An Upper Power Domain Construction in terms of Strongly Compact Sets

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Abstract

A novel upper power domain construction is defined by means of *strongly compact sets*. Its power domains contain less elements than the classical ones in terms of compact sets, but still admit all necessary operations, i.e. they contain less junk. The notion of strong compactness allows a proof of stronger properties than compactness would, e.g. an intrinsic universal property of the upper power construction, and its commutation with the lower construction.

1 Introduction

In 1978, Smyth [Smy78] defined the *upper power domain* $\mathcal{U}\mathbf{X}$ for countably algebraic domains $\mathbf{X}$ as the set of all upper *finitely generable sets* of $\mathbf{X}$ (see section 3.1) ordered by comparison of finite approximants. In [Smy83], he replaced the finitely generable sets by compact sets. In our opinion, the novel topological notion of *strongly compact sets* more faithfully topologizes the concept of finitely generable sets.

Some properties of the construction $\mathcal{U}_S$ with strongly compact sets are analogous to properties of the classical construction $\mathcal{U}_K$ in terms of compact sets. When applied to *admitting* dcpo's $\mathbf{X}$ and $\mathbf{Y}$, the 'power domains' $\mathcal{U}\mathbf{X}$ are again dcpo's which are equipped with a singleton map $\iota : \mathbf{X} \to \mathcal{U}\mathbf{X}$, binary union of members of $\mathcal{U}\mathbf{X}$, and the possibility to extend functions $\mathbf{X} \to \mathcal{U}\mathbf{Y}$ to additive[1] functions $\mathcal{U}\mathbf{X} \to \mathcal{U}\mathbf{Y}$ (see Theorem 3.9).

The class of $\mathcal{U}_S$-admitting domains[2] is potentially larger than that of $\mathcal{U}_K$-admitting ones, and contains all domains with property M. This is one of the advantages of $\mathcal{U}_S$ over $\mathcal{U}_K$.

Whenever $\mathcal{U}_K\mathbf{X}$ is defined, $\mathcal{U}_S\mathbf{X}$ is a subset of $\mathcal{U}_K\mathbf{X}$, and the operations of $\mathcal{U}_S\mathbf{X}$ are restrictions of those of $\mathcal{U}_K\mathbf{X}$. This means that compact, but not strongly compact sets cannot be reached by any

[1] Additive functions are homomorphisms for finite unions.

[2] In this paper, we use the word 'domain' for a directed-complete partial order, so that we can speak of 'power domain constructions' instead of 'power dcpo constructions'.

reasonable operation, i.e. they form a kind of junk. The $\mathcal{U}_S$-powerdomains avoid this junk, whence they have potentially better properties. Indeed, we were able to prove theorems about $\mathcal{U}_S$, whose analogues for $\mathcal{U}_K$ could neither be proved nor be refuted. Thus, $\mathcal{U}_S$ seems to be simpler than $\mathcal{U}_K$; at least, it is better adapted to the proof techniques of the author.

$\mathcal{U}_S$ enjoys an intrinsic universal property: any additive continuous function between two power domains $\mathcal{U}_S\mathbf{X}$ and $\mathcal{U}_S\mathbf{Y}$ is uniquely determined by its values on the singletons of $\mathcal{U}_S\mathbf{X}$ (see Corollary 5.4). This implies that the additive extension mentioned above is unique (see Corollary 5.5).

Another main result of this paper is that the upper construction $\mathcal{U}_S$ and the lower construction $\mathcal{L}$ commute on the large class of $\mathcal{U}_S$-conformal domains (see Theorem 6.1). These domains are defined by a certain close relationship between the Scott topologies of $\mathbf{X}$ and $\mathcal{U}_S\mathbf{X}$. All continuous domains are $\mathcal{U}_S$-conformal.

Our commutation theorem thus largely generalizes the results of [FM90] where commutation was shown for bounded complete algebraic domains only by means of the theory of information systems. In addition, we show that a further generalization using $\mathcal{U}_S$ is impossible: commutation implies conformity.

The paper starts by providing the necessary background in domain theory and topology in section 2. Compactness and strong compactness are defined and investigated in section 3.1. These notions are then used to define the two upper power constructions $\mathcal{U}_K$ and $\mathcal{U}_S$ in section 3.2. An example of a compact set which is not strongly compact is presented in section 7.

Section 4 is concerned with various domain classes related to the two upper power constructions. In section 4.1, continuous domains are characterized in a purely topological way. By generalization, one obtains the class of multi-continuous domains introduced in section 4.2. Our notion of multi-continuity is equivalent to the quasicontinuity of [GLS83]. In multi-continuous domains, the notions of compactness and strong compactness coincide. In sections 4.3 and 4.4, we define and investigate the classes of admitting and conformal domains. All multi-continuous domains are $\mathcal{U}_S$-conformal. In section 4.5, we show that $\mathcal{U}_S\mathbf{X}$ is continuous if and only if the domain $\mathbf{X}$ is multi-continuous.

Section 5 is devoted to those properties of $\mathcal{U}_S$ whose analogues for $\mathcal{U}_K$ could not be decided, i.e. neither proven nor refuted. In section 5.1, we show that domains with property M are $\mathcal{U}_S$-admitting. Then, the intrinsic universal property of $\mathcal{U}_S$ is derived in section 5.3: if $\mathbf{X}$ and $\mathbf{Y}$ are $\mathcal{U}_S$-admitting, then every function from $\mathbf{X}$ to $\mathcal{U}_S\mathbf{Y}$ has a unique additive extension from $\mathcal{U}_S\mathbf{X}$ to $\mathcal{U}_S\mathbf{Y}$.

The commutation theorem is presented in section 6. For $\mathcal{U}_S$-conformal domains $\mathbf{X}$, the two power domains $\mathcal{L}(\mathcal{U}_S\mathbf{X})$ and $\mathcal{U}_S(\mathcal{L}\mathbf{X})$ are isomorphic such that corresponding nested singletons are related. The isomorphism is unique. Conversely, the existence of such an isomorphism implies the $\mathcal{U}_S$-conformity of $\mathbf{X}$. Thus, our theorem cannot be generalized further.

2 Domain-theoretical and topological background

In this section, we introduce some notions and notations from domain theory and topology. We also include a collection of facts from both theories that will be needed in the remainder of this paper. In addition, we present a formal definition of power constructions in section 2.4. Finally, we define the standard lower power construction in section 2.5.

For a set $A \subseteq X$, $\mathsf{co}\, A = \mathsf{co}_X A$ is the complement of A in X. For $f : X \to Y$ and $A \subseteq X$ and $B \subseteq Y$, $f[A]$ is the image of A and $f^{-1}[B]$ the inverse image of B.

2.1 Domains

A *poset* (partially ordered set) is a set P together with a reflexive, antisymmetric, and transitive relation '$\leq$'. We often identify the poset $\mathbf{P} = (P, \leq)$ with its carrier P.

For $A \subseteq \mathbf{P}$, let $\downarrow A$ be the set of all points *below* some point of A, and correspondingly $\uparrow A$ the set of all points *above* some point of A. Often, we abbreviate $\downarrow\{x\}$ by $\downarrow x$ and $\uparrow\{x\}$ by $\uparrow x$. We refrain from doing so if x happens to be a set. $A \subseteq \mathbf{P}$ is a *lower set* iff $\downarrow A = A$, and an *upper set* iff $\uparrow A = A$. A set F is *finitary* iff there is a finite set E such that $F = \uparrow E$.

We refer to the standard notions of upper bound, least upper bound (join) denoted by '$\bigsqcup$', greatest lower bound (meet), directed set, and monotonic function. A *domain* is a non-empty poset where every directed set has a join. There is no requirement for a least element or algebraicity.

A monotonic function $f : \mathbf{X} \to \mathbf{Y}$ between two domains is *continuous* iff $f(\bigsqcup D) \leq \bigsqcup f[D]$ holds for all directed sets $D \subseteq \mathbf{X}$. $[\mathbf{X} \to \mathbf{Y}]$ is the domain of all continuous functions from $\mathbf{X}$ to $\mathbf{Y}$ ordered pointwise. The notation $f : [\mathbf{X} \to \mathbf{Y}]$ includes the continuity of f.

A domain is *complete* iff every subset has a join. Complete domains $\mathbf{X}$ have both a least element ($\bigsqcup \emptyset$) and a greatest element ($\bigsqcup \mathbf{X}$). A domain is *bounded complete* if every bounded subset has a join. A domain has property M [Jun88] iff it is finitary as a whole, and for all points x and y, the set of common upper bounds $\uparrow x \cap \uparrow y$ is finitary. Obviously, every bounded complete domain has property M.

2.2 The Scott topology

A subset of a domain $\mathbf{X}$ is called (Scott) *closed* iff it is a lower set closed w.r.t. joins of directed subsets. Lower cones $\downarrow x$ are obviously closed. Finite unions and arbitrary intersections of closed sets are closed. Hence, every set A has a least closed superset, its *closure* $\mathsf{cl}\, A$. The complements of the closed sets are called (Scott) *open*. Open sets are upper sets that meet every directed set whose join they contain. The set of all open supersets of a set A is denoted by $\mathcal{O}(A)$. $\mathcal{O}(\{x\})$ is often abbreviated to $\mathcal{O}(x)$.

By these notions, every domain becomes a topological space. One can show that the notion of continuity introduced above coincides with topological continuity, i.e. $f : \mathbf{X} \to \mathbf{Y}$ is continuous iff the inverse image $f^{-1}[O]$ of all open sets O of $\mathbf{Y}$ is open in $\mathbf{X}$.

Two upper sets may be compared by comparing their open environments:

Lemma 2.1 For two subsets A, B of $\mathbf{X}$, $\mathcal{O}(A) \subseteq \mathcal{O}(B)$ holds iff $\uparrow A \supseteq \uparrow B$.

Proof: Let $\uparrow A \supseteq \uparrow B$. If $A \subseteq O$ for an open set O then $B \subseteq \uparrow A \subseteq O$ since open sets are upper.

Let $\mathcal{O}(A) \subseteq \mathcal{O}(B)$, and let b be a point of B and assume $b \notin \uparrow A$. Then there is no point of A below b, i.e. A does not meet the closed set $\downarrow b$. Hence, its complement $\mathbf{X} \setminus \downarrow b$ is in $\mathcal{O}(A) \subseteq \mathcal{O}(B)$. Because of $b \in B$, $b \in \mathbf{X} \setminus \downarrow b$ follows contradicting reflexivity of '$\leq$'. $\square$

This lemma will be heavily used. To prove an inclusion $A \subseteq B$ for two upper sets A and B, we may assume $B \subseteq O$ for some open set O and have to conclude $A \subseteq O$ from this assumption.

Similarly, two points may be compared:

Lemma 2.2 For two points x and y of $\mathbf{X}$, $\mathcal{O}(x) \subseteq \mathcal{O}(y)$ holds iff $x \leq y$.

Proof: By Lemma 2.1, $\mathcal{O}(x) \subseteq \mathcal{O}(y)$ is equivalent to $\uparrow x \supseteq \uparrow y$. This in turn holds iff $x \leq y$. $\square$

2.3 Rudin's Lemma

In [GLS83] and [Jun88], an interesting lemma is stated and proved that in its original weaker form dates back to [Rud80]. In the version of Theorem 4.11 in [Jun88], the lemma reads as follows:

Lemma 2.3 Let $\mathbf{X}$ be a poset and $\mathcal{E}$ a set of non-empty finite sets of $\mathbf{X}$ such that $\uparrow[\mathcal{E}] = \{\uparrow E \mid E \in \mathcal{E}\}$ is $\supseteq$-directed. Then there is a directed subset D of $\bigcup \mathcal{E}$ that meets all members of $\mathcal{E}$.

In this paper, we only need the following consequence of the lemma:

Lemma 2.4 Let $\mathbf{X}$ be a domain, $\mathcal{F}$ a $\supseteq$-directed set of finitary subsets of $\mathbf{X}$, and O an open set of $\mathbf{X}$ such that $\bigcap \mathcal{F} \subseteq O$ holds. Then $F \subseteq O$ holds for some $F \in \mathcal{F}$.

Proof: Assume $F \not\subseteq O$ for all $F \in \mathcal{F}$. Let C be the complement of O. Then $F \cap C \neq \emptyset$ follows for all $F \in \mathcal{F}$. For every $F \in \mathcal{F}$, let E_F be a finite set such that $\uparrow E_F = F$ holds. Let $\mathcal{E} = \{E_F \cap C \mid F \in \mathcal{F}\}$. Since C is lower, $\uparrow E \cap C \neq \emptyset$ implies $E \cap C \neq \emptyset$, and $\uparrow E \supseteq \uparrow E'$ implies $\uparrow(E \cap C) \supseteq \uparrow(E' \cap C)$. Hence $\mathcal{E}$ consists of non-empty finite sets, and $\uparrow[\mathcal{E}]$ is $\supseteq$-directed. By the lemma above, there is a directed set $D \subseteq \bigcup \mathcal{E}$. All members of $\mathcal{E}$ are subsets of C, whence $D \subseteq \bigcup \mathcal{E} \subseteq C$, and thus $\bigsqcup D$ is in C. In addition, D meets all members of $\mathcal{E}$. Thus, $\bigsqcup D \in \uparrow(E_F \cap C) \subseteq F$ holds for all $F \in \mathcal{F}$, whence $\bigsqcup D \in \bigcap \mathcal{F}$.

