Separation Classes of Equilogical Spaces

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Abstract

The category TOP of topological spaces is not cartesian closed, but can be embedded into the cartesian closed category ASSM of *assemblies* over algebraic lattices, which is a generalisation of Scott's category EQU of equilogical spaces. In this paper, we identify cartesian closed subcategories of assemblies which correspond to well-known separation properties of topology: $\mathcal{T}_0$, $\mathcal{T}_1$, Hausdorff, completely Hausdorff, totally disconnected, completely regular, zero-dimensional.

1 Introduction

It is well-known that the categories TOP of *topological spaces* and TOP_0 of $\mathcal{T}_0$ topological spaces are not cartesian closed, i.e., do not admit a function space construction such that typed λ-calculus could be interpreted in the category. In December 1996, Dana Scott proposed to remedy the situation by extending TOP_0 to the cartesian closed category EQU of equilogical spaces [12, 1, 4]. There are several equivalent descriptions of EQU; one of them is by *modest sets over algebraic lattices* [1]. *Assemblies* are an immediate generalisation of modest sets; the cartesian closed category ASSM of assemblies over algebraic lattices contains the whole of TOP, not just TOP_0.

In Section 2, we recall the definitions of assemblies and modest sets from [1]. The corresponding categories are complete, co-complete, and cartesian closed [1]. Here, we show a slightly stronger property of ASSM: it is a concrete category with initial structures for all sources and final structures for all sinks.

In Section 3, we consider the classes of (S, T)-*separated assemblies.* These classes are parametrised by a topological $\mathcal{T}_0$-space S and a subset T of S. All these classes are closed under subspace, product, and exponentiation. Suitable choices of S and

T yield classes whose intersections with TOP yield the classes of $\mathcal{T}_0$, $\mathcal{T}_1$, Hausdorff, completely Hausdorff, and totally disconnected spaces.

(S, S)-separated spaces correspond to the S-spaces [11] or S-posets [13] in Synthetic Domain Theory. These are also characterised by the fact that the canonical map $\varepsilon_{\mathcal{X}} : \mathcal{X} \to [[\mathcal{X} \to S] \to S]$ is injective. People working with cartesian closed topological categories usually are more interested in the class of spaces $\mathcal{X}$ where $\varepsilon_{\mathcal{X}}$ is initial [5, 8]. In Section 4, we study both notions (and many more) at once by considering classes of S-$\mathcal{E}$-spaces characterised by the fact that $\varepsilon_{\mathcal{X}}$ is in some class $\mathcal{E}$ of arrows. If $\mathcal{E}$ has suitable properties, S-$\mathcal{E}$-spaces are closed under product and exponentiation. The class of Ω-initial spaces (where Ω is Sierpinski space) contains all topological spaces and forms the cartesian closed topological hull of TOP [8]. In the final Section 5, we present some remarks on injective objects in this category.

Notation. For $f : X \to Y$ and $A \subseteq X$, let $f^+A = \{fa \mid a \in A\}$, and for $B \subseteq Y$, let $f^-B = \{a \in X \mid fa \in B\}$.

2 The Category of Assemblies

In [12], Dana Scott introduced *equilogical spaces* as equivalence relations on topological $\mathcal{T}_0$ spaces. In [1], Scott's group showed that the category EQU of equilogical spaces is equivalent to the category of *modest sets* (on algebraic lattices), and also considered the larger category ASSM of *assemblies* (on algebraic lattices). In contrast to the original description of EQU, morphisms in these categories are (set-theoretic) functions. The category ASSM has the further advantage that it includes all topological spaces, not just the $\mathcal{T}_0$ spaces as EQU does. Hence, we work with ASSM in this paper.

2.1 Assemblies

The following definition is taken from [1]. An *assembly* $\mathcal{X} = (X, L, E)$ consists of a *carrier set* X, an algebraic lattice L, and a function E from X to the powerset of L such that $E(x)$ is non-empty for all x in X. The elements of $E(x)$ are *realisers* for x. For $A \subseteq X$, let $E(A) = \bigcup_{x \in A} E(x)$; in particular, $E(X)$ is the set of all realisers.

Remark: One could also employ continuous lattices as done by Hyland in [9, Section 2], or else use algebraic or continuous bounded-complete dcpo's D with the additional requirement that $E(X)$ be dense in D (see [1]). □

Given two assemblies (X, L, E) and (Y, M, F), a function $f : X \to Y$ is *realisable* if there is a (Scott) continuous function $\varphi : L \to M$ such that $a \in E(x)$ implies $\varphi a \in F(fx)$. Let ASSM be the category of assemblies and realisable functions.

Two assemblies $\mathcal{X}$ and $\mathcal{Y}$ are *equivalent* ($\mathcal{X} \cong \mathcal{Y}$) if they have the same carrier set, and the set-theoretic identity on this set is realisable in both directions. Two equivalent assemblies are isomorphic in the usual category-theoretic sense, but not vice-versa.

In the definition of equivalence, it is not required that the two realisers compose to the identities on the two lattices. Hence, the lattices of two equivalent assemblies may be of radically different size and shape. This is quite different from many familiar categories such as topological spaces or groups where equivalent objects would be equal.

