Similarity, Topology, and Uniformity

Dedicated to Dieter Spreen on the occasion of his 60th birthday

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Abstract

We generalize various notions of generalized metrics even further to one general concept comprising them all. For convenience, we turn around the ordering in the target domain of the generalized metrics so that we speak of similarity instead of distance. Starting from an extremely general situation without axioms, we examine which axioms or additional properties are needed to obtain useful results. For instance, we shall see that commutativity and associativity of the generalized version of addition occurring in the triangle inequality is not really needed, nor do we require a generalized version of subtraction.

Each similarity space comes with its own domain of possible similarity values. Therefore, we consider non-expanding functions modulo some rescaling between different domains of similarity values. We show that non-expanding functions with locally varying rescaling functions correspond to topologically continuous functions, while non-expanding functions with a globally fixed rescaling generalize uniformly continuous functions.

Key words: Generalized metric, topology, quasi-uniformity, continuous lattices

1 Introduction

Metric spaces have been generalized in many different ways: symmetry has been dropped, self-distances need not be 0, and the target domain of the distance function has been generalized from $\mathbb{R}^+$ to more general domains (a survey of some classes of generalized metrics can be found in Section 2). We generalize these generalizations even further to a state without axioms and with arbitrary topological $\mathcal{T}_0$ spaces S as possible target domains. For convenience, we order these target domains by their specialization relation, which

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corresponds to the opposite of the usual ordering in case of $\mathbb{R}^+$ (see Section 2.5). Therefore, we speak of similarity instead of distance (if a point x moves toward a point y, the distance between x and y shrinks, but their similarity increases). We then pursue the two following main goals:

- We study the properties of a single similarity space to find out which hypotheses are needed to prove results known from more familiar classes of generalized metrics. For instance, to show that the so-called open balls are really open, a weak form of triangle inequality is sufficient in which the binary operation that takes over the role of addition is not required to be commutative or associative and may even vary dependent on the "middle point" of the triangle (see Section 6). Moreover, we never need an analogue of subtraction in this paper (an inverse, partial inverse, or adjoint of addition).
- We study categories of similarity spaces in which each space can have its own target domain of possible similarity values. The morphisms are some analogues of non-expanding functions, but modulo some rescaling that is needed to compare the similarity values of different spaces (see Section 5). We call such morphisms globally continuous if a globally fixed rescaling is used, and locally continuous if there may be different rescalings at different points of the space. In Section 10 and Section 11 we show how to characterize these functions differently without using rescalings: locally continuous functions correspond to topologically continuous functions, while globally continuous functions generalize uniformly continuous functions.

Section 2 presents some known classes of generalized metrics and the motivation for switching from distance to similarity. Section 3 contains some background material: a brief introduction to topological spaces and the more general and less familiar neighborhood spaces. Section 4 introduces generalized similarity systems and their possible properties such as being symmetric or self-uniform (having a uniform value for self-similarities). We also define left and right pre-open balls and the neighborhood structures derived from them. Preopen balls generalize the usual open balls, but are not necessarily open without further axioms.

Globally and locally continuous functions are defined in Section 5 as generalizations of non-expanding functions modulo some global or local rescaling of the similarity values. Section 6 then studies generalizations of the triangle inequality that are motivated by the wish to stay as general as possible, but to be able to conclude that pre-open balls are open, and hence their induced neighborhood structures are topologies. This leads to the notions of locally and globally transitive similarity systems; a globally transitive system has a single operation playing the role of addition in the triangle inequality, while a locally transitive system has a possibly different operation for each "middle point" of the triangle. Section 7 shows how the familiar notions of generalized metrics induce similarity systems, and how rescalings act in these familiar cases.

In the later sections, the possible domains of similarity values are restricted to be continuous lattices, and various powerful theorems are proved with domain-theoretic methods in this special case. Section 8 is another background section, presenting material on algebraic and continuous lattices, including a generalization of the well-known injectivity property of continuous lattices. This property is used several times in the remainder of the paper to obtain suitable rescalings for various purposes. First, similarity spaces are introduced as equivalence classes of similarity systems in Section 9.

The categories of locally and globally transitive similarity spaces with *locally* continuous functions are related to bitopological spaces and pairwise continuous functions in Section 10. Every locally transitive similarity space induces a bitopological space such that local continuity is equivalent to pairwise continuity. Conversely, every bitopological space is induced by some globally transitive similarity space. In case of symmetry, the two induced topologies are identical, and the prefix "bi" can be dropped in these statements.

Finally the category of globally transitive similarity spaces with *globally* continuous functions is related to generalizations of uniform spaces and uniformly continuous functions in Section 11. In particular, self-uniform globally transitive similarity spaces correspond to quasi-uniform spaces, and symmetric self-uniform globally transitive similarity spaces to uniform spaces. In these cases, global continuity of functions w.r.t. similarity spaces is equivalent to uniform continuity w.r.t. the corresponding (quasi-)uniform spaces. Section 12 contains a conclusion and ideas for future work.

Notational conventions. The composition of $f : X \to Y$ and $g : Y \to Z$ is $g \circ f = (x \mapsto g(fx)) : X \to Z$. For a given function $f : X \to Y$, we denote the image of a set $A \subseteq X$ by $f^+A = \{fx \mid x \in A\}$, and the inverse image of a set $B \subseteq Y$ by $f^-B = \{x \in X \mid fx \in B\}$.

2 Generalized Metrics and Similarities

We first recall some notions of generalized metrics known from the literature. For the beginning, let X be a set (the set of points) and $\delta : X \times X \to \mathbb{R}^+ = \{r \in \mathbb{R} \mid r \geq 0\}$ (a distance function).

2.1 Metric and Pseudo-Metric

The distance function δ is a *pseudo-metric* if it satisfies the following axioms:

(S0) $\delta(x, x) = 0$ (self-distances are 0);

(Sym)	$\delta(x,y) \;=\; \delta(y,x)$	(symmetry);
(Tr_+)	$\delta(x,z) \;\le\; \delta(x,y) + \delta(y,z)$	(triangle inequality).

A *metric* additionally satisfies the following separation property:

$(\mathrm{Sep_M})$ $\quad \delta(x,y) \;=\; 0 \;\Rightarrow\; x = y.$

For a (pseudo-)ultrametric, (Tr_+) is strengthened to

$(\mathrm{Tr}_\vee)$ $\quad \delta(x,z) \;\le\; \delta(x,y) \vee \delta(y,z)$

where '$\vee$' denotes maximum in $\mathbb{R}^+$.

For every x in X and $r > 0$, the *open ball* $\mathrm{B}(x,r)$ about x with radius r is defined as $\mathrm{B}(x,r) = \{y \in X \mid \delta(x,y) < r\}$. The *induced topology* is defined by saying that a set $U \subseteq X$ is *open* if for all $x \in U$ there is $r > 0$ such that $x \in \mathrm{B}(x,r) \subseteq U$. The open balls form a base for the induced topology.

2.2 Quasi-Metric and Pseudo-Quasi-Metric

A *pseudo-quasi-metric* is a distance function that satisfies all properties of a pseudo-metric except for symmetry. Thus the axioms of a pseudo-quasi-metric are (S0) and (Tr_+). Open balls and the induced topology are defined exactly in the same way as for pseudo-metrics. A difference is that the induced topology may have a non-symmetric specialization preorder, namely

$$x \le y \;\Leftrightarrow\; \delta(x,y) = 0.$$

Originally, the separation property $(\mathrm{Sep_M})$ of metrics was used literally for quasi-metrics. Later, it was often weakened to

$(\mathrm{Sep_Q})$ $\quad \delta(x,y) \;=\; \delta(y,x) \;=\; 0 \;\Rightarrow\; x = y.$

The stronger property $(\mathrm{Sep_M})$ is equivalent to the $\mathcal{T}_1$ property of the induced topology, while $(\mathrm{Sep_Q})$ is equivalent to $\mathcal{T}_0$.

2.3 Pseudo-Partial Metric and Partial Metric

Here, symmetry is retained, but self-distances may be non-zero, and a corresponding correction term is introduced in the triangle inequality. Usually, it is still required that self-distances be not larger than distances to other points. The resulting axioms for pseudo-partial metrics are the following:

(SSD)	$\delta(x,x) \;\le\; \delta(x,y)$	(small self-distances);
(Sym)	$\delta(x,y) \;=\; \delta(y,x)$	(symmetry);
$(\mathrm{Tr_P})$	$\delta(x,z) \;\le\; \delta(x,y) + \delta(y,z) - \delta(y,y)$	(modified triangle inequality).

A generalization without (SSD) was considered in [5]. The additional term $-\,\delta(y, y)$ in ($\mathrm{Tr_P}$) is usually not further motivated (see also Section 7.3).

The induced topology is based on open balls as usual. Its specialization preorder is

$$x \le y \;\Leftrightarrow\; \delta(x, x) = \delta(x, y).$$

Hence, the induced topology is $\mathcal{T}_0$ iff

($\mathrm{Sep_P}$) $\quad \delta(x, x) \;=\; \delta(x, y) \;=\; \delta(y, y) \;\Rightarrow\; x = y.$

This property is taken as the separation property for partial metrics.

2.4 Generalized Value Domains

All the approaches presented above use $\mathbb{R}^+$ as the target domain of the distance function. A modest generalization is to use $[0, \infty]$ instead, as for instance proposed in [1] for pseudo-quasi-metrics (under the name "generalized metric spaces"). O'Neill [10,11] proposes to extend the value domain for partial metrics to $\mathbb{R}$, thus allowing negative distances. More substantial generalizations are $[0, \infty)^I$, $[0, \infty]^I$, or $[0, 1]^I$ for some index set I.

Value lattices and value quantales are even more general. These two related concepts are defined in [8] as special complete lattices V with a commutative monoid structure $(V, +, 0)$ whose neutral element 0 is the least element of V. Addition $+$ has to preserve arbitrary infima to get an adjoint that takes over the role of subtraction. For a value lattice, V is merely required to be the opposite of a continuous lattice. For a value quantale, V has to be completely distributive in such a way that the set of elements well-above 0 is filtered. This condition is very restrictive; it rules out powers $[0, \infty]^I$ of $[0, \infty]$ with more than one component (such powers are value lattices, however). Value quantales are also studied in [3] and [13]. The so-called value quantales in [2] are however value lattices in the terminology of [8].

2.5 Distance vs. Similarity

The induced topology of a metric, pseudo-metric, pseudo-quasi-metric, or pseudo-partial metric is defined via open balls of the form $\mathrm{B}(x, r) = \{y \in X \mid \delta(x, y) < r\}$. An open ball $\mathrm{B}(x, r)$ can also be written as $\{y \in X \mid \delta(x, y) \in U\}$ where $U = [0, r) = \{r' \mid r' < r\}$ is a "co-Scott-open" set of $\mathbb{R}^+$, i.e. a Scott-open set in the *opposite* ordering of $\mathbb{R}^+$. This points to a notational difficulty: while the triangle inequality and the (SSD) axiom of pseudo-partial metrics are naturally employing the usual ordering of $\mathbb{R}^+$, a topological and/or domain-theoretical approach would more naturally employ the opposite ordering of $\mathbb{R}^+$. This point becomes more prominent when generalized value domains are considered; recall that value lattices are *opposites* of continuous lattices (with additional properties).

Therefore, we turn around the orderings of the value domains in this paper. Then we should not speak of distance δ, but of similarity σ: If a point x moves towards a point y, then the distance $\delta(x, y)$ between x and y gets smaller, but the similarity of x and y gets larger.

Of course, distances and similarities are two views of the same thing. In this paper, I prefer to use the similarity view since I will use domain-theoretic methods and hence prefer to work with Scott-open sets and continuous lattices instead of co-Scott-open sets and the opposites of continuous lattices. I also believe that the similarity view is more natural in the generalized setting that I consider in this paper. A slight drawback is that the familiar look of the triangle inequality is lost when '$\leq$' is replaced by '$\geq$'.

3 Topological Spaces and Neighborhood Spaces

Our similarity systems will be so general that their induced "topology" actually will merely be a neighborhood structure. Since neighborhood spaces are less widely known than topological spaces, we include here a brief introduction of both concepts, concentrating on their mutual relationship.

A *topology* τ on a set X is a set of subsets of X closed under arbitrary union and finite intersection. The sets in τ are called *open sets*. A *topological space* (X, τ) is a set X with a topology τ on X. A *base* of (X, τ) is a subset $\mathcal{B}$ of τ such that for all O in τ and x in O, there is some B in $\mathcal{B}$ with $x \in B \subseteq O$. A topological space is *countably based* if it has a countable base.

Given two topological spaces (X, τ_X) and (Y, τ_Y), a function $f : X \to Y$ is *continuous* if the inverse image of every open set of Y is open in X ($V \in \tau_Y \Rightarrow f^- V \in \tau_X$). This defines the category Top of topological spaces.

A *neighborhood space* $(X, \mathcal{N})$ is a set X with an assignment of a *neighborhood filter* $\mathcal{N}(x) \subseteq \mathcal{P}X$ to every x of X; the elements of $\mathcal{N}(x)$ are called *neighborhoods* of X. The axioms of a neighborhood filter $\mathcal{N}(x)$ are the following:

- $\mathcal{N}(x)$ is upward closed: $A \in \mathcal{N}(x)$, $A \subseteq A' \Rightarrow A' \in \mathcal{N}(x)$;
- $\mathcal{N}(x)$ is closed under finite intersection;
- all neighborhoods of x contain x: $A \in \mathcal{N}(x) \Rightarrow x \in A$.

Given two neighborhood spaces $(X, \mathcal{N}_X)$ and $(Y, \mathcal{N}_Y)$, a function $f : X \to Y$ is *continuous* if $B \in \mathcal{N}_Y(fx)$ implies $f^- B \in \mathcal{N}_X(x)$, or equivalently, if for every B in $\mathcal{N}_Y(fx)$ there is $A \in \mathcal{N}_X(x)$ such that $f^+ A \subseteq B$. This defines the category Nbh of neighborhood spaces.

Neighborhood spaces are a generalization of topological spaces in the following sense: every topological space (X, τ) defines a neighborhood space $(X, \mathcal{N}_\tau)$ by saying that A is in $\mathcal{N}_\tau(x)$ iff there is an O in τ such that $x \in O \subseteq A$. A function $f : (X, \tau_X) \to (Y, \tau_Y)$ is Top-continuous if and only if $f : (X, \mathcal{N}_{\tau_X}) \to (Y, \mathcal{N}_{\tau_Y})$ is Nbh-continuous. Thus, the above construction provides a full embedding of Top into Nbh.

For the opposite direction, define a subset O of a neighborhood space to be open if it is a neighborhood of all its elements ($x \in O$ implies $O \in \mathcal{N}(x)$). These open sets do form a topology $\tau_\mathcal{N}$, and this topology is the original topology if $\mathcal{N}$ was induced by a topology ($\tau_{\mathcal{N}_\tau} = \tau$ always holds). On the other hand, this topology does not always give back the original neighborhood structure (in general, $\mathcal{N}_{\tau_\mathcal{N}}$ is different from $\mathcal{N}$).

An *open base* of a neighborhood space $(X, \mathcal{N})$ is a subset $\mathcal{B}$ of $\mathcal{P}X$ such that

(1) All B in $\mathcal{B}$ are open: $\forall B \in \mathcal{B}\ \forall x \in B\colon\ B \in \mathcal{N}(x)$;
(2) For $A \in \mathcal{N}(x)$, there is B in $\mathcal{B}$ such that $x \in B \subseteq A$.

Not every neighborhood space has an open base. In fact, the following are equivalent for $(X, \mathcal{N})$:

(1) $(X, \mathcal{N})$ has an open base;
(2) $(X, \mathcal{N})$ is topological, i.e. there is a topology τ on X such that $\mathcal{N} = \mathcal{N}_\tau$;
(3) $\mathcal{N} = \mathcal{N}_{\tau_\mathcal{N}}$.

In this case, $\mathcal{B}$ is a base of $\tau_\mathcal{N}$ in the topological sense.

A useful relation to consider in a general topological space (X, τ) is the *specialization preorder* defined by $x \leq_\tau x'$ if for all $O \in \tau$, $x \in O$ implies $x' \in O$. Clearly, this *is* a preorder (a reflexive and transitive relation), and continuous functions preserve that preorder.

The corresponding relation for a neighborhood space $(X, \mathcal{N})$ is the *specialization relation* defined by $x \to_\mathcal{N} x'$ if $A \in \mathcal{N}(x)$ implies $x' \in A$. This relation is still reflexive, but transitivity is lost in general. Nevertheless, it is still preserved by continuous functions ($x \to_{\mathcal{N}_X} x' \Rightarrow fx \to_{\mathcal{N}_Y} fx'$), and it agrees with the specialization preorder on topological spaces, i.e. $x \to_{\mathcal{N}_\tau} x' \Leftrightarrow x \leq_\tau x'$.

4 Generalized Similarity Systems and Their Neighborhood Structures

4.1 Definition of Generalized Similarity Systems

As shown in Section 2, the classical notion of metric space has been generalized in many directions: axioms such as symmetry and $\delta(x, x) = 0$ have been dropped, and the domain of distance values has been generalized from $\mathbb{R}^+$ to more general structures. For our similarities, we follow this line to the end by dropping *all* axioms (at least initially) and admitting arbitrary $\mathcal{T}_0$ spaces as domains of similarity values.