But, the fact that $\bigsqcup D \in C \cap \bigcap \mathcal{F}$ contradicts $\bigcap \mathcal{F} \subseteq O$. □

2.4 Power constructions

In [Hec90, Hec91b], power constructions are defined as functions $\mathcal{P}$ mapping certain *ground domains* $\mathbf{X}$ to *power domains* $\mathcal{P}\mathbf{X}$. *Total* constructions are defined for all domains, whereas *partial* ones are defined on a certain class of domains only. There is no special structure required for ground domains, whereas power domains have to be equipped with the following continuous operations:

- $\theta \in \mathcal{P}\mathbf{X}$ for every domain $\mathbf{X}$, representing a formal empty set;
- $\uplus : [\mathcal{P}\mathbf{X} \times \mathcal{P}\mathbf{X} \to \mathcal{P}\mathbf{X}]$ for every $\mathbf{X}$, representing binary union;
- $\iota : [\mathbf{X} \to \mathcal{P}\mathbf{X}]$ for every $\mathbf{X}$, representing singleton formation;
- $\overline{\mathcal{P}} : [[\mathbf{X} \to \mathcal{P}\mathbf{Y}] \to [\mathcal{P}\mathbf{X} \to \mathcal{P}\mathbf{Y}]]$ for every two domains $\mathbf{X}$ and $\mathbf{Y}$. $\overline{\mathcal{P}}$ is a second order function extending set-valued functions defined on the ground domain $\mathbf{X}$ to set-valued functions defined on the power domain $\mathcal{P}\mathbf{X}$.

Special symbols 'θ' and '$\uplus$' are used because there are power constructions where these operations are not concretely given as empty set and set union.

In the first papers about power constructions, the empty set was excluded from the resulting 'powerdomains' [Plo76, Smy78]. In [Smy83], Smyth considered 'powerspaces' with empty set as well as 'powerdomains' without empty set. There are applications where the empty set is needed as the semantic value for failure or abnormal termination. Mathematically, the omission of the empty set from $\mathcal{U}_S$, $\mathcal{U}_K$, and $\mathcal{L}$ would be artificial and clutter up our proofs unnecessarily.

The operations listed above have to satisfy certain axioms: formal union '$\uplus$' must be commutative, associative, and idempotent with neutral element θ. Both the extended maps $\overline{\mathcal{P}}f$ and extension $\overline{\mathcal{P}}$ itself have to be additive, i.e. homomorphic for θ and '$\uplus$'. In addition, $(\mathcal{P}, \iota, \overline{\mathcal{P}})$ has to be a Kleisli triple, i.e. $\overline{\mathcal{P}}f \circ \iota = f$, $\overline{\mathcal{P}}\iota = id$, and $\overline{\mathcal{P}}g \circ \overline{\mathcal{P}}f = \overline{\mathcal{P}}(\overline{\mathcal{P}}g \circ f)$ have to hold. The first equation $\overline{\mathcal{P}}f \circ \iota = f$ means that $\overline{\mathcal{P}}f$ extends f from formal singletons to all power domain elements.

The axioms imply that each power construction becomes a functor by the definition $\mathcal{P}f = \overline{\mathcal{P}}(\iota \circ f)$. The functions $\mathcal{P}f$ are additive, and $\mathcal{P}f \circ \iota = \iota \circ f$ holds, i.e. ι is a natural transformation. Many further properties of power constructions are contained in [Hec90, Hec91b].

Power constructions may be classified according to their effect on the one-point domain $\mathbf{1} = \{\diamond\}$. *Lower power constructions* yield $\mathcal{P}\mathbf{1} = \{\theta < \iota\diamond\}$, whereas *upper power constructions* yield $\mathcal{P}\mathbf{1} = \{\iota\diamond < \theta\}$. Except for the slight variation of eventually leaving out $\emptyset$, we know of only one lower power construction, which is presented in the next section. In contrast to this, there are several different upper constructions, as is shown by the paper at hand.

2.5 The lower power domain construction

There is only one known lower power construction $\mathcal{L}$. It is defined for all domains, and may be concretely represented in terms of closed sets of the ground domain [Smy83].

Definition 2.5
For all domains $\mathbf{X}$, $\mathcal{L}\mathbf{X}$ is the set of closed subsets of $\mathbf{X}$ ordered by inclusion '$\subseteq$'.

Proposition 2.6 For all domains $\mathbf{X}$, $\mathcal{L}\mathbf{X}$ is a complete domain. The join of a family $(C_i)_{i\in I}$ of members of $\mathcal{L}\mathbf{X}$ is given by $\mathsf{cl}\bigcup_{i\in I} C_i$. Finite joins are finite unions.

Now, we show that $\mathcal{L}$ satisfies the definition of power constructions given in the previous section. Concerning extension, we can even do better: $\overline{\mathcal{L}}$ is defined for functions f to all complete domains, and its result is the *unique* additive extension of f.

Theorem 2.7
For arbitrary domains $\mathbf{X}$ and $\mathbf{Y}$, there are the following continuous operations:

- Empty set: $\theta = \emptyset \in \mathcal{L}\mathbf{X}$,
- Binary union: $\uplus = \cup : [\mathcal{L}\mathbf{X} \times \mathcal{L}\mathbf{X} \to \mathcal{L}\mathbf{X}]$,
- Singleton set: $\iota : [\mathbf{X} \to \mathcal{L}\mathbf{X}]$ defined by $\iota x = {\downarrow}x$,
- Function extension: $\overline{\mathcal{L}} : [[\mathbf{X} \to \mathbf{Z}] \to [\mathcal{L}\mathbf{X} \to \mathbf{Z}]]$ for every complete domain $\mathbf{Z}$, where $\overline{\mathcal{L}}fC = \bigsqcup_{x\in C} fx = \bigsqcup f[C]$ for $C \in \mathcal{L}\mathbf{X}$.
  For every $f : [\mathbf{X} \to \mathbf{Z}]$, the map $\overline{\mathcal{L}}f : [\mathcal{L}\mathbf{X} \to \mathbf{Z}]$ preserves finite joins, i.e. $\overline{\mathcal{L}}f(\emptyset) = \bot$ and $\overline{\mathcal{L}}f(A \cup B) = \overline{\mathcal{L}}fA \sqcup \overline{\mathcal{L}}fB$, and extends f, i.e. $\overline{\mathcal{L}}f \circ \iota = f$ holds. It is unique with these properties.
- Function mapping: $\mathcal{L} : [[\mathbf{X} \to \mathbf{Y}] \to [\mathcal{L}\mathbf{X} \to \mathcal{L}\mathbf{Y}]]$ is given by $\mathcal{L}f\,C = \mathsf{cl}\,f[C]$ for $C \in \mathcal{L}\mathbf{X}$.

All axioms of power constructions are satisfied.

Proof: The empty set is obviously closed. The union of two closed sets is closed, and it is the join of these closed sets. In a complete domain, the operation of binary join is always continuous.

${\downarrow}x$ is closed, and ι is monotonic since $x \leq y$ implies ${\downarrow}x \subseteq {\downarrow}y$. For continuity, we have to show ${\downarrow}(\bigsqcup D) \leq \bigsqcup {\downarrow}[D]$, or equivalently ${\downarrow}(\bigsqcup D) \subseteq \mathsf{cl}\bigcup_{d\in D}{\downarrow}d$, or $\bigsqcup D \in \mathsf{cl}\,{\downarrow}D$. This holds by definition of Scott closedness.

Extension is a bit more difficult. Because $\mathbf{Z}$ is a complete domain, one may define $\hat{f}A = \bigsqcup f[A]$ for every subset A of $\mathbf{X}$. We show three main properties of $\hat{f}$.

(1) $\hat{f}({\downarrow}x) = fx$ follows from the monotonicity of f.

(2) $\hat{f}(\bigcup_{i\in I} A_i) = \bigsqcup_{i\in I}\hat{f}A_i$
Proof: $\hat{f}(\bigcup_{i\in I} A_i) = \bigsqcup f[\bigcup_{i\in I} A_i] = \bigsqcup\bigcup_{i\in I} f[A_i] = \bigsqcup_{i\in I}\bigsqcup f[A_i] = \bigsqcup_{i\in I}\hat{f}A_i$
Hence, $\hat{f}$ in particular preserves finite joins and is monotonic w.r.t. '$\subseteq$'.

(3) $\hat{f}(\mathsf{cl}\,A) = \hat{f}A$ for all $A \subseteq \mathbf{X}$.
'$\geq$' follows from $\mathsf{cl}\,A \supseteq A$. For the opposite direction, let $y = \hat{f}A$, and assume there is $x \in \mathsf{cl}\,A$ with $fx \not\leq y$. Then fx is in the open set $O = \mathbf{Z} \setminus {\downarrow}y$. By continuity of f, $O' = f^{-1}[O]$ is an open

environment of x. Since x is in the closure of A, O' meets A, i.e. there is a in A with $fa \in O$, i.e. $fa \not\leq y = \bigsqcup f[A]$. This contradiction implies $fx \leq y$ for all $x \in \mathsf{cl}\, A$, whence $\hat{f}(\mathsf{cl}\, A) \leq y$.

Now, we restrict $\hat{f}$ to $\overline{\mathcal{L}}f : \mathcal{L}\mathbf{X} \to \mathbf{Z}$. $\overline{\mathcal{L}}f$ still preserves finite joins and satisfies $\overline{\mathcal{L}}f \circ \iota = f$ by (1). It is continuous because

$$\overline{\mathcal{L}}f\,(\bigsqcup_{i\in I} A_i) = \hat{f}(\mathsf{cl} \bigcup_{i\in I} A_i) \overset{(3)}{=} \hat{f}(\bigcup_{i\in I} A_i) \overset{(2)}{=} \bigsqcup_{i\in I} \hat{f} A_i = \bigsqcup_{i\in I} \overline{\mathcal{L}}f A_i$$

Let $F_1, F_2 : [\mathcal{L}\mathbf{X} \to \mathbf{Z}]$ be two extensions of f preserving finite joins. Then F_1 and F_2 coincide for all sets $\downarrow E$ where E is finite. If C is an arbitrary closed set, then $\mathcal{D} = \{\downarrow E \mid E \text{ is a finite subset of } C\}$ is directed and has union (and thus limit) C. Hence, $F_1 = F_2$ holds by continuity.

If $\mathbf{Z}$ is chosen as $\mathcal{L}\mathbf{Y}$, then $\mathcal{L}f$ is defined as $\overline{\mathcal{L}}(\iota \circ f)$. Thus, we obtain $\overline{\mathcal{L}}(\iota \circ f)A = \mathsf{cl} \bigcup_{a\in A} \downarrow\{fa\} = \mathsf{cl} \downarrow \bigcup_{a\in A}\{fa\} = \mathsf{cl}\, f[A]$.

The proofs of continuity of $\overline{\mathcal{L}}$ itself and of the remaining axioms are omitted. □

3 Upper power domains in terms of (strongly) compact sets

In this section, we introduce *compact* sets and *strongly compact* sets. Two *upper power domain constructions* $\mathcal{U}_K$ and $\mathcal{U}_S$ are defined in terms of these topological notions. In contrast to the lower construction, these upper ones are not applicable to all domains, but only to *admitting* ones.

3.1 Compact and strongly compact sets

Upper power domains are often defined in terms of compact upper sets [Smy83]. Compactness is a standard topological notion.

Definition 3.1 A subset K of a domain $\mathbf{X}$ is called (Scott) *compact* iff whenever K is covered by a family $(O_i)_{i\in I}$ of open sets, i.e. $K \subseteq \bigcup_{i\in I} O_i$, there is a finite subset F of I such that $K \subseteq \bigcup_{i\in F} O_i$.

In [Smy78], Smyth used finitely generable sets to define the first upper power construction. He modeled a non-deterministic computation as a finitely branching tree with infinite branches only. The nodes of this tree are labeled by domain elements such that the labels of parents are smaller than the labels of their children. These labels represent partial results of the computation. The actual shape and labeling of the tree are considered irrelevant. What really matters, are the labels of the infinite border of the tree, i.e. the set of limits of the labelings of all branches of the tree. All the sets that may be obtained in this manner are called *finitely generable*.

Intuitively, a finitely generable set S is approximated by the sequence of horizontal tree cuts. All these cuts E are finite, and $S \subseteq \uparrow E$ holds. This suggests the following definition of the more abstract notion of strong compactness. In this paper, we shall use this novel notion to define upper power domains.

Definition 3.2 A set S in a domain $\mathbf{X}$ is *strongly compact* iff for all open sets O with $S \subseteq O$ there is a finitary set F with $S \subseteq F \subseteq O$.

Hence, every environment O of a strongly compact set S contains a finitary set F that approximates S. Thus, we are able to reduce the potentially infinite structure of a strongly compact set to a finite structure. Technically, this possibility is often used when applying Lemma 2.1 to show inclusion of strongly compact upper sets by considering their environments.

The name 'strong compactness' was chosen since the properties of this notion are analogous to those of compactness. For the following propositions, we prove only the statements about strong compactness. The statements about compactness are well known from topology.

Proposition 3.3
Every finite set is strongly compact, and every strongly compact set is compact.

Proof: If S is finite, then $S \subseteq \uparrow S \subseteq O$ holds for all open sets O.

Let S be strongly compact, and $S \subseteq \bigcup \mathcal{O}$ where $\mathcal{O}$ is a set of open sets. Because the union is open, too, there is a finite set E such that $S \subseteq \uparrow E \subseteq \bigcup \mathcal{O}$. Every e in E is in some open set $O_e \in \mathcal{O}$. Hence, $S \subseteq \bigcup_{e \in E} \uparrow e \subseteq \bigcup_{e \in E} O_e$. □

Proposition 3.4 A is (strongly) compact iff $\uparrow A$ is so.

Proof: Since open sets are upper sets, and $A \subseteq \uparrow E$ iff $\uparrow A \subseteq \uparrow E$. □

Proposition 3.5 If A and B are (strongly) compact, then so is $A \cup B$.