As an example, note that all assemblies $(\{x\}, L, E)$ with the same singleton carrier are equivalent. A realiser from $(\{x\}, L, E)$ to $(\{x\}, M, F)$ is given by a constant function from L to M with a value in the non-empty set $F(x) \subseteq M$.

An assembly (X, L, E) is *modest* (a modest set) if all the sets $E(x)$ are disjoint, i.e., if $E(x) \cap E(x') \neq \emptyset$ implies $x = x'$. If $\mathcal{X} \cong \mathcal{Y}$ and $\mathcal{X}$ is modest, then so is $\mathcal{Y}$. The full subcategory MOD of modest sets (on algebraic lattices) is equivalent to Scott's category EQU of equilogical spaces [1].

2.2 Initial Assemblies

In [1], it is shown that ASSM has finite limits, i.e., finite products and equalisers. The construction of finite products obviously generalises to a construction of infinite products, but even more can be said:

Proposition 2.1 In ASSM, there are initial structures for all sources. In MOD, there are initial structures for jointly injective sources.

Proof: Let Y be a set, $(\mathcal{Z}_i)_{i\in I} = (Z_i, N_i, G_i)_{i\in I}$ a family of assemblies, and $(g_i : Y \to Z_i)_{i\in I}$ a family of functions. Then define $\mathcal{Y} = (Y, M, F)$ with $M = \prod_{i\in I} N_i$ and $F(y) = \prod_{i\in I} G_i(g_i y)$ for y in Y. By choice, the sets $F(y)$ are non-empty. The functions $g_i : \mathcal{Y} \to \mathcal{Z}_i$ are realised by the projections $\pi_i : M \to N_i$. If $\mathcal{X} = (X, L, E)$ is another assembly, and $f : \mathcal{X} \to \mathcal{Y}$ is a function such that all $g_i \circ f : \mathcal{X} \to \mathcal{Z}_i$ are realised by $\psi_i : L \to N_i$, then f is realised by the corresponding tuple function $\psi : L \to M$.

If all $\mathcal{Z}_i$ are modest, and $g_i y_1 = g_i y_2$ for all i in I implies $y_1 = y_2$ (joint injectivity), then $\mathcal{Y}$ is modest. □

This property is part of the definition of a *topological category* (see e.g., [10]). The other requirements are that for any set X, the subcategory of objects with carrier X is equivalent to a small category, and even to a one-object category if X is a singleton. While the latter holds for ASSM (all singleton assemblies are equivalent), the former does not hold as ASSM is not well-powered.

Products are special cases of initial structures with $Y = \prod_{i\in I} Z_i$ and $g_i = \pi_i$, and so are *subspaces* ($I = \{1\}$, $Y \subseteq Z_1$, g_1 the inclusion map). The *equaliser* of $f, g : \mathcal{X} \to \mathcal{Y}$ can be obtained as the subspace $\{x \in \mathcal{X} \mid fx = gx\}$ of $\mathcal{X}$.

In ASSM, a morphism is monic if and only if the underlying set-function is *injective*. A morphism $g : \mathcal{Y} \to \mathcal{Z}$ is *initial* if $\mathcal{Y}$ is the initial structure for g, i.e., if for all assemblies $\mathcal{X}$ and $f : \mathcal{X} \to \mathcal{Y}$, the realisability of $g \circ f$ implies the realisability of f. (In TOP, the initial morphisms $g : Y \to Z$ are exactly those continuous functions such that each open of Y is the inverse image of an open of Z.)

An *embedding* is an injective initial morphism. Surjective embeddings are isomorphisms. Each subspace embedding is an embedding, and conversely, if $f : \mathcal{X} \to \mathcal{Y}$ is an embedding, then $\mathcal{X}$ is isomorphic to the subspace $f^+(\mathcal{X})$ of $\mathcal{Y}$.

2.3 Exponentials

For $\mathcal{X} = (X, L, E)$ and $\mathcal{Y} = (Y, M, F)$, the function space $[\mathcal{X} \to \mathcal{Y}]$ is $(Z, [L \to M], G)$ where Z is the set of realisable functions from $\mathcal{X}$ to $\mathcal{Y}$, $[L \to M]$ is the algebraic lattice of Scott continuous functions from L to M, and $G(f)$ is the set of realisers of f. With this construction, ASSM becomes a cartesian closed category [1].

MOD is closed under exponentiation, i.e., if $\mathcal{X}$ is in ASSM and $\mathcal{Y}$ in MOD, then $[\mathcal{X} \to \mathcal{Y}]$ (formed in ASSM) is in MOD.

2.4 Final Assemblies

In a topological category, final structures for all sinks automatically exist [10]. As ASSM is not quite a topological category, the existence of final structures has to be verified directly.

Proposition 2.2 In ASSM and MOD, there are final structures for all sinks.

