Power Domains and Second Order Predicates

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Abstract

Lower, upper, sandwich, mixed, and convex power domains are isomorphic to domains of second order predicates mapping predicates on the ground domain to logical values in a semiring. The various power domains differ in the nature of the underlying semiring logic and in logical constraints on the second order predicates.

1 Introduction

A power domain construction maps every domain $\mathbf{X}$ of some distinguished class of domains into a so-called power domain over $\mathbf{X}$ whose points represent sets of points of the ground domain. Power domain constructions were originally proposed to model the semantics of non-deterministic programming languages [15, 16, 8, 14]. Other motivations are the semantic representation of a set data type [6], or of relational data bases [2, 3].

In 1976, Plotkin [15] proposed the *convex* power domain construction. A short time later, Smyth [16] introduced a simpler construction, the *upper* power construction. In [17], a third power domain construction occurs, the *lower* construction, that completes the trio of classical power domain constructions.

Starting from problems in data base theory, Buneman et al. [2] proposed to combine lower and upper power domain to a so-called *sandwich* power domain. By extending Plotkin's domain in a natural way, Gunter developed the *mixed* power domain [3, 4].

Given at least five different power domain constructions, the question arises what is the essence of these constructions. In [7], we defined power domain constructions algebraically by axioms concerning existence and properties of the basic operations of empty set, singleton, binary union, and function extension.

The resulting algebraic theory of power constructions, which is summarized in section 3, shows that every construction has a *characteristic semiring* reflecting the inherent logic of the construction. 0 stands for 'false', 1 for 'true', addition means disjunction, and multiplication represents conjunction.

The general algebraic theory provides a *final* power domain construction for every semiring R. It is explicitly given by mapping ground domains $\mathbf{X}$ to the space of linear second order R-predicates over $\mathbf{X}$ (see Th. 3.5). A (first order) R-predicate over $\mathbf{X}$ maps members of $\mathbf{X}$ to logical values in R, whereas a second order R-predicate maps first order predicates to R. All these results were published in [7]. They are repeated in section 3 of the paper at hand for convenience of the reader.

The present paper handles the five known power domain constructions mentioned at the beginning in the framework of the algebraic theory. Section 5 contains an overview of the five constructions. To investigate the relations among them, we consider products of power constructions and sub-constructions in section 4. In section 6, we show the lower power construction $\mathcal{L}$ to be final for the semiring $\mathbf{L} = \{0 < 1\}$. An analogous result is shown for the upper construction $\mathcal{U}$ in terms of compact upper sets in section 7. In section 8, the sandwich construction $\mathcal{S}$ is shown to be final for semiring $\mathbf{B} = \{\perp, 0, 1\}$. Thus, these three constructions are isomorphic to spaces of linear second order predicates.

Although the mixed power construction $\mathcal{M}$ and Plotkin's construction $\mathcal{C}$ are not final, they may also be described in terms of second order predicates because their power domains are subsets of the sandwich power domains. In section 9, we present the logical conditions that characterize the predicates corresponding to mixed or Plotkin domain members among all members of $\mathcal{S}\mathbf{X}$.

Both [7] and the paper in hand are extracts of the comprehensive thesis [5] containing more details and background information.

2 Theoretical background

In this section, we introduce some notions and notations from domain theory, algebra, and topology. $A \subseteq_f B$ means A is a finite subset of B. For $f : X \to Y$ and $A \subseteq X$ and $B \subseteq Y$, $f[A]$ is the image of A and $f^{-1}[B]$ the inverse image of B. fx means application of f to x, and accordingly, $g \circ f$ means $\lambda x.\, g\,(fx)$. Gfx parses as $(Gf)\,x$.

2.1 Posets and domains

A *poset* (partially ordered set) is a set P together with a reflexive, antisymmetric, and transitive relation '$\leq$'. We often identify the poset $\mathbf{P} = (P, \leq)$ with its carrier P.