Definition 1 *A* generalized similarity system *or shortly* gss *is a tuple* $\mathcal{X} = (X, S, \sigma)$ *where* $X = |\mathcal{X}|$ *is a set (the set of points),* $S = S_{\mathcal{X}}$ *is a* $\mathcal{T}_0$ *topological space (the value domain), and* $\sigma = \sigma_{\mathcal{X}} : X \times X \to S$ *is a function (the similarity function).*

The term "generalized" refers to the fact that S is an arbitrary $\mathcal{T}_0$ space; later, we shall restrict S to be a continuous lattice. We could have done this from the beginning, coming closer to the value lattices mentioned in Section 2.4, but many results hold in the more general setting of arbitrary $\mathcal{T}_0$ topological spaces. The usage of "system" instead of "space" will become clear in Section 9.2.

Definition 2 *The* opposite $(X, S, \sigma)^{\mathrm{op}}$ *of* (X, S, σ) *is* $(X, S, \sigma^{\mathrm{op}})$ *with* $\sigma^{\mathrm{op}}(x, x') = \sigma(x', x)$.

When we want to compare values in S, we use the *specialization preorder* induced by the topology of S, i.e. $s \leq s'$ iff $s \in u \Rightarrow s' \in u$ for all open sets u of S. Metrics, pseudo-metrics, quasi-metrics, and partial metrics induce generalized similarity systems with $S = \mathbb{R}^+$, topologized with the co-Scott topology, i.e. open sets of the form $[0, r)$. The specialization preorder induced by this topology is the *opposite* of the natural order on $\mathbb{R}^+$. This is the reason why we are drawn to the similarity view.

The various kinds of metrics have special properties: metrics and partial metrics are symmetric in the sense $\delta(x, x') = \delta(x', x)$; metrics and quasi-metrics satisfy $\delta(x, x) = 0$, i.e. there is a uniform value 0 for self-distances, and partial metrics have "small self-distances" $\delta(x, x) \leq \delta(x, x')$. With the necessary order reversal in the last case, these properties motivate the following definition:

Definition 3 *A generalized similarity system* (X, S, σ) *is* symmetric *if* $\sigma(x, x') = \sigma(x', x)$ *holds for all* $x, x' \in X$. *It is* self-uniform *if there is a uniform value for self-similarities, i.e. if* $\sigma(x, x) = \sigma(x', x')$ *holds for all* $x, x' \in X$. *It is* selfish *if each point is more similar to itself than to other points, i.e.* $\sigma(x, x') \leq \sigma(x, x)$ *and* $\sigma(x, x') \leq \sigma(x', x')$ *hold for all* $x, x' \in X$.

Clearly, if a gss $\mathcal{X}$ is symmetric / self-uniform / selfish, then its opposite $\mathcal{X}^{\mathrm{op}}$ has the same property, and $\mathcal{X}$ is symmetric iff $\mathcal{X}^{\mathrm{op}} = \mathcal{X}$.

4.2 The Induced Neighborhoods

We first define the "pre-open" balls of a gss in analogy to the open balls of a (generalized) metric space. We call them pre-open balls since they are not necessarily open without further axioms. As hinted at in Section 2.5, we use open sets of S as the "radii" of the pre-open balls. Since similarities are not symmetric in general, there are actually two kinds of pre-open balls.

Definition 4 *Let (X, S, σ) be a generalized similarity system, and let x be a point of X and u an open set of S. The* right pre-open ball $\mathrm{B}^{\mathrm{R}}(x,u)$ *about x with radius u is defined as* $\mathrm{B}^{\mathrm{R}}(x,u) = \{x' \in X \mid \sigma(x,x') \in u\}$*. The corresponding* left pre-open ball *is* $\mathrm{B}^{\mathrm{L}}(x,u) = \{x' \in X \mid \sigma(x',x) \in u\}$.

Note that, unlike in the (quasi-)metric case, the condition $x \in \mathrm{B}^{\mathrm{R}}(x,u)$, i.e. $\sigma(x,x) \in u$, is not automatically satisfied. At least, in selfish gss non-empty pre-open balls contain their center (if $\sigma(x,x') \in u$, then $\sigma(x,x) \in u$).

We could now proceed as in the metric case by defining that a set $U \subseteq X$ is right open if for all x in U there is an open set u of S such that $x \in \mathrm{B}^{\mathrm{R}}(x,u) \subseteq U$. These open sets would form a topology, but the pre-open balls would not be open in general, and not much could be said about this topology. Thus, we prefer to define neighborhoods instead.

Definition 5 *A set $A \subseteq X$ is a right neighborhood of a point x if there is an open set u of S such that $x \in \mathrm{B}^{\mathrm{R}}(x,u) \subseteq A$. The resulting neighborhood filter is called $\mathcal{N}^{\mathrm{R}}(x)$, and the neighborhood space $(X, \mathcal{N}^{\mathrm{R}})$ is called* $\mathrm{N}^{\mathrm{R}}(X,S,\sigma)$*. Left neighborhoods, $\mathcal{N}^{\mathrm{L}}(x)$, and $\mathrm{N}^{\mathrm{L}}(X,S,\sigma)$ are defined analogously.*

The right topology mentioned before Def. 5 is a posteriori obtained as the topology induced by $\mathcal{N}^{\mathrm{R}}$ as described in Section 3. If the right pre-open balls are right open, they form an open basis in the sense of Section 3, and the right neighborhood space is actually a topological space. In Section 6, we present some axioms that include the familiar generalized metric cases and ensure that the pre-open balls are open. Section 6.4 contains an example in which the pre-open balls are *not* open, and the induced neighborhood spaces are *not* topological.

With the help of the two unary functions $\sigma^{\mathrm{R}}_x = (x' \mapsto \sigma(x,x')) : X \to S$ and $\sigma^{\mathrm{L}}_x = (x' \mapsto \sigma(x',x)) : X \to S$, the pre-open balls can be written as inverse images: $\mathrm{B}^{\mathrm{R}}(x,u) = \sigma^{\mathrm{R}\,-}_x u$ and $\mathrm{B}^{\mathrm{L}}(x,u) = \sigma^{\mathrm{L}\,-}_x u$. In the symmetric case, $\sigma^{\mathrm{R}}_x = \sigma^{\mathrm{L}}_x$ holds for all x in X, whence $\mathrm{B}^{\mathrm{R}}(x,u) = \mathrm{B}^{\mathrm{L}}(x,u)$, and therefore the right and left neighborhood structures coincide. In this case, the qualifiers R and L can be omitted. In the general case, switching to the opposite gss interchanges the two neighborhood structures: $\mathrm{N}^{\mathrm{R}}(\mathcal{X}^{\mathrm{op}}) = \mathrm{N}^{\mathrm{L}}\mathcal{X}$ and $\mathrm{N}^{\mathrm{L}}(\mathcal{X}^{\mathrm{op}}) = \mathrm{N}^{\mathrm{R}}\mathcal{X}$.

4.3 Examples

Let $X = \{0,1\}$ be a two-point set, and $\mathbb{S}$ Sierpinski space, i.e. the two-point set $\{0,1\}$ with $\{1\}$ as the only non-trivial open set. Consider the following two similarity functions $\sigma^s, \sigma^u : X \times X \to \mathbb{S}$:

σ^s	0	1
0	0	0
1	0	1

σ^u	0	1
0	1	1
1	0	1

As indicated by the superscripts, $\mathbb{S}^s = (X, \mathbb{S}, \sigma^s)$ is symmetric (but not self-uniform), and $\mathbb{S}^u = (X, \mathbb{S}, \sigma^u)$ is self-uniform (but not symmetric). Both systems are selfish since the specialization preorder of $\mathbb{S}$ is $0 < 1$.

In $\mathbb{S}^s$, we have $\mathrm{B}(1, \{1\}) = \{1\}$, whence $\{1\}$ is a (left and right) neighborhood of 1. On the other hand, $\mathrm{B}(0, \{1\}) = \emptyset$ and $\mathrm{B}(0, \{0, 1\}) = \{0, 1\}$, whence the smallest neighborhood of 0 is $\{0, 1\}$. These neighborhoods are open so that the resulting neighborhood space is topological; it actually is Sierpinski space: $\mathrm{N^R}\,\mathbb{S}^s = \mathrm{N^L}\,\mathbb{S}^s = \mathbb{S}$.

In $\mathbb{S}^u$, we have $\mathrm{B^R}(1, \{1\}) = \{1\}$ and $\mathrm{B^R}(0, \{1\}) = \{0, 1\}$, whence $\mathrm{N^R}\,\mathbb{S}^u = \mathbb{S}$, too. Yet $\mathrm{B^L}(1, \{1\}) = \{0, 1\}$ and $\mathrm{B^L}(0, \{1\}) = \{0\}$, whence $\{0\}$ is left open, but $\{1\}$ is not, i.e. $\mathrm{N^L}\,\mathbb{S}^u = \mathbb{S}^{\mathrm{op}}$, the opposite Sierpinski space ($\mathbb{S}$ with 0 and 1 interchanged).

Both $\mathbb{S}^s$ and $\mathbb{S}^u$ are instances of larger families of examples (see Section 10.4 for $\mathbb{S}^s$ and 6.4 for $\mathbb{S}^u$). Other examples relating to various kinds of generalized metrics are presented in Section 7.

5 Uniformly/Globally/Locally Continuous Functions

Guidelines for defining morphisms between generalized similarity systems are given by the definitions of uniformly continuous functions and non-expanding functions between metric spaces.

5.1 Uniformly Continuous Functions

A function $f : (X, \delta_X) \to (Y, \delta_Y)$ between metric spaces is uniformly continuous if for all $s > 0$ there is $r > 0$ such that $\delta_X(x_1, x_2) < r \Rightarrow \delta_Y(fx_1, fx_2) < s$. When this is translated to our gss setting, the condition $a < r$ is written as $a \in [0, r)$ where $[0, r)$ is a non-empty Scott-open set of $\mathbb{R}^{+\mathrm{op}}$. Such open sets contain all self-distances $\delta_X(x, x) = 0$. When generalizing this to arbitrary gss, and one has to take into account that the self-similarities are not necessarily contained in all non-empty open sets. This leads to the following definition:

Definition 6 *Let $\mathcal{X} = (X, S_{\mathcal{X}}, \sigma_{\mathcal{X}})$ and $\mathcal{Y} = (Y, S_{\mathcal{Y}}, \sigma_{\mathcal{Y}})$ be two generalized similarity systems. By a function $f : \mathcal{X} \to \mathcal{Y}$, we mean a function $f : X \to Y$. Such a function is* uniformly continuous (UC) *w.r.t. $\mathcal{X}$ and $\mathcal{Y}$ if for every x in X and open v of $S_{\mathcal{Y}}$ containing $\sigma_{\mathcal{Y}}(fx, fx)$, there is an open u of $S_{\mathcal{X}}$ containing $\sigma_{\mathcal{X}}(x, x)$ with the property $\sigma_{\mathcal{X}}(x_1, x_2) \in u \Rightarrow \sigma_{\mathcal{Y}}(fx_1, fx_2) \in v$.*

Proposition 7 *Every uniformly continuous function is continuous w.r.t. both neighborhood structures.*

PROOF. For continuity w.r.t. the right neighborhood structures, assume $fx \in \mathrm{B}^{\mathrm{R}}(fx, v)$ for some open v of $S_{\mathcal{Y}}$. This means $\sigma_{\mathcal{Y}}(fx, fx) \in v$, and thus uniform continuity of f yields an open u of $S_{\mathcal{X}}$ such that $\sigma_{\mathcal{X}}(x, x) \in u$, and $\sigma_{\mathcal{X}}(x_1, x_2) \in u \Rightarrow \sigma_{\mathcal{Y}}(fx_1, fx_2) \in v$. The first of these two properties means $x \in \mathrm{B}^{\mathrm{R}}(x, u)$, and when specialized to $x_1 = x$, the second property implies $\mathrm{B}^{\mathrm{R}}(x, u) \subseteq f^{-}(\mathrm{B}^{\mathrm{R}}(fx, v))$.

Continuity w.r.t. the left neighborhood structures then also holds because the definition of uniform continuity is symmetric. $\square$

5.2 Non-Expanding Functions Modulo Rescaling

We also want to define analogues of non-expanding functions. For each $\mathcal{T}_0$ topological space S, one could set up a category of S-systems (X, S, σ), i.e. systems using this S as the space of similarity values. This is done for instance in [2,3] for value lattices or value quantales V, with non-expanding functions as the morphisms ($\delta_Y(fx, fx') \le \delta_X(x, x')$). In the similarity view, we would take functions that increase similarity, i.e. a morphism $f : \mathcal{X} \to \mathcal{Y}$ would be a function $f : |\mathcal{X}| \to |\mathcal{Y}|$ satisfying $\sigma_{\mathcal{Y}}(fx, fx') \ge \sigma_{\mathcal{X}}(x, x')$.

Yet, as suggested by the notation (X, S, σ), we want to supply each system with its own space S of similarity values. To characterize morphisms $f : \mathcal{X} \to \mathcal{Y}$, we need a kind of rescaling $\varphi : S_{\mathcal{X}} \to S_{\mathcal{Y}}$ to compare the similarity values of $\mathcal{X}$ with those of $\mathcal{Y}$. To take the structures of $S_{\mathcal{X}}$ and $S_{\mathcal{Y}}$ as topological spaces into account, it is reasonable to require that φ be continuous. Thus our first proposal is to define that $f : \mathcal{X} \to \mathcal{Y}$ is a morphism if there is a continuous "rescaling" $\varphi : S_{\mathcal{X}} \to S_{\mathcal{Y}}$ such that $\sigma_{\mathcal{Y}}(fx, fx') \ge \varphi(\sigma_{\mathcal{X}}(x, x'))$. Yet this definition is too weak; if $S_{\mathcal{Y}}$ has a least element $\perp$, then $\varphi = (a \mapsto \perp)$ would prove that all functions $f : |\mathcal{X}| \to |\mathcal{Y}|$ are morphisms.

A natural requirement is that morphisms be continuous w.r.t. the left and right neighborhood structures. We thus look for conditions on $f : |\mathcal{X}| \to |\mathcal{Y}|$ and $\varphi : S_{\mathcal{X}} \to S_{\mathcal{Y}}$ ensuring that $f^{-}V \in \mathcal{N}^{\mathrm{R}}_{\mathcal{X}}(x)$ for every $V \in \mathcal{N}^{\mathrm{R}}_{\mathcal{Y}}(fx)$. Given $V \in \mathcal{N}^{\mathrm{R}}_{\mathcal{Y}}(fx)$, we must find an open u of $S_{\mathcal{X}}$ such that $x \in \mathrm{B}^{\mathrm{R}}(x, u) \subseteq f^{-}V$. Yet $V \in \mathcal{N}^{\mathrm{R}}_{\mathcal{Y}}(fx)$ means there is an open v of $S_{\mathcal{Y}}$ such that $fx \in \mathrm{B}^{\mathrm{R}}(fx, v) \subseteq V$. Using continuity of φ, a natural candidate for u is $\varphi^{-}v$. Thus we need conditions ensuring that $x \in \mathrm{B}^{\mathrm{R}}(x, \varphi^{-}v) \subseteq f^{-}V$ under the hypothesis $fx \in \mathrm{B}^{\mathrm{R}}(fx, v) \subseteq V$, which certainly holds if

$$\mathrm{B}^{\mathrm{R}}(x, \varphi^{-}v) \subseteq f^{-}(\mathrm{B}^{\mathrm{R}}(fx, v)) \tag{1}$$

and

$$fx \in \mathrm{B}^{\mathrm{R}}(fx, v) \;\Rightarrow\; x \in \mathrm{B}^{\mathrm{R}}(x, \varphi^{-}v)\,. \tag{2}$$

Condition (1) is equivalent to $\varphi(\sigma_{\mathcal{X}}(x,x')) \in v \;\Rightarrow\; \sigma_{\mathcal{Y}}(fx,fx') \in v$. This condition for all opens v of $S_{\mathcal{Y}}$ is equivalent to

$$\varphi(\sigma_{\mathcal{X}}(x,x')) \;\leq\; \sigma_{\mathcal{Y}}(fx,fx') \tag{3}$$

in the specialization preorder of $S_{\mathcal{Y}}$, which is the same condition as we proposed above.

Condition (2) is equivalent to

$$\sigma_{\mathcal{Y}}(fx,fx) \in v \;\Rightarrow\; \varphi(\sigma_{\mathcal{X}}(x,x)) \in v.$$

This condition for all opens v of $S_{\mathcal{Y}}$ is equivalent to

$$\sigma_{\mathcal{Y}}(fx,fx) \;\leq\; \varphi(\sigma_{\mathcal{X}}(x,x)) \tag{4}$$

in the specialization preorder of $S_{\mathcal{Y}}$. Together with $\geq$ coming from (3), we get equality in (4) since $S_{\mathcal{Y}}$ is a $\mathcal{T}_0$-space.