Proof: $A \cup B \subseteq O$ open implies $A \subseteq \uparrow E \subseteq O$ and $B \subseteq \uparrow F \subseteq O$ for some finite sets E and F, whence $A \cup B \subseteq \uparrow (E \cup F) \subseteq O$. □

Proposition 3.6 If A is (strongly) compact in $\mathbf{X}$ and $f : [\mathbf{X} \to \mathbf{Y}]$ is continuous, then $f[A]$ is (strongly) compact in $\mathbf{Y}$.

Proof: $f[A] \subseteq O$ implies $A \subseteq f^{-1}[O]$ whence $A \subseteq \uparrow E \subseteq f^{-1}[O]$ for some finite E. This implies $f[A] \subseteq \uparrow f[E] \subseteq O$. □

Despite these analogous properties, strongly compact sets can often be handled more easily than compact sets. In sections 5 and 6, we shall present a number of theorems for strongly compact sets whose analogues for compact sets could not be proved or refuted by us.

For a large class of domains, one can show that all compact sets are strongly compact (see Theorem 4.8). Beyond this class, there are domains with compact sets that are not strongly compact. An example is provided in section 7.

Remark: Strongly compact sets may be defined for arbitrary topological spaces where $\uparrow E$ means upper closure in the specification pre-order. In (T1)-spaces where the pre-order degenerates to equality, a set is strongly compact iff it is finite. Thus, the notion of strong compactness is not considered in mainstream topology.

3.2 The topological upper power constructions

We now introduce two upper power constructions: $\mathcal{U}_S$ using strongly compact sets, and $\mathcal{U}_K$ using compact sets. As far as possible, we try to derive properties that hold for *both* constructions. In these common properties, they are referred to as $\mathcal{U}_\Box$, and their members as $\Box$-compact sets.

In contrast to the lower power domains, the upper power domains cannot be formed for arbitrary ground domains. Instead, we need a particular condition on $\mathbf{X}$ such that $\mathcal{U}_\Box \mathbf{X}$ is a domain and the operations of Theorem 3.9 are continuous. In Theorem 3.10, we shall show that this condition is not only sufficient, but also necessary.

Definition 3.7 For an arbitrary domain $\mathbf{X}$, let $\mathcal{U}_\Box \mathbf{X}$ be the set of all $\Box$-compact upper sets of $\mathbf{X}$ ordered by inverse inclusion '$\supseteq$'. The domain $\mathbf{X}$ is $\mathcal{U}_\Box$*-admitting* iff for all directed sets $\mathcal{D}$ in $\mathcal{U}_\Box \mathbf{X}$, whenever $\bigcap \mathcal{D} \subseteq O$ holds for some open set O in $\mathbf{X}$, then $K \subseteq O$ holds for some $K \in \mathcal{D}$.

For open sets O of $\mathbf{X}$, let $\overline{\mathcal{U}}_\Box(O)$ be the set of all $\Box$-compact upper subsets of O.

In view of the convention above, note that the definition defines $\mathcal{U}_S\mathbf{X}$, $\mathcal{U}_K\mathbf{X}$, $\mathcal{U}_S$-admitting, $\mathcal{U}_K$-admitting, $\overline{\mathcal{U}}_S(O)$, and $\overline{\mathcal{U}}_K(O)$. Notice also the formal similarity between $\mathcal{U}_\Box$-admissibility and Rudin's Lemma 2.4. Whereas this lemma holds for all domains, there are domains that are not $\mathcal{U}_\Box$-admitting (see section 4.6).

Proposition 3.8 If $\mathbf{X}$ is $\mathcal{U}_\Box$-admitting, then $\mathcal{U}_\Box\mathbf{X}$ is a domain. The join of a directed set $\mathcal{D}$ of $\mathcal{U}_\Box\mathbf{X}$ is $\bigcap\mathcal{D}$. For open sets O of $\mathbf{X}$, $\overline{\mathcal{U}}_\Box(O)$ is an open set of $\mathcal{U}_\Box\mathbf{X}$.

Proof: To show that the upper set $\bigcap\mathcal{D}$ is a member of $\mathcal{U}_\Box\mathbf{X}$, we have to differentiate between $\mathcal{U}_K$ and $\mathcal{U}_S$.

In case of $\mathcal{U}_K$, let $\bigcap\mathcal{D} \subseteq \bigcup_{i\in I} O_i$ be an open cover. Since $\mathbf{X}$ is $\mathcal{U}_K$-admitting, there is a member A of $\mathcal{D}$ such that $A \subseteq \bigcup_{i\in I} O_i$. Since A is compact, there is a finite subset F of I such that $\bigcap\mathcal{D} \subseteq A \subseteq \bigcup_{i\in F} O_i$.

In case of $\mathcal{U}_S$, let $\bigcap\mathcal{D} \subseteq O$ where O is open. By $\mathcal{U}_S$-admissibility, there is a member A of $\mathcal{D}$ such that $A \subseteq O$. As A is strongly compact, there is a finitary set F such that $\bigcap\mathcal{D} \subseteq A \subseteq F \subseteq O$.

The remainder of the proof can be done for $\mathcal{U}_K$ and $\mathcal{U}_S$ together. $\overline{\mathcal{U}}_\Box(O)$ is an upper set in $\mathcal{U}_\Box\mathbf{X}$ since $A \leq B$ means $A \supseteq B$ whence $A \subseteq O$ implies $B \subseteq A \subseteq O$. $\overline{\mathcal{U}}_\Box(O)$ is Scott open since $\bigsqcup\mathcal{D} = \bigcap\mathcal{D}$ and $\mathbf{X}$ is $\mathcal{U}_\Box$-admitting. $\Box$

Next, we show that both constructions $\mathcal{U}_\Box$ are partial power constructions in the sense of section 2.4.

Theorem 3.9 For $\mathcal{U}_\Box$-admitting domains $\mathbf{X}$ and $\mathbf{Y}$, there are the following operations:

- Empty set: $\theta = \emptyset \in \mathcal{U}_\Box\mathbf{X}$,
- Binary union: $\cup : [\mathcal{U}_\Box\mathbf{X} \times \mathcal{U}_\Box\mathbf{X} \to \mathcal{U}_\Box\mathbf{X}]$,
- Singleton set: $\iota : [\mathbf{X} \to \mathcal{U}_\Box\mathbf{X}]$ defined by $\iota x = \uparrow x$,
- Function extension: $\overline{\mathcal{U}}_\Box : [[\mathbf{X} \to \mathcal{U}_\Box\mathbf{Y}] \to [\mathcal{U}_\Box\mathbf{X} \to \mathcal{U}_\Box\mathbf{Y}]]$ defined by $\overline{\mathcal{U}}_\Box f A = \bigcup_{a\in A} fa$,
- Function mapping: $\mathcal{U}_\Box : [[\mathbf{X} \to \mathbf{Y}] \to [\mathcal{U}_\Box\mathbf{X} \to \mathcal{U}_\Box\mathbf{Y}]]$ defined by $\mathcal{U}_\Box f A = \uparrow f[A]$.

All axioms demanded in section 2.4 are satisfied.

Proof: The empty set is $\Box$-compact. The union of two $\Box$-compact sets is $\Box$-compact by Proposition 3.5. Union is obviously monotonic. Continuity needs only be shown w.r.t. one argument. Then it holds since $A \cup \bigcap_{i\in I} B_i = \bigcap_{i\in I}(A \cup B_i)$.

$\uparrow x$ is $\Box$-compact by Proposition 3.3 and 3.4. ι is monotonic since $x \leq y$ implies $\uparrow x \supseteq \uparrow y$. For continuity, we have to show $\uparrow(\bigsqcup D) \leq \bigsqcup \uparrow[D]$, or equivalently $\uparrow(\bigsqcup D) \supseteq \bigcap_{d\in D} \uparrow d$. This holds because $\bigsqcup D$ is the least upper bound of D.

Again, extension is the most difficult operation. In contrast to the lower power construction, it is difficult to show that the result is indeed $\Box$-compact. We have to do so for compactness and strong compactness separately.

In case of $\mathcal{U}_S$, assume $\bigcup_{a\in A} fa \subseteq O$ holds for some open set O. Then, $fa \subseteq O$ or equivalently $fa \in \overline{\mathcal{U}}_S(O)$ holds for all a in A. Thus, $A \subseteq f^{-1}[\overline{\mathcal{U}}_S(O)]$ follows. The latter set is open by continuity of f. By strong compactness of A, there is a finite set E such that $A \subseteq \uparrow E \subseteq f^{-1}[\overline{\mathcal{U}}_S(O)]$. For all e in E, $fe \subseteq O$ follows, whence there are finitary sets F_e such that $fe \subseteq F_e \subseteq O$. Let F be the finitary set $\bigcup_{e\in E} F_e$. For every a in A, there is some e in E below a, whence $fa \subseteq fe \subseteq F_e \subseteq F \subseteq O$. Thus, $\bigcup_{a\in A} fa \subseteq F \subseteq O$ holds as required.

In case of $\mathcal{U}_K$, let $\bigcup_{a\in A} fa \subseteq \bigcup_{i\in I} O_i$ be an open cover. Then $fa \subseteq \bigcup_{i\in I} O_i$ holds for all a in A. Since fa is compact, there are finite subsets E_a of I such that $fa \subseteq \bigcup_{i\in E_a} O_i$. Calling these finite unions U_a, we obtain $fa \subseteq U_a$ or equivalently $a \in f^{-1}[\overline{\mathcal{U}}_K(U_a)]$ for all a in A, whence $A \subseteq \bigcup_{a\in A} f^{-1}[\overline{\mathcal{U}}_K(U_a)]$. By compactness of A, there is a finite subset E of A such that $A \subseteq \bigcup_{e\in E} f^{-1}[\overline{\mathcal{U}}_K(U_a)]$. Hence, for every a in A there is e in E such that $fa \subseteq U_e$. Thus, we finally obtain the finite sub-cover $\bigcup_{a\in A} fa \subseteq \bigcup_{e\in E} U_e = \bigcup_{e\in E} \bigcup_{i\in E_e} O_i$ of the original open cover.

Continuity of $\overline{\mathcal{U}}_S f$ and $\overline{\mathcal{U}}_K f$ can be shown together. $\overline{\mathcal{U}}_\Box f$ is monotonic since $A \subseteq B$ implies $\bigcup_{a\in A} fa \subseteq \bigcup_{b\in B} fb$. For continuity, we have to show $\overline{\mathcal{U}}_\Box f(\bigsqcup_{i\in I} A_i) \le \bigsqcup_{i\in I}(\overline{\mathcal{U}}_\Box f A_i)$, or $\overline{\mathcal{U}}_\Box f(\bigcap_{i\in I} A_i) \supseteq \bigcap_{i\in I} \overline{\mathcal{U}}_\Box f A_i$, or $\bigcup_{a\in\bigcap_{i\in I} A_i} fa \supseteq \bigcap_{i\in I} \bigcup_{a\in A_i} fa$. We employ Lemma 2.1 and assume $\bigcup_{a\in\bigcap_{i\in I} A_i} fa \subseteq O$. Then $fa \subseteq O$ holds for all a in $\bigcap_{i\in I} A_i$, whence $\bigcap_{i\in I} A_i \subseteq f^{-1}[\overline{\mathcal{U}}_\Box(O)]$. By $\mathcal{U}_\Box$-admissibility, there is some k in I such that $A_k \subseteq f^{-1}[\overline{\mathcal{U}}_\Box(O)]$. Thus, $fa \subseteq O$ holds for all a in A_k, whence $O \supseteq \bigcup_{a\in A_k} fa \supseteq \bigcap_{i\in I} \bigcup_{a\in A_i} fa$ as required by Lemma 2.1.

Function mapping is derived from extension: $\overline{\mathcal{U}}_\Box(\iota \circ f)A = \bigcup_{a\in A} \uparrow\{fa\} = \uparrow f[A]$.

The proofs of the continuity of $\overline{\mathcal{U}}_\Box$ itself and of the axioms of section 2.4 are omitted. □

Function extension of $\mathcal{U}_\Box$ is much weaker than function extension of $\mathcal{L}$. We can only extend functions to a power domain of the same kind, and there is no statement of uniqueness. For $\mathcal{U}_S$ at least, uniqueness may be shown (cf. Corollary 5.4). This leads to the intrinsic universal property of $\mathcal{U}_S$ as stated in Corollary 5.5.

We use the same symbol $\overline{\mathcal{U}}_\Box$ for extension of functions from $[\mathbf{X} \to \mathcal{U}_\Box\mathbf{Y}]$ to $[\mathcal{U}_\Box\mathbf{X} \to \mathcal{U}_\Box\mathbf{Y}]$ and 'extension' of open sets in $\mathbf{X}$ to those in $\mathcal{U}_\Box\mathbf{X}$ since both notions coincide if open sets are considered functions to the codomain $\{\bot < \top\} = \{\{\diamond\} < \emptyset\} = \mathcal{U}_\Box\{\diamond\}$. We do not include the proof of this claim, but refer to the next theorem that contains a similar statement.

The condition of $\mathcal{U}_\Box$-admissibility is the most general one such that the statements above hold:

Theorem 3.10 Let $\mathbf{2} = \{\bot < \top\}$. If $\mathbf{X}$ is a domain such that $\mathcal{U}_\Box\mathbf{X}$ is a domain again[3] and $\mathcal{U}_\Box f : \mathcal{U}_\Box\mathbf{X} \to \mathcal{U}_\Box\mathbf{2}$ is continuous for every continuous $f : [\mathbf{X} \to \mathbf{2}]$, then $\mathbf{X}$ is $\mathcal{U}_\Box$-admitting.