For $A \subseteq \mathbf{P}$, let $\downarrow A$ be the set of all points *below* some point of A, and correspondingly $\uparrow A$ the set of all points *above* some point of A. We use the abbreviations $\downarrow x = \downarrow\{x\}$ and $\uparrow x = \uparrow\{x\}$. A set $A \subseteq \mathbf{P}$ is a *lower* set iff $\downarrow A = A$, and an *upper* set iff $\uparrow A = A$.

We refer to the standard notions of upper bound, least upper bound (lub) denoted by '$\bigsqcup$', directed set, monotonic and continuous function. A *domain* is a poset where every directed set has a lub, also called *limit*. A domain need neither have a least element, nor be algebraic. Continuous functions between domains are sometimes called *morphisms*. For domains $\mathbf{X}$ and $\mathbf{Y}$, $\mathbf{X} \times \mathbf{Y}$ denotes the domain of all pairs of points of $\mathbf{X}$ and $\mathbf{Y}$, and $[\mathbf{X} \to \mathbf{Y}]$ is the domain of all morphisms from $\mathbf{X}$ to $\mathbf{Y}$ ordered pointwise. The notation $f : [\mathbf{X} \to \mathbf{Y}]$ includes the continuity of f.

A point a in a domain $\mathbf{X}$ is *way below* a point b iff for all directed sets $D \subseteq \mathbf{X}$ with $b \leq \bigsqcup D$, there is an element d in D such that $a \leq d$. A point a is *isolated* iff it is way-below itself. A domain $\mathbf{X}$ is *algebraic* iff every point of $\mathbf{X}$ is the lub of a directed set of isolated points. The set $\mathbf{X}^0$ of all isolated points of $\mathbf{X}$ is called the *base*. $\mathbf{X}$ is *continuous* iff every point x of $\mathbf{X}$ is the lub of a directed set of points that all are way-below x.

An M-*domain* is an algebraic domain whose base has property M [12, 15], i.e. for any finite subset E of the base there is a finite set F of upper bounds of E with the property that there is a point in F below every upper bound of E.

2.2 Monoids, semirings, and modules

Monoids, semirings, and modules are well-known algebraic concepts. Here, we define variants of these notions where the carrier is a domain, and all operations are continuous.

Definition 2.1 (Monoid domains and additive maps)
A *monoid domain* (or simply monoid) $(M, +, 0)$ is a domain M together with an associative operation $+ : [M \times M \to M]$ and an element 0 of (the carrier of) M which is the neutral element of '$+$'. The monoid is *commutative* iff '$+$' is.
A map $f : [X \to Y]$ between two monoids is *additive* iff it is a *monoid homomorphism*, i.e. $f(0_X) = 0_Y$ and $f(a + b) = fa + fb$ hold.

Definition 2.2 (Semirings)
A *semiring* (domain) $(R, +, 0, \cdot, 1)$ is a domain R with continuous operations such that $(R, +, 0)$ is a commutative monoid, $(R, \cdot, 1)$ is a monoid, and multiplication '$\cdot$' is additive in both arguments.

Semiring homomorphisms are continuous mappings that preserve the semiring operations.

Semirings are generalizations of both rings and distributive lattices. These in turn are generalizations of fields and Boolean algebras. Hence, both the notations $(R, +, 0, \cdot, 1)$ of the definition above and a logical notation $(R, \vee, \mathsf{F}, \wedge, \mathsf{T})$ seem to be adequate.

Assuming the logical interpretation, morphisms from a domain $\mathbf{X}$ to a semiring R are called *(R-)predicates*. Second order predicates are then morphisms in $[[\mathbf{X} \to R] \to R]$.

Definition 2.3 (Modules)
Let $R = (R, +, 0, \cdot, 1)$ be a semiring domain. $M = (M, +, 0, \cdot)$ is a *(left) R-module* iff $(M, +, 0)$ is a commutative monoid domain, and $\cdot : [R \times M \to M]$ is additive in both arguments and satisfies $1_R \cdot A = A$ and $a \cdot (b \cdot C) = (a \cdot b) \cdot C$.

