \mathsf{V}^{\mathrm{N}}(X \times Y)$ and $\otimes : \mathsf{V}^{\mathrm{N}}X \otimes \mathsf{V}^{\mathrm{N}}Y \to \mathsf{V}^{\mathrm{N}}(X \otimes Y)$.

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