5.3 Definition of GC and LC Functions

The two conditions (3) and (4) that imply continuity w.r.t. the right neighborhoods are symmetric and thus also guarantee continuity w.r.t. the left neighborhoods. Yet one may wonder why there should be a single rescaling φ; there could be different rescalings for different parts of the space, indeed for different points. These considerations lead to the following definitions:

Definition 8 *Let $\mathcal{X} = (X, S_{\mathcal{X}}, \sigma_{\mathcal{X}})$ and $\mathcal{Y} = (Y, S_{\mathcal{Y}}, \sigma_{\mathcal{Y}})$ be two generalized similarity systems. A function $f : \mathcal{X} \to \mathcal{Y}$ is* globally continuous (GC) *w.r.t. $\mathcal{X}$ and $\mathcal{Y}$ if there is a continuous $\varphi : S_{\mathcal{X}} \to S_{\mathcal{Y}}$ such that*
(GC1) $\varphi(\sigma_{\mathcal{X}}(x,x')) \leq \sigma_{\mathcal{Y}}(fx,fx') \quad \forall\, x,x' \in X$, *and*
(GC2) $\varphi(\sigma_{\mathcal{X}}(x,x\,)) = \sigma_{\mathcal{Y}}(fx,fx\,) \quad \forall\, x \in X$.

It is right-locally continuous (RLC) *if there is a family $(\varphi^{\mathrm{R}}_x)_{x\in X}$ of continuous functions $\varphi^{\mathrm{R}}_x : S_{\mathcal{X}} \to S_{\mathcal{Y}}$ such that*
(RLC1) $\varphi^{\mathrm{R}}_x(\sigma_{\mathcal{X}}(x,x')) \leq \sigma_{\mathcal{Y}}(fx,fx') \quad \forall\, x,x' \in X$, *and*
(RLC2) $\varphi^{\mathrm{R}}_x(\sigma_{\mathcal{X}}(x,x\,)) = \sigma_{\mathcal{Y}}(fx,fx\,) \quad \forall\, x \in X$.

It is left-locally continuous (LLC) *if there is a family $(\varphi^{\mathrm{L}}_x)_{x\in X}$ of continuous functions $\varphi^{\mathrm{L}}_x : S_{\mathcal{X}} \to S_{\mathcal{Y}}$ such that*
(LLC1) $\varphi^{\mathrm{L}}_x(\sigma_{\mathcal{X}}(x',x)) \leq \sigma_{\mathcal{Y}}(fx',fx) \quad \forall\, x,x' \in X$, *and*
(LLC2) $\varphi^{\mathrm{L}}_x(\sigma_{\mathcal{X}}(x\,,x)) = \sigma_{\mathcal{Y}}(fx\,,fx) \quad \forall\, x \in X$.

It is locally continuous (LC) *if it is both RLC and LLC.*

There is no requirement that the families $(\varphi^{\mathrm{R}}_x)_{x\in X}$ and $(\varphi^{\mathrm{L}}_x)_{x\in X}$ be continuous in x in any way.

Proposition 9 *Every RLC function is continuous w.r.t. the right neighborhood structure, and every LLC function is continuous w.r.t. the left neighborhood structure. Every GC function is UC and LC, and all UC functions and all LC functions are continuous w.r.t. both neighborhood structures.*

PROOF. Part of this is obvious. To show that GC functions are UC, assume f is GC witnessed by φ, and $\sigma_{\mathcal{Y}}(fx, fx) \in v$ for some open v of $S_{\mathcal{Y}}$. Let $u = \varphi^{-}v$, which is open in $S_{\mathcal{X}}$. Since $\varphi(\sigma_{\mathcal{X}}(x,x)) = \sigma_{\mathcal{Y}}(fx, fx) \in v$ by (GC2), we have $\sigma_{\mathcal{X}}(x,x) \in u$. If $\sigma_{\mathcal{X}}(x_1,x_2) \in u$, then $\varphi(\sigma_{\mathcal{X}}(x_1,x_2)) \in v$. Since $\varphi(\sigma_{\mathcal{X}}(x_1,x_2)) \leq \sigma_{\mathcal{Y}}(fx_1, fx_2)$ by (GC1), $\sigma_{\mathcal{Y}}(fx_1, fx_2) \in v$ follows as required.

The proof that RLC functions are continuous w.r.t. the right neighborhood structure follows the arguments presented in the motivating section 5.2, only with φ replaced by φ_x^{R} throughout. The corresponding property for UC functions was Prop. 7. $\square$

The next proposition is obvious from the definitions.

Proposition 10 *A function $f : \mathcal{X} \to \mathcal{Y}$ is RLC iff $f : \mathcal{X}^{\mathrm{op}} \to \mathcal{Y}^{\mathrm{op}}$ is LLC, and vice versa. A function $f : \mathcal{X} \to \mathcal{Y}$ is LC iff $f : \mathcal{X}^{\mathrm{op}} \to \mathcal{Y}^{\mathrm{op}}$ is LC, and same for GC and UC.*

All our continuity notions can be used to set up categories.

Proposition 11 *The identity* $\mathrm{id} : \mathcal{X} \to \mathcal{X}$ *is GC, hence UC, LC, RLC, and LLC. The properties GC, UC, LC, RLC, and LLC are preserved under composition. Every constant function is GC, hence UC, LC, RLC, and LLC.*

PROOF. Global continuity of the identity on $\mathcal{X}$ is witnessed by the identity on $S_{\mathcal{X}}$, and global continuity of the constant function $(x \mapsto y) : \mathcal{X} \to \mathcal{Y}$ is witnessed by the constant function $(a \mapsto \sigma_{\mathcal{Y}}(y,y)) : S_{\mathcal{X}} \to S_{\mathcal{Y}}$. If $f : \mathcal{X} \to \mathcal{Y}$ is RLC witnessed by $(\varphi_x^{\mathrm{R}})_{x \in |\mathcal{X}|}$ and $g : \mathcal{Y} \to \mathcal{Z}$ is RLC witnessed by $(\psi_y^{\mathrm{R}})_{y \in |\mathcal{Y}|}$, then $g \circ f$ is RLC witnessed by $(\psi_{fx}^{\mathrm{R}} \circ \varphi_x^{\mathrm{R}})_{x \in |\mathcal{X}|}$. $\square$

6 Transitive Systems and their Variants

6.1 Motivation of the Definitions

We now look for conditions that ensure that pre-open balls are open in (X, S, σ). In the cases of (quasi-)metrics and partial metrics, this follows from the triangle inequality. Thus we look for a generalization of the triangle inequality to gss. With the necessary order reversal, a first candidate is $\sigma(x,y) * \sigma(y,z) \leq \sigma(x,z)$ for some continuous binary operation $* : S \times S \to S$. To be able to cover partial metrics with their correction term $-\sigma(y,y)$ that depends on y

(see Section 7.3), we generalize to a family of $*$-operations $(*_y)_{y \in X}$ such that

$$\sigma(x,y) \;*_y\; \sigma(y,z) \;\leq\; \sigma(x,z)\,. \tag{5}$$

Yet like in the case of morphisms, a second condition is needed since (5) is always satisfied if $*_y$ is chosen such that $a *_y b = \bot$ for all $a, b \in S$. To find that condition, let us try to prove that $\mathrm{B}^{\mathrm{R}}(x,u)$ is open in $\mathrm{N}^{\mathrm{R}}(X,S,\sigma)$ under the assumption that (5) holds.

We have to show that $\mathrm{B}^{\mathrm{R}}(x,u)$ is a right neighborhood of all its elements. For $y \in \mathrm{B}^{\mathrm{R}}(x,u)$, i.e. $\sigma(x,y) \in u$, we have to find an open v of S such that $y \in \mathrm{B}^{\mathrm{R}}(y,v) \subseteq \mathrm{B}^{\mathrm{R}}(x,u)$. This means $\sigma(y,y) \in v$ and

$$\sigma(y,z) \in v \;\;\Rightarrow\;\; \sigma(x,z) \in u\,. \tag{6}$$

Here, (6) can be ensured with the help of (5) if

$$\sigma(y,z) \in v \;\;\Rightarrow\;\; \sigma(x,y) *_y \sigma(y,z) \in u\,. \tag{7}$$

This implication in turn can be ensured by choosing v to be the inverse image of u under the continuous function $(b \mapsto \sigma(x,y) *_y b) : S \to S$. (For this, it is not even needed that $*_y$ is continuous in both arguments; it suffices that it is continuous in its second argument.)

To obtain $\sigma(y,y) \in v$, we have then to ensure $\sigma(x,y) *_y \sigma(y,y) \in u$, which follows from $\sigma(x,y) \in u$ if

$$\sigma(x,y) \;\leq\; \sigma(x,y) \;*_y\; \sigma(y,y)\,. \tag{8}$$

Together with (5), we obtain equality in (8).

To show also that left pre-open balls are open in the left neighborhood structure, one needs that the functions $*_y$ are continuous in their left argument, and the dual of property (8), i.e. $\sigma(y,x) = \sigma(y,y) *_y \sigma(y,x)$.

6.2 Transitive and Locally Transitive Systems

The above considerations motivate the following definition, which also includes weaker and stronger variants.

Definition 12 *A generalized similarity system* (X,S,σ) *is* locally transitive *if there is a family* $(*_y)_{y \in X}$ *of functions* $*_y : S \times S \to S$ *continuous in each argument separately such that*

(LTr1) $\quad \sigma(x,y) \;*_y\; \sigma(y,z) \;\leq\; \sigma(x,z)$
(LTr2R) $\quad \sigma(x,y) \;*_y\; \sigma(y,y) \;=\; \sigma(x,y)$
(LTr2L) $\quad \sigma(y,y) \;*_y\; \sigma(y,z) \;=\; \sigma(y,z)\,.$

It is weakly locally transitive *if (LTr1) holds, and (LTr2R) and (LTr2L) are* *replaced by the weaker equation*

(LTr2W) $\sigma(y,y) *_y \sigma(y,y) \ = \ \sigma(y,y)\,.$

It is (weakly) globally transitive *if it is (weakly) locally transitive in a way that all the operations $*_y$ for $y \in X$ are identical (then the index y can be dropped). We write (Tr1), (Tr2R), (Tr2L), and (Tr2W) in this case.*

It is (weakly) boolean transitive *if it is (weakly) globally transitive with binary meet $\wedge_S$ as the composition operation $*$.*

The defining (in)equations for (weakly) boolean transitive systems can be simplified.

Proposition 13 *A gss (X,S,σ) is weakly boolean transitive if each pair $a,b \in S$ has an infimum, $a \wedge b$, the function $\wedge : S \times S \to S$ is separately continuous, and*

(BTr1) $\quad \sigma(x,y) \wedge \sigma(y,z) \ \leq \ \sigma(x,z)\,.$

It is boolean transitive iff it is weakly boolean transitive and selfish.

PROOF. Condition (Tr2W) is redundant since binary meet is idempotent, and (Tr2R) $\sigma(x,y) \wedge \sigma(y,y) = \sigma(x,y)$ is equivalent to $\sigma(x,y) \leq \sigma(y,y)$. $\square$

The motivating considerations in Section 6.1 immediately lead to the following proposition:

Proposition 14 *In a locally transitive system, the right pre-open balls are open in the right neighborhood structure, in fact form an open base of that neighborhood structure, and likewise for left pre-open balls and left neighborhood structure.*

Corollary 15 *The neighborhood spaces $\mathrm{N}^{\mathrm{R}}\mathcal{X}$ and $\mathrm{N}^{\mathrm{L}}\mathcal{X}$ of a locally transitive system $\mathcal{X}$ are topological. We replace the names $\mathrm{N}^{\mathrm{R}}\mathcal{X}$ by $\mathrm{T}^{\mathrm{R}}\mathcal{X}$ and $\mathrm{N}^{\mathrm{L}}\mathcal{X}$ by $\mathrm{T}^{\mathrm{L}}\mathcal{X}$ in this case, and introduce the symbols $\tau^{\mathrm{R}}_{\mathcal{X}}$ and $\tau^{\mathrm{L}}_{\mathcal{X}}$ for their respective topologies.*

An interesting point is that the proposition and its corollary do not require that the operations $*_y$ are jointly continuous (continuous as functions $S \times S \to S$), nor any algebraic properties such as commutativity or associativity. (Yet note that the distinction between separate continuity and joint continuity disappears when we restrict S to the class of continuous lattices starting from Section 8).

Proposition 16 *If $\mathcal{X}$ is (weakly) locally / globally / boolean transitive, then so is $\mathcal{X}^{\mathrm{op}}$.*

PROOF. This is fairly obvious, but one has to replace $*_y$ by $*_y^{\mathrm{op}}$ with $a *_y^{\mathrm{op}} b = b *_y a$. $\square$

6.3 The Specialization Relation of the Induced Neighborhood Spaces

The specialization relation of the induced neighborhood spaces can be characterized in general. For locally transitive systems, this characterization can be strengthened.

Proposition 17 *Let $\mathcal{X} = (X, S, \sigma)$ be a gss. The specialization relation of $\mathrm{N}^{\mathrm{R}}\mathcal{X}$ is given by $y \to^{\mathrm{R}} z \Leftrightarrow \sigma(y, y) \leq \sigma(y, z)$ where $\leq$ is the specialization preorder of S. If $\mathcal{X}$ is locally transitive, this is equivalent to $\forall x \in X\colon\ \sigma(x, y) \leq \sigma(x, z)$. Analogously, $y \to^{\mathrm{L}} z$ iff $\sigma(y, y) \leq \sigma(z, y)$ (iff $\forall x \in X:\ \sigma(y, x) \leq \sigma(z, x)$ in case of locally transitive gss).*

PROOF. The relation $y \to^{\mathrm{R}} z$ means that z is contained in every right neighborhood of y (cf. Section 3). Thus it is equivalent to $y \in \mathrm{B}^{\mathrm{R}}(y, u) \Rightarrow z \in \mathrm{B}^{\mathrm{R}}(y, u)$ for all opens u of S, or $\sigma(y, y) \in u \Rightarrow \sigma(y, z) \in u$ for all u, or $\sigma(y, y) \leq \sigma(y, z)$ in the specialization preorder of S. If $\mathcal{X}$ is locally transitive, $\sigma(y, y) \leq \sigma(y, z)$ implies $\sigma(x, y) = \sigma(x, y) *_y \sigma(y, y) \leq \sigma(x, y) *_y \sigma(y, z) \leq \sigma(x, z)$. The opposite direction is obtained by setting $x = y$. ☐

Proposition 18 *For any gss, the relations $\to^{\mathrm{R}}$ and $\to^{\mathrm{L}}$ are reflexive. They are transitive for locally transitive gss. Relation $\to^{\mathrm{R}}$ is preserved by RLC functions and $\to^{\mathrm{L}}$ by LLC functions.*

PROOF. This is partly obvious and partly follows from the general properties of specialization relations (see Section 3). ☐

Proposition 19 *Relation $\to^{\mathrm{R}}$ of $\mathcal{X}^{\mathrm{op}}$ is $\to^{\mathrm{L}}$ of $\mathcal{X}$ and vice versa. In a symmetric gss, $\to^{\mathrm{R}}$ and $\to^{\mathrm{L}}$ are identical. In a self-uniform gss, $\to^{\mathrm{R}}$ is the opposite of $\to^{\mathrm{L}}$.*

PROOF. Only the last statement deserves a proof. Let $\mathcal{X}$ be self-uniform. Relation $y \to^{\mathrm{R}} z$ means $\sigma(y, y) \leq \sigma(y, z)$. By self-uniformity, $\sigma(y, y)$ equals $\sigma(z, z)$. Hence, $y \to^{\mathrm{R}} z$ is equivalent to $\sigma(z, z) \leq \sigma(y, z)$, i.e. $z \to^{\mathrm{L}} y$. ☐

6.4 Similarities from Reflexive Relations

Let $\to$ be a reflexive relation on a set X. Then we define a gss $\mathcal{X}_{\to} = (X, \mathbb{S}, \sigma)$ where $\mathbb{S} = \{0, 1\}$ is Sierpinski space and $\sigma(x, y) = 1$ if $x \to y$, and $= 0$ otherwise. By reflexivity of $\to$, $\mathcal{X}_{\to}$ is self-uniform and selfish; it is symmetric iff $\to$ is symmetric. The gss $\mathbb{S}^u$ of Section 4.3 is the gss induced by the reflexive relation $\to$ on $\{0, 1\}$ with $0 \to 0$, $0 \to 1$, $1 \to 1$, but not $1 \to 0$.

Proposition 20 *The derived relation $\to^{\mathrm{R}}$ of $\mathcal{X}_{\to}$ is the original relation $\to$, and $\to^{\mathrm{L}}$ is its opposite.*

PROOF. $x \to^{\mathrm{R}} y$ holds iff $\sigma(x, x) \leq \sigma(x, y)$, iff $1 \leq \sigma(x, y)$, iff $x \to y$. Since $\mathcal{X}_{\to}$ is self-uniform, $\to^{\mathrm{L}}$ is the opposite of $\to^{\mathrm{R}}$ by Prop. 19. ☐

Proposition 21 *For a reflexive relation $\to$, the following are equivalent:*

(1) Relation $\to$ is transitive.
(2) $\sigma(x, y) \wedge \sigma(y, z) \leq \sigma(x, z)$ where $\wedge$ is the meet operation of $\mathbb{S}$ (or conjunction in logical view).
(3) $\mathcal{X}_\to$ is boolean transitive.
(4) $\mathcal{X}_\to$ is globally transitive.
(5) $\mathcal{X}_\to$ is locally transitive.
(6) The right pre-open balls $\mathrm{B}^{\mathrm{R}}(x, u)$ are right open.
(7) The right neighborhood space $\mathrm{N}^{\mathrm{R}}\mathcal{X}_\to$ is topological.
(8) (Same with left.)