Let M_1 and M_2 be two R-modules. A morphism $f : [M_1 \to M_2]$ is *(left) (R-)linear* iff

$$f(A + B) = fA + fB \quad \text{and} \quad f(r \cdot A) = r \cdot fA$$

We speak of *right modules* and *right linear* morphisms if the semiring factor occurs to the right, i.e. $\cdot : [M \times R \to M]$. The axioms are analogous to the ones above.

2.3 Scott topology

A subset of a domain $\mathbf{X}$ is called *(Scott) closed* iff it is a lower set closed w.r.t. lubs of directed subsets. Lower cones $\downarrow x$ are obviously closed. Arbitrary intersections and finite unions of closed sets are closed. Hence, every set A has a least closed superset, the *closure* $\mathsf{cl}\, A$. The complements of the closed sets are called *open.* The set of all open supersets of a set A is denoted by $\mathcal{O}(A)$. We abbreviate $\mathcal{O}(\{x\})$ by $\mathcal{O}(x)$. $\Omega\mathbf{X}$ denotes the domain of open sets of $\mathbf{X}$ ordered by inclusion. A subset K of a domain $\mathbf{X}$ is called *(Scott) compact* iff whenever K is covered by a family $(O_i)_{i\in I}$ of open sets, i.e. $K \subseteq \bigcup_{i\in I} O_i$, there is a finite subset F of I such that $K \subseteq \bigcup_{i\in F} O_i$.

In the remainder of this paper, we use some properties of the notions introduced above. These properties are collected now. We assume to be in a fixed domain $\mathbf{X}$ always. The proofs of particularly well-known properties are omitted.

Proposition 2.4 a is isolated iff $\uparrow a$ is open.

Proposition 2.5 If $\mathbf{X}$ is algebraic, and $a, b \leq x$ holds for $a, b \in \mathbf{X}^0$ and $x \in \mathbf{X}$, then there is $c \in \mathbf{X}^0$ such that $a, b \leq c \leq x$.

Proposition 2.6 If an open set O meets $\mathsf{cl}\, A$, then it meets A itself.

Proposition 2.7 x is in $\mathsf{cl}\, A$ iff every O in $\mathcal{O}(x)$ meets A.

Proposition 2.8 Let $E \subseteq \mathbf{X}^0$. Then $E \subseteq \mathsf{cl}\, A$ implies $E \subseteq \downarrow A$.

Proof: Let e be in E. Since e is isolated, $\uparrow e$ is open by Prop. 2.4. Since e is in $\mathsf{cl}\, A$, $\uparrow e$ meets A by Prop. 2.7. Hence, there is $x \geq e$ with $x \in A$, i.e. $e \in \downarrow A$. □

Proposition 2.9 K is compact iff $\mathcal{O}(K)$ is open in $\Omega\mathbf{X}$.

Proposition 2.10 For two subsets A, B of $\mathbf{X}$, $\mathcal{O}(A) \subseteq \mathcal{O}(B)$ holds iff $\uparrow A \supseteq \uparrow B$.

Proof: Let $\uparrow A \supseteq \uparrow B$. If $A \subseteq O$ then $B \subseteq \uparrow A \subseteq O$ since open sets are upper sets.
Let $\mathcal{O}(A) \subseteq \mathcal{O}(B)$, and let b be a point of B and assume $b \notin \uparrow A$. Then there is no point of A below b, i.e. A does not meet the closed set $\downarrow b$. Hence, its complement $\mathbf{X} \setminus \downarrow b$ is in $\mathcal{O}(A) \subseteq \mathcal{O}(B)$. Because of $b \in B$, $b \in \mathbf{X} \setminus \downarrow b$ follows contradicting reflexivity of '$\leq$'. □

3 Power domain constructions

In this section, we present a short summary of the algebraic theory of power constructions as contained in [7, 5].

3.1 Specification of power constructions

A *power (domain) construction* $\mathcal{P}$ maps *ground domains* $\mathbf{X}$ into *power domains* over $\mathbf{X}$. The power domains have to satisfy the following axioms:

Empty set: There is a distinguished element θ in every power domain $\mathcal{P}\mathbf{X}$.