Proof: Let $\mathcal{D}$ be a $\supseteq$-directed set of $\Box$-compact upper sets and O an open set such that $\mathcal{D} \subseteq O$ holds. $\mathcal{D}$ is directed in the domain $\mathcal{U}_\Box\mathbf{X}$, whence it has a join $\bigsqcup\mathcal{D}$. All members of $\mathcal{D}$ are supersets of $\bigsqcup\mathcal{D}$, whence $\bigsqcup\mathcal{D} \subseteq \bigcap\mathcal{D} \subseteq O$ follows.

The open set O induces a continuous function f from $\mathbf{X}$ to $\mathbf{2}$ which maps all elements of O to $\top$ and all other points to $\bot$. By hypothesis, $F = \mathcal{U}_\Box f : [\mathcal{U}_\Box\mathbf{X} \to \mathcal{U}_\Box\mathbf{2}]$ is continuous. $\mathcal{U}_\Box\mathbf{2}$ is $\{\{\bot,\top\} < \{\top\} < \emptyset\}$. By continuity, $\mathcal{O} = F^{-1}\{\{\top\},\emptyset\}$ is open in $\mathcal{U}_\Box\mathbf{X}$. We claim $S \subseteq O$ iff $S \in \mathcal{O}$ for all sets S in $\mathcal{U}_\Box\mathbf{X}$. This holds due to the following chain of equivalences:

$$S \subseteq O \quad\text{iff}\quad \forall x \in S : fx = \top \quad\text{iff}\quad f[S] \subseteq \{\top\} \quad\text{iff}\quad \uparrow f[S] \in \{\{\top\},\emptyset\} \quad\text{iff}\quad S \in \mathcal{O}$$

Thus, $\bigsqcup\mathcal{D} \subseteq O$ implies $\bigsqcup\mathcal{D} \in \mathcal{O}$, whence some member M of $\mathcal{D}$ is in $\mathcal{O}$ because $\mathcal{O}$ is open. Hence, $M \subseteq O$ holds for some M in $\mathcal{D}$. □

4 Classes of domains

In this section, we introduce a hierarchy of domains in order to relate the classes of 'admitting' domains to more well-known classes. We start by translating the condition of *continuity* of domains into topological terms. Continuity is then generalized to *multi-continuity* and *local (strong) compactness*. We investigate the *$\mathcal{U}_\Box$-admitting* domains more closely, and then turn to the more restricted notion of *$\mathcal{U}_\Box$-conformity*. For $\mathcal{U}_\Box$-conformal domains, the Scott topologies of $\mathcal{U}_\Box\mathbf{X}$ and of $\mathbf{X}$ are closely related. Finally, we present conditions for the continuity of the upper power domains $\mathcal{U}_\Box\mathbf{X}$.

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[3] Remember domains are just directed-complete partial orders.

4.1 Continuous domains

Continuity of domains is a standard notion [GHK+80, Law88]. We first recall its order-theoretic standard definition and then provide an equivalent purely topological definition. This equivalence seems to be novel although it is not difficult to derive.

Definition 4.1 A point a in a domain $\mathbf{X}$ is *way below* a point b, $a \ll b$, iff for all directed sets $D \subseteq \mathbf{X}$ with $b \leq \bigsqcup D$, there is an element d in D such that $a \leq d$. A domain $\mathbf{X}$ is *continuous* iff every point x of $\mathbf{X}$ is the join of a directed set of points that all are way-below x.

Concerning the lower power domain construction, we claim

Theorem 4.2 If $\mathbf{X}$ is continuous, then $\mathcal{L}\mathbf{X}$ is too.

A proof is contained in [Hec90]. It is not included here since the paper is concerned with upper power domains. For these, Theorem 4.13 provides even stronger results.

We now reach at the topological characterization of continuity of domains.

Theorem 4.3 A domain $\mathbf{X}$ is continuous iff for all x in $\mathbf{X}$ and open sets O with $x \in O$, there are a point y and an open set O' such that $x \in O' \subseteq \uparrow y \subseteq O$.

Proof: The implication from left to right directly follows from Proposition 1.14 in [Jun88].

For the opposite direction, assume $\mathbf{X}$ is a domain satisfying the topological criterion. Let x be a point of $\mathbf{X}$, and let D be the set of all points d such that an open set O_d exists with $x \in O_d \subseteq \uparrow d$. We show: all points of D are way-below x, D is directed, and $\bigsqcup D$ is x.

Assume $x \in O_d \subseteq \uparrow d$, and let E be directed. Then $x \leq \bigsqcup E$ implies $\bigsqcup E \in O_d$, whence some point e of E is in O_d as O_d is open. By $O_d \subseteq \uparrow d$, $d \leq e$ follows. Thus, we have shown $d \ll x$.

Since $\mathbf{X}$ itself is open, $x \in \mathbf{X}$ implies there are a point y and an open set O_y such that $x \in O_y \subseteq \uparrow y$. Thus, D is not empty. Let u and v be two points of D. We look for a common upper bound of u and v in D. u and v in D implies there are open sets O_u and O_v such that $x \in O_u \subseteq \uparrow u$ and $x \in O_v \subseteq \uparrow v$. Then $x \in O_u \cap O_v$ follows, whence by hypothesis, there are a point w and an open set O_w such that $x \in O_w \subseteq \uparrow w \subseteq O_u \cap O_v$. Then w is in D, and $\uparrow w \subseteq O_u \subseteq \uparrow u$ implies $w \geq u$. Similarly, $w \geq v$ follows. Now, we know D is directed.

We have to show $\bigsqcup D = x$. For every d in D, $x \in O_d \subseteq \uparrow d$ holds, whence $d \leq x$. $\bigsqcup D \leq x$ follows. For the opposite relation, we apply Lemma 2.2. Let x be a point of an open set O. We have to show $\bigsqcup D \in O$. By hypothesis, there are a point y and an open set O_y such that $x \in O_y \subseteq \uparrow y \subseteq O$. y in D and y in O follow. $y \leq \bigsqcup D$ implies $\bigsqcup D \in O$. $\square$

4.2 Multi-continuity and local compactness

The topological criterion of Theorem 4.3 suggests some generalizations. Replacing the single point y by a finite set E, we obtain the class of *multi-continuous* domains. Further generalization leads to *local compactness* as known from topology, and analogously to *local strong compactness*.

Definition 4.4

- $\mathbf{X}$ is *multi-continuous* iff for every point x in an open set O, there are an open set O' and a finitary set F such that $x \in O' \subseteq F \subseteq O$.
- $\mathbf{X}$ is *locally* $\square$*-compact* iff for every point x in an open set O, there are an open set O' and a $\square$-compact set K such that $x \in O' \subseteq K \subseteq O$.

Multi-continuity is equivalent to the notion of *quasicontinuity* as introduced in [GLS83]. The implication of multi-continuity by quasicontinuity can be seen by the proof of Corollary 4.2 in [GLS83], whereas the opposite implication follows from part (3) of Theorem 4.7 in the paper at hand.

The locality properties defined above form a hierarchy of domain classes:

Proposition 4.5 For domains, continuity implies multi-continuity, which is equivalent to local strong compactness, which implies local compactness.

Proof: The claimed implications follow from Propositions 3.3 and 3.4. Local strong compactness is equivalent to multi-continuity since $S \subseteq O$ for strongly compact set S and open set O implies $S \subseteq F \subseteq O$ for some finitary set F by definition of strong compactness. □

We now come to the main theorem about multi-continuous domains. It provides several criteria equivalent to multi-continuity. Before we state and prove it, we define some auxiliary notions.

Definition 4.6 Let $\mathbf{X}$ be a domain. For a subset A of $\mathbf{X}$, let $\mathcal{F}(A)$ be the set of all finitary upper environments of A, i.e. the set of all finitary upper sets F such that there is an open set O with $A \subseteq O \subseteq F$. For points x, $\mathcal{F}(\{x\})$ is abbreviated by $\mathcal{F}(x)$.

Analogously, let $\mathcal{K}(A)$ be the set of all compact upper environments of A. Again, $\mathcal{K}(\{x\})$ is abbreviated by $\mathcal{K}(x)$.

Theorem 4.7 For a domain $\mathbf{X}$, the following statements are equivalent:

(1) $\mathbf{X}$ is multi-continuous.

(2) For all points x and open sets O of $\mathbf{X}$, $x \in O$ implies there is F in $\mathcal{F}(x)$ such that $F \subseteq O$.

(3) For all points x, the set $\mathcal{F}(x)$ is $\supseteq$-directed and $\bigcap \mathcal{F}(x) = \uparrow x$ holds.

(4) For all compact sets K and open sets O of $\mathbf{X}$, $K \subseteq O$ implies there is F in $\mathcal{F}(K)$ such that $F \subseteq O$.

(5) For all compact sets K, the set $\mathcal{F}(K)$ is $\supseteq$-directed and $\bigcap \mathcal{F}(K) = \uparrow K$ holds.

Proof: Statements (1) and (2) are equivalent by the definitions of multi-continuity and $\mathcal{F}(.)$. The remaining equivalences are shown in the order (2) $\Rightarrow$ (4) $\Rightarrow$ (5) $\Rightarrow$ (3) $\Rightarrow$ (2).

(2) $\Rightarrow$ (4) : Let $K \subseteq O$ where K is compact and O is open. By (2), for every x in K, there are a finitary upper set F_x and an open set O_x such that $x \in O_x \subseteq F_x \subseteq O$. Thus, $K \subseteq \bigcup_{x \in K} O_x$ whence by compactness there is a finite set $E \subseteq K$ such that $K \subseteq \bigcup_{e \in E} O_e \subseteq \bigcup_{e \in E} F_e \subseteq O$ where $\bigcup_{e \in E} F_e$ is finitary upper and in $\mathcal{F}(K)$.

(4) $\Rightarrow$ (5) : K is a subset of the open set $\mathbf{X}$, whence $\mathcal{F}(K)$ is not empty by (4). Let F_1 and F_2 be two members of $\mathcal{F}(K)$. Then there are open sets O_1 and O_2 such that $K \subseteq O_i \subseteq F_i$. $K \subseteq O_1 \cap O_2$ follows, whence there is a set F in $\mathcal{F}(K)$ by (4) such that $K \subseteq F \subseteq O_1 \cap O_2$. $K \subseteq F \subseteq F_i$ follows, whence $\mathcal{F}(K)$ is $\supseteq$-directed.

All members of $\mathcal{F}(K)$ are upper supersets of K. Hence, $\uparrow K \subseteq F$ holds for all F in $\mathcal{F}(K)$, whence $\uparrow K \subseteq \bigcap \mathcal{F}(K)$. For the opposite inclusion $\bigcap \mathcal{F}(K) \subseteq \uparrow K$, we apply Lemma 2.1. Let $\uparrow K \subseteq O$ for some open set O. We have to show $\bigcap \mathcal{F}(K) \subseteq O$. $\uparrow K \subseteq O$ implies $K \subseteq O$ whence by (4) there is F in $\mathcal{F}(K)$ such that $F \subseteq O$. Thus, $\bigcap \mathcal{F}(K) \subseteq F \subseteq O$ follows.

(5) $\Rightarrow$ (3) is immediate since $\{x\}$ is compact.

(3) $\Rightarrow$ (2) : Let x be in an open set O. By (3), $\bigcap \mathcal{F}(x) = \uparrow x \subseteq O$ holds. Since $\mathcal{F}(x)$ is a $\supseteq$-directed set of finitary upper sets, Rudin's Lemma 2.4 applies and states $F \subseteq O$ for some F in $\mathcal{F}(x)$. □

As an immediate conclusion of statement (4) of this theorem, one obtains:

Proposition 4.8 In a multi-continuous domain every compact set is strongly compact.

For local compactness, a similar, but weaker theorem holds:

Theorem 4.9 For a $\mathcal{U}_K$-*admitting* domain $\mathbf{X}$, the following statements are equivalent:

(1) $\mathbf{X}$ is locally compact.

(2) For all points x and open sets O of $\mathbf{X}$, $x \in O$ implies there is K' in $\mathcal{K}(x)$ such that $K' \subseteq O$.

(3) For all points x, the set $\mathcal{K}(x)$ is $\supseteq$-directed and $\bigcap \mathcal{K}(x) = \uparrow x$ holds.

(4) For all compact sets K and open sets O of $\mathbf{X}$, $K \subseteq O$ implies there is K' in $\mathcal{K}(K)$ such that $K' \subseteq O$.

(5) For all compact sets K, the set $\mathcal{K}(K)$ is $\supseteq$-directed and $\bigcap \mathcal{K}(K) = \uparrow K$ holds.

Proof: The proof is analogous to that of Theorem 4.7. The only exception is proof step '(3) $\Rightarrow$ (2)' where Rudin's Lemma cannot be employed. It must be replaced by $\mathcal{U}_K$-admissibility, which thus has to be supposed. □

4.3 $\mathcal{U}_S$- and $\mathcal{U}_K$-admitting domains

In this section, we want to investigate how the novel classes of $\mathcal{U}_S$-admitting and $\mathcal{U}_K$-admitting domains fit into the domain hierarchy.

Theorem 4.10 Every multi-continuous domain is $\mathcal{U}_K$-admitting, and every $\mathcal{U}_K$-admitting domain is $\mathcal{U}_S$-admitting.

Proof: The second statement holds since $\mathcal{U}_K$-admissibility demands some property for all directed sets of compact sets, whereas $\mathcal{U}_S$-admissibility requires the same property for all directed sets of strongly compact sets, and these are fewer.

For the first implication, we use the following properties of the operator $\mathcal{F}(.)$ mapping sets to the set of their finitary upper environments.

(1) If $A \leq B$, then $\mathcal{F}(A) \subseteq \mathcal{F}(B)$ since $A \subseteq O \subseteq F$ implies $B \subseteq A \subseteq O \subseteq F$.