Proof: Let $(\mathcal{X}_i)_{i \in I} = (X_i, L_i, E_i)_{i \in I}$ be a family of assemblies, Y a set, and $(f_i : X_i \to Y)_{i \in I}$ a family of functions. For the moment, let us assume that this family is jointly surjective, i.e., $Y = \bigcup_{i \in I} f_i^+(X_i)$. Let $M = \mathcal{P}I \times \prod_{i \in I} L_i$ where $\mathcal{P}I$ is the powerset of I. For i in I and a in L_i, let $\sigma_i a = (\{i\}, (i \mapsto a, j \neq i \mapsto \bot)) \in M$. Finally, for y in Y, let $F(y) = \bigcup_{i \in I} \sigma_i^+(E_i(f_i^- \{y\}))$, which is non-empty by joint surjectivity. Then $\mathcal{Y} = (Y, M, F)$ is an assembly, and each f_i is realised by the continuous function σ_i. Note that $\mathcal{Y}$ is modest if all $\mathcal{X}_i$ are modest. If $\mathcal{Z} = (Z, N, G)$ is another assembly, and $g : \mathcal{Y} \to \mathcal{Z}$ a function such that $g \circ f_i : \mathcal{X}_i \to \mathcal{Z}$ are realised by continuous functions $\psi_i : L_i \to N$, then g is realised by the continuous function $\psi : M \to N$ defined by $\psi(S, b) = \bigvee_{i \in S} \psi_i b_i$.

If the sink $(f_i : X_i \to Y)$ is not jointly surjective, it can be extended to a jointly surjective sink by adding all constant functions $\lambda * .y : \{*\} \to Y$. Since constant functions are automatically realisable, the final assembly for the extended sink is also final for the original sink. $\square$

Special cases of final structures are *disjoint sums* (coproducts) and *quotients*. *Final maps* are defined dually to initial maps, and *quotient maps* are surjective final maps.

2.5 Topological Spaces as Assemblies

The following generalises the embedding of $\mathcal{T}_0$ topological spaces into modest sets from [1].

For a topological space $(X, \mathcal{O})$, we use the powerset $\mathcal{PO}$ as the algebraic lattice (actually, the powerset of a subbasis would suffice). The basic open sets of the Scott topology $\mathcal{S}$ of $\mathcal{PO}$ are the sets $\uparrow\{O_1, \ldots, O_n\}$ with O_i in $\mathcal{O}$.

Define a function $e : X \to \mathcal{PO}$ by $ex = \mathcal{O}(x) = \{O \in \mathcal{O} \mid x \in O\}$. We have $ex \in \uparrow\{O_1, \ldots, O_n\}$, iff all O_i are in ex, iff $x \in O_1 \cap \cdots \cap O_n$, and thus, $e : (X, \mathcal{O}) \to (\mathcal{PO}, \mathcal{S})$ is continuous, and $e^- : \mathcal{S} \to \mathcal{O}$ is surjective, i.e., e is initial (in TOP).

It is well-known that for any embedding $e : X \to Y$ in TOP, any continuous function from X to a continuous lattice L can be extended to a continuous function from Y to L [6]. It is easy to see that this extension property holds for initial mappings as well.

Using e, the assembly $\mathrm{A_T}(X, \mathcal{O})$ belonging to $(X, \mathcal{O})$ is defined as $(X, \mathcal{PO}, E)$ where $E(x) = \{ex\}$. The above-mentioned extension property ensures that any continuous $f : X \to Y$ is realisable from $\mathrm{A_T}X$ to $\mathrm{A_T}Y$. Since continuity of $e_Y \circ f$ implies continuity of f by initiality, the converse also holds; therefore, $\mathrm{A_T}$ is a full and faithful functor from TOP to ASSM. Clearly, $\mathrm{A_T}X$ is modest if and only if X is $\mathcal{T}_0$. Hence, $\mathrm{A_T}$ cuts down to a full and faithful functor from TOP_0 to MOD.

The particular choice of e and $\mathcal{PO}$ does not matter. If $e_1 : X \to L_1$ and $e_2 : X \to L_2$ are two initial mappings to algebraic lattices L_1 and L_2, then the extension property implies the equivalence of the two assemblies $(X, L_1, x \mapsto \{e_1x\})$ and $(X, L_2, x \mapsto \{e_2x\})$. In particular, it follows that for any algebraic lattice L with its Scott topology, $\mathrm{A_T}L$ is equivalent to the assembly $(L, L, x \mapsto \{x\})$.

In the remainder of this paper, we shall often identify X and $\mathrm{A_T}X$ (or something equivalent to $\mathrm{A_T}X$). We use the symbol Ω for the Sierpinski space $\{0, 1\}$ with open sets $\emptyset$, $\{1\}$, and Ω, for the algebraic lattice $\{0, 1\}$ with $0 \leq 1$, and for the assembly $(\Omega, \Omega, x \mapsto \{x\})$. This assembly can be used to define a topology on (the carrier set of) an equilogical space $\mathcal{X}$ as follows: A subset O of $\mathcal{X}$ is *open* iff its characteristic function $\chi_O : \mathcal{X} \to \Omega$ is realisable. We call the resulting topological space $\mathrm{T_A}\mathcal{X}$.

If $f : \mathcal{X} \to \mathcal{Y}$ is realisable, then $f : \mathrm{T_A}\mathcal{X} \to \mathrm{T_A}\mathcal{Y}$ is continuous. The resulting functor $\mathrm{T_A} : \mathrm{ASSM} \to \mathrm{TOP}$ is *left adjoint* to $\mathrm{A_T} : \mathrm{TOP} \to \mathrm{ASSM}$. Beware that $\mathrm{T_A}\mathcal{X}$ is not necessarily $\mathcal{T}_0$ if $\mathcal{X}$ is modest, i.e., $\mathrm{T_A}$ does *not* cut down to a functor from MOD to TOP_0.