Binary union: There is a continuous operation $\uplus : [\mathcal{P}\mathbf{X} \times \mathcal{P}\mathbf{X} \to \mathcal{P}\mathbf{X}]$ in every power domain. '$\uplus$' is commutative and associative, and θ is its neutral element.

Singleton sets: There is a continuous mapping $\iota : [\mathbf{X} \to \mathcal{P}\mathbf{X}]$, $x \mapsto \{\!|x|\!\}$ for every ground domain $\mathbf{X}$.

Extension of functions: For every two domains $\mathbf{X}$ and $\mathbf{Y}$, there is a higher order function $ext : [[\mathbf{X} \to \mathcal{P}\mathbf{Y}] \to [\mathcal{P}\mathbf{X} \to \mathcal{P}\mathbf{Y}]]$ mapping set-valued functions on ground domain elements into set-valued functions on sets. The intuitive meaning of $ext\, f\, A$ is $\bigcup_{a \in A} fa$. Extension has to satisfy the following axioms:

- $ext\, f\, A$ is additive in both A and f.
- $ext\, f \circ \iota = f$ and $ext\, \iota = id$.
- For every two morphisms $f : [\mathbf{X} \to \mathcal{P}\mathbf{Y}]$ and $g : [\mathbf{Y} \to \mathcal{P}\mathbf{Z}]$,

  $$ext\, g\, (ext\, f\, A) = ext\, (\lambda a.\, ext\, g\, (fa))\, A$$

  holds for all A in $\mathcal{P}\mathbf{X}$, or: $ext\, g \circ ext\, f = ext\, (ext\, g \circ f)$.

$$\begin{tikzcd}
\mathcal{P}\mathbf{X} \arrow[r, "ext\, f"] & \mathcal{P}\mathbf{Y} \arrow[r, "ext\, g"] & \mathcal{P}\mathbf{Z} \\
\mathbf{X} \arrow[u, "\iota"] \arrow[ur, "f"'] & \mathbf{Y} \arrow[u, "\iota"] \arrow[ur, "g"'] &
\end{tikzcd}$$

3.2 The algebraic properties of power domains

The operations as specified above allow to derive many other operations with useful algebraic properties. Among these, there are *map* and big union turning the power construction into a monad. We here include the most important ones only; for the other ones, we refer to [7, 5].

Extension depends on two ground domains, $\mathbf{X}$ and $\mathbf{Y}$. Particularly interesting instances of extension are obtained if one of $\mathbf{X}$ and $\mathbf{Y}$ is the one-point domain $\mathbf{1} = \{\diamond\}$. In case $\mathbf{X} = \mathbf{1}$, extension has functionality $ext : [[\mathbf{1} \to \mathcal{P}\mathbf{Y}] \to [\mathcal{P}\mathbf{1} \to \mathcal{P}\mathbf{Y}]]$. Dropping the obsolete argument in $\mathbf{1}$, uncurrying, and twisting arguments leads to a morphism $* : [\mathcal{P}\mathbf{1} \times \mathcal{P}\mathbf{Y} \to \mathcal{P}\mathbf{Y}]$. The definition is $b * S = ext\, (\lambda\diamond.\, S)\, b$. If we additionally choose $\mathbf{Y} = \mathbf{1}$, then '$*$' becomes an inner operation of $\mathcal{P}\mathbf{1}$.

The axioms of power constructions suffice to prove the following theorem:

Theorem 3.1 Let $\mathcal{P}$ be a power construction. Then $(\mathcal{P}\mathbf{1}, \uplus, \theta, *, \{\!|\diamond|\!\})$ is a semiring domain, the *characteristic semiring* of $\mathcal{P}$. $(\mathcal{P}\mathbf{X}, \uplus, \theta, *)$ is a left $\mathcal{P}\mathbf{1}$-module for all domains $\mathbf{X}$. For $f : [\mathbf{X} \to \mathcal{P}\mathbf{Y}]$, the extension $ext : [\mathcal{P}\mathbf{X} \to \mathcal{P}\mathbf{Y}]$ is linear.