(2) $\mathcal{F}(A)$ is $\supseteq$-directed for all compact sets A.

(3) $\bigcap \mathcal{F}(A) = A$ for all compact upper sets A.

Properties (2) and (3) are due to statement (5) of Theorem 4.7.

Now let $(A_i)_{i \in I}$ be a directed family of compact upper sets, and let $\bigcap_{i \in I} A_i \subseteq O$ for some open set O. Then, $\bigcap_{i \in I} A_i = \bigcap_{i \in I}(\bigcap \mathcal{F}(A_i)) = \bigcap(\bigcup_{i \in I} \mathcal{F}(A_i))$ where the first equation is due to (3) and the second one due to general set theory. Now, $\bigcup_{i \in I} \mathcal{F}(A_i)$ is a directed union (by (1)) of directed sets (by (2)), and hence directed itself. By Rudin's Lemma 2.4, there is some F in $\bigcup_{i \in I} \mathcal{F}(A_i)$ with $F \subseteq O$. Thus, there is some k in I with $F \in \mathcal{F}(A_k)$ and $A_k = \bigcap \mathcal{F}(A_k) \subseteq F \subseteq O$. □

By Proposition 4.5, the theorem above may be rephrased to

- locally strongly compact $\Rightarrow$ $\mathcal{U}_S$-admitting

The analogous implication

- locally compact $\Rightarrow$ $\mathcal{U}_K$-admitting

cannot be shown analogously, since the usage of Rudin's Lemma in the proof above must be replaced by an application of $\mathcal{U}_K$-admissibility when replacing $\mathcal{F}(.)$ by $\mathcal{K}(.)$ — the property which has to be shown. In fact, we do not know whether this implication holds.

4.4 $\mathcal{U}_S$- and $\mathcal{U}_K$-conformal domains

$\mathcal{U}_\square$-admissibility allows mapping open sets of the ground domain to open sets of the power domain by means of $\overline{\mathcal{U}}_\square(.)$ (see Proposition 3.8). If the opposite is also possible, we speak of $\mathcal{U}_\square$-conformity.

Definition 4.11 A domain $\mathbf{X}$ is $\mathcal{U}_\square$-*conformal* iff it is $\mathcal{U}_\square$-admitting, and for all $K \in \mathcal{O}$ where $\mathcal{O}$ is an open set of $\mathcal{U}_\square\mathbf{X}$, there is an open set O in $\mathbf{X}$ such that $K \in \overline{\mathcal{U}}_\square(O) \subseteq \mathcal{O}$.

Topologically speaking, $\mathcal{U}_\square$-conformity means that the Scott topology of $\mathcal{U}_\square\mathbf{X}$ has as a base the sets $\overline{\mathcal{U}}_\square(O)$ where O is open in $\mathbf{X}$. A surprising equivalent criterion for $\mathcal{U}_S$-conformity is provided by the commutation theorem 6.1.

Theorem 4.12 Every locally $\square$-compact and $\mathcal{U}_\square$-admitting domain is $\mathcal{U}_\square$-conformal. Hence, every multi-continuous domain is $\mathcal{U}_S$-conformal.

Proof: The second statement holds since by Proposition 4.5, multi-continuous means locally strongly compact, and by Theorem 4.10, this implies $\mathcal{U}_S$-admissibility.

We prove the first statement for the case $\mathcal{U}_S$. Let $A \in \mathcal{O}$, where $\mathcal{O}$ is open in $\mathcal{U}_S\mathbf{X}$. From part (5) of Theorem 4.7, we know that A is the join of the directed set $\mathcal{F}(A)$. Since $\mathcal{O}$ is open, there is some $F \in \mathcal{F}(A)$ with $F \in \mathcal{O}$. From $F \in \mathcal{F}(A)$, we get an open set O with $A \subseteq O \subseteq F$. Then $A \in \overline{\mathcal{U}}_S(O)$ holds, and $B \in \overline{\mathcal{U}}_S(O)$ implies $B \subseteq O \subseteq F$, i.e. $B \geq F$, whence $B \in \mathcal{O}$. Thus, we obtain $A \in \overline{\mathcal{U}}_S(O) \subseteq \mathcal{O}$.

The proof for $\mathcal{U}_K$ is obtained by replacing $\mathcal{F}(.)$ by $\mathcal{K}(.)$. This poses no difficulty since Rudin's Lemma is not applied here. $\square$

4.5 Continuity of upper power domains

In this section, we present necessary and sufficient criteria for the continuity of the upper power domains $\mathcal{U}_S\mathbf{X}$ and $\mathcal{U}_K\mathbf{X}$.

Theorem 4.13 Let $\mathbf{X}$ be $\mathcal{U}_\square$-admitting. Then $\mathcal{U}_\square\mathbf{X}$ is continuous iff $\mathbf{X}$ is locally $\square$-compact.

Proof: We use the topological characterization of continuity given by Theorem 4.3.

Let $\mathbf{X}$ be locally $\square$-compact. Then $\mathbf{X}$ is $\mathcal{U}_\square$-conformal by Theorem 4.12. Let $A \in \mathcal{O}$ where $\mathcal{O}$ is open in $\mathcal{U}_\square\mathbf{X}$. Then there is an open set O of $\mathbf{X}$ such that $A \in \overline{\mathcal{U}}_\square(O) \subseteq \mathcal{O}$. By Theorems 4.7 and 4.9, $A \subseteq O$ implies the existence of an open set O' and a $\square$-compact set K such that $A \subseteq O' \subseteq K \subseteq O$. First, A is in $\overline{\mathcal{U}}_\square(O')$. Second, $B \in \overline{\mathcal{U}}_\square(O')$ implies $B \subseteq O' \subseteq K$, i.e. $B \geq K$. Thus, $\overline{\mathcal{U}}_\square(O') \subseteq \uparrow\{K\}$ holds. Third, $K \subseteq O$ implies $K \in \overline{\mathcal{U}}_\square(O) \subseteq \mathcal{O}$, whence $\uparrow\{K\} \subseteq \mathcal{O}$. Summarizing, we obtain $A \in \overline{\mathcal{U}}_\square(O') \subseteq \uparrow\{K\} \subseteq \mathcal{O}$ as required.

Now, let $\mathcal{U}_\square\mathbf{X}$ be continuous. Let x be a point of an open set O of $\mathbf{X}$. Then $\uparrow x \in \overline{\mathcal{U}}_\square(O)$, whence by continuity of $\mathcal{U}_\square\mathbf{X}$, there are an open set $\mathcal{O}$ of $\mathcal{U}_\square\mathbf{X}$ and a member A of $\mathcal{U}_\square\mathbf{X}$ such that $\uparrow x \in \mathcal{O} \subseteq \uparrow\{A\} \subseteq \overline{\mathcal{U}}_\square(O)$. Let $O' = \iota^{-1}[\mathcal{O}]$. By continuity of ι, O' is an open set of $\mathbf{X}$. First, $\iota x = \uparrow x \in \mathcal{O}$ implies $x \in O'$. Second, $y \in O'$ means $\uparrow y \in \mathcal{O} \subseteq \uparrow\{A\}$, whence $\uparrow y \geq A$, i.e. $\uparrow y \subseteq A$. Thus, we obtain $O' \subseteq A$. Third, $\uparrow\{A\} \subseteq \overline{\mathcal{U}}_\square(O)$ implies $A \in \overline{\mathcal{U}}_\square(O)$, whence $A \subseteq O$. Summarizing, we obtain $x \in O' \subseteq A \subseteq O$ for a $\square$-compact set A as required. $\square$

4.6 Further classes and open problems

In [Smy83], Smyth introduced the class of *sober* domains. Every multi-continuous domain is sober [Hec90], and all sober domains are $\mathcal{U}_K$-admitting [Hec90], whence $\mathcal{U}_S$-admitting, too. The non-sober domain of [Joh81] is also not $\mathcal{U}_S$-admitting. There are sober domains that are not multi-continuous. Further relationships among the notions of multi-continuity, $\mathcal{U}_\Box$-conformity, sobriety, and $\mathcal{U}_\Box$-admissibility are however unknown, e.g. neither do we know of examples of $\mathcal{U}_K$-admitting domains not being sober, nor can we prove the equivalence of these notions. Likewise, we do not know whether the two power domain constructions $\mathcal{U}_\Box$ preserve $\mathcal{U}_\Box$-conformity or $\mathcal{U}_\Box$-admissibility.

5 Properties unique to $\mathcal{U}_S$

All developments of the previous section were done for compact and strongly compact sets in parallel. There were only a few differences, e.g. the implication of $\mathcal{U}_\Box$-admissibility by local $\Box$-compactness (Theorem 4.10) could be shown for the case $\mathcal{U}_S$ only. In this section, we present results which could only be shown for the case of strong compactness. We were not able to prove or refute the analogous results for the case of compactness.

5.1 Ground domains with property M

In section 2.1, we defined that a domain has property M iff it is finitary as a whole, and for all points x and y, the set of common upper bounds $\uparrow x \cap \uparrow y$ is finitary. Obviously, every bounded complete domain has property M. We show some consequences of M for the upper power domain in terms of strongly compact sets.

Theorem 5.1 If $\mathbf{X}$ has property M, then it is $\mathcal{U}_S$-admitting, and $\mathcal{U}_S\mathbf{X}$ is complete. All joins are given by intersection.

Proof: For strongly compact upper set A, let $\mathcal{F}'(A)$ be the set of all finitary supersets of A.[4] The operator $\mathcal{F}'(.)$ has the following properties:

(1) If $A \le B$, then $\mathcal{F}'(A) \subseteq \mathcal{F}'(B)$ since $A \subseteq F$ implies $B \subseteq A \subseteq F$.

(2) $\mathcal{F}'(A)$ is directed.

$\mathcal{F}'(A)$ is not empty since $A \subseteq \mathbf{X}$ and $\mathbf{X}$ is finitary by M. $\mathcal{F}'(A)$ is directed because the intersection of two finitary sets is finitary: for finite sets E and E', $\uparrow E \cap \uparrow E' = \bigcup_{a\in E, b\in E'} \uparrow a \cap \uparrow b$ is a finite union of finitary sets by M.

(3) $\bigcap \mathcal{F}'(A) = A$.

The inclusion '$\supseteq$' is obvious by definition of $\mathcal{F}'(.)$. For the opposite, we employ Lemma 2.1. If $A \subseteq O$ where O is open, then by strong compactness of A there is a finitary set F such that $A \subseteq F \subseteq O$. Then $F \in \mathcal{F}'(A)$ whence $\bigcap \mathcal{F}'(A) \subseteq F \subseteq O$.

From these three properties, $\mathcal{U}_S$-admissibility follows analogously as $\mathcal{U}_K$-admissibility follows from the properties of $\mathcal{F}(.)$ in the proof of Theorem 4.10.

As $\mathbf{X}$ itself is finitary, it is also strongly compact by Proposition 3.3 and Proposition 3.4. $\mathbf{X}$ is the least element of $\mathcal{U}_S\mathbf{X}$, and $\emptyset$ the greatest one.

Let A and B be two strongly compact upper sets. We show that $A \cap B$ is strongly compact, and hence forms the join of A and B. Let $\mathcal{F} = \{F_1 \cap F_2 \mid F_1 \in \mathcal{F}'(A) \text{ and } F_2 \in \mathcal{F}'(B)\}$. Since $\mathbf{X}$ has property M, $\mathcal{F}$ consists of finitary, whence strongly compact, upper sets. $\mathcal{F}$ is directed since $\mathcal{F}'(A)$ and $\mathcal{F}'(B)$ are directed (by (2) above). $\bigcap \mathcal{F}'(A) = A$ and $\bigcap \mathcal{F}'(B) = B$ imply $\bigcap \mathcal{F} = A \cap B$. Hence, $A \cap B$ is strongly compact as a directed intersection by Proposition 3.8.

[4] $\mathcal{F}'(A)$ consists of all finitary *supersets*, whereas $\mathcal{F}(A)$ consists of all finitary *environments*.

A domain with least element and join of every two points is complete. By Proposition 3.8, directed joins are given by intersection, too, whence all joins are intersections. □

By this theorem, the hierarchy w.r.t. $\mathcal{U}_S$ extends to:

- Continuous $\Rightarrow$ multi-continuous $\Rightarrow$ $\mathcal{U}_S$-conformal $\Rightarrow$ $\mathcal{U}_S$-admitting
- Complete $\Rightarrow$ bounded complete $\Rightarrow$ property M $\Rightarrow$ $\mathcal{U}_S$-admitting

The class of $\mathcal{U}_S$-admitting domains is thus quite large.

5.2 A criterion for open sets in $\mathcal{U}_S\mathbf{X}$

By Proposition 3.8, the sets $\overline{\mathcal{U}}_S(O) = \{S \in \mathcal{U}_S\mathbf{X} \mid S \subseteq O\}$ are open in $\mathcal{U}_S\mathbf{X}$ if O is open in $\mathbf{X}$. In the next lemma, we characterize all open sets of $\mathcal{U}_S\mathbf{X}$ that are obtained by $\overline{\mathcal{U}}_S(.)$.

Lemma 5.2
Let $\mathbf{X}$ be $\mathcal{U}_S$-admitting. For a subset $\mathcal{O}$ of $\mathcal{U}_S\mathbf{X}$, the following two statements are equivalent:

(1) There is an open set O of $\mathbf{X}$ such that $\mathcal{O} = \overline{\mathcal{U}}_S(O)$.

(2) $\mathcal{O}$ is Scott open, and closed w.r.t. binary union.

Proof: The sets $\overline{\mathcal{U}}_S(O)$ are open by Proposition 3.8. They are closed w.r.t. union since $S_1, S_2 \subseteq O$ implies $S_1 \cup S_2 \subseteq O$.