PROOF. (1) $\Rightarrow$ (2) : If one of $\sigma(x, y)$ and $\sigma(y, z)$ is 0, the left hand side is 0. Otherwise, both are 1, i.e. $x \to y$ and $y \to z$ hold, hence $x \to z$ by transitivity, hence $\sigma(x, z) = 1$, too.
(2) $\Rightarrow$ (3) : Recall that $\mathcal{X}_\to$ is selfish. Hence (2) means that $\mathcal{X}_\to$ is boolean transitive by Prop. 13.
(3) $\Rightarrow$ (4) $\Rightarrow$ (5) are obvious generalizations.
(5) $\Rightarrow$ (6) is Prop. 14.
(6) $\Rightarrow$ (7) : By (6), the right pre-open balls form an open base of $\mathrm{N}^{\mathrm{R}}\mathcal{X}_\to$, which is therefore topological (see Section 3).
(7) $\Rightarrow$ (1) : By Prop. 20, relation $\to$ is the specialization relation of $\mathrm{N}^{\mathrm{R}}\mathcal{X}_\to$. The specialization relation of a topological space is transitive.
Equivalence with the corresponding "left" statements follows from replacing $\to$ by its opposite and the fact that a relation is transitive iff its opposite is transitive. □

Since there are non-transitive reflexive relations in every set with at least three elements, the above proposition provides examples for gss in which pre-open balls are not open, for gss whose induced neighborhood structures are not topological, and for gss that are not locally transitive.

6.5 Continuity of the Similarity Function

In a locally transitive gss $\mathcal{X}$, the local composition functions $*_y : S_\mathcal{X} \times S_\mathcal{X} \to S_\mathcal{X}$ are continuous in each argument separately (separately continuous for short). This is sufficient to show the same property for $\sigma_\mathcal{X} : |\mathcal{X}| \times |\mathcal{X}| \to S_\mathcal{X}$ if an appropriate topology is chosen for $|\mathcal{X}| \times |\mathcal{X}|$.

Proposition 22 *If $\mathcal{X}$ is locally transitive (with separately continuous composition functions), then $\sigma_\mathcal{X} : \mathrm{T}^{\mathrm{L}}\mathcal{X} \times \mathrm{T}^{\mathrm{R}}\mathcal{X} \to S_\mathcal{X}$ is separately continuous.*

PROOF. Here, $\mathrm{T}^{\mathrm{L}}\mathcal{X}$ and $\mathrm{T}^{\mathrm{R}}\mathcal{X}$ are the names for the induced neighborhood spaces, which are actually topological in this case (see Cor. 15). For fixed x in $|\mathcal{X}|$ and each open u of $S_\mathcal{X}$, the inverse image of u under $y \mapsto \sigma(x, y)$ is $\mathrm{B}^{\mathrm{R}}(x, u)$, which is open in $\mathrm{T}^{\mathrm{R}}\mathcal{X}$ by Prop. 14, and the inverse image of u under

$y \mapsto \sigma(y, x)$ is $\mathrm{B^L}(x, u)$, which is open in $\mathrm{T^L}\mathcal{X}$. $\square$

If the local composition functions $*_y : S_\mathcal{X} \times S_\mathcal{X} \to S_\mathcal{X}$ are even continuous in the product topology (jointly continuous), then so is $\sigma_\mathcal{X}$.

Proposition 23 *If $\mathcal{X}$ is locally transitive with jointly continuous composition functions, then $\sigma_\mathcal{X} : \mathrm{T^L}\mathcal{X} \times \mathrm{T^R}\mathcal{X} \to S_\mathcal{X}$ is jointly continuous.*

PROOF. Let w be an open set of $S_\mathcal{X}$, and $\sigma(x, y) \in w$. From (LTr2L) and (LTr2R), we get the fact

$$(\sigma(x, x) *_x \sigma(x, y)) *_y \sigma(y, y) \; = \; \sigma(x, y)\,.$$

Let $\gamma : S \times S \to S$ be defined by $\gamma(a, b) = (a *_x \sigma(x, y)) *_y b$. Then γ is continuous, and $\gamma(\sigma(x, x), \sigma(y, y)) = \sigma(x, y) \in w$. By continuity of γ, there are open sets u and v of S such that $\sigma(x, x) \in u$, $\sigma(y, y) \in v$, and $\gamma^+(u \times v) \subseteq w$. Hence $x \in \mathrm{B^L}(x, u) =: U$ and $y \in \mathrm{B^R}(y, v) =: V$, and thus $(x, y) \in U \times V$, which is an open set of $\mathrm{T^L}\mathcal{X} \times \mathrm{T^R}\mathcal{X}$.

Let (x', y') be in $U \times V$. Then $\sigma(x', x) \in u$ and $\sigma(y, y') \in v$, hence $w \ni \gamma(\sigma(x', x), \sigma(y, y')) = (\sigma(x', x) *_x \sigma(x, y)) *_y \sigma(y, y') \leq \sigma(x', y')$ using the local triangle inequality (LTr1) for similarities. Thus, $\sigma(x', y') \in w$ for all $(x', y') \in U \times V$, and so $U \times V \subseteq \sigma^- w$. $\square$

7 Similarities and (Generalized) Metrics

In this section, we compare various kinds of gss $\mathcal{X}$ with fixed similarity domain $S_\mathcal{X}$ with generalized metrics as defined in Section 2.

7.1 Non-Negative Reals with Addition

Let (X, S, σ) be a weakly globally transitive gss with $S = \mathbb{R}^{+\mathrm{op}}$, i.e. the non-negative reals with the Scott topology of the opposite ordering, and addition '+' as the global composition operation. The defining properties of a weakly globally transitive system are in this case

(Tr1) $\quad \sigma(x, y) + \sigma(y, z) \;\leq_S\; \sigma(x, z)$
(Tr2W) $\quad \sigma(x, x) + \sigma(x, x) \;=\; \sigma(x, x)$

where '$\leq_S$' is the specialization preorder of S, i.e. the *opposite* of the order of $\mathbb{R}^+$. From (Tr2W), $\sigma(x, x) = 0$ follows, and so the properties are equivalent to

$$\begin{aligned} \sigma(x, x) \; &= \; 0 \\ \sigma(x, z) \; &\leq_\mathbb{R} \; \sigma(x, y) + \sigma(y, z), \end{aligned}$$

which are exactly the defining properties of a pseudo-quasi-metric (see Section 2.2). Note that such a gss is automatically self-uniform, selfish, and globally transitive.

7.2 Non-Negative Reals with Maximum

We now consider the case where addition is replaced by the maximum operation $\vee_{\mathbb{R}^+}$ of $\mathbb{R}^+$, which is binary meet $\wedge_S$ in the specialization preorder of S. The defining properties of a weakly globally transitive system are then

(1) $\sigma(x,z) \;\leq_{\mathbb{R}}\; \sigma(x,y) \vee_{\mathbb{R}^+} \sigma(y,z)$
(2) $\sigma(x,x) \vee_{\mathbb{R}^+} \sigma(x,x) \;=\; \sigma(x,x)$.

Here, (2) is redundant, and (1) is the triangle inequality for ultrametrics. Such a system is not automatically globally transitive. The missing properties are

$$\sigma(x,x) \vee_{\mathbb{R}^+} \sigma(x,y) \;=\; \sigma(x,y) \quad \text{and} \quad \sigma(x,y) \vee_{\mathbb{R}^+} \sigma(y,y) \;=\; \sigma(x,y).$$

which are equivalent to $\sigma(x,x) \leq_{\mathbb{R}} \sigma(x,y)$ and $\sigma(y,y) \leq_{\mathbb{R}} \sigma(x,y)$. There is no reason why $\sigma(x,x)$ should be 0 in this case. Apart from the lack of symmetry, such systems could be called pseudo-partial ultrametric, but there seems to be no official definition of that notion.

7.3 Pseudo-Partial Metrics

Here we show that pseudo-partial metrics can be considered as selfish symmetric locally transitive gss. Recall the definition of pseudo-partial metrics from Section 2.3:

(SSD)	$\delta(x,x) \;\leq_{\mathbb{R}}\; \delta(x,y)$	(small self-distances);
(Sym)	$\delta(x,y) \;=\; \delta(y,x)$	(symmetry);
(Tr$_{\mathrm{P}}$)	$\delta(x,z) \;\leq_{\mathbb{R}}\; \delta(x,y) + \delta(y,z) - \delta(y,y)$	(modified triangle inequality).

Using the specialization preorder of $S = \mathbb{R}^{+\mathrm{op}}$, (SSD) and (Sym) just mean that the resulting gss is selfish and symmetric. The triangle inequality can be written as $\delta(x,y) *_y \delta(y,z) \leq_S \delta(x,z)$ where $a *_y b = (a+b) \dot{-} \delta(y,y) = \max(0, a+b-\delta(y,y))$ defines a continuous local composition operator. The remaining two axioms for such an operator are satisfied:

$$\delta(x,y) *_y \delta(y,y) \;=\; (\delta(x,y) + \delta(y,y)) \dot{-} \delta(y,y) \;=\; \delta(x,y)$$
$$\delta(y,y) *_y \delta(y,z) \;=\; (\delta(y,y) + \delta(y,z)) \dot{-} \delta(y,y) \;=\; \delta(y,z)$$

In my opinion, this is the real reason for the correction term $-\,\delta(y,y)$: it ensures that the resulting gss is locally transitive, and so pre-open balls are open.

7.4 The System of Finite and Infinite Sequences

Papers on partial metrics, e.g., [10], often contain the following example: X is the set of finite and countably infinite sequences over some alphabet Σ, with partial metric $p(x, y) = 2^{-l(x \wedge y)}$ where $x \wedge y$ is the longest common prefix of x and y and $l(x \wedge y)$ its length. Of course, this example can be considered as a locally transitive gss as presented in Section 7.3, but there is actually a more elegant and natural method: Let $S = \mathbb{N}_0^\infty$, the chain of natural numbers with a top element ∞, endowed with the Scott topology, and $\sigma(x, y) = l(x \wedge y)$, the length of the longest common prefix of x and y. In contrast to the gss of Section 7.3, this gss is even globally transitive with binary minimum as composition operation, i.e. boolean transitive. For the triangle inequality

$$\min\left(l(x \wedge y),\, l(y \wedge z)\right) \;\leq\; l(x \wedge z)$$

let $a = x \wedge y$ and $b = y \wedge z$, and assume w.l.o.g. that $l(a) \leq l(b)$ so that the left-hand side is $l(a)$. Since both a and b are prefixes of y and $l(a) \leq l(b)$, a must be a prefix of b and thus also of z. Hence a is a common prefix of x and z, and so $l(a) \leq l(x \wedge z)$, the right-hand side. The two other axioms

$$\min\left(l(x \wedge x),\, l(x \wedge y)\right) = l(x \wedge y) \text{ and } \min\left(l(x \wedge y),\, l(y \wedge y)\right) = l(x \wedge y)$$

hold since $l(x \wedge y) \leq l(x)$ and $\leq l(y)$. The induced topology of this system is clearly the same as the partial metric topology, namely the Scott topology of the domain of finite and infinite sequences with prefix ordering.

7.5 Morphisms between (Pseudo-Quasi-)Metric Spaces

Let us return to the case of pseudo-quasi-metric spaces considered in Section 7.1, in particular $\mathbb{R}$ with its standard metric. Although the gss coming from these spaces all have the same value domain $S = \mathbb{R}^{+\mathrm{op}}$, our GC and LC functions still use a rescaling. In this case, $f : \mathcal{X} \to \mathcal{Y}$ is GC if there is a continuous function $\varphi : S \to S$, i.e. an upper semi-continuous function $\varphi : \mathbb{R}^+ \to \mathbb{R}^+$ such that $\varphi(\delta(x, x')) \leq_S \delta(fx, fx')$, i.e. $\delta(fx, fx') \leq_{\mathbb{R}} \varphi(\delta(x, x'))$, and $\varphi(\delta(x, x)) = \delta(fx, fx)$, i.e. $\varphi(0) = 0$.

- Non-expanding functions f are characterized by $\delta(fx, fx') \leq_{\mathbb{R}} \delta(x, x')$. Such functions are GC with $\varphi = \mathrm{id}$.
- More generally, Lipschitz functions f are characterized by the existence of some $c \in \mathbb{R}^+$ such that $\delta(fx, fx') \leq_{\mathbb{R}} c \cdot \delta(x, x')$. Such functions are GC with $\varphi = (a \mapsto c \cdot a)$.
- The square-root function $\sqrt{\cdot} : \mathbb{R}^+ \to \mathbb{R}^+$ is not a Lipschitz function because of its infinite slope at 0. Nevertheless, it is a GC function. This is witnessed by the rescaling $\varphi = (r \mapsto \sqrt{r})$, which is continuous and satisfies $\varphi(0) = 0$. To prove $\delta(\sqrt{x}, \sqrt{x'}) \leq_{\mathbb{R}} \varphi(\delta(x, x'))$, assume w.l.o.g. $x' \geq x$ and let $r = x' - x$. Then we have to show $\sqrt{x + r} - \sqrt{x} \leq \sqrt{r}$, or $\sqrt{x + r} \leq \sqrt{x} + \sqrt{r}$, which is true for $x, r \geq 0$.
- One might now believe that with a suitable rescaling, all continuous functions are GC. This is however wrong: The square function $(x \mapsto x^2) : \mathbb{R}^+ \to \mathbb{R}^+$ is only LC, not GC. Assume there is an upper semi-continuous function

φ such that $\varphi(0) = 0$ and $\delta(x^2, x'^2) \leq_{\mathbb{R}} \varphi(\delta(x, x'))$. Again assume w.l.o.g. $x' \geq x$ and let $r = x' - x$. Then the inequality becomes $(x+r)^2 - x^2 \leq \varphi(r)$, or $2rx + r^2 \leq \varphi(r)$. For $r > 0$, the left-hand side is unbounded as x increases. So the only way to satisfy the inequality is to include ∞ in the value domain and to assume $\varphi(r) = \infty$ for $r > 0$, but this contradicts upper semi-continuity and $\varphi(0) = 0$.
To show that the square function is LC, let $x' = x + r$ with $r \in \mathbb{R}$ (may be < 0). Then $\delta(x^2, x'^2) \leq_{\mathbb{R}} \varphi_x(\delta(x, x'))$ becomes $|(x+r)^2 - x^2| \leq \varphi_x(|r|)$. The left-hand side is $|2rx+r^2| = |2x+r| \cdot |r| \leq (2x+|r|) \cdot |r|$, so $\varphi_x(d) = (2x+d) \cdot d$ does the job.

8 Algebraic and Continuous Lattices

In the remainder of this paper, the spaces of similarity values will be restricted to continuous lattices. This enables us to obtain characterizations of LC functions (Section 10) and GC functions (Section 11) that do not involve rescaling.

Most of the material in this section is well-known. The generalization of the injectivity property of continuous lattices presented in Section 8.5 forms an exception.

8.1 Characterization of Algebraic and Continuous Lattices

We do not present here the "official" definitions of algebraic and continuous lattices (see [4]), but an equivalent topological characterization.

A *continuous lattice* D with basis B is a complete lattice such that for every Scott-open set $u \subseteq D$ and every point $a \in u$, there are a point $b \in B$ and a Scott-open set $v \subseteq D$ such that $a \in v \subseteq \uparrow b \subseteq u$. The continuous lattice is *ω-continuous* if it has a countable basis.

An *algebraic lattice* D with basis B is a complete lattice whose Scott topology has a base of open sets $\uparrow b$, $b \in B$, i.e. for every point a in a Scott-open set u, there is $b \in B$ such that $\uparrow b$ is Scott open and $a \in \uparrow b \subseteq u$. An *$\omega$-algebraic* lattice has a countable basis. Every (ω-)algebraic lattice is (ω-)continuous.

8.2 Binary Operations in Continuous Lattices

A useful property of continuous lattices D is that every binary operation $* : D \times D \to D$ that is continuous in each argument separately is automatically jointly continuous. Hence, $\sigma_{\mathcal{X}} : \mathrm{T}^{\mathrm{L}}\mathcal{X} \times \mathrm{T}^{\mathrm{R}}\mathcal{X} \to S_{\mathcal{X}}$ is (jointly) continuous for all locally transitive systems with a continuous lattice $S_{\mathcal{X}}$ (Prop. 23). Specific continuous binary operations in all continuous lattices D are binary meet $\wedge : D \times D \to D$ and binary join $\vee : D \times D \to D$.

8.3 Construction of Algebraic Lattices

Algebraic lattices can be obtained by ideal completion. We sketch here the dual filter construction that matches our needs more closely.

Let M be a meet-semilattice, by which we mean a poset $(M, \leq)$ with binary greatest lower bounds (meets) $a \wedge b$ and a greatest element 1. A *filter* in M is a subset A of M that is upward closed ($a \in A, a \leq a' \Rightarrow a' \in A$), contains 1, and is closed under binary meet ($a_1, a_2 \in A \Rightarrow a_1 \wedge a_2 \in A$).

Proposition 24 *When ordered by subset inclusion $\subseteq$, the filters on M form an algebraic lattice D. For every $a \in M$, $\uparrow a = \{b \in M \mid a \leq b\}$ is a filter, and $\{\uparrow a \mid a \in M\}$ is a basis of D. For every $a \in M$, $\langle a \rangle = \{A \in D \mid a \in A\}$ is a Scott-open set of D, and $\{\langle a \rangle \mid a \in M\}$ is a base for the Scott topology of D.*

PROOF. Arbitrary intersections and directed unions of filters are filters. Thus D is a complete lattice, and $\langle a \rangle$ is Scott open since directed join is set union. The sets $\uparrow a$ are obviously filters. We have $\langle a \rangle = \uparrow\{\uparrow a\}$ because $a \in B \Leftrightarrow \uparrow a \subseteq B$ for every filter B.