For any topological space X, $\mathrm{T_A}(\mathrm{A_T}X)$ is X. Thus TOP is equivalent to a reflective subcategory of ASSM, consisting of *topological* assemblies characterised by $\mathcal{X} \cong \mathrm{A_T}(\mathrm{T_A}\mathcal{X})$. An assembly (X, L, E) with the property that all sets $E(x)$ are singletons is topological, and any topological assembly is equivalent to an assembly with this property.

The subcategory of topological assemblies is closed under initial structures, hence closed under product and subspace, and closed under disjoint sums. It is *not* closed under quotients; on the contrary we have:

Proposition 2.3 Each assembly is a quotient of a topological assembly.

Proof: Given $\mathcal{X} = (X, L, E)$, let Y be the set $\sum_{x \in X} E(x)$, and $F(x, a) = \{a\}$ for a in $E(x)$. Then $\mathcal{Y} = (Y, L, F)$ is a topological assembly, and $q : \mathcal{Y} \to \mathcal{X}$ with $q(x, a) = x$ is a quotient map realised by the identity on L. □

Clearly, $\mathcal{Y}$ is modest ($= \mathcal{T}_0$) if $\mathcal{X}$ is modest. Thus, the proposition generalises the well-known fact that each equilogical space (in the sense of Scott) is a quotient of a topological $\mathcal{T}_0$ space.

3 (S, T)-separated Assemblies

Several notions of separation can be described in a uniform way as follows: Let S be a $\mathcal{T}_0$ topological space, or rather, the corresponding modest set, and let T be a subset of (the carrier set) of S. An assembly $\mathcal{X}$ is (S, T)-*separated* if for all $x \neq x'$ in $\mathcal{X}$, there is a realisable function $f : \mathcal{X} \to S$ such that $fx \neq fx'$ and $fx, fx' \in T$.

In case of $T = S$, one need not refer to T at all. The resulting notion of (S, S)-separated assemblies is analogous to the notion of S-spaces [11] or S-posets [13] in Synthetic Domain Theory.

Since $\mathrm{T_A}$ is left adjoint to $\mathrm{A_T}$ and S is topological, we immediately obtain:

Proposition 3.1 $\mathcal{X}$ is (S, T)-separated iff $\mathrm{A_T}(\mathrm{T_A}\mathcal{X})$ is (S, T)-separated.

Thus, (S, T)-separation of $\mathcal{X}$ really is a property of the induced topological space $\mathrm{T_A}\mathcal{X}$. In the definition given above, $\mathcal{X}$ may be replaced by $\mathrm{T_A}\mathcal{X}$, and 'realisable function' by 'continuous function'. With this observation, several (S, T)-separation classes can be characterised:

- Let Ω be Sierpinski space. $\mathcal{X}$ is (Ω, Ω)-separated iff the points of $\mathrm{T_A}\mathcal{X}$ can be distinguished by continuous functions to Ω, i.e., open sets. Thus, $\mathcal{X}$ is (Ω, Ω)-separated iff $\mathrm{T_A}\mathcal{X}$ is $\mathcal{T}_0$.
- Let S be the four-point lattice $\{\bot, 0, 1, \top\}$ with incomparable elements 0 and 1, topologised by the Alexandroff or Scott topology (thus, $S \cong \Omega \times \Omega$), and let $T = \{0, 1\}$. If $\mathcal{X}$ is (S, T)-separated, then $\mathrm{T_A}\mathcal{X}$ is $\mathcal{T}_1$. For, if $x \neq x'$, then there is a continuous $f : \mathrm{T_A}\mathcal{X} \to S$ such that w.l.o.g. $fx = 0$ and $fx' = 1$. Then the open set $f^{-}\{0, \top\}$ contains x but not x', and the open set $f^{-}\{1, \top\}$ contains x' but not x. Conversely, if $\mathrm{T_A}\mathcal{X}$ is $\mathcal{T}_1$, then for $x \neq x'$, $\{x\}$ and $\{x'\}$ are disjoint closed sets, and the function f that maps x to 0, x' to 1, and the remainder to $\top$ is a continuous (S, T)-separating function.
- The last argument shows that $\bot$ is not neeeded. Thus, $\mathcal{X}$ is $(\{\bot, 0, 1, \top\}, \{0, 1\})$-separated, iff it is $(\{0, 1, \top\}, \{0, 1\})$-separated, iff $\mathrm{T_A}\mathcal{X}$ is $\mathcal{T}_1$.