Conversely, let $\mathcal{O}$ be an open set of $\mathcal{U}_S\mathbf{X}$ that is closed w.r.t. union. Let $O = \iota^{-1}[\mathcal{O}] = \{x \in \mathbf{X} \mid {\uparrow}x \in \mathcal{O}\}$. O is open as the inverse image of $\mathcal{O}$ by ι. We claim $\mathcal{O} = \overline{\mathcal{U}}_S(O)$.

Let S be in $\mathcal{O}$. x in S implies ${\uparrow}x \subseteq S$, i.e. $S \leq {\uparrow}x$. Hence, ${\uparrow}x$ is in $\mathcal{O}$, whence x is in O. Thus, $S \subseteq O$ holds. Conversely, $S \subseteq O$ implies $S \subseteq {\uparrow}E \subseteq O$ for some finite set E by strong compactness. By ${\uparrow}E \subseteq O$, every e in E is in O, whence ${\uparrow}e \in \mathcal{O}$ for all e in E. Thus, $S \geq {\uparrow}E = \bigcup_{e \in E} {\uparrow}e \in \mathcal{O}$ because $\mathcal{O}$ is closed w.r.t. finite union. □

Since we used strong compactness explicitly in this proof, an analogous proof for $\mathcal{U}_K$ is impossible. We may however prove a similar criterion where 'binary union' is replaced by 'arbitrary union'. This would not be sufficient to prove the results of the next section.

5.3 The intrinsic universal property of $\mathcal{U}_S$

Theorem 2.7 stated a universal property of $\mathcal{L}$: every continuous function f from a domain $\mathbf{X}$ to a complete domain $\mathbf{Z}$ may be uniquely extended to a continuous function from $\mathcal{L}\mathbf{X}$ to $\mathbf{Z}$ preserving all finite joins. For $\mathcal{U}_S$, a similar property holds involving finite meets. Instead of an arbitrary complete domain, the target must be an upper power domain generated by $\mathcal{U}_S$ itself. Thus, we call this an intrinsic universal property.

For functions between two power domains created by $\mathcal{U}_S$, preservation of finite meets just means additivity, i.e. preservation of empty set and binary union. We start by investigating additive continuous functions.

Lemma 5.3 Let $\mathbf{X}$ and $\mathbf{Y}$ be two $\mathcal{U}_S$-admitting domains. If $F : [\mathcal{U}_S\mathbf{X} \to \mathcal{U}_S\mathbf{Y}]$ is additive, then $FB = \bigcap\{FA \mid A \in \mathcal{F}'(B)\}$ holds for all B in $\mathcal{U}_S\mathbf{X}$, where $\mathcal{F}'(B)$ is the set of all finitary upper supersets of B.

Proof: If $A \supseteq B$, then $FA \supseteq FB$. Thus, FB is a subset of the intersection.

Conversely, we have to show $T = \bigcap\{FA \mid A \in \mathcal{F}'(B)\} \subseteq FB$. Applying Lemma 2.1, we have to show that $FB \subseteq O$ implies $T \subseteq O$ for all open sets O of $\mathbf{Y}$. Let $FB \subseteq O$. Then FB is in $\overline{\mathcal{U}}_S(O)$, whence B is in the open set $\mathcal{O} = F^{-1}[\overline{\mathcal{U}}_S(O)]$. $\mathcal{O}$ is closed w.r.t. binary union since F is additive: $A, B \in \mathcal{O}$ implies $FA, FB \in \overline{\mathcal{U}}_S(O)$, whence $F(A \cup B) = FA \cup FB \in \overline{\mathcal{U}}_S(O)$, whence $A \cup B \in \mathcal{O}$. By Lemma 5.2, there is some open set O' of $\mathbf{X}$ such that $\mathcal{O} = \overline{\mathcal{U}}_S(O')$. $B \subseteq O'$ implies $B \subseteq A \subseteq O'$ for some finitary upper set A by strong compactness of B. $A \subseteq O'$ means $FA \subseteq O$, whence $T \subseteq O$.□

The lemma shows that additive continuous functions from $\mathcal{U}_S\mathbf{X}$ to $\mathcal{U}_S\mathbf{Y}$ only depend on their values on finitary upper sets. All these sets are finite unions of upper cones, whence we obtain

Corollary 5.4 Let $\mathbf{X}$ and $\mathbf{Y}$ be two $\mathcal{U}_S$-admitting domains. If $F, G : [\mathcal{U}_S\mathbf{X} \to \mathcal{U}_S\mathbf{Y}]$ are two additive functions with $F \circ \iota = G \circ \iota$, then $F = G$ follows.

By combination with Theorem 3.9, we obtain

Corollary 5.5 Let $\mathbf{X}$ and $\mathbf{Y}$ be two $\mathcal{U}_S$-admitting domains. For every function $f : [\mathbf{X} \to \mathcal{U}_S\mathbf{Y}]$, there is a unique additive function $F : [\mathcal{U}_S\mathbf{X} \to \mathcal{U}_S\mathbf{Y}]$ with $F \circ \iota = f$. F is explicitly given by

$$FB = \bigcup_{x \in B} fx = \bigcap \{ \bigcup_{e \in E} fe \mid E \text{ finite and } B \subseteq \uparrow E \}.$$

We were not able to prove analogous theorems for $\mathcal{U}_K$. On the other hand, we were not able to find any counterexamples.

6 The commutation theorem

In [FM90], it was shown that $\mathcal{U}(\mathcal{L}\mathbf{X})$ and $\mathcal{L}(\mathcal{U}\mathbf{X})$ are isomorphic for bounded complete algebraic domains $\mathbf{X}$. The longish proof was done by means of the theory of information systems. We were able to largely generalize this statement up to the class of $\mathcal{U}_S$-conformal domains if $\mathcal{U}$ is generalized to $\mathcal{U}_S$. Our proof is still longish, but completely differs from the proof in [FM90] in that it is done by topological means. Further generalization is impossible: $\mathcal{U}_S$-conformity is also necessary for commutation of $\mathcal{L}$ and $\mathcal{U}_S$.

Theorem 6.1 Let $\mathbf{X}$ be $\mathcal{U}_S$-admitting. Then the following two statements are equivalent:

(1) $\mathbf{X}$ is $\mathcal{U}_S$-conformal.

(2) $\mathcal{U}_S(\mathcal{L}\mathbf{X})$ and $\mathcal{L}(\mathcal{U}_S\mathbf{X})$ are isomorphic under a mapping which sends $\iota_{\mathcal{U}}(\iota_{\mathcal{L}}x)$ of $\mathcal{U}_S(\mathcal{L}\mathbf{X})$ to $\iota_{\mathcal{L}}(\iota_{\mathcal{U}}x)$ in $\mathcal{L}(\mathcal{U}_S\mathbf{X})$, for each x in $\mathbf{X}$.

The isomorphism of (2) is unique.

Before we prove this theorem, we note that it allows an elegant proof of the preservation of multi-continuity under $\mathcal{L}$.

Theorem 6.2 If $\mathbf{X}$ is multi-continuous, then so is $\mathcal{L}\mathbf{X}$.

Proof: If $\mathbf{X}$ is multi-continuous, then it is $\mathcal{U}_S$-conformal by Theorem 4.12. $\mathcal{U}_S\mathbf{X}$ is continuous by Theorem 4.13, whence $\mathcal{L}(\mathcal{U}_S\mathbf{X})$ is continuous by Theorem 4.2. By isomorphism, $\mathcal{U}_S(\mathcal{L}\mathbf{X})$ is continuous, too, and by Theorem 4.13, $\mathcal{L}\mathbf{X}$ is multi-continuous. □

The proof of Theorem 6.1 consists of two parts contained in the two subsections of this section. In the first part, we show that conformity implies commutation. Uniqueness of the isomorphism and the inverse implication are shown in the second part.

6.1 Conformity implies commutation

The proof of this implication proceeds in several steps. The precondition of $\mathcal{U}_S$-conformity is not needed until the very last step, so that we can use the results of the first steps even when proving the opposite implication in the next subsection.

Thus, we start by assuming only that $\mathbf{X}$ is $\mathcal{U}_S$-admitting. Then $\mathcal{U}_S\mathbf{X}$ is a domain, and $\mathcal{L}(\mathcal{U}_S\mathbf{X})$ is a complete domain by Proposition 2.6. In contrast, $\mathcal{U}_S(\mathcal{L}\mathbf{X})$ is a complete domain even for arbitrary $\mathbf{X}$ since $\mathcal{L}\mathbf{X}$ is complete by Proposition 2.6 and complete domains are $\mathcal{U}_S$-admitting and produce complete domains under $\mathcal{U}_S$ by Theorem 5.1.

Step 1: A function $\varphi' : \mathcal{L}\mathbf{X} \to \mathcal{L}(\mathcal{U}_S\mathbf{X})$

For a closed set C of $\mathbf{X}$, we define $\varphi' C = \{K \in \mathcal{U}_S\mathbf{X} \mid C \cap K \neq \emptyset\}$. Its complement $\mathsf{co}\,(\varphi' C) = \{K \in \mathcal{U}_S\mathbf{X} \mid K \subseteq \mathsf{co}\, C\} = \overline{\mathcal{U}}_S(\mathsf{co}\, C)$ is open by Proposition 3.8, whence $\varphi' C$ is closed.

Step 2: A monotonic function $\varphi : \mathcal{U}_S(\mathcal{L}\mathbf{X}) \to \mathcal{L}(\mathcal{U}_S\mathbf{X})$

For a strongly compact upper set $\mathcal{K}$ of $\mathcal{L}\mathbf{X}$, we define $\varphi\mathcal{K} = \bigcap_{C\in\mathcal{K}} \varphi' C = \{K \in \mathcal{U}_S\mathbf{X} \mid C \cap K \neq \emptyset \ \forall C \in \mathcal{K}\}$. As an intersection of closed sets, this is indeed a closed set of $\mathcal{U}_S\mathbf{X}$.

$\mathcal{K} \leq \mathcal{K}'$ means $\mathcal{K} \supseteq \mathcal{K}'$. If K is in $\varphi\mathcal{K}$, then K meets all members of $\mathcal{K}$, and thus it also meets all members of $\mathcal{K}'$. Thus, $\mathcal{K} \leq \mathcal{K}'$ implies $\varphi\mathcal{K} \leq \varphi\mathcal{K}'$.

Step 3: A continuous function $\psi' : \mathcal{U}_S\mathbf{X} \to \mathcal{U}_S(\mathcal{L}\mathbf{X})$

For a strongly compact upper set K of $\mathcal{U}_S\mathbf{X}$, we define $\psi' K = \{C \in \mathcal{L}\mathbf{X} \mid K \cap C \neq \emptyset\}$. We have to show that $\psi' K$ is a strongly compact upper set of $\mathcal{L}\mathbf{X}$.

We claim $\psi' = \mathcal{U}_S\, \iota_{\mathcal{L}}$.[5] This equality implies $\psi' : [\mathcal{U}_S\mathbf{X} \to \mathcal{U}_S(\mathcal{L}\mathbf{X})]$ as required.

$\mathcal{U}_S \iota_{\mathcal{L}} K = \uparrow(\iota_{\mathcal{L}}[K]) = \uparrow\{\downarrow x \mid x \in K\}$ holds. Let C be a member of this set. Then $C \geq \downarrow x$, i.e. $C \supseteq \downarrow x$, for some $x \in K$, whence $x \in C \cap K$. Conversely, assume there is some x in $C \cap K$. Then $\downarrow x \subseteq C$ since C is closed, whence $C \geq \downarrow x$ where $x \in K$.

Step 4: A continuous function $\psi : \mathcal{L}(\mathcal{U}_S\mathbf{X}) \to \mathcal{U}_S(\mathcal{L}\mathbf{X})$

From Theorem 2.7, we get a continuous extension $\psi = \overline{\mathcal{L}}\psi'$ that preserves finite joins and satisfies $\psi \circ \iota_{\mathcal{L}} = \psi'$. For a closed set $\mathcal{C}$ of $\mathcal{U}_S\mathbf{X}$, $\psi\mathcal{C}$ is explicitly given by $\psi\mathcal{C} = \bigcap_{K\in\mathcal{C}} \psi' K = \{C \in \mathcal{L}\mathbf{X} \mid K \cap C \neq \emptyset \ \forall K \in \mathcal{C}\}$.[6]

The definition implies $\psi \circ \iota_{\mathcal{L}} \circ \iota_{\mathcal{U}} = \psi' \circ \iota_{\mathcal{U}} = \mathcal{U}_S \iota_{\mathcal{L}} \circ \iota_{\mathcal{U}} = \iota_{\mathcal{U}} \circ \iota_{\mathcal{L}}$ by naturality of $\iota_{\mathcal{U}}$. Thus, ψ maps nested singletons of $\mathcal{L}(\mathcal{U}_S\mathbf{X})$ to nested singletons of $\mathcal{U}_S(\mathcal{L}\mathbf{X})$ as required.

Step 5: Relations between φ and ψ

Let $\mathcal{K}$ be a member of $\mathcal{U}_S(\mathcal{L}\mathbf{X})$, and let C in turn be a member of $\mathcal{K}$. By definition of φ, C meets all members of $\varphi\mathcal{K}$. By definition of ψ, C is then in $\psi(\varphi\mathcal{K})$. Thus, we get $\mathcal{K} \subseteq \psi(\varphi\mathcal{K})$, or in terms of order $\mathcal{K} \geq \psi(\varphi\mathcal{K})$, or by abstraction $\psi \circ \varphi \leq id$.