Every filter A is the directed join (union) of $\{\uparrow a \mid a \in A\}$. Hence, if A is in a Scott-open set U, then $\uparrow a \in U$ for some $a \in A$, and so $A \in \langle a \rangle = \uparrow\{\uparrow a\} \subseteq U$, which shows that D is algebraic with the claimed bases for D and for its Scott topology. □

8.4 Pre-Embedding Topological Spaces into Algebraic Lattices

A *pre-embedding* $e : X \to Y$ of topological spaces X and Y is a continuous function such that every open set of X is the inverse image of an open set of Y. An *embedding* is an injective pre-embedding. If X is $\mathcal{T}_0$, then every pre-embedding $e : X \to Y$ actually is an embedding.

Proposition 25 *Every (countably based) topological space can be pre-embedded into an (ω-)algebraic lattice.*

PROOF. Let $\mathcal{B}$ be a (countable) base of the topological space X. Close $\mathcal{B}$ under finite intersections; this includes the empty intersection X. (This operation preserves countability.) The resulting (countable) meet-semilattice $\mathcal{B}^\cap$ induces an (ω-)algebraic lattice D by Prop. 24. Let $e : X \to D$ be defined by $e\,x = \{B \in \mathcal{B}^\cap \mid x \in B\}$. Consider the basic Scott-open sets $\langle B \rangle$ of D, for B in $\mathcal{B}^\cap$. From $e\,x \in \langle B \rangle \Leftrightarrow B \in e\,x \Leftrightarrow x \in B$, it follows at once that e is continuous and that e is a pre-embedding since $\mathcal{B}^\cap$ is a base for the topology of X. □

It is well-known that continuous lattices are injective w.r.t. topological pre-embeddings: if $e : X \to Y$ is a pre-embedding of topological spaces and $f : X \to D$ a continuous function from X to a continuous lattice D, then there is a continuous "extension" $F : Y \to D$ satisfying $F \circ e = f$. Here, we prove a generalization of this injectivity property that will be used in the characterization of LC and GC functions in Sections 10 and 11.

Below, let X be a set (no topology required), (Y, τ) a topological space, L a complete lattice endowed with the Scott topology, and $g : X \to Y$ and $f : X \to L$ two functions.

Proposition 26 *The extension $E_g f$ of f along g defined as*

$$E_g\, f\, y \;=\; \bigvee \{\bigwedge f^+(g^- U) \mid U \in \tau,\, U \ni y\}$$

is a continuous function $E_g f : Y \to L$ satisfying $E_g f \circ g \leq f$.

PROOF. $E_g f$ is well-defined since L is a complete lattice, and the join $\bigvee$ in its definition is directed. Hence, for every Scott-open set V of L, $E_g f y \in V$ implies $z = \bigwedge f^+(g^- U) \in V$ for some open set U containing y. If $y' \in U$, then $E_g f y' \geq z \in V$. Hence, $y \in U \subseteq (E_g f)^- V$, which proves continuity of $E_g f$. To show $E_g f(gx) \leq fx$ for all x in X, let $E_g f(gx)$ be in some Scott-open set V of L. As above, there is an open set U containing gx such that $\bigwedge f^+(g^- U) \in V$. Then fx is in $f^+(g^- U)$, whence $fx \geq \bigwedge f^+(g^- U) \in V$. $\square$

The proposition above is not very impressive on its own; the choice $E_g f = (y \mapsto \bot_L)$ would have provided the same result more easily. The real power comes from the following addendum:

Proposition 27 *We now require that the complete lattice L is continuous. If x is a point of X such that for every Scott-open set V of L containing fx there is an open set U of Y containing gx satisfying $g^- U \subseteq f^- V$, then $E_g f \circ g$ and f coincide at x, i.e. $E_g f(gx) = fx$ holds for this point x.*

PROOF. Since $E_g f \circ g \leq f$ is already known from Prop. 26, we only need to show $fx \leq E_g f(gx)$. Let $fx \in W$ for some Scott-open set W of L. Since L is continuous, there are b in L and a Scott-open set V of L such that $fx \in V \subseteq \uparrow b \subseteq W$. By hypothesis, there is an open set U of Y such that $gx \in U$ and $g^- U \subseteq f^- V$, or $f^+(g^- U) \subseteq V$, hence $\bigwedge f^+(g^- U) \in \uparrow b$. Since $gx \in U$, we have $E_g f(gx) \geq \bigwedge f^+(g^- U) \in \uparrow b \subseteq W$. $\square$

The above proposition is stronger than injectivity of continuous lattices.

Corollary 28 *If X and Y are topological spaces, $e : X \to Y$ a pre-embedding, L a continuous lattice, and $f : X \to L$ a continuous function, then $E_g f \circ g = f$ holds.*

PROOF. To show the hypothesis of Prop. 27, assume $fx \in V$ open. Then $x \in f^{-}V$ open since f is continuous, and $f^{-}V = g^{-}U$ for some open U of Y since g is a pre-embedding. $\square$

It is also possible to show a generalized converse of Prop. 27.

Proposition 29 *Let X be a set, Y and Z topological spaces, $g : X \to Y$ and $f : X \to Z$ functions, and $\varphi : Y \to Z$ a continuous function such that $\varphi \circ g \le f$ holds, and in addition $(\varphi \circ g)(x) = fx$ for a specific point x. Then for every open set V of Z containing fx there is an open set U of Y containing gx such that $g^{-}U \subseteq f^{-}V$.*

PROOF. Let $U = \varphi^{-}V$, which is open since φ is continuous. Since $\varphi(gx) = fx \in V$, $gx \in U$ follows. Whenever $gx' \in U$, then $\varphi(gx') \in V$, hence $fx' \in V$ since $\varphi(gx') \le fx'$. This shows $g^{-}U \subseteq f^{-}V$. $\square$

9 Similarity Systems and Similarity Spaces

9.1 Similarity Systems

By a *similarity system*, we mean a generalized similarity system (gss) $\mathcal{X}$ whose space $S_{\mathcal{X}}$ of similarity values is a continuous lattice, called the *value lattice*. (These value lattices are more general than those of [8] since no "addition" is required.) A similarity system is *countably based* if its value lattice is ω-continuous. The gss $\mathbb{S}^s$ and $\mathbb{S}^u$ of Section 4.3, the gss from reflexive relations (Section 6.4), and the gss of finite and infinite sequences (Section 7.4) actually are countably based similarity systems. The various kinds of generalized metrics induce countably based similarity systems with $S_{\mathcal{X}} = [0, \infty]^{\mathrm{op}}$.

9.2 Similarity Spaces

The following definitions could be applied to gss as well. Given two similarity systems $\mathcal{X}$ and $\mathcal{X}'$ with the same point set $X = |\mathcal{X}| = |\mathcal{X}'|$, we say $\mathcal{X}$ is *finer* than $\mathcal{X}'$, written as $\mathcal{X} \to \mathcal{X}'$, if the identity function id_X is GC as a function from $\mathcal{X}$ to $\mathcal{X}'$. Unfolding Definition 8, this means that there is a Scott-continuous function $\varphi : S_{\mathcal{X}} \to S_{\mathcal{X}'}$ such that $\varphi(\sigma_{\mathcal{X}}(x_1, x_2)) \le \sigma_{\mathcal{X}'}(x_1, x_2)$ and $\varphi(\sigma_{\mathcal{X}}(x, x)) = \sigma_{\mathcal{X}'}(x, x)$. We say $\mathcal{X}$ and $\mathcal{X}'$ are *equivalent*, written as $\mathcal{X} \leftrightarrow \mathcal{X}'$, if $\mathcal{X} \to \mathcal{X}'$ and $\mathcal{X}' \to \mathcal{X}$.

The value lattices of equivalent systems may look quite different. For instance, all systems whose point set is a fixed singleton set are equivalent no matter how large or small their value lattice is. The reason is that constant functions are GC (Prop. 11).

As the name suggests, equivalence is an equivalence relation on the class of similarity systems with fixed point set X. The equivalence classes of this equivalence relation are called *similarity spaces*, and the systems belonging to such a class are called *representations* of that space.

The relationship between similarity spaces and their representing systems is similar to the relationship between topological spaces or domains and their bases. We shall later construct a similarity system from the base of a topological space. Each base gives a different system, but these systems will all be equivalent and thus represent a single space so that a well-defined map from topological spaces to similarity systems will result.

The common point set of all representations of a space can be taken as the point set of the space, but a space does not have a fixed value lattice since each of its representations may have a different one. The same is true for the similarity function. On the other hand, GC functions are continuous w.r.t. the induced neighborhood structures (Prop. 9). Hence, if $\mathcal{X}$ is finer than $\mathcal{X}'$, its neighborhood structures are finer than those of $\mathcal{X}'$, and equivalent similarity systems share the same neighborhood structures, which can thus be attributed to the similarity space they represent.

For two similarity spaces $\mathbf{X}$ and $\mathbf{Y}$, a function $f : \mathbf{X} \to \mathbf{Y}$ is a function between the respective point sets ($f : |\mathbf{X}| \to |\mathbf{Y}|$). For such a function, the following are equivalent:

(1) There are representations $\mathcal{X}$ of $\mathbf{X}$ and $\mathcal{Y}$ of $\mathbf{Y}$ such that $f : \mathcal{X} \to \mathcal{Y}$ is GC (UC, LC, RLC, LLC).
(2) For *all* representations $\mathcal{X}$ of $\mathbf{X}$ and $\mathcal{Y}$ of $\mathbf{Y}$, the function $f : \mathcal{X} \to \mathcal{Y}$ is GC (UC, LC, RLC, LLC).

In this case, we say $f : \mathbf{X} \to \mathbf{Y}$ is GC (UC, LC, RLC, LLC).

9.3 Properties of Similarity Spaces

We say that a similarity space has a property such as self-uniform, symmetric, locally transitive, or countably based if at least one of its representing systems has this property (there might be other representing systems without this property). A disadvantage of this definition is that if a space $\mathbf{X}$ has two properties P_1 and P_2, this merely means that it has representing systems $\mathcal{X}_1$ and $\mathcal{X}_2$ such that $\mathcal{X}_1$ satisfies P_1 and $\mathcal{X}_2$ satisfies P_2. Therefore, we say that $\mathbf{X}$ *jointly* has properties P_1 and P_2 if it has a representing system $\mathcal{X}$ that satisfies both P_1 and P_2.

9.4 Self-Uniform Spaces

More can be said in case of the property of self-uniformity.

Proposition 30 *For two similarity systems $\mathcal{X}$ and $\mathcal{X}'$, if $\mathcal{X} \to \mathcal{X}'$ and $\mathcal{X}$ is* *self-uniform, then so is $\mathcal{X}'$.*

PROOF. Let φ be a witness for $\mathcal{X} \to \mathcal{X}'$. Then $\sigma_{\mathcal{X}'}(x,x) = \varphi(\sigma_{\mathcal{X}}(x,x))$. Hence, if all $\sigma_{\mathcal{X}}(x,x)$ are equal, then so are all $\sigma_{\mathcal{X}'}(x,x)$. $\square$

Corollary 31 *If $\mathcal{X}$ and $\mathcal{X}'$ are equivalent, then $\mathcal{X}$ is self-uniform if and only if $\mathcal{X}'$ is self-uniform. Hence, all representations of a self-uniform space are self-uniform.*

10 Characterization of Locally Continuous Functions

The goal of this section is to characterize LC functions without an existential statement over witnesses. Section 11 does the same for GC functions.

10.1 Right Locally Continuous Functions

We start with RLC functions.

Theorem 32 *Let $\mathcal{X}$ and $\mathcal{Y}$ be two similarity systems. A function $f : \mathcal{X} \to \mathcal{Y}$ is right locally continuous if and only if $f : \mathrm{N}^{\mathrm{R}}\mathcal{X} \to \mathrm{N}^{\mathrm{R}}\mathcal{Y}$ is continuous in the sense of neighborhood spaces.*

PROOF. A RLC function is continuous by Prop. 9. For the opposite direction, we employ Prop. 26 and 27 using the fact that $S_{\mathcal{Y}}$ is a continuous lattice. Fix $x \in |\mathcal{X}|$. Consider $\sigma_x = (x' \mapsto \sigma_{\mathcal{X}}(x,x')) : |\mathcal{X}| \to S_{\mathcal{X}}$ and $\sigma_{fx} \circ f = (x' \mapsto \sigma_{\mathcal{Y}}(fx, fx')) : |\mathcal{X}| \to S_{\mathcal{Y}}$. By Prop. 26, there is a continuous function $\varphi_x = E_{\sigma_x}(\sigma_{fx} \circ f) : S_{\mathcal{X}} \to S_{\mathcal{Y}}$ satisfying $\varphi_x \circ \sigma_x \leq \sigma_{fx} \circ f$, i.e. $\varphi_x(\sigma_{\mathcal{X}}(x,x')) \leq \sigma_{\mathcal{Y}}(fx, fx')$, which is property (RLC1).

To show (RLC2) $\varphi_x(\sigma_{\mathcal{X}}(x,x)) = \sigma_{\mathcal{Y}}(fx,fx)$, we apply Prop. 27 to the point x. Let v be a Scott-open set of $S_{\mathcal{Y}}$ containing $(\sigma_{fx} \circ f)(x) = \sigma_{\mathcal{Y}}(fx,fx)$. Then $V = \mathrm{B}^{\mathrm{R}}(fx, v)$ is a neighborhood of fx in $\mathrm{N}^{\mathrm{R}}\mathcal{Y}$. Since $f : \mathrm{N}^{\mathrm{R}}\mathcal{X} \to \mathrm{N}^{\mathrm{R}}\mathcal{Y}$ is continuous, $f^{-}V$ is a neighborhood of x in $\mathrm{N}^{\mathrm{R}}\mathcal{X}$. Hence, there is an open set u of $S_{\mathcal{X}}$ such that $x \in \mathrm{B}^{\mathrm{R}}(x,u) \subseteq f^{-}V$. We need to show $\sigma_x^{-}(u) \subseteq (\sigma_{fx} \circ f)^{-}(v)$. We have $\sigma_x^{-}(u) = \mathrm{B}^{\mathrm{R}}(x,u) \subseteq f^{-}V = f^{-}(\mathrm{B}^{\mathrm{R}}(fx,v)) = f^{-}(\sigma_{fx}^{-}v) = (\sigma_{fx} \circ f)^{-}(v)$. $\square$

Corollary 33 *The functor N^{R} embeds the category $\mathsf{Sim}_{\mathrm{RLC}}$ of similarity spaces with right locally continuous functions as a full subcategory into the category Nbh of neighborhood spaces.*

10.2 Locally Continuous Functions

A *bineighborhood space* is a set with two unrelated neighborhood structures. Similarly, a *bitopological space* is a set with two unrelated topologies. A function between such spaces is *pairwise continuous* if it is continuous w.r.t. both neighborhood structures / topologies. This gives the categories BiNbh and BiTop. We say that a bitopological space is *countably based* if both topologies have a countable base.

For a similarity system $\mathcal{X}$, we can combine the neighborhood spaces $\mathrm{N}^{\mathrm{R}}\mathcal{X} = (|\mathcal{X}|, \mathcal{N}^{\mathrm{R}}\mathcal{X})$ and $\mathrm{N}^{\mathrm{L}}\mathcal{X} = (|\mathcal{X}|, \mathcal{N}^{\mathrm{L}}\mathcal{X})$ into the bineighborhood space $\mathrm{N}^{\mathrm{B}}\mathcal{X} = (|\mathcal{X}|, \mathcal{N}^{\mathrm{L}}\mathcal{X}, \mathcal{N}^{\mathrm{R}}\mathcal{X})$. From Theorem 32 and its dual for left locally continuous functions, we obtain:

Theorem 34 *Let $\mathcal{X}$ and $\mathcal{Y}$ be two similarity systems. A function $f : \mathcal{X} \to \mathcal{Y}$ is locally continuous if and only if $f : \mathrm{N}^{\mathrm{B}}\mathcal{X} \to \mathrm{N}^{\mathrm{B}}\mathcal{Y}$ is pairwise continuous.*

Corollary 35 *The functor N^{B} embeds the category $\mathsf{Sim}_{\mathrm{LC}}$ of similarity spaces with locally continuous functions as a full subcategory into the category BiNbh of bineighborhood spaces.*

The induced neighborhood structures of a locally transitive similarity system $\mathcal{X}$ are topological by Cor. 15. Hence, the bineighborhood space $\mathrm{N}^{\mathrm{B}}\mathcal{X}$ actually is a bitopological space. In this case, we write $\mathrm{T}^{\mathrm{B}}\mathcal{X}$ instead of $\mathrm{N}^{\mathrm{B}}\mathcal{X}$.