- On the other hand, the omission of the top element from the four-point lattice yields *Hausdorff* separation. If $\mathcal{X}$ is $(\{\perp, 0, 1\}, \{0,1\})$-separated, and f is a continuous separating function for $x \neq x'$, then $f^{-}\{0\}$ and $f^{-}\{1\}$ are the required disjoint opens. Conversely, if $x \in O$ and $x' \in O'$ with $O \cap O' = \emptyset$, then the function mapping O to 0, O' to 1, and the remainder to $\perp$ is well-defined, continuous, and separating.
- Now let S be the unit interval $[0,1]$ with the standard Hausdorff topology. In this case, it does not matter whether $T = S$ or $T = \{0,1\}$. In both cases, $\mathcal{X}$ is (S,T)-separated iff $\mathrm{T_A}\mathcal{X}$ is *completely Hausdorff.* This is a less well-known property that strengthens Hausdorff separation in the same way as complete regularity strengthens regularity.
- Finally, if $\mathbf{2}$ is the discrete space $\{0,1\}$, then $\mathcal{X}$ is $(\mathbf{2},\mathbf{2})$-separated iff $\mathrm{T_A}\mathcal{X}$ is *totally disconnected*, i.e., for every $x \neq x'$ there is a clopen (closed and open) set which contains x, but not x'.

It is easy to see that the separation classes enumerated above contain each other in the order of enumeration ((Ω,Ω)-separated assemblies form the largest class and $(\mathbf{2},\mathbf{2})$-separated assemblies the smallest). Since for topological spaces X, $\mathrm{T_A}(\mathrm{A_T}X) = X$ holds, the intersections of the enumerated (S,T)-separation classes with TOP yield exactly the corresponding topological separation classes.

Proposition 3.2 Each (Ω,Ω)-separated assembly is modest. There are modest sets which are not (Ω,Ω)-separated.

Proof: If $\mathcal{X} = (X, L, E)$ and $f : X \to \Omega$ is realised by $\varphi : L \to \Omega$, then $a \in E(x) \cap E(x')$ implies $\varphi a \in E(fx) \cap E(fx')$ and thus $fx = fx'$ as Ω is modest. Since this holds for all realisable $f : \mathcal{X} \to \Omega$, $x = x'$ follows from the (Ω,Ω)-separation property of $\mathcal{X}$.

The assembly $(\{a,b\}, \{0 < 1 < 2\}, E)$ with $E(a) = \{0,2\}$ and $E(b) = \{1\}$ is modest, but a and b cannot be separated by any realisable function to Ω. □

This proposition shows that the adjunction $\mathrm{T_A}/\mathrm{A_T}$ between TOP and ASSM *does not* cut down to an adjunction between TOP_0 and MOD $\cong$ EQU, but *does* cut down to an adjunction between TOP_0 and the full subcategory of (Ω,Ω)-separated assemblies. This full subcategory is equivalent to the full subcategory EQU_0 of EQU which was introduced in [2].

Proposition 3.3 For any (S,T), the class of (S,T)-separated assemblies is closed under subspace, product, and exponentiation.

Proof: This is fairly easy. For subspaces, employ the subspace inclusion, for products, employ the projections, and for function spaces $[\mathcal{X} \to \mathcal{Y}]$, use the (realisable) point evaluations $\lambda f.\, fx : [\mathcal{X} \to \mathcal{Y}] \to \mathcal{Y}$. (Here, only $\mathcal{Y}$ must be separated; $\mathcal{X}$ can be any assembly.) □

Thus, we were able to identify cartesian closed complete subcategories of ASSM, the intersection of which with TOP yields the well-known classes of $\mathcal{T}_0$, $\mathcal{T}_1$, Hausdorff, completely Hausdorff, and totally disconnected spaces.

4 S-$\mathcal{E}$-assemblies

In this section, we generalise the notion of S-space from Reus' and Streicher's General Synthetic Domain Theory [11], and connect it with similar notions from the theory of cartesian closed topological categories [5, 8].

4.1 The General Theory

The following definitions and simple facts are taken from [11, Section 3.2] (their true origin is probably much older). All occurring functions are realisable without explicit statements.

Let S be a fixed assembly. We write $S\mathcal{X}$ for $[\mathcal{X} \to S]$, $S^2\mathcal{X}$ for $[[\mathcal{X} \to S] \to S]$ etc. The object map $\mathcal{X} \to S\mathcal{X}$ becomes a contravariant functor by defining $Sf = \lambda h^{S\mathcal{Y}}.\, h \circ f : S\mathcal{Y} \to S\mathcal{X}$ for $f : \mathcal{X} \to \mathcal{Y}$. Hence, S^2 is a covariant functor, S^3 a contravariant functor etc. There is a natural transformation ε from the identity to S^2 with $\varepsilon_\mathcal{X} : \mathcal{X} \to S^2\mathcal{X}$ defined as $\varepsilon_\mathcal{X} = \lambda x^\mathcal{X}.\, \lambda p^{S\mathcal{X}}.\, px$. For any assembly $\mathcal{X}$, we have $S\varepsilon_\mathcal{X} \circ \varepsilon_{S\mathcal{X}} = \mathrm{id}_{S\mathcal{X}}$ for $\varepsilon_{S\mathcal{X}} : S\mathcal{X} \to S^3\mathcal{X}$ and $S\varepsilon_\mathcal{X} : S^3\mathcal{X} \to S\mathcal{X}$. Then Reus and Streicher go on to define that $\mathcal{X}$ is an S-space if $\varepsilon_\mathcal{X}$ is *monic* (*injective*). On the other hand, people working in cartesian closed topological categories are more interested in those $\mathcal{X}$ where $\varepsilon_\mathcal{X}$ is *initial* [5, 8].