Analogously, $\mathcal{C} \subseteq \varphi(\psi\mathcal{C})$ holds for all $\mathcal{C}$ in $\mathcal{L}(\mathcal{U}_S\mathbf{X})$. This means $\mathcal{C} \leq \varphi(\psi\mathcal{C})$, whence $\varphi \circ \psi \geq id$. The two relations $\psi \circ \varphi \leq id$ and $\varphi \circ \psi \geq id$ imply $\varphi \circ \psi \circ \varphi = (\varphi \circ \psi) \circ \varphi \geq \varphi$ and $\varphi \circ \psi \circ \varphi = \varphi \circ (\psi \circ \varphi) \leq \varphi$, whence $\varphi \circ \psi \circ \varphi = \varphi$. Similarly, $\psi \circ \varphi \circ \psi = \psi$ holds. In Step 6, we show that φ is injective, whence $\varphi \circ \psi \circ \varphi = \varphi$ implies $\psi \circ \varphi = id$. In the final Step 7, we show that ψ is injective, too, whence also $\varphi \circ \psi = id$.

Step 6: φ is injective: $\varphi\mathcal{K} \leq \varphi\mathcal{K}'$ implies $\mathcal{K} \leq \mathcal{K}'$

Let $\mathcal{K}$ and $\mathcal{K}'$ be two strongly compact upper sets of $\mathcal{L}\mathbf{X}$. $\varphi\mathcal{K} \leq \varphi\mathcal{K}'$ means $\varphi\mathcal{K} \subseteq \varphi\mathcal{K}'$. We have to conclude $\mathcal{K} \leq \mathcal{K}'$, i.e. $\mathcal{K} \supseteq \mathcal{K}'$. Applying Lemma 2.1, we assume $\mathcal{K} \subseteq \mathcal{O}$ where $\mathcal{O}$ is an open set of $\mathcal{L}\mathbf{X}$, and have to show $\mathcal{K}' \subseteq \mathcal{O}$.

Since $\mathcal{K}$ is strongly compact, $\mathcal{K} \subseteq \mathcal{O}$ implies there is a finitary set $\mathcal{F}$ in $\mathcal{L}\mathbf{X}$ such that $\mathcal{K} \subseteq \mathcal{F} \subseteq \mathcal{O}$. As $\mathcal{F}$ is finitary, there is a finite set $\mathcal{E}$ of closed sets $E_1, \ldots, E_n$ such that $\mathcal{F} = \uparrow\mathcal{E}$. We show $\mathcal{K}' \subseteq \mathcal{F}$, whence $\mathcal{K}' \subseteq \mathcal{O}$ immediately follows by $\mathcal{F} \subseteq \mathcal{O}$.

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[5]This was the way the author found the mapping ψ'. The mapping φ' may be analogously derived.

[6]Interestingly, the definitions φ and ψ only differ in the domain and codomain.

Let C be a member of $\mathcal{K}'$. C is closed, whence its complement $O = \mathsf{co}\, C$ is open. The sets $O_i = \mathsf{co}\, E_i$ are also open. With these notions, we first show an auxiliary property: $\overline{\mathcal{U}}_S(O) \subseteq \overline{\mathcal{U}}_S(O_1) \cup \cdots \cup \overline{\mathcal{U}}_S(O_n)$. This property will later be used to show that there is some k such that $O \subseteq O_k$. By complementing, this implies $C \supseteq E_k$, i.e. $C \geq E_k$ for some k, whence $C \in \uparrow\mathcal{E} = \mathcal{F}$.

Proof of $\overline{\mathcal{U}}_S(O) \subseteq \overline{\mathcal{U}}_S(O_1) \cup \cdots \cup \overline{\mathcal{U}}_S(O_n)$:
Let K be a member of $\overline{\mathcal{U}}_S(O)$, i.e. a strongly compact upper subset of O. Then K does not meet O's complement C, whence K is not in $\varphi\mathcal{K}'$. Thus, K is neither in $\varphi\mathcal{K}$ because of the precondition $\varphi\mathcal{K} \subseteq \varphi\mathcal{K}'$. Hence, there is a member A of $\mathcal{K}$ such that $A \cap K = \emptyset$. Because of $\mathcal{K} \subseteq \uparrow\mathcal{E}$, there is some E_i in $\mathcal{E}$ such that $A \geq E_i$, i.e. $A \supseteq E_i$. Hence, $E_i \cap K = \emptyset$ also holds, whence $K \subseteq O_i$, i.e. $K \in \overline{\mathcal{U}}_S(O_i)$.

Proof of $O \subseteq O_k$ for some k:
Assume $O \not\subseteq O_k$ for all k. Then for all k there is some x_k in O with $x_k \notin O_k$. Let $K = \uparrow\{x_1, \ldots, x_n\} \subseteq O$. By finiteness, this is a strongly compact set. Hence, $K \in \overline{\mathcal{U}}_S(O)$, whence there is some i such that $K \subseteq O_i$, i.e. in particular $x_i \in K \subseteq O_i$. Thus, we derived a contradiction.

Step 6 is now concluded. Notice that we did not use $\mathcal{U}_S$-conformity so far, i.e. all results proved till now are valid for general $\mathcal{U}_S$-admitting ground domains $\mathbf{X}$.

Step 7: ψ is injective: $\psi\mathcal{C} \leq \psi\mathcal{C}'$ implies $\mathcal{C} \leq \mathcal{C}'$

Let $\mathcal{C}$ and $\mathcal{C}'$ be two closed sets of $\mathcal{U}_S\mathbf{X}$. $\psi\mathcal{C} \leq \psi\mathcal{C}'$ means $\psi\mathcal{C} \supseteq \psi\mathcal{C}'$. We have to conclude $\mathcal{C} \leq \mathcal{C}'$, i.e. $\mathcal{C} \subseteq \mathcal{C}'$.

Let K be in $\mathcal{C}$, and assume K is not in $\mathcal{C}'$. Then K is in the complement $\mathcal{O}'$ of $\mathcal{C}'$. Since $\mathcal{O}'$ is open in $\mathcal{U}_S\mathbf{X}$, there is an open set O of $\mathbf{X}$ such that $K \in \overline{\mathcal{U}}_S(O) \subseteq \mathcal{O}'$ by $\mathcal{U}_S$-conformity. $K \subseteq O$ implies $K \cap \mathsf{co}\, O = \emptyset$. Hence, $\mathsf{co}\, O$ is not in $\psi\mathcal{C}$, whence it is not in $\psi\mathcal{C}'$, either. This means there is some $K' \in \mathcal{C}'$ such that $K' \cap \mathsf{co}\, O = \emptyset$. Thus, $K' \subseteq O$, whence $K' \in \overline{\mathcal{U}}_S(O) \subseteq \mathcal{O}'$ contradicting $K' \in \mathcal{C}'$.

6.2 Commutation implies conformity

We now assume that $\mathcal{L}(\mathcal{U}_S\mathbf{X})$ and $\mathcal{U}_S(\mathcal{L}\mathbf{X})$ are isomorphic via an isomorphism ϑ which relates the respective nested singletons, i.e. $\vartheta \circ \iota_{\mathcal{L}} \circ \iota_{\mathcal{U}} = \iota_{\mathcal{U}} \circ \iota_{\mathcal{L}}$. From the previous subsection, we know of an explicitly given continuous function $\psi : [\mathcal{L}(\mathcal{U}_S\mathbf{X}) \to \mathcal{U}_S(\mathcal{L}\mathbf{X})]$ with an analogous property. We first show $\vartheta = \psi$, which implies the uniqueness statement of the commutation theorem. Thereafter, we conclude the $\mathcal{U}_S$-conformity of $\mathbf{X}$ from the bijectivity of ψ.

Uniqueness:

Let $\vartheta' = \vartheta \circ \iota_{\mathcal{L}}$ and $\vartheta'' = \vartheta' \circ \iota_{\mathcal{U}}$, and ψ' and ψ'' analogously. We know $\vartheta'' = \iota_{\mathcal{U}} \circ \iota_{\mathcal{L}} = \psi''$. We want to employ Theorem 5.4 to show $\vartheta' = \psi' : [\mathcal{U}_S\mathbf{X} \to \mathcal{U}_S(\mathcal{L}\mathbf{X})]$, and then conclude $\vartheta = \psi$ by Theorem 2.7.

To apply Theorem 5.4, both ϑ' and ψ' must be additive. $\psi' = \mathcal{U}_S\iota_{\mathcal{L}}$ is additive by Theorem 3.9. Additivity between $\mathcal{U}_S$-powerdomains means preservation of finite meets. As an isomorphism, ϑ preserves all meets. $\iota_{\mathcal{L}} : [\mathcal{U}_S\mathbf{X} \to \mathcal{L}(\mathcal{U}_S\mathbf{X})]$ preserves finite meets since

- $\iota_{\mathcal{L}}(\top) = \downarrow\{\emptyset\} = \{K \mid K \supseteq \emptyset\} = \top$
- $\iota_{\mathcal{L}}(A \sqcap B) = \downarrow\{A \cup B\} = \{K \mid K \supseteq A \cup B\} = \{K \mid K \supseteq A\} \cap \{K \mid K \supseteq B\} = \iota_{\mathcal{L}}(A) \sqcap \iota_{\mathcal{L}}(B)$

Thus, their composition $\vartheta' = \vartheta \circ \iota_{\mathcal{L}}$ preserves finite meets.

To apply Theorem 2.7, both ϑ and ψ must preserve finite joins. ϑ preserves all joins as an isomorphism, and $\psi = \overline{\mathcal{L}}\psi'$ preserves finite joins by Theorem 2.7.

Now, we have shown that whenever an isomorphism between $\mathcal{L}(\mathcal{U}_S\mathbf{X})$ and $\mathcal{U}_S(\mathcal{L}\mathbf{X})$ relating nested singletons exists, it is concretely given by ψ. Thus, we have shown the uniqueness statement of the commutation theorem.

Conformity:

We now conclude $\mathcal{U}_S$-conformity of $\mathbf{X}$ from the fact that ψ is an isomorphism. Let K be a strongly compact set, $\mathcal{O}$ an open set of $\mathcal{U}_S\mathbf{X}$, and $K \in \mathcal{O}$. Let $\mathcal{C} = \mathsf{co}\,\mathcal{O}$. Then $K \notin \mathcal{C}$, or equivalently $\downarrow\{K\} \not\leq \mathcal{C}$ in $\mathcal{L}(\mathcal{U}_S\mathbf{X})$. Applying the isomorphism ψ, one obtains $\psi(\downarrow\{K\}) \not\leq \psi\mathcal{C}$ in $\mathcal{U}_S(\mathcal{L}\mathbf{X})$, or equivalently $\psi'K \not\supseteq \psi\mathcal{C}$. Hence, there is a member C of $\psi\mathcal{C}$ such that $C \notin \psi'K$. Let O be the complement of C.

$C \notin \psi'K$ means $K \cap C = \emptyset$, i.e. $K \subseteq O$. On the other hand,

$$\begin{array}{llll} C \in \psi\mathcal{C} & \text{iff} \quad (\forall K' \in \mathcal{C} : K' \cap C \neq \emptyset) & \text{iff} & (K' \cap C = \emptyset \Rightarrow K' \in \mathcal{O}) \\ & \text{iff} \quad (K' \subseteq O \Rightarrow K' \in \mathcal{O}) & \text{iff} & \overline{\mathcal{U}}_S(O) \subseteq \mathcal{O} \end{array}$$

Together, we obtain $K \in \overline{\mathcal{U}}_S(O) \subseteq \mathcal{O}$ as required.

7 Compactness is weaker than strong compactness

In this section, we present a domain with a compact subset which is not strongly compact. This example shows that the notion of strong compactness does not coincide with compactness, and that the two upper power domain constructions $\mathcal{U}_S$ and $\mathcal{U}_K$ differ.

A basic constituent of our example domain is the domain of 'lazy naturals'. Its carrier set is $\{\alpha_1, \alpha_2, \alpha_3, \ldots, a, b_1, b_2, b_3, \ldots\}$ ordered by $\alpha_1 < \alpha_2 < \alpha_3 < \cdots < a$ and $\alpha_n < b_m$ iff $n \leq m$. Graphically, this domain looks like

$$\begin{array}{ccccccc} b_1 & & b_2 & & b_3 & & \\ \uparrow & & \uparrow & & \uparrow & & \\ \alpha_1 & \rightarrow & \alpha_2 & \rightarrow & \alpha_3 & \rightarrow \cdots & a \end{array}$$

In the following, we denote sequences and chains such as $(b_n)_{n\in\mathbf{N}}$ and $(\alpha_n)_{n\in\mathbf{N}}$ by b_* and α_* respectively.

Every open set containing a also has to contain some end of the chain α_*. Since open sets are upper, it also has to contain some end of the sequence b_* of incomparable maximal points. We abbreviate the description of this situation by saying that the point a *induces* the sequence b_* via the chain α_*.