Corollary 36 *The functor T^{B} embeds the category $\mathsf{Sim}^{\mathrm{LT}}_{\mathrm{LC}}$ of locally transitive similarity spaces with locally continuous functions as a full subcategory into the category BiTop of bitopological spaces.*

10.3 Similarity Spaces from Bitopological Spaces

We do not consider the general case $\mathrm{N}^{\mathrm{B}} : \mathsf{Sim}_{\mathrm{LC}} \hookrightarrow \mathsf{BiNbh}$ any further, but concentrate on $\mathrm{T}^{\mathrm{B}} : \mathsf{Sim}^{\mathrm{LT}}_{\mathrm{LC}} \hookrightarrow \mathsf{BiTop}$ and show that this embedding actually is an equivalence of categories by constructing a functor in the opposite direction. We start with the following lemma:

Lemma 37 *Let $(X, \tau^{\mathrm{L}}, \tau^{\mathrm{R}})$ be a bitopological space with bases $\mathcal{B}^{\mathrm{L}}$ and $\mathcal{B}^{\mathrm{R}}$ for its topologies. Then there is a globally transitive similarity system $\mathcal{S}(X, \mathcal{B}^{\mathrm{L}}, \mathcal{B}^{\mathrm{R}})$ depending on the bases such that $\mathrm{T}^{\mathrm{B}}\mathcal{S}(X, \mathcal{B}^{\mathrm{L}}, \mathcal{B}^{\mathrm{R}}) = (X, \tau^{\mathrm{L}}, \tau^{\mathrm{R}})$. If the bases are countable, $\mathcal{S}(X, \mathcal{B}^{\mathrm{L}}, \mathcal{B}^{\mathrm{R}})$ is countably based.*

PROOF. We apply the construction of Prop. 25 to the topological spaces $X^{\mathrm{L}} = (X, \tau^{\mathrm{L}})$ with base $\mathcal{B}^{\mathrm{L}}$ and $X^{\mathrm{R}} = (X, \tau^{\mathrm{R}})$ with base $\mathcal{B}^{\mathrm{R}}$. This leads to algebraic lattices D^{L} and D^{R} with pre-embeddings $e^{\mathrm{L}} : X^{\mathrm{L}} \to D^{\mathrm{L}}$ and $e^{\mathrm{R}} : X^{\mathrm{R}} \to D^{\mathrm{R}}$. (The lattices are ω-algebraic if the bases are countable.) We now form $D^{\mathrm{B}} = D^{\mathrm{L}} \times D^{\mathrm{R}}$, which is again an algebraic lattice (ω-algebraic if the bases are countable), and define $\sigma^{\mathrm{B}}(x, y) = (e^{\mathrm{L}}x, e^{\mathrm{R}}y) \in D^{\mathrm{B}}$.

As global composition, we use $* : D^{\mathrm{B}} \times D^{\mathrm{B}} \to D^{\mathrm{B}}$ defined by $(a^{\mathrm{L}}, a^{\mathrm{R}}) * (b^{\mathrm{L}}, b^{\mathrm{R}}) = (a^{\mathrm{L}}, b^{\mathrm{R}})$, which is continuous because it is built from projections. The equality $\sigma^{\mathrm{B}}(x, y) * \sigma^{\mathrm{B}}(y, z) = \sigma^{\mathrm{B}}(x, z)$ holds for all $x, y, z \in X$, which shows at once the triangle inequality (Tr1) and the two equations (Tr2L) and

(Tr2R).

Finally, we have to show that $\mathrm{T^L}(X, D^{\mathrm{B}}, \sigma^{\mathrm{B}}) = (X, \tau^{\mathrm{L}})$ and $\mathrm{T^R}(X, D^{\mathrm{B}}, \sigma^{\mathrm{B}}) = (X, \tau^{\mathrm{R}})$. We concentrate on the latter. Let τ' be the topology of $\mathrm{T^R}(X, D^{\mathrm{B}}, \sigma^{\mathrm{B}})$.

For $\tau' \subseteq \tau^{\mathrm{R}}$, we show that every right open ball of $(X, D^{\mathrm{B}}, \sigma^{\mathrm{B}})$ is in τ^{R}. Let $x \in X$, w be an open set of D^{B}, and $y \in \mathrm{B^R}(x, w)$. Then $\sigma^{\mathrm{B}}(x, y) \in w$, whence there are open sets u of D^{L} and v of D^{R} with $\sigma^{\mathrm{B}}(x, y) = (e^{\mathrm{L}}x, e^{\mathrm{R}}y) \in u \times v \subseteq w$. Hence $e^{\mathrm{R}}y \in v$, or $y \in e^{\mathrm{R}-}v$. If $y' \in e^{\mathrm{R}-}v$, then $\sigma^{\mathrm{B}}(x, y') = (e^{\mathrm{L}}x, e^{\mathrm{R}}y') \in u \times v \subseteq w$, and so $y' \in \mathrm{B^R}(x, w)$. This shows $y \in e^{\mathrm{R}-}v \subseteq \mathrm{B^R}(x, w)$.

For $\tau^{\mathrm{R}} \subseteq \tau'$, let V be in τ^{R}. Since $e^{\mathrm{R}} : X^{\mathrm{R}} \to D^{\mathrm{R}}$ is a pre-embedding, there is an open set v of D^{R} such that $V = e^{\mathrm{R}-}v$. Let $w = D^{\mathrm{L}} \times v$, which is an open set of $D^{\mathrm{B}} = D^{\mathrm{L}} \times D^{\mathrm{R}}$. Then for $x, y \in X$, $y \in \mathrm{B^R}(x, w) \Leftrightarrow (e^{\mathrm{L}}x, e^{\mathrm{R}}y) \in D^{\mathrm{L}} \times v \Leftrightarrow e^{\mathrm{R}}y \in v \Leftrightarrow y \in V$. Hence for every $x \in V$, we have $x \in \mathrm{B^R}(x, w) = V$, which shows $V \in \tau'$. $\square$

The next lemma shows how this construction interacts with functions. For sets X and functions f, we abbreviate $X \times X$ by X^2 and $f \times f$ by f^2. Hence, $f : X \to Y$ induces $f^2 : X^2 \to Y^2$ defined by $f^2(x_1, x_2) = (fx_1, fx_2)$. From f^2, we get image $f^{2+} : \mathcal{P}X^2 \to \mathcal{P}Y^2$ and inverse image $f^{2-} : \mathcal{P}Y^2 \to \mathcal{P}X^2$ as usual.

Lemma 38 *Let $\Xi = (X, \tau^{\mathrm{L}}, \tau^{\mathrm{R}})$ be a bitopological space with bases $\mathcal{B}^{\mathrm{L}}$ and $\mathcal{B}^{\mathrm{R}}$, $\mathcal{X} = (X, D^{\mathrm{B}}, \sigma^{\mathrm{B}}) = \mathcal{S}(X, \mathcal{B}^{\mathrm{L}}, \mathcal{B}^{\mathrm{R}})$ the similarity system constructed in Lemma 37, and $\mathcal{Y} = (Y, S, \sigma)$ a weakly locally transitive similarity system. Then $f : \mathcal{X} \to \mathcal{Y}$ is GC if and only if it is LC if and only if $f : \Xi \to \mathrm{N^B}\mathcal{Y}$ is pairwise continuous.*

PROOF. Every GC function is LC by Prop. 9. If $f : \mathcal{X} \to \mathcal{Y}$ is LC, then $f : \mathrm{N^B}\mathcal{X} \to \mathrm{N^B}\mathcal{Y}$ is pairwise continuous by Prop. 9 again, and $\mathrm{N^B}\mathcal{X} = \mathrm{T^B}\mathcal{X} = \Xi$ by Lemma 37. Thus, we only have to show that $f : \mathcal{X} \to \mathcal{Y}$ is GC if $f : \Xi \to \mathrm{N^B}\mathcal{Y}$ is pairwise continuous. We apply Prop. 26 and 27 using the fact that S is a continuous lattice. Consider $\sigma^{\mathrm{B}} : X^2 \to D^{\mathrm{B}}$ and $\sigma \circ f^2 : X^2 \to Y^2 \to S$. By Prop. 26, there is a continuous function $\varphi = E_{\sigma^{\mathrm{B}}}(\sigma \circ f^2) : D^{\mathrm{B}} \to S$ satisfying $\varphi \circ \sigma^{\mathrm{B}} \le \sigma \circ f^2$, i.e. $\varphi(\sigma^{\mathrm{B}}(x, x')) \le \sigma(fx, fx')$, which is property (GC1).

To show (GC2) $\varphi(\sigma^{\mathrm{B}}(x, x)) = \sigma(fx, fx)$, we apply Prop. 27 to (x, x). Let w be a Scott-open set of S containing $(\sigma \circ f^2)(x, x) = \sigma(fx, fx)$. Since $\mathcal{Y}$ is weakly locally transitive, there is a continous function $*_{fx} : S \times S \to S$ such that among other things $\sigma(fx, fx) = \sigma(fx, fx) *_{fx} \sigma(fx, fx) \in w$ holds. Hence there is an open set v of S such that $\sigma(fx, fx) \in v$ and $v *_{fx} v \subseteq w$.

From $\sigma(fx, fx) \in v$, we obtain $fx \in \mathrm{B^R}(fx, v) =: V^{\mathrm{R}}$, which is a right neighborhood of fx in $\mathrm{N^R}\mathcal{Y}$. Since f is pairwise continuous, $f^- V^{\mathrm{R}}$ is a right neighborhood of x in $\mathrm{N^R}\mathcal{X} = \mathrm{T^R}\mathcal{X}$. Hence, there is an open set $U^{\mathrm{R}} \in \tau^{\mathrm{R}}$ such that $x \in U^{\mathrm{R}} \subseteq f^- V^{\mathrm{R}}$. Since $e^{\mathrm{R}} : X^{\mathrm{R}} \to D^{\mathrm{R}}$ is a pre-embedding, there is an open set u^{R} of D^{R} such that $U^{\mathrm{R}} = e^{\mathrm{R}-}u^{\mathrm{R}}$. Similarly, $V^{\mathrm{L}} = \mathrm{B^L}(fx, v)$ is a left neighborhood of fx, which gives $U^{\mathrm{L}} \in \tau^{\mathrm{L}}$ and an open set u^{L} of D^{L} with analogous properties. Then $u = u^{\mathrm{L}} \times u^{\mathrm{R}}$ is an open set of $D^{\mathrm{B}} = D^{\mathrm{L}} \times D^{\mathrm{R}}$, and we claim $\sigma^{\mathrm{B}-}(u) \subseteq (\sigma \circ f^2)^{-}(w)$. First note that

$$\begin{aligned} \sigma^{\mathrm{B}-}(u) \;&=\; (e^{\mathrm{L}} \times e^{\mathrm{R}})^{-}(u^{\mathrm{L}} \times u^{\mathrm{R}}) \;=\; e^{\mathrm{L}-}u^{\mathrm{L}} \times e^{\mathrm{R}-}u^{\mathrm{R}} \\ &=\; U^{\mathrm{L}} \times U^{\mathrm{R}} \;\subseteq\; f^{-}V^{\mathrm{L}} \times f^{-}V^{\mathrm{R}} \;=\; f^{2-}(V^{\mathrm{L}} \times V^{\mathrm{R}}) \end{aligned}$$

and $(\sigma \circ f^2)^{-}(w) = f^{2-}(\sigma^{-}w)$, so we are done once $V^{\mathrm{L}} \times V^{\mathrm{R}} \subseteq \sigma^{-}w$ is proved. Let $(y^{\mathrm{L}}, y^{\mathrm{R}}) \in V^{\mathrm{L}} \times V^{\mathrm{R}}$. Then $\sigma(y^{\mathrm{L}}, fx) \in v$ and $\sigma(fx, y^{\mathrm{R}}) \in v$, whence $\sigma(y^{\mathrm{L}}, y^{\mathrm{R}}) \geq \sigma(y^{\mathrm{L}}, fx) *_{fx} \sigma(fx, y^{\mathrm{R}}) \in v *_{fx} v \subseteq w$. □

Lemma 38 has several interesting consequences.

Theorem 39 *The construction $\mathcal{S}$ of Lemma 37 defines a functor $\mathbf{S}$ that embeds BiTop as a coreflective full subcategory into the categories $\mathsf{Sim}^{\mathrm{GT}}_{\mathrm{GC}}$ of globally transitive similarity spaces and $\mathsf{Sim}^{\mathrm{LT}}_{\mathrm{GC}}$ of locally transitive similarity spaces with globally continuous functions. Countably based bitopological spaces are mapped to countably based similarity spaces. The coreflector is $\mathrm{T}^{\mathrm{B}} : \mathsf{Sim}^{\mathrm{LT}}_{\mathrm{GC}} \to \mathsf{BiTop}$.*

PROOF. Let $\Xi = (X, \tau^{\mathrm{L}}, \tau^{\mathrm{R}})$ be a bitopological space. Depending on the chosen bases $\mathcal{B}^{\mathrm{L}}$ and $\mathcal{B}^{\mathrm{R}}$, the similarity systems $\mathcal{S}(X, \mathcal{B}^{\mathrm{L}}, \mathcal{B}^{\mathrm{R}})$ may vary, but all these similarity systems are equivalent by Lemma 38 and thus representations of the same similarity space $\mathbf{S}\,\Xi$. Lemma 37 yields $\mathrm{T}^{\mathrm{B}}\,\mathbf{S}\,\Xi = \Xi$.

Given two bitopological spaces Ξ and Υ, we can apply Lemma 38 to Ξ and a representation $\mathcal{Y}$ of $\mathbf{S}\,\Upsilon$ to see that $f : \Xi \to \Upsilon = \mathrm{T}^{\mathrm{B}}\,\mathbf{S}\,\Upsilon$ is pairwise continuous iff $f : \mathbf{S}\,\Xi \to \mathbf{S}\,\Upsilon$ is GC. This shows that $\mathbf{S} : \mathsf{BiTop} \to \mathsf{Sim}^{\mathrm{GT}}_{\mathrm{GC}}$ is a full embedding functor with left inverse T^{B}. Lemma 38 also shows that for $\Xi \in \mathsf{BiTop}$ and $\mathbf{Y} \in \mathsf{Sim}^{\mathrm{LT}}$, $f : \mathbf{S}\,\Xi \to \mathbf{Y}$ is GC iff $f : \Xi \to \mathrm{T}^{\mathrm{B}}\mathbf{Y}$ is pairwise continuous. Hence T^{B} is a coreflector. We can use either $\mathsf{Sim}^{\mathrm{LT}}_{\mathrm{GC}}$ or $\mathsf{Sim}^{\mathrm{GT}}_{\mathrm{GC}}$ since $\mathbf{S}\,\Xi$ is globally transitive, but local transitivity is sufficient to get a bitopological space from N^{B} by Cor. 15. □

Theorem 40 *The categories $\mathsf{Sim}^{\mathrm{GT}}_{\mathrm{LC}}$ of globally transitive similarity spaces and $\mathsf{Sim}^{\mathrm{LT}}_{\mathrm{LC}}$ of locally transitive similarity spaces with locally continuous functions are equivalent to BiTop.*

PROOF. The equivalence is given by the functors $\mathrm{T}^{\mathrm{B}} : \mathsf{Sim}^{\mathrm{LT}}_{\mathrm{LC}} \to \mathsf{BiTop}$ (Cor. 15 and Prop. 9), $\mathbf{S} : \mathsf{BiTop} \to \mathsf{Sim}^{\mathrm{GT}}_{\mathrm{LC}}$ (Lemma 38), and the embedding of $\mathsf{Sim}^{\mathrm{GT}}_{\mathrm{LC}}$ into $\mathsf{Sim}^{\mathrm{LT}}_{\mathrm{LC}}$. For $\Xi \in \mathsf{BiTop}$, $\mathrm{T}^{\mathrm{B}}\,\mathbf{S}\,\Xi = \Xi$ holds by Lemma 37. On the other hand, we have $\mathrm{T}^{\mathrm{B}}\,\mathbf{S}\,\mathrm{T}^{\mathrm{B}}\,\mathbf{X} = \mathrm{T}^{\mathrm{B}}\,\mathbf{X}$ for $\mathbf{X} \in \mathsf{Sim}^{\mathrm{LT}}_{\mathrm{LC}}$ by Lemma 37 again, whence $\mathbf{S}\,\mathrm{T}^{\mathrm{B}}\,\mathbf{X} \cong_{\mathrm{LC}} \mathbf{X}$ by Theorem 34. □

10.4 The Symmetric Case

The two neighborhood structures induced by a symmetric similarity space are identical. Clearly, the full subcategory of BiNbh consisting of spaces $(X, \mathcal{N}, \mathcal{N})$

with two identical neighborhood structures is equivalent to the category Nbh of spaces $(X, \mathcal{N})$ with a single neighborhood structure. Thus Cor. 35 reduces to the following:

Theorem 41 *The category* $\mathsf{Sim}_{\mathrm{LC}}^{\mathrm{Sym}}$ *of symmetric similarity spaces with locally continuous functions embeds as a full subcategory into the category* Nbh *of neighborhood spaces.*

For locally transitive symmetric similarity spaces, the same argument allows us to reduce BiTop to Top. Yet there is still an obstacle: the similarity system $\mathcal{S}(X, \mathcal{B}^{\mathrm{L}}, \mathcal{B}^{\mathrm{R}})$ of Lemma 37 is never symmetric, not even in the case $\mathcal{B}^{\mathrm{L}} = \mathcal{B}^{\mathrm{R}}$. However, the similarity *space* represented by such a system *is* symmetric, i.e. $\mathcal{S}(X, \mathcal{B}, \mathcal{B})$ is equivalent to a symmetric system $\mathcal{S}(X, \mathcal{B})$.