This result connects our work with that of Main [14] where power domains are introduced as free semiring modules. There are however some differences: our constructions may create non-free modules, and our singleton function ι need not be strict.

Notice that power domains contain much more algebraic structure than just modules. In deriving the module product, we only used instances of extension where $\mathbf{X}$ is the one-point domain $\mathbf{1}$. Thus, we did not use the full power of extension for arbitrary domains $\mathbf{X}$ and $\mathbf{Y}$.

3.3 Power homomorphisms and the category PC

Homomorphisms between algebraic structures are mappings preserving all operations of these structures. Power constructions may be considered algebraic structures on a higher level. Thus, it is also possible and useful to define corresponding homomorphisms.

A power homomorphism $H : \mathcal{P} \dot{\to} \mathcal{Q}$ between two power constructions $\mathcal{P}$ and $\mathcal{Q}$ is a 'family' of morphisms $H = (H_{\mathbf{X}})_{\mathbf{X}} : [\mathcal{P}\mathbf{X} \to \mathcal{Q}\mathbf{X}]$ commuting over all power operations, i.e.

- The empty set in $\mathcal{P}\mathbf{X}$ is mapped to the empty set in $\mathcal{Q}\mathbf{X}$: $H\theta = \theta$.
- The image of a union is the union of the images: $H(A \uplus B) = (HA) \uplus (HB)$.
- Singletons in $\mathcal{P}\mathbf{X}$ are mapped to singletons in $\mathcal{Q}\mathbf{X}$: $H\{\!|x|\!\}_{\mathcal{P}} = \{\!|x|\!\}_{\mathcal{Q}}$.
- Let $f : [\mathbf{X} \to \mathcal{P}\mathbf{Y}]$. Then $H \circ f : [\mathbf{X} \to \mathcal{Q}\mathbf{Y}]$, and $H(ext_{\mathcal{P}}\, f\, A) = ext_{\mathcal{Q}}(H \circ f)(HA)$ has to hold for all A in $\mathcal{P}\mathbf{X}$.

The above axioms allow to prove the following laws:

- $H(a * B) = Ha * HB$ for a in $\mathcal{P}\mathbf{1}$ and B in $\mathcal{P}\mathbf{X}$.
- $H_{\mathbf{1}} : [\mathcal{P}\mathbf{1} \to \mathcal{Q}\mathbf{1}]$ is a semiring homomorphism.

It is easily seen that power homomorphisms may be composed, and there are also identity power homomorphisms. Thus, we get the category PC of power constructions as objects and power homomorphisms as arrows.

A power isomorphism between two constructions $\mathcal{P}$ and $\mathcal{Q}$ is a family of isomorphisms $H = H_{\mathbf{X}} : [\mathcal{P}\mathbf{X} \to \mathcal{Q}\mathbf{X}]$ such that both $(H_{\mathbf{X}})_{\mathbf{X}}$ and $(H_{\mathbf{X}}^{-1})_{\mathbf{X}}$ are power homomorphisms.

Proposition 3.2 If $H : \mathcal{P} \dot{\to} \mathcal{Q}$ is a power homomorphism such that the individual maps $H_{\mathbf{X}} : [\mathcal{P}\mathbf{X} \to \mathcal{Q}\mathbf{X}]$ are all bijective, and their inverses are monotonic, then H is a power isomorphism.

3.4 Linear power homomorphisms and the categories $\mathsf{PC}(R)$

If the two constructions $\mathcal{P}$ and $\mathcal{Q}$ share the same characteristic semiring, then one can define: A power homomorphism is linear iff all the functions $H_{\mathbf{X}}$ are linear. To be more flexible, we do not require $\mathcal{P}\mathbf{1} = \mathcal{Q}\mathbf{1}$, but only an isomorphism $\mathcal{P}\mathbf{1} \cong \mathcal{Q}\mathbf{1}$.

Definition 3.3 Let R be a semiring. An *R-construction* is a pair $(\mathcal{P}, \varphi)$ of a power construction $\mathcal{P}$ and a semiring isomorphism $\varphi : [R \to \mathcal{P}\mathbf{1}]$.

We shall often omit the isomorphism φ if it is obvious from the context, and speak of the R-construction $\mathcal{P}$. The power domains of an R-construction become R-modules by defining $r \cdot A = \varphi r * A$ for r in R and A in $\mathcal{P}\mathbf{X}$.

Definition 3.4
Let $(\mathcal{P}, \varphi)$ and $(\mathcal{Q}, \varphi')$ be two R-constructions. $H : (\mathcal{P}, \varphi) \dot{\to} (\mathcal{Q}, \varphi')$ is an *R-linear power homomorphism* iff $H : \mathcal{P} \dot{\to} \mathcal{Q}$ is a power homomorphism and $H_{\mathbf{1}} \circ \varphi = \varphi'$ holds.

The name R-linear is appropriate since $H_{\mathbf{1}} \circ \varphi = \varphi'$ is equivalent to the R-linearity of all $H_{\mathbf{X}}$.

The category of R-constructions and R-linear power homomorphisms is denoted by $\mathsf{PC}(R)$. Notice that linear power homomorphisms are considerably more special than just families of linear mappings because they have to respect extension in its full generality.

3.5 Final power constructions

For every semiring R, the category $\mathsf{PC}(R)$ has an initial object $\mathcal{P}_i^R$ as well as a final object $\mathcal{P}_f^R$. An R-construction $\mathcal{P}$ is *final* iff for every R-construction $\mathcal{Q}$ there is exactly one R-linear power homomorphism $\mathcal{Q} \dot{\to} \mathcal{P}$. Initiality is defined dually.

Initial and final R-constructions are shown to exist and investigated to some extent in [7, 5]. In the present paper, we do not consider initial constructions except briefly in section 10. Final R-constructions on the other hand allow to understand power domains in terms of second order predicates. They were never proposed in the literature, probably because the notion of a power homomorphism was missing.

The explicit representation of the final R-construction was found by considering the morphism $\mathcal{E} : [\mathcal{Q}\mathbf{X} \to [[\mathbf{X} \to R] \to R]]$ defined by $\mathcal{E}A = \lambda p. \rho^{-1}\, (ext_{\mathcal{Q}}\, (\rho \circ p)\, A)$ for R-constructions $(\mathcal{Q}, \rho)$. By $\mathcal{E}$, members of $\mathcal{Q}\mathbf{X}$ are mapped into second order predicates. Intuitively, $\mathcal{E}$ denotes existential quantification: given a set A and an R-predicate p, $\mathcal{E} A\, p$ tells whether some member of A satisfies p.

Some of the axioms of extension easily translate into the following properties of $\mathcal{E}$:

- $\mathcal{E}\, \theta = \lambda p.\, \theta$
- $\mathcal{E}\, (A \uplus B) = \lambda p.\, (\mathcal{E}\, A\, p) + (\mathcal{E}\, B\, p)$
- $\mathcal{E}\, \{\!|x|\!\} = \lambda p.\, p\, x$
- $\mathcal{E}\, (ext\, f\, A) = \lambda p.\, \mathcal{E} A\, (\lambda a.\, \mathcal{E}\, (f a)\, p)$

Additionally, one can show that $\mathcal{E}A$ is right linear, i.e. $\mathcal{E}$ maps from $\mathcal{Q}\mathbf{X}$ to $[[\mathbf{X} \to R] \stackrel{rlin}{\to} R]$. These properties suggest the following explicit representation of $\mathcal{P}_f^R$:

Theorem 3.5 Let R be a given semiring. The final R-construction $\mathcal{P}_f^R = (\mathcal{P}, \varphi)$ is explicitly given by $\mathcal{P}\mathbf{X} = [[\mathbf{X} \to R] \stackrel{rlin}{\to} R]$ and the isomorphism $\varphi(r) = \lambda p.\, r \cdot p\diamond$. Its operations are defined by

- $\theta = \lambda p.\, 0$
- $A \uplus B = \lambda p.\, Ap + Bp$
- $\{\!|x|\!\} = \lambda p.\, p\, x$ for $x \in \mathbf{X}$.
- $ext\, f\, A = \lambda p.\, A\,(\lambda a_{\in \mathbf{X}}\,.\, f a p)$ for $f : [\mathbf{X} \to \mathcal{P}\mathbf{Y}]$ and $A \in \mathcal{P}\mathbf{X}$.