After these preliminaries, we now introduce the actual example. Following the convention established above, we denote maximal points by Latin letters and members of approximating chains by Greek letters. The example domain contains

(1) a point a approximated by a chain α_*,

(2) a sequence b_* of incomparable points where each point b_i is approximated by a chain β_{i*},

(3) a matrix of incomparable points c_{**}, i.e. $(c_{ij})_{i\in\mathbf{N},j\in\mathbf{N}}$,

(4) a point d approximated by an uncountable number of mutually unrelated chains δ_{f*} indexed by the set of functions $f : \mathbf{N} \rightarrow \mathbf{N}$

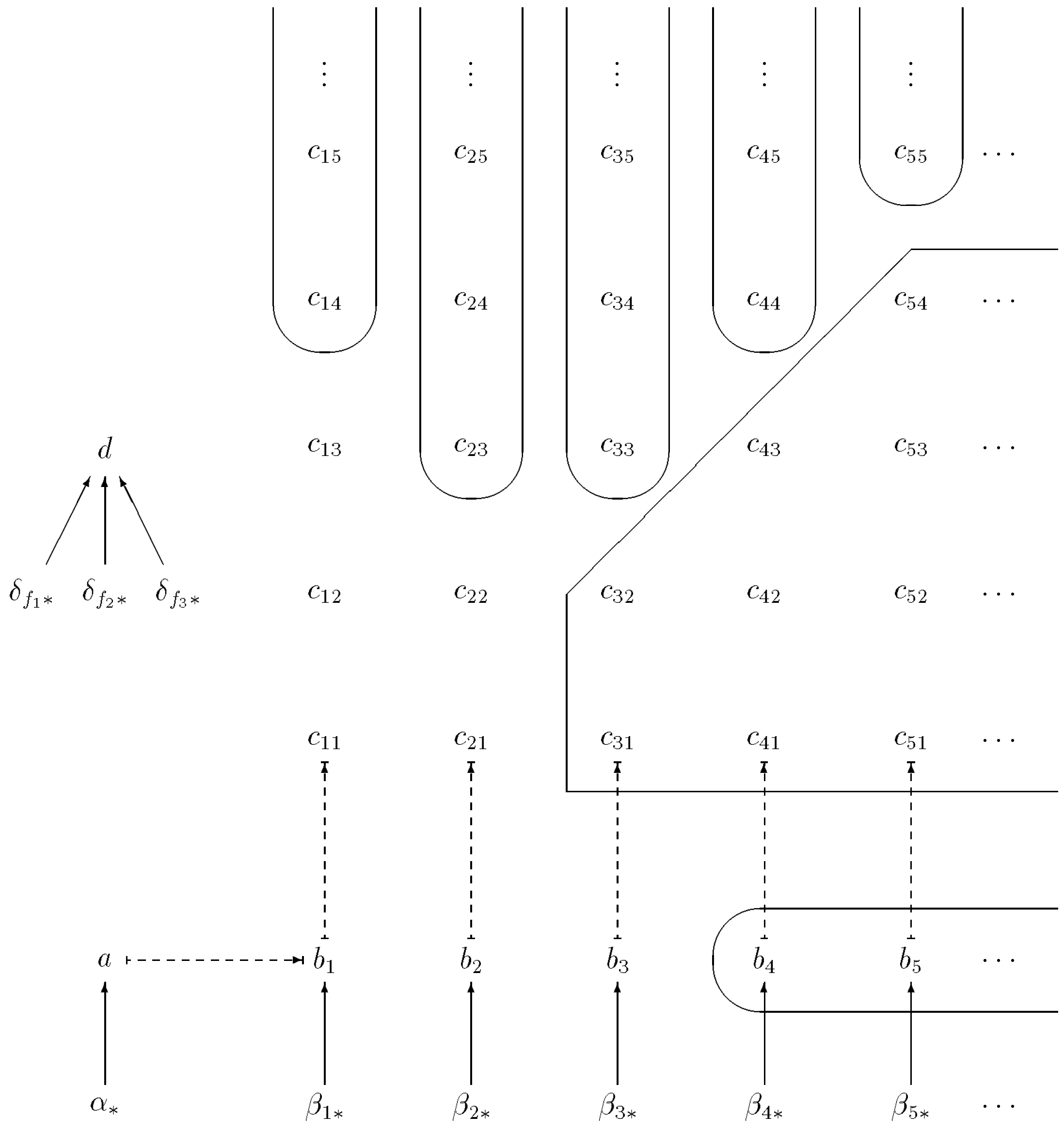
together with the following additional relationships:

(1) a induces b_* via α_*, i.e. $\alpha_k \leq b_{k'}$ whenever $k \leq k'$,

(2) every point b_i induces the sequence c_{i*} via β_{i*}, i.e. $\beta_{ik} \leq c_{ik'}$ whenever $k \leq k'$,

(3) for every function $f : \mathbf{N} \to \mathbf{N}$, the region below the graph of f is induced by d via δ_{f*}, i.e. for every $f : \mathbf{N} \to \mathbf{N}$, $\delta_{fk} \leq c_{ij}$ whenever $k \leq i$ and $j < f(i)$.

It is the double induction from a over b_* to c_{**} that is the heart of this example. The poset thus defined is visualized by the following picture:



The arrows denote the approximation of points by ascending chains. The additional relationships of the 'inductions' via the α- and β-chains are indicated by the dashed arrows. The 'inductions' via the δ-chains are not shown. In contrast, we show the induced coverings: the picture contains some sample regions contained in the upper cones $\uparrow x$ of some sample points. Regions of this principal shape make up the finite open cover constructed below.

The poset $\mathbf{X}$ introduced above is a domain since directed sets without maximal points are subsets of a single chain. The point approximated by a chain is its join since it is the only upper bound of the whole chain. This holds for all chains. Sobriety of $\mathbf{X}$ can be checked by a thorough case analysis. It implies both $\mathcal{U}_K$- and $\mathcal{U}_S$-admissibility as shown in [Hec90].

Let K be the set of maximal points of the domain $\mathbf{X}$. K consists of a, d, the b-points, and the c-points. We claim that K is compact, but not strongly compact.

Let $K \subseteq \bigcup_{i \in I} O_i$ be an open cover of K. Then a is in some open set O_0. Since a induces b_* via

α_*, some end of the sequence b_* has to be in O_0. Thus, there is a number k such that $b_i \in O_0$ for all $i \geq k$. Since the points b_i induce the sequences c_{i*} via β_{i*}, some end of all sequences c_{i*} is in O_0, too. Hence, for all $i \geq k$, there is a number $f(i)$ such that $c_{ij} \in O_0$ for all $j \geq f(i)$. The remaining b-points b_1 through b_{k-1} are in some open sets O_1 through O_{k-1}. Again, for all $i < k$, there is a number $f(i)$ such that $c_{ij} \in O_i$ for all $j \geq f(i)$. We now have covered a, all b-points, and the points c_{ij} where $j \geq f(i)$ by the open sets O_0 through O_{k-1}. The function f constructed above is now used to cover almost all remaining points. Point d is in some open set O_k, whence some end of the chain δ_{f*} is in O_k, too. Thus, there is a number l such that all points c_{ij} with $i \geq l$ and $j < f(i)$ are in O_k. Now, the open sets O_0 through O_k cover all points of K except the points c_{ij} with $i < l$ and $j < f(i)$. This is a finite number of points that is covered by a finite number of further open sets.

If K were strongly compact, then there would be a finite set F such that $K \subseteq \uparrow F$. We say that F covers a point x iff $x \in \uparrow F$. We show that a finite set F cannot cover all c-points. Let F_δ be the set of all δ-points contained in F, and similarly F_α, F_β, F_a, F_b, and F_c. The only parts of F which cover some c-points are F_δ, F_β, and F_c. Let $F_\delta = \{\delta_{f_1 k_1}, \ldots, \delta_{f_n k_n}\}$, and let $f = max\ (f_1, \ldots, f_n)$. Then F_δ can cover at most the points c_{ij} with $j < f(i)$. Let $F_\beta = \{\beta_{l_1 j_1}, \ldots, \beta_{l_m j_m}\}$, and let $l = max\ (l_1, \ldots, l_m)$. Then F_β covers at most the points c_{ij} with $i < l$. The infinitely many remaining c-points c_{ij} with $i \geq l$ and $j \geq f(i)$ cannot be covered by the finite set F_c.

For this example domain $\mathbf{X}$, we have shown that $\mathcal{U}_S\mathbf{X}$ is a proper subset of $\mathcal{U}_K\mathbf{X}$. We claim that $\mathcal{U}_S\mathbf{X}$ is enough: it contains $\emptyset$ and all 'singletons' $\uparrow x$, and is closed w.r.t. binary union and directed joins by Theorem 3.9. Moreover, it is closed w.r.t. all the further set operations introduced in [Hec90, Hec91b]. Thus, the compact, but not strongly compact sets of $\mathbf{X}$ constitute a kind of 'junk' that is not reachable by any reasonable set operation.

Unfortunately, our example domain cannot be used to decide whether the analogue of Theorem 5.4 holds for $\mathcal{U}_K$. One can show that for this specific domain $\mathbf{X}$, every additive continuous map to any admitted $\mathcal{U}_K\mathbf{Y}$ is uniquely determined by its values on the 'singletons' $\uparrow x$. The commutation $\mathcal{U}_K(\mathcal{L}\mathbf{X}) \cong \mathcal{L}(\mathcal{U}_K\mathbf{X})$ could not be checked for our example domain $\mathbf{X}$ because of its inherent complexity.

8 Conclusion

In order to generalize the notion of finitely generable sets as it occurred in [Smy78], we introduced the topological notion of strong compactness. The upper power construction $\mathcal{U}_S$ in terms of strongly compact upper sets provides an alternative to the upper construction $\mathcal{U}_K$ in terms of compact sets. Its power domains contain less junk than those of $\mathcal{U}_K$, and still admit all reasonable operations. $\mathcal{U}_S$ enjoys a host of mathematical properties: it admits unique extension of functions defined on the ground domain to additive functions on the power domain, and it commutes with the lower construction $\mathcal{L}$ on a large class of ground domains. This class cannot be further extended.

The original construction $\mathcal{U}_{f.g.}$ of [Smy78] was only defined for countably algebraic ground domains. In [Smy83], it was generalized to $\mathcal{U}_K$ defined for sober domains by Smyth himself. In [HP79] however, an initial upper power construction $\mathcal{U}_i$ was proposed that also generalized $\mathcal{U}_{f.g.}$. $\mathcal{U}_i\mathbf{X}$ is defined as the free commutative idempotent upper monoid over $\mathbf{X}$ where 'upper monoid' refers to the axiom $x+y \leq x$. For continuous ground domains, all three of $\mathcal{U}_i$, $\mathcal{U}_S$, and $\mathcal{U}_K$ coincide, but on general domains, they all differ. For $\mathcal{U}_S$ and $\mathcal{U}_K$, this is shown in section 7. The 'topological' constructions $\mathcal{U}_S$ and $\mathcal{U}_K$ have the advantage to be concretely given, whereas $\mathcal{U}_i\mathbf{X}$ is in general not constructively described. $\mathcal{U}_S$ is superior to $\mathcal{U}_K$ since it is potentially more defined (cf. Theorems 4.10 and 5.1), and its power domains contain less junk and satisfy an intrinsic universal property (Corollary 5.5) similar to that of $\mathcal{U}_i\mathbf{X}$.

We also showed that $\mathcal{U}_S$ and the lower power construction $\mathcal{L}$ commute exactly on the class of $\mathcal{U}_S$-conformal domains. We were not able to show an analogous theorem for $\mathcal{U}_K$ — except the obvious fact that $\mathcal{U}_K$ and $\mathcal{L}$ commute on multi-continuous domains since $\mathcal{U}_K$ and $\mathcal{U}_S$ coincide for these. For $\mathcal{U}_i$ however, we can do better: $\mathcal{U}_i$ and $\mathcal{L}$ commute on *all* domains as shown in [Hec91a] by algebraic methods. The two commutation theorems are independent since $\mathcal{U}_i$ and $\mathcal{U}_S$ even differ on multi-continuous domains (they are equal on continuous domains). Moreover, their proofs are performed by completely different methods.

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References

[FM90] K.E. Flannery and J.J. Martin. The Hoare and Smyth power domain constructors commute under composition. *Journal of Computer and System Sciences*, 40:125–135, 1990.

[GHK+80] G. Gierz, K.H. Hofmann, K. Keimel, J. D. Lawson, M. Mislove, and D. S. Scott. *A Compendium of Continuous Lattices*. Springer-Verlag, 1980.

[GLS83] G. Gierz, J.D. Lawson, and A. Stralka. Quasicontinuous posets. *Houston Journal of Mathematics*, 9:191–208, 1983.

[Hec90] R. Heckmann. *Power Domain Constructions*. PhD thesis, Universität des Saarlandes, 1990.

[Hec91a] R. Heckmann. Lower and upper power domain constructions commute on all cpos. *Information Processing Letters*, 40(1):7–11, 1991.

[Hec91b] R. Heckmann. Power domain constructions. *Science of Computer Programming*, 1991. to appear.

[HP79] M.C.B Hennessy and G.D. Plotkin. Full abstraction for a simple parallel programming language. In J. Becvar, editor, *Foundations of Computer Science*, pages 108–120. *Lecture Notes in Computer Science 74*, Springer-Verlag, 1979.

[Joh81] P. Johnstone. Scott is not always sober. In Banaschewski and Hoffmann, editors, *Continuous Lattices, Bremen 1979. Lecture Notes in Mathematics 871*, Springer-Verlag, 1981.

[Jun88] A. Jung. *Cartesian Closed Categories of Domains*. PhD thesis, FB Mathematik, Technische Hochschule Darmstadt, 1988.

[Law88] J.D. Lawson. The versatile continuous order. In Michael G. Main, A. Melton, Michael Mislove, and D. Schmidt, editors, *Mathematical Foundations of Programming Language Semantics (MFPLS '87)*, pages 565–622. *Lecture Notes in Computer Science 298*, Springer-Verlag, 1988.

[Plo76] G.D. Plotkin. A powerdomain construction. *SIAM Journal on Computing*, 5(3):452–487, 1976.

[Rud80] Mary E. Rudin. Directed sets which converge. In McAuley and Rao, editors, *General Topology and Modern Analysis (Riverside)*, pages 305–307, 1980.

[Smy78] M.B. Smyth. Power domains. *Journal of Computer and System Sciences*, 16:23–36, 1978.

[Smy83] M.B. Smyth. Power domains and predicate transformers: A topological view. In J. Diaz, editor, *ICALP '83*, pages 662–676. *Lecture Notes in Computer Science 154*, Springer-Verlag, 1983.