Lemma 42 *Let* (X, τ) *be a topological space with base* $\mathcal{B}$*. Then the similarity system* $\mathcal{S}(X, \mathcal{B}, \mathcal{B})$ *of Lemma 37 is equivalent to a symmetric selfish boolean transitive system* $\mathcal{S}(X, \mathcal{B})$*.*

PROOF. Let $\Xi = (X, \tau)$ and let $e : \Xi \to D$ be the pre-embedding constructed from $\mathcal{B}$ as in Prop. 25. Then Lemma 37 yields the system $(X, D^{\mathrm{B}}, \sigma^{\mathrm{B}})$ with $D^{\mathrm{B}} = D \times D$ and $\sigma^{\mathrm{B}}(x, y) = (ex, ey)$. Here, we construct $(X, D, \sigma^{\wedge})$ with $\sigma^{\wedge}(x, y) = ex \wedge ey$. This system is clearly symmetric and selfish. It is boolean transitive by Prop. 13 since $\sigma^{\wedge}(x, y) \wedge \sigma^{\wedge}(y, z) = (ex \wedge ey) \wedge (ey \wedge ez) \leq ex \wedge ez = \sigma^{\wedge}(x, z)$.

To show $(X, D^{\mathrm{B}}, \sigma^{\mathrm{B}}) \to (X, D, \sigma^{\wedge})$, use $\varphi = ((a, b) \mapsto a \wedge b)$, i.e. $\wedge : D \times D \to D$, which is continuous in an algebraic lattice. Then $\varphi(\sigma^{\mathrm{B}}(x, y)) = \varphi(ex, ey) = ex \wedge ey = \sigma^{\wedge}(x, y)$ holds.

For $(X, D, \sigma^{\wedge}) \to (X, D^{\mathrm{B}}, \sigma^{\mathrm{B}})$, use $\varphi = (a \mapsto (a, a)) : D \to D \times D$. Then $\varphi(\sigma^{\wedge}(x, y)) = \varphi(ex \wedge ey) = (ex \wedge ey, ex \wedge ey) \leq (ex, ey) = \sigma^{\mathrm{B}}(x, y)$ and $\varphi(\sigma^{\wedge}(x, x)) = (ex \wedge ex, ex \wedge ex) = (ex, ex) = \sigma^{\mathrm{B}}(x, x)$ hold as required. $\square$

Corollary 43 *The construction of Lemma 42 embeds* Top *as a coreflective full subcategory into the categories* $\mathsf{Sim}_{\mathrm{GC}}^{P}$ *of* P*-similarity spaces with GC functions, where* P *is any property between "jointly symmetric, selfish, and boolean transitive" and "symmetric and locally transitive".*

Corollary 44 *The categories* $\mathsf{Sim}_{\mathrm{LC}}^{P}$ *of* P*-similarity spaces with LC functions (*P *as in Cor. 43) are equivalent to* Top*.*

When applied to Sierpinski space $\mathbb{S}$, the construction of Lemma 42 yields the similarity space $\mathbb{S}^s$ from Section 4.3. In case of $\mathbb{R}$, one has to distinguish between the similarity space $\mathbb{R}_t$ constructed from the standard topology of $\mathbb{R}$ and the similarity space $\mathbb{R}_d$ given by the Euclidean metric. These two spaces are different since $\mathbb{R}_d$ is self-uniform, but $\mathbb{R}_t$ is not (cf. Cor. 31), and $(x \mapsto x^2) : \mathbb{R}_t \to \mathbb{R}_t$ is GC by Lemma 38 / Cor. 43, but $(x \mapsto x^2) : \mathbb{R}_d \to \mathbb{R}_d$ is *not* GC as shown in Section 7.5.

In [7], Kopperman has shown that every topological space (X, τ) can be obtained from a generalized metric, which in our language corresponds to a self-uniform similarity system $\mathcal{X}$ that is not symmetric in general and satisfies $\mathrm{T}^{\mathrm{R}}\mathcal{X} = (X, \tau)$ while $\mathrm{T}^{\mathrm{L}}\mathcal{X}$ is different. In contrast, our construction yields a symmetric system $\mathcal{X}$ that is not self-uniform in general and satisfies $\mathrm{T}^{\mathrm{R}}\mathcal{X} = \mathrm{T}^{\mathrm{L}}\mathcal{X} = (X, \tau)$. In Section 10.3, we have shown how to obtain any two topologies from a system that is non-symmetric in general.

11 The Characterization of Globally Continuous Functions

We first characterize GC functions without referring to the existence of some witness, and then give an equivalent description of the category $\mathsf{Sim}_{\mathrm{GC}}$ of similarity spaces with GC functions without using value lattices.

11.1 Globally Continuous and Uniformly Continuous Functions

In the general case of gss, each GC function is UC by Prop. 9. The opposite implication holds for similarity systems.

Theorem 45 *A function between two similarity systems is globally continuous (GC) if and only if it is uniformly continuous (UC). Thus* $\mathsf{Sim}_{\mathrm{GC}} = \mathsf{Sim}_{\mathrm{UC}}$.

PROOF. Let $f : \mathcal{X} \to \mathcal{Y}$ be uniformly continuous. We employ Prop. 26 and 27 to obtain a witness φ for global continuity of f. Consider $\sigma_{\mathcal{X}} : X^2 \to S_{\mathcal{X}}$ and $\sigma_{\mathcal{Y}} \circ f^2 : X^2 \to Y^2 \to S_{\mathcal{Y}}$. By Prop. 26, there is a continuous function $\varphi = E_{\sigma_{\mathcal{X}}}(\sigma_{\mathcal{Y}} \circ f^2) : S_{\mathcal{X}} \to S_{\mathcal{Y}}$ satisfying $\varphi \circ \sigma_{\mathcal{X}} \leq \sigma_{\mathcal{Y}} \circ f^2$, i.e. $\varphi(\sigma_{\mathcal{X}}(x, x')) \leq \sigma_{\mathcal{Y}}(fx, fx')$, which is property (GC1). To show (GC2) $\varphi(\sigma_{\mathcal{X}}(x, x)) = \sigma_{\mathcal{Y}}(fx, fx)$, we apply Prop. 27 to (x, x). Let v be a Scott-open set of $S_{\mathcal{Y}}$ containing $(\sigma_{\mathcal{Y}} \circ f^2)(x, x) = \sigma_{\mathcal{Y}}(fx, fx)$. Since f is uniformly continuous, there is a Scott-open set u of $S_{\mathcal{X}}$ containing $\sigma_{\mathcal{X}}(x, x)$ such that $\sigma_{\mathcal{X}}(x_1, x_2) \in u$ implies $\sigma_{\mathcal{Y}}(fx_1, fx_2) \in v$. The implication is equivalent to $\sigma_{\mathcal{X}}^{-} u \subseteq (\sigma_{\mathcal{Y}} \circ f^2)^{-} v$. This is what is required by the hypothesis of Prop. 27. $\square$

11.2 Square-Neighborhood Spaces and Square-Topological Spaces

Next, we want to characterize GC/UC functions further without using the value lattices at all. In the general case, this requires new kinds of structures, the square-neighborhood spaces and square-topological spaces. The following should be compared with the description of neighborhood spaces and topological spaces in Section 3.

A *square-neighborhood space* $(X, \mathcal{N}^2)$ is a set X with a map $\mathcal{N}^2$ from points x of X to neighborhood filters $\mathcal{N}^2(x) \subseteq \mathcal{P}X^2$ of (x, x). Thus, for each x of X, $\mathcal{N}^2(x)$ is a filter consisting of subsets of X^2 that all contain the pair (x, x). (There are no neighborhood filters for pairs (x_1, x_2) with $x_1 \neq x_2$.)

A square-neighborhood space $(X, \mathcal{N}^2)$ is *symmetric* if for all x in X, $R \in \mathcal{N}^2(x)$ implies $R^{\mathrm{op}} \in \mathcal{N}^2(x)$, where $R^{\mathrm{op}} = \{(y, x) \mid (x, y) \in R\}$ is the opposite relation of R.

A function $f : (X, \mathcal{N}^2_X) \to (Y, \mathcal{N}^2_Y)$ is *uniformly continuous* (UC) if for all x in X, $B \in \mathcal{N}^2_Y(fx)$ implies $f^{2-}B \in \mathcal{N}^2_X(x)$. This defines the category SqNbh of square-neighborhood spaces. We shall later see that square-neighborhood spaces generalize quasi-uniform spaces, which is the reason for the name "uniformly continuous". We shall also see that the UC functions between square-neighborhood spaces are closely related to the UC functions between similarity spaces so that having the same name for both is justified.

Let $(X, \mathcal{N}^2)$ be a square-neighborhood space. A subset O of X^2 is *open* if it is a neighborhood of all the diagonal elements in it, i.e. if $(x, x) \in O$ implies $O \in \mathcal{N}^2(x)$. An *open base* of $(X, \mathcal{N}^2)$ is a subset $\mathcal{B}$ of $\mathcal{P}X^2$ such that

(1) All B in $\mathcal{B}$ are open;
(2) For $A \in \mathcal{N}^2(x)$, there is B in $\mathcal{B}$ such that $x \in B \subseteq A$.

Not every square-neighborhood space has an open base. We say that $(X, \mathcal{N}^2)$ is *topological* if it has an open base. In this case, we speak of a *square-topological space*, but we refrain from describing it by something like a topology on X^2. Such a topology would not be uniquely determined since uniform continuity only refers to the points of the diagonal.

The category of square-topological spaces and uniformly continuous functions is called SqTop. A square-topological space is *countably based* if it has a countable open base.

11.3 Square-Topological Spaces from Similarity Spaces

From a similarity system $\mathcal{X} = (X, S, \sigma)$, we construct a square-neighborhood space $\mathrm{N}^2\mathcal{X} = (X, \mathcal{N}^2_{\mathcal{X}})$ by defining that A is in $\mathcal{N}^2_{\mathcal{X}}(x)$ if there is an open set u of S such that $(x, x) \in \sigma^- u \subseteq A$. This space is topological with open base $\{\sigma^- u \mid u \in \tau_S\}$. More generally, $\{\sigma^- u \mid u \in \mathcal{B}\}$ is an open base of $\mathrm{N}^2\mathcal{X}$ for every base $\mathcal{B}$ of the topology of S. Hence, $\mathrm{N}^2\mathcal{X}$ is countably based if S is ω-continuous.

If $\mathcal{X}$ is symmetric, then so is $\mathrm{N}^2\mathcal{X}$. For, $\sigma(x, y) = \sigma(y, x)$ implies $(\sigma^- u)^{\mathrm{op}} = \sigma^- u$, and so $(x, x) \in \sigma^- u \subseteq A$ implies $(x, x) \in \sigma^- u = (\sigma^- u)^{\mathrm{op}} \subseteq A^{\mathrm{op}}$.

Theorem 46 *Let $\mathcal{X}$ and $\mathcal{Y}$ be two similarity systems. A function $f : \mathcal{X} \to \mathcal{Y}$ is uniformly continuous if and only if $f : \mathrm{N}^2\mathcal{X} \to \mathrm{N}^2\mathcal{Y}$ is uniformly continuous.*

PROOF. First, let $f : \mathcal{X} \to \mathcal{Y}$ be UC (see Def. 6). For $A \in \mathcal{N}^2_{\mathcal{Y}}(fx)$, there is an open set v of $S_{\mathcal{Y}}$ such that $(fx, fx) \in \sigma^-_{\mathcal{Y}} v \subseteq A$. Then $\sigma_{\mathcal{Y}}(fx, fx) \in v$. By uniform continuity in the sense of similarity systems, there is an open set u

of $S_{\mathcal{X}}$ containing $\sigma_{\mathcal{X}}(x,x)$ and satisfying $\sigma_{\mathcal{X}}(x_1,x_2) \in u \Rightarrow \sigma_{\mathcal{Y}}(fx_1,fx_2) \in v$. Hence $(x,x) \in \sigma_{\mathcal{X}}^{-} u \subseteq f^{2-}(\sigma_{\mathcal{Y}}^{-} v) \subseteq f^{2-}A$, which shows $f^{2-}A \in \mathcal{N}^2_{\mathcal{X}}(x)$.

For the opposite direction, let $f : \mathrm{N}^2\mathcal{X} \to \mathrm{N}^2\mathcal{Y}$ be uniformly continuous. If $\sigma_{\mathcal{Y}}(fx,fx) \in v$ for some open v of $S_{\mathcal{Y}}$, then $(fx,fx) \in \sigma_{\mathcal{Y}}^{-} v$, whence $\sigma_{\mathcal{Y}}^{-} v \in \mathcal{N}^2_{\mathcal{Y}}(fx)$. By uniform continuity in the sense of square-neighborhood spaces, $f^{2-}(\sigma_{\mathcal{Y}}^{-} v)$ is in $\mathcal{N}^2_{\mathcal{X}}(x)$. This means that there is an open set u of $S_{\mathcal{X}}$ such that $(x,x) \in \sigma_{\mathcal{X}}^{-} u \subseteq f^{2-}(\sigma_{\mathcal{Y}}^{-} v)$. Thus $\sigma_{\mathcal{X}}(x,x) \in u$ and $\sigma_{\mathcal{X}}(x_1,x_2) \in u \Rightarrow \sigma_{\mathcal{Y}}(fx_1,fx_2) \in v$. □

If $\mathcal{X}$ and $\mathcal{X}'$ are equivalent similarity systems with point set X, Theorem 46 implies that $\mathrm{id}_X : \mathrm{N}^2\mathcal{X} \to \mathrm{N}^2\mathcal{X}'$ and $\mathrm{id}_X : \mathrm{N}^2\mathcal{X}' \to \mathrm{N}^2\mathcal{X}$ are uniformly continuous and thus $\mathrm{N}^2\mathcal{X}$ and $\mathrm{N}^2\mathcal{X}'$ are identical. Hence, N^2 is well-defined for similarity spaces, and we obtain:

Corollary 47 *The construction* N^2 *as defined above yields a full and faithful functor from* $\mathsf{Sim}_{\mathrm{GC}} = \mathsf{Sim}_{\mathrm{UC}}$ *to* SqTop.

11.4 Similarity Spaces from Square-Topological Spaces

Our ultimate goal is to show that N^2 is an isomorphism. Therefore, we now introduce a construction in the opposite direction.

Lemma 48 *For every (symmetric) square-topological space* $(X,\mathcal{N}^2)$ *with open base* $\mathcal{B}$*, there is a (symmetric) similarity system* $\mathcal{X} = \mathcal{S}^2(X,\mathcal{B})$ *such that* $\mathrm{N}^2\mathcal{X} = (X,\mathcal{N}^2)$*. If* $\mathcal{B}$ *is countable, then* $\mathcal{X}$ *is countably based.*

PROOF. In the symmetric case, we first replace $\mathcal{B}$ by $\mathcal{B}' = \{B \cap B^{\mathrm{op}} \mid B \in \mathcal{B}\}$. The sets $B \cap B^{\mathrm{op}}$ are still open since $(x,x) \in B \cap B^{\mathrm{op}}$ implies $(x,x) \in B$, hence $B \in \mathcal{N}^2(x)$, hence also $B^{\mathrm{op}} \in \mathcal{N}^2(x)$ and $B \cap B^{\mathrm{op}} \in \mathcal{N}^2(x)$. Since $B \cap B^{\mathrm{op}} \subseteq B$, $\mathcal{B}'$ is again an open base of $(X,\mathcal{N}^2)$. In the non-symmetric case, let $\mathcal{B}' = \mathcal{B}$.

Let $\mathcal{B}^{\cap}$ be the closure of $\mathcal{B}'$ under finite intersections, which is again an open base of $(X,\mathcal{N}^2)$. Let D be the algebraic lattice constructed from the meet-semilattice $(\mathcal{B}^{\cap},\subseteq)$ according to Prop. 24. For $x,x' \in X$, let $\sigma(x,x') = \{B \in \mathcal{B}^{\cap} \mid (x,x') \in B\}$, which is a filter. In the symmetric case, $\mathcal{B}'$ consists of symmetric relations, hence so does $\mathcal{B}^{\cap}$. Thus, in this case for all $B \in \mathcal{B}^{\cap}$, $(x,x') \in B$ iff $(x',x) \in B$, and so $\sigma(x',x) = \sigma(x,x')$.

Let $\mathcal{X} = (X,D,\sigma)$. Since $\{\langle B\rangle \mid B \in \mathcal{B}^{\cap}\}$ is a base for the Scott topology of D, $\{\sigma^{-}\langle B\rangle \mid B \in \mathcal{B}^{\cap}\}$ is an open base for $\mathrm{N}^2\mathcal{X}$. Now $(x,x') \in \sigma^{-}\langle B\rangle \Leftrightarrow \sigma(x,x') \in \langle B\rangle \Leftrightarrow B \in \sigma(x,x') \Leftrightarrow (x,x') \in B$. Hence, $\sigma^{-}\langle B\rangle = B$ and $\{\sigma^{-}\langle B\rangle \mid B \in \mathcal{B}^{\cap}\} = \mathcal{B}^{\cap}$. Thus, $\mathcal{B}^{\cap}$ is an open base of $\mathrm{N}^2\mathcal{X}$ and of the original space $(X,\mathcal{N}^2)$. Therefore, $\mathrm{N}^2\mathcal{X}$ and $(X,\mathcal{N}^2)$ are identical. □

For each open base $\mathcal{B}$ of $(X, \mathcal{N}^2)$, there is a different similarity system $\mathcal{X}_{\mathcal{B}}$ such that $\mathrm{N}^2 \mathcal{X}_{\mathcal{B}} = (X, \mathcal{N}^2)$. By Theorem 46, all these similarity systems are equivalent and thus represent the same similarity space, which we call $\mathbf{S}^2(X, \mathcal{N}^2)$. For every square-topological space Ξ, $\mathrm{N}^2\, \mathbf{S}^2\, \Xi = \Xi$ holds. On the other hand, for every similarity space $\mathbf{X}$, we have $\mathrm{N}^2\, \mathbf{S}^2\, \mathrm{N}^2\, \mathbf{X} = \mathrm{N}^2\, \mathbf{X}$, which implies $\mathbf{S}^2\, \mathrm{N}^2\, \mathbf{X} = \mathbf{X}$ by Theorem 46. By the same theorem, $f : \mathbf{S}^2\, \Xi \to \mathbf{S}^2\, \Upsilon$ is GC iff $f : \mathrm{N}^2\, \mathbf{S}^2\, \Xi \to \mathrm{N}^2\, \mathbf{S}^2\, \Upsilon$ is uniformly continuous, iff $f : \Xi \to \Upsilon$ is uniformly continuous.