In order to deal with both notions at once, we parametrise not only on S, but also on the class $\mathcal{E}$ of arrows $\varepsilon_\mathcal{X}$ is required to belong to. To obtain sufficiently strong results, we require the class $\mathcal{E}$ to have the following properties:

($\mathcal{E}1$) $\mathcal{E}$ contains all identities and is closed under composition.

($\mathcal{E}2$) Let $f : \mathcal{X} \to \mathcal{Y}$ and $g : \mathcal{Y} \to \mathcal{Z}$. If $g \circ f$ is in $\mathcal{E}$, then f is in $\mathcal{E}$.

($\mathcal{E}3$) If all $f_i : \mathcal{X}_i \to \mathcal{Y}_i$ are in $\mathcal{E}$, then the product map $\prod_{i\in I} f_i : \prod_{i\in I} \mathcal{X}_i \to \prod_{i\in I} \mathcal{Y}_i$ is in $\mathcal{E}$.

($\mathcal{E}4$) If $f : \mathcal{Y} \to \mathcal{Z}$ is in $\mathcal{E}$, then $(\mathcal{X} \to f) : [\mathcal{X} \to \mathcal{Y}] \to [\mathcal{X} \to \mathcal{Z}]$ is in $\mathcal{E}$ for any $\mathcal{X}$.

Examples of such classes are the classes of

1. monic arrows (= injective realisable functions),
2. initial arrows,
3. embeddings,
4. sections (i.e., those $e : \mathcal{X} \to \mathcal{Y}$ such that there is an arrow $r : \mathcal{Y} \to \mathcal{X}$—called retraction—with $r \circ e = \mathrm{id}_\mathcal{X}$).

Every section is an embedding, and the class of embeddings is exactly the intersection of the class of monics with the class of initial arrows. The verification of the required properties is not difficult. For the class of initial arrows, some properties are stated in [8]. Note that by ($\mathcal{E}1$) and ($\mathcal{E}2$), $\mathcal{E}$ necessarily contains all sections; i.e., the class of sections is the smallest possible class $\mathcal{E}$.

Given a class $\mathcal{E}$ of arrows, we say that $\mathcal{X}$ is an *S-$\mathcal{E}$-space* if $\varepsilon_{\mathcal{X}}$ is in $\mathcal{E}$. For the four special classes listed above, we use the names *S-separated*, *S-initial*, *S-embedded*, and *S-retract*. Any S-retract is S-embedded. An assembly is S-embedded if and only if it is S-separated and S-initial.

Reus and Streicher present a 'representation theorem' for S-separated spaces [11, Theorem 3.10] which can easily be extended to the general case. The additional statement about a topological $\mathcal{Y}$ is inspired by [8].

Theorem 4.1 An assembly $\mathcal{X}$ is an S-$\mathcal{E}$-space (i.e., $\varepsilon_{\mathcal{X}} : \mathcal{X} \to S^2\mathcal{X}$ is in $\mathcal{E}$) if and only if there is some $\mathcal{E}$-arrow $e : \mathcal{X} \to S\mathcal{Y}$ for some assembly $\mathcal{Y}$. If the class $\mathcal{E}$ contains all embeddings, $\mathcal{Y}$ may be assumed as *topological.*

Proof: For the non-trivial direction of the equivalence assume an $\mathcal{E}$-arrow $e : \mathcal{X} \to S\mathcal{Y}$. We compose $\varepsilon_{\mathcal{X}} : \mathcal{X} \to S^2\mathcal{X}$, $S^2e : S^2\mathcal{X} \to S^3\mathcal{Y}$, and $S\varepsilon_{\mathcal{Y}} : S^3\mathcal{Y} \to S\mathcal{Y}$:

$$S\varepsilon_{\mathcal{Y}} \circ S^2 e \circ \varepsilon_{\mathcal{X}} \;=\; S\varepsilon_{\mathcal{Y}} \circ \varepsilon_{S\mathcal{Y}} \circ e \;=\; e$$

using naturality of ε and the ε-equality mentioned in the beginning. Since the composed map is $e \in \mathcal{E}$, $\varepsilon_{\mathcal{X}}$ is in $\mathcal{E}$ by ($\mathcal{E}2$).

For the second statement, assume an $\mathcal{E}$-arrow $e : \mathcal{X} \to S\mathcal{Y}$ to an arbitrary assembly $\mathcal{Y}$. By Prop. 2.3, $\mathcal{Y}$ is a quotient of a topological assembly $\mathcal{Z}$. By [8, Lemma 6], functor S transforms the quotient map $q : \mathcal{Z} \to \mathcal{Y}$ into an embedding $Sq : S\mathcal{Y} \to S\mathcal{Z}$. □

Corollary 4.2

1. All S-powers $S\mathcal{X} = [\mathcal{X} \to S]$ are S-$\mathcal{E}$-spaces. In particular, $S \cong S\mathbf{1}$ is an S-$\mathcal{E}$-space.
2. If $e : \mathcal{X} \to \mathcal{Y}$ is in $\mathcal{E}$ and $\mathcal{Y}$ is an S-$\mathcal{E}$-space, then so is $\mathcal{X}$. In particular, the classes of S-separated spaces, S-initial spaces, and S-embedded spaces are closed under subspaces, hence equalisers, and the class of S-retracts is closed under retracts.