The inverse of φ is given by $\psi(A) = A\,(\lambda\diamond.1)$. The unique R-linear power homomorphism from another R-construction $\mathcal{Q}$ to $\mathcal{P}_f^R$ is given by $\mathcal{E}$ as defined above.

4 Creating new power constructions

Whereas the previous section summarized the relevant results of [7], the contents of this section and the subsequent ones are not yet published except as part of the thesis [5].

In this section, we present two methods to create new power constructions from existing ones. Given a family of power constructions, there is a product power construction, i.e. the category PC has arbitrary products. Product formation preserves finality: the product of final R_i-constructions is a final $(\prod_{i \in I} R_i)$-construction. We further consider sub-constructions of power constructions. Given an R-construction $\mathcal{P}$ and a sub-semiring R' of R, the greatest R'-construction $\mathcal{P}'$ that is a sub-construction of $\mathcal{P}$ may be explicitly characterized in terms of second order predicates.

This general theory is useful when considering the known power constructions. Convex and mixed construction are sub-constructions of the sandwich construction, which in turn is a sub-construction of the product of the lower and the upper power construction.

4.1 Products of power constructions

Given a family $(\mathcal{P}_i)_{i \in I}$ of power constructions, we may build a product construction $\mathcal{P} = \prod_{i \in I} \mathcal{P}_i$:

- $\mathcal{P}\mathbf{X} = \prod_{i \in I} \mathcal{P}_i\mathbf{X}$ for all ground domains $\mathbf{X}$
- $\theta = (\theta_i)_{i \in I}$
- $(A_i)_{i \in I} \uplus (B_i)_{i \in I} = (A_i \uplus B_i)_{i \in I}$
- $\{\!|x|\!\} = (\{\!|x|\!\}_i)_{i \in I}$ for all x in $\mathbf{X}$
- For $f : [\mathbf{X} \to \mathcal{P}\mathbf{Y}]$ let $f_i = \pi_i \circ f$. Then $ext\, f\,(A_i)_{i \in I} = (ext_i\, f_i\, A_i)_{i \in I}$ where ext_i denotes the extension functional of $\mathcal{P}_i$. Here, π_i denotes projection to component i.

The verification of the power axioms for $\mathcal{P}$ is straightforward since the power operations work independently in all dimensions. The characteristic semiring of $\mathcal{P}$ is the product of the characteristic semirings of the $\mathcal{P}_i$. It is also immediate that the projections induce power homomorphisms $\pi_k : \prod_{i\in I}\mathcal{P}_i \dot{\to} \mathcal{P}_k$, and that $\prod_{i\in I}\mathcal{P}_i$ forms a categorical product in the category PC.

In this paper, we are particularly interested in final power constructions described by second order predicates. The notion of finality nicely coexists with the notion of product:

Theorem 4.1 If $\mathcal{P}_i$ are final R_i-constructions for all $i \in I$, then the product $\mathcal{P} = \prod_{i\in I}\mathcal{P}_i$ is a final R-construction where $R = \prod_{i\in I} R_i$.

$$\prod_{i\in I}[[\mathbf{X} \to R_i] \overset{rlin}{\to} R_i] \;\cong\; [[\mathbf{X} \to \prod_{i\in I} R_i] \overset{rlin}{\to} \prod_{i\in I} R_i]$$

Proof: Let $\mathcal{P} = \prod_{i\in I}\mathcal{P}_i$ and let $\mathcal{Q}$ be the final R-construction where $R = \prod_{i\in I} R_i$, i.e. $\mathcal{Q}\mathbf{X} = [[\mathbf{X} \to R] \overset{rlin}{\to} R]$. We have to show that $\mathcal{P}$ and $\mathcal{Q}$ are isomorphic. For $k \in I$, let $\eta_k : [R_k \to R]$ be the mapping where all components of $\eta_k x$ are 0 except the component k which is x.

Since $\mathcal{Q}$ is final, there is a (unique) linear power homomorphism $\mathcal{E} : \mathcal{P} \dot{\to} \mathcal{Q}$. Abbreviating $\pi_i \circ p$ by p_i, we obtain $\mathcal{E}A = \lambda p.\,(\mathcal{E}_i\, A_i\, p_i)_{i\in I}$ where $\mathcal{E}_i$ is the unique linear power homomorphism from $\mathcal{P}_i$ to itself, i.e. is the identity. Thus, $\mathcal{E}A = \lambda p.\,(A_i\, p_i)_{i\in I}$. We have to show that $\mathcal{E}$ is a power isomorphism. By Prop. 3.2, it suffices to show that $\mathcal{E}$ is a surjective embedding.

Assume $\mathcal{E}A \le \mathcal{E}B$ holds for $A, B \in \mathcal{P}\mathbf{X}$. For all $k \in I$ and all $q : [\mathbf{X} \to R_k]$, let $p = \eta_k \circ q : [\mathbf{X} \to R]$. $\mathcal{E}A \le \mathcal{E}B$ implies $(A_i\, p_i)_{i\in I} = \mathcal{E}Ap \le \mathcal{E}B\, p = (B_i\, p_i)_{i\in I}$. This in particular holds for dimension k. Thus, $A_k\, q \le B_k\, q$ holds for all k in I and $q : [\mathbf{X} \to R_k]$, whence $A_k \le B_k$ for all k, whence $A \le B$.

For surjectivity, let $Q : [[\mathbf{X} \to R] \overset{rlin}{\to} R]$. Then let $Q_i : [[\mathbf{X} \to R_i] \to R_i]$ be defined by $Q_i = \lambda q.\,\pi_i\,(Q\,(\eta_i \circ q))$. The proof of right linearity of Q_i is straight forward. Thus, $P = (Q_i)_{i\in I}$ is a member of $\mathcal{P}\mathbf{X}$. We claim $\mathcal{E}P = Q$.
$\mathcal{E}P\, p = (Q_i p_i)_{i\in I} = (\pi_i\,(Q\,(\eta_i \circ p_i)))_{i\in I}$ holds where $p_i = \pi_i \circ p$. Note that $\eta_i(\pi_i r) = r \cdot \eta_i\, 1_i$ holds for all r in R. Thus,

$$\begin{aligned}\pi_i\,(Q\,(\eta_i \circ \pi_i \circ p)) \;&=\; \pi_i\,(Q\,(p \cdot \eta_i\, 1_i)) \;=\; \pi_i\,(Qp \cdot \eta_i\, 1_i) \\ &=\; \pi_i\,(Qp) \cdot \pi_i\,(\eta_i\, 1_i) \;=\; \pi_i\,(Qp) \cdot 1_i \;=\; \pi_i\,(Qp)\end{aligned}$$

whence $(\pi_i\,(Q\,(\eta_i \circ p_i)))_{i\in I} = Qp$. □

4.2 Sub-constructions

Let $\mathcal{P}$ be a given power construction. $\mathcal{Q}$ is called a *sub-construction* of $\mathcal{P}$ iff $\mathcal{Q}$ maps ground domains $\mathbf{X}$ into subsets of $\mathcal{P}\mathbf{X}$ such that

- $\theta \in \mathcal{Q}\mathbf{X}$,
- If A and B are in $\mathcal{Q}\mathbf{X}$, then $A \uplus B$ is in $\mathcal{Q}\mathbf{X}$,
- $\{|x|\}$ is in $\mathcal{Q}\mathbf{