Corollary 49 *The category* $\mathsf{Sim}_{\mathrm{GC}} = \mathsf{Sim}_{\mathrm{UC}}$ *of similarity spaces with globally or uniformly continuous functions is isomorphic to the category* SqTop *of square-topological spaces with uniformly continuous functions. The isomorphism restricts to the countably based spaces and to the symmetric spaces on both sides.*

11.5 The Self-Uniform Case

The self-uniform similarity spaces correspond to square-topological spaces in which all neighborhood filters $\mathcal{N}^2(x)$ are identical.

Lemma 50 *If* $\mathcal{X}$ *is a self-uniform similarity system, then* $\mathrm{N}^2\mathcal{X}$ *has the property* $\mathcal{N}^2(x) = \mathcal{N}^2(x')$ *for all* $x, x' \in |\mathcal{X}|$.

PROOF. Let $A \in \mathcal{N}^2(x)$. Then there is an open u of $S_{\mathcal{X}}$ such that $(x, x) \in \sigma^-\, u \subseteq A$. The membership $(x, x) \in \sigma^-\, u$ means $\sigma(x, x) \in u$. By self-uniformity, $\sigma(x', x') \in u$ follows, whence $(x', x') \in \sigma^-\, u \subseteq A$, and so $A \in \mathcal{N}^2(x')$. □

Lemma 51 *Let* $(X, \mathcal{N}^2)$ *be a square-neighborhood space such that all neighborhood filters* $\mathcal{N}^2(x)$ *are identical. Then* $(X, \mathcal{N}^2)$ *is topological, and for every (countable) open base* $\mathcal{B}$ *of* $(X, \mathcal{N}^2)$*, there is a (countable) open base* $\mathcal{B}'$ *such that* $(X, D, \sigma) = \mathcal{S}^2(X, \mathcal{B}')$ *has the property* $\sigma(x, x) = \top_D$ *for all* x *in* X*. Such a system is self-uniform and selfish.*

PROOF. Let $\mathcal{N}$ be the common value of $\mathcal{N}^2(x)$ if $X \neq \emptyset$, and $\mathcal{N} = \{\emptyset\}$ if $X = \emptyset$. Then all sets $O \in \mathcal{N}$ are open since $(x, x) \in O$ implies $O \in \mathcal{N} = \mathcal{N}^2(x)$ (and there is nothing to show in case $X = \emptyset$). Thus, $\mathcal{N}$ is an open base of $(X, \mathcal{N}^2)$. If $\mathcal{B}$ is an arbitrary (countable) open base, then $\mathcal{B}' = \mathcal{B}^{\cap} \cap \mathcal{N}$ is a (countable) open base, too, which is closed under finite intersection. The construction of Lemma 48 applied to $\mathcal{B}'$ yields $\sigma(x, x) = \{B \in \mathcal{B}' \mid (x, x) \in B\}$. Since $\mathcal{B}' \subseteq \mathcal{N} = \mathcal{N}^2(x)$, $(x, x) \in B$ holds for all $B \in \mathcal{B}'$, whence $\sigma(x, x) = \mathcal{B}'$, which is the top element of the algebraic lattice D obtained from $\mathcal{B}'$ by Prop. 24. □

Lemmas 50 and 51 show that the isomorphism of Cor. 49 relates self-uniform similarity spaces to those square-topological spaces in which all neighborhood filters $\mathcal{N}^2(x)$ are identical. The description of the latter can be simplified by noting only the common value of $\mathcal{N}^2(x)$ (or $\{\emptyset\}$ if $X = \emptyset$). This is a filter consisting of subsets of X^2 that all contain all pairs (x,x), $x \in X$, i.e. contain the entire diagonal.

Definition 52 *A* diagonal-neighborhood space $(X, \mathcal{N}^\Delta)$ *is given by a point set X and a neighborhood filter $\mathcal{N}^\Delta \subseteq \mathcal{P}X^2$ of the diagonal Δ_X of X^2. It is symmetric if $R \in \mathcal{N}^\Delta$ implies $R^{\mathrm{op}} \in \mathcal{N}^\Delta$. A* base $\mathcal{B}$ *of $(X, \mathcal{N}^\Delta)$ is a subset of $\mathcal{N}^\Delta$ such that for all U in $\mathcal{N}^\Delta$, there is B in $\mathcal{B}$ with $B \subseteq U$. A function $f : (X, \mathcal{N}^\Delta_X) \to (Y, \mathcal{N}^\Delta_Y)$ is* uniformly continuous *if $B \in \mathcal{N}^\Delta_Y$ implies $f^{2-}B \in \mathcal{N}^\Delta_X$. The resulting category is called* $\mathsf{DiagNbh}$.

It is quite obvious that $\mathsf{DiagNbh}$ is isomorphic to the subcategory of square-neighborhood spaces in which all neighborhood filters $\mathcal{N}^2(x)$ are identical. Such square-neighborhood spaces are topological by Lemma 51. Combining everything one obtains:

Corollary 53 *The category $\mathsf{Sim}^{\mathrm{SUn}}_{\mathrm{GC}}$ of self-uniform similarity spaces with globally continuous functions is isomorphic to the category $\mathsf{DiagNbh}$ of diagonal-neighborhood spaces with uniformly continuous functions. The isomorphism restricts to the countably based spaces and to the symmetric spaces on both sides.*

We write N^Δ for the functor from $\mathsf{Sim}^{\mathrm{SUn}}_{\mathrm{GC}}$ to $\mathsf{DiagNbh}$.

When going from a self-uniform system to the induced diagonal-neighborhood space and then back via Lemma 51, one obtains an equivalent system in which the common value of $\sigma(x,x)$ is the top element of the value lattice.

Proposition 54 *Every (symmetric) self-uniform similarity system is equivalent to a (symmetric) system (X, S, σ) such that $\sigma(x,x) = \top_S$ for all x in X. Every (symmetric) self-uniform similarity space is jointly (symmetric and) self-uniform and selfish.*

11.6 Quasi-Uniform Spaces

The diagonal-neighborhood spaces of Def. 52 are an obvious generalization of quasi-uniform spaces.

Definition 55 *A* quasi-uniform space *is a diagonal-neighborhood space $(X, \mathcal{N}^\Delta)$ with the additional condition that for all U in $\mathcal{N}^\Delta$, there is V in $\mathcal{N}^\Delta$ such that $V \circ V \subseteq U$. In this case, $\mathcal{N}^\Delta$ is called a* quasi-uniformity. *The category of quasi-uniform spaces with uniformly continuous functions is called* QUnif. *A* uniform space *is a symmetric quasi-uniform space. In this case, we speak of a uniformity and the corresponding category is called* Unif.

Here, '$\circ$' denotes relational composition. Composition on $\mathcal{P}X^2$ is associative and monotonic w.r.t. $\subseteq$, and $\Delta = \{(x,x) \mid x \in X\}$ is its neutral element. For all $U, V \in \mathcal{N}^\Delta$, $U = U \circ \Delta \subseteq U \circ V$ and $V = \Delta \circ V \subseteq U \circ V$ holds.

Proposition 56 *If $\mathcal{X}$ is a self-uniform weakly globally transitive similarity system, then $\mathrm{N}^{\Delta}\mathcal{X}$ is a quasi-uniform space.*

PROOF. Let $\mathcal{X} = (X, S, \sigma)$ and $\mathrm{N}^{\Delta}\mathcal{X} = (X, \mathcal{N}^{\Delta})$. If $X = \emptyset$, then $\mathcal{N}^{\Delta} = \{\emptyset\}$ is a quasi-uniformity. Otherwise, let $a \in S$ be the common value of $\sigma(x, x)$, $x \in X$. Since $\mathcal{X}$ is weakly globally transitive, there is a continuous operation $* : S \times S \to S$ such that (Tr1) $\sigma(x, y) * \sigma(y, z) \le \sigma(x, z)$ and (Tr2W) $\sigma(x, x) * \sigma(x, x) = \sigma(x, x)$, i.e. $a * a = a$.

For $U \in \mathcal{N}^{\Delta}$, there is an open u of S such that $a \in u$ and $\sigma^{-}\, u \subseteq U$. Since $a = a * a \in u$, there is an open v of S such that $a \in v$ and $v * v \subseteq u$. Let $V = \sigma^{-}\, v$. Then $V \in \mathcal{N}^{\Delta}$, and we show $V \circ V \subseteq U$. If $(x, z) \in V \circ V$, there is y such that $(x, y) \in V$ and $(y, z) \in V$, hence $\sigma(x, y) \in v$ and $\sigma(y, z) \in v$, and so $\sigma(x, y) * \sigma(y, z) \in v * v \subseteq u$. By (Tr1), $\sigma(x, z) \in u$ follows, hence $(x, z) \in U$. □

It is well-known that every pseudo-quasi-metric $\delta : X \times X \to \mathbb{R}^{+}$ induces a quasi-uniformity $\mathcal{U}$ on X by saying that $U \in \mathcal{U}$ if there is $r > 0$ such that $\{(x, y) \mid \delta(x, y) < r\} \subseteq U$. This is a special case of our general construction N^{Δ} since $\{(x, y) \mid \delta(x, y) < r\} = \delta^{-}[0, r)$, and the sets $[0, r)$ form a base of the Scott topology of $[0, \infty]^{\mathrm{op}}$.

For the opposite direction, i.e. the construction of a similarity space from a quasi-uniform space, we employ classical results on metrization of uniform spaces.

Define a *generalized pseudo-quasi-metric* on X to be a function $\delta : X \times X \to {\mathbb{R}^{+}}^{I}$ for some index set I with the properties $\delta(x, x) = 0$ and $\delta(x, z) \le \delta(x, y) + \delta(y, z)$ (understood coordinate-wise with the ordering of $\mathbb{R}^{+}$). A *generalized pseudo-metric* is in addition symmetric. The case $I = \mathbf{1}$ (a singleton set) leads back to ordinary pseudo-(quasi-)metrics.

We say that a generalized pseudo-quasi-metric is *bounded* if $\delta(x, y) \le 1$ (coordinate-wise) for all $x, y \in X$. A bounded generalized pseudo-quasi-metric space can be considered as a self-uniform globally transitive similarity system in two equivalent ways: with $S = ([0, \infty]^{\mathrm{op}})^{I}$ and addition as global composition $*$, or with $S = ([0, 1]^{\mathrm{op}})^{I}$ and truncated addition $a * b = (\min_{\mathbb{R}}(1, a_i + b_i))_{i \in I}$. In both cases, $*$ is commutative, associative, and has neutral element 0. In the non-generalized case ($I = \mathbf{1}$), these similarity systems are countably based.

Kelley [6, Chapter 6, Lemma 12] presents the following "Metrization Lemma":

Lemma 57 *Let X be a set and $(U_n)_{n \in \mathbb{N}}$ a sequence of subsets of X^2 such that each U_n includes the diagonal, $U_0 = X^2$, and $U_{n+1} \circ U_{n+1} \circ U_{n+1} \subseteq U_n$ for each n in $\mathbb{N}$. Then there is a bounded pseudo-quasi-metric $\delta : X^2 \to [0, 1]$ such that $U_n \subseteq \delta^{-}[0, 2^{-n}) \subseteq U_{n-1}$ for all $n > 0$. If each U_n is symmetric, then δ can be chosen as a bounded pseudo-metric.* □

Theorem 58 *Every (quasi-)uniformity is induced by a generalized pseudo-(quasi-)metric. Every countably based (quasi-)uniformity is induced by an ordinary pseudo-(quasi-)metric.*

PROOF. (In this proof, $\leq$ and $<$ always refer to the standard (strict) ordering of $\mathbb{R}$.) Let $(X, \mathcal{U})$ be the given (quasi-)uniform space. Let $\{(U_n^i)_{n\in\mathbb{N}} \mid i \in I\}$ be a set of sequences as in Lemma 57 such that all U_n^i are in $\mathcal{U}$ and $\{U_n^i \mid n \in \mathbb{N}, i \in I\}$ is a base of $\mathcal{U}$. Such a set exists since for every U in $\mathcal{U}$, there is a suitable sequence $(U_n)_{n\in\mathbb{N}}$ with $U_1 = U$. If $\mathcal{U}$ has a countable base $\{B_1, B_2, \ldots\}$, then a single sequence suffices (i.e. $I = \mathbf{1}$ can be chosen): take $U_0 = X^2$ and $U_{n+1} \in \mathcal{U}$ such that $U_{n+1} \circ U_{n+1} \circ U_{n+1} \subseteq U_n$ and $U_{n+1} \subseteq B_{n+1}$. If $(X, \mathcal{U})$ is symmetric, then all U_n^i can be chosen to be symmetric (replace U_n^i by $U_n^i \cap (U_n^i)^{\mathrm{op}}$).

Now choose a bounded pseudo-(quasi-)metric $\delta_i : X^2 \to [0,1]$ for every sequence $(U_n^i)_{n\in\mathbb{N}}$ as in the Metrization Lemma 57, and let $\delta : X^2 \to [0,1]^I$ be given by $\delta(x,y) = (\delta_i(x,y))_{i\in I}$, which defines a bounded generalized pseudo-(quasi-)metric. Let $\mathcal{U}'$ be the (quasi-)uniformity induced by δ. We claim $\mathcal{U}' = \mathcal{U}$.

For $\mathcal{U}' \subseteq \mathcal{U}$, it is sufficient to show $\delta^- u \in \mathcal{U}$ for all u taken from a subbase of $[0,1]^I$. A suitable subbase is $\{u_r^i \mid i \in I, r > 0\}$ where $u_r^i = \{x \in [0,1]^I \mid x_i < r\}$. Let n be large enough such that $2^{-n} \leq r$. Then $\delta^- u_r^i = \delta_i^- [0, r) \supseteq \delta_i^- [0, 2^{-n}) \supseteq U_n^i \in \mathcal{U}$, hence $\delta^- u_r^i \in \mathcal{U}$.

For $\mathcal{U} \subseteq \mathcal{U}'$, it is sufficient to show $\{U_n^i \mid n \in \mathbb{N}, i \in I\} \subseteq \mathcal{U}'$. Since $U_n^i \supseteq \delta_i^- [0, 2^{-(n+1)}) = \delta^- u^i_{2^{-(n+1)}} \in \mathcal{U}'$, $U_n^i \in \mathcal{U}'$ follows. $\square$

Prop. 56 and Theorem 58 together show how to "improve" a given self-uniform weakly globally transitive similarity system with no particular algebraic properties for its composition operation $*$: Going to the induced quasi-uniform space by Prop. 56 and back by Theorem 58 yields an equivalent selfish self-uniform globally transitive similarity system with value lattice $S = ([0,1]^{\mathrm{op}})^I$ or $S = ([0,\infty]^{\mathrm{op}})^I$, and (truncated) addition as composition, which is commutative, associative, and has neutral element $0 = \top_S$, which is also the common value of $\sigma(x,x)$.

Prop. 56 and Theorem 58 also imply the following categorical isomorphisms:

Corollary 59 *The category of self-uniform (weakly) globally transitive similarity spaces with globally continuous functions is isomorphic to the category* QUnif *of quasi-uniform spaces with uniformly continuous functions. The isomorphism restricts to the countably based spaces and to the symmetric spaces on both sides. (Symmetric quasi-uniform spaces are uniform spaces.)*

12 Conclusion and Future Work

Similarity spaces in their full generality are probably too general to be useful. For many applications, properties such as global or local transitivity are certainly needed, and sometimes additional properties such as symmetry and self-uniformity will be useful. Effective versions of the theory will require a countable base, and maybe a generalization of the separability property known from metric spaces.

Our locally transitive similarity spaces include quasi-uniform spaces, generalized pseudo-quasi-metric spaces and partial metric spaces. All these classes come with notions of convergence, completeness, and completion (see [2,3] for generalized quasi-metrics and [10,11] for partial metrics). These notions should be extended to some class of transitive similarity spaces and thereby unified if possible. (Note however that already for ordinary quasi-metric and quasi-uniform spaces various different notions of completeness and completion exist [9,12].)

The category of similarity spaces with globally continuous functions and its various subcategories given by symmetric spaces, globally transitive spaces etc. should be examined for constructions such as products, subspaces, sums, quotients, power spaces, and function spaces. This has been done to some extent in [2,3,13], but for categories with a fixed value lattice and non-expanding functions [2,3], or a restricted class of non-expanding functions [13].

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