Theorem 4.3 All S-$\mathcal{E}$-classes are closed under product and exponentiation.

Proof: [11] has a proof for S-separated spaces that uses the logic of an 'extension of the Extended Calculus of Constructions'. [5] has a classical proof for S-initial spaces. We present a new simple and generic proof, applying Theorem 4.1.

- Let $(\mathcal{X}_i)_{i\in I}$ be a family of S-$\mathcal{E}$-spaces, and $e_i : \mathcal{X}_i \to S\mathcal{Y}_i$ a corresponding family of $\mathcal{E}$-arrows. With ($\mathcal{E}3$), $\prod_{i\in I} e_i : \prod_{i\in I} \mathcal{Z}_i \to \prod_{i\in I} S\mathcal{Y}_i \cong S(\sum_{i\in I} \mathcal{Y}_i)$ provides an $\mathcal{E}$-arrow from the product to an S-power.
- If $e : \mathcal{X} \to S\mathcal{Y}$ is an $\mathcal{E}$-arrow, then with ($\mathcal{E}4$), $(\mathcal{Z} \to e) : [\mathcal{Z} \to \mathcal{X}] \to [\mathcal{Z} \to S\mathcal{Y}] \cong S(\mathcal{Z} \times \mathcal{Y})$ provides an $\mathcal{E}$-arrow from $[\mathcal{Z} \to \mathcal{X}]$ to an S-power. Here, no particular properties of $\mathcal{Z}$ are required. □

Corollary 4.4 If S is an S'-$\mathcal{E}$-space, then every S-$\mathcal{E}$-space is an S'-$\mathcal{E}$-space.

For every $\mathcal{X}$, there is an S-separated reflection which is obtained as the quotient of $\mathcal{X}$ under the equivalence $x \approx x'$ iff $\varepsilon_{\mathcal{X}} x = \varepsilon_{\mathcal{X}} x'$. There is also an S-initial reflection which has the same carrier as $\mathcal{X}$, but the initial structure induced by $\varepsilon_{\mathcal{X}}$ [5]. Finally, there is an S-embedded reflection, namely the subspace $\varepsilon^{+}_{\mathcal{X}}(\mathcal{X})$ of $S^2\mathcal{X}$.

4.2 S-separated Assemblies

The S-separated assemblies defined above are nothing else than the (S, S)-separated assemblies of the previous section. This is easily seen as follows: For x, x' in $\mathcal{X}$, $\varepsilon_{\mathcal{X}} x = \varepsilon_{\mathcal{X}} x'$ holds in $S^2\mathcal{X}$ iff $fx = \varepsilon_{\mathcal{X}} x f = \varepsilon_{\mathcal{X}} x' f = f'x$ for all f in $S\mathcal{X}$, i.e., for all realisable functions $f : \mathcal{X} \to S$.

In the previous section, we had considered the three special cases $S = \Omega$ ($\mathrm{T_A}\mathcal{X}$ is $\mathcal{T}_0$), $S = [0, 1]$ ($\mathrm{T_A}\mathcal{X}$ is completely Hausdorff), and $S = \mathbf{2}$ ($\mathrm{T_A}\mathcal{X}$ is totally disconnected).

4.3 S-initial Assemblies

Let S and X be topological spaces (considered as assemblies). The function e_X from X to the product $\prod_{f \in SX} S$ defined as $e_X x = (f \mapsto fx)$ is continuous, hence realisable. Note that $\prod_{f \in SX} S$ is not the same as S^2X; in the product, the assembly structure of SX is ignored, while it is essential in the construction of S^2X. Nevertheless, we may conclude that X is S-initial provided that e_X is initial (in TOP and thus in ASSM) since S is S-initial and this property is preserved by products and inverse images under initial maps.

In case of $S = \Omega$, e_X is initial for all topological spaces X. For $S = I = [0, 1]$, e_X is initial iff X is completely regular and for $S = \mathbf{2}$, e_X is initial iff X is zero-dimensional (has a basis of clopen sets). (We do not require $\mathcal{T}_0$ in these notions.)

Proposition 4.5 Every topological assembly is Ω-initial. A topological assembly is Ω-embedded, iff it is $\mathcal{T}_0$, iff it is modest. Every completely regular topological space is I-initial, and every zero-dimensional topological space is $\mathbf{2}$-initial.

It may be possible that ε_X is initial without e_x being initial. Thus, the following questions remain open.

Problem 1 Is every I-initial topological space completely regular?

Problem 2 Is every $\mathbf{2}$-initial topological space zero-dimensional?

Since $\mathbf{2}$ is a subspace of I, and I is Ω-embedded as a $\mathcal{T}_0$ topological space, we obtain from Cor. 4.4:

Proposition 4.6 Every **2**-initial space is I-initial, and every I-initial space is Ω-initial. The same holds with 'initial' replaced by 'embedded' or 'separated'.

4.4 Ω-initial Assemblies

An assembly (X, L, E) is *join-closed* if all sets $E(x)$ are closed under non-empty joins of L. It is *order-convex* if all sets $E(x)$ are order-convex, i.e., if $a, c \in E(x)$ and $a \leq b \leq c$, then b is in $E(x)$. These properties are preserved by initial structures and exponentiation, and satisfied by Ω (and in fact by $\mathrm{A_T}X$ for every topological space X). Hence, every Ω-initial space is (equivalent to) a join-closed order-convex assembly.

While these properties are not sufficient to characterise Ω-initial spaces completely, their relevance is as follows: The full subcategory of join-closed order-convex assemblies is equivalent to a category of *filter spaces* (see [7] which elaborates remarks from [9, Section 2]). Therefore, there is a representative *set* for the class of join-closed order-convex assemblies with a given carrier set. Hence, this class forms a *cartesian closed topological (CCT) category*, and this property is inherited by the class of Ω-initial spaces.

In [8], a CCT category $\mathcal{C}'$ is called a *CCT hull* of a full subcategory $\mathcal{C}$ if $\mathcal{C}$ is closed under finite products of $\mathcal{C}'$, and all objects of $\mathcal{C}'$ are final structures of some $\mathcal{C}$-objects X_i, and initial structures of some $\mathcal{C}'$-powers $[Y_i \to Z_i]$ of $\mathcal{C}$-objects Y_i, Z_i. Our results so far show that the Ω-initial assemblies form a CCT hull of TOP (the requirements about final and initial structures are due to Prop. 2.3 and the second statement of Theorem 4.1). Here, we could also write *the* CCT hull of TOP since any two CCT hulls of a given category are equivalent [8, Theorem 12]. The CCT hull of TOP was already identified as a category of filter spaces in [5], the objects of which were called *epitopological* or *Antoine spaces*.

5 Some Notes on Injective Spaces

In a category, an object Z is *injective* for an arrow $f : X \to Y$ if for every arrow $g : X \to Z$, there is some (not necessarily unique) 'extension' $h : Y \to Z$ such that $h \circ f = g$.

The following simple facts are taken from [6, Lemma II-3.2]: If Z is injective for f, then so is any retract of Z. If some Z_i are injective for f, then the product $\prod_{i \in I} Z_i$ is injective for f. We add a little lemma:

Lemma 5.1 In a cartesian closed category, $SZ = [Z \to S]$ is injective for $f : X \to Y$ if and only if S is injective for $f \times \mathrm{id}_Z : X \times Z \to Y \times Z$.

A topological space is injective for all initial maps in TOP if and only if it is a continuous lattice with the Scott topology (this is a slight modification of the results of [6, Section II-3]). In contrast, we have the following:

Proposition 5.2 For all continuous lattices L (with Scott topology) and topological spaces Z, considered as assemblies, the assembly $[Z \to L]$ is injective for all initial maps in TOP $\subseteq$ ASSM.

Proof: Given an initial $f : X \to Y$ in TOP, $f \times \mathrm{id}_Z : X \times Z \to Y \times Z$ is initial in TOP as well. Hence, the continuous lattice L is injective for $f \times \mathrm{id}_Z$. By Lemma 5.1, $[Z \to L]$ is injective for f. $\square$

Hence, there are assemblies which are injective for all initial maps in TOP, but not continuous lattices, e.g., the assembly $\Omega Z = [Z \to \Omega]$ for a topological space Z which is not core-compact.

Thus, there are a plenty of assemblies which are injective for all initial maps in TOP. On the other hand, the terminal object (= singleton assembly) is the only modest set which is injective for all embeddings in MOD [3]. The question is what happens in between.

Problem 3 Are there non-trivial Ω-initial assemblies which are injective for all initial maps between Ω-initial assemblies? In particular, is Ω such an assembly?

At least, some partial answers can be derived easily.

Proposition 5.3 An Ω-initial assembly which is injective for all initial maps between Ω-initial assemblies must be an Ω-retract.

Proof: Let $\mathcal{Z}$ be such an assembly. Then $\mathcal{Z}$ is in particular injective for the initial map $\varepsilon_{\mathcal{Z}} : \mathcal{Z} \to \Omega^2\mathcal{Z}$. The extension of $\mathrm{id}_{\mathcal{Z}}$ is a map $r : \Omega^2\mathcal{Z} \to \mathcal{Z}$ with $r \circ \varepsilon_{\mathcal{Z}} = \mathrm{id}_{\mathcal{Z}}$. $\square$

Proposition 5.4 If Ω is injective for all initial maps between Ω-initial assemblies, then all Ω-retracts have this property.

Proof: Let $f : \mathcal{X} \to \mathcal{Y}$ be an initial map between Ω-initial assemblies, and let $\mathcal{Z}$ be an Ω-retract. The function space $\Omega\mathcal{Z}$ is Ω-initial, whence $f \times \mathrm{id}_{\Omega\mathcal{Z}}$ is an initial map between Ω-initial assemblies. By hypothesis, Ω is injective for $f \times \mathrm{id}_{\Omega\mathcal{Z}}$, and by Lemma 5.1, injectivity of $\Omega^2\mathcal{Z}$ follows, which implies injectivity of its retract $\mathcal{Z}$. $\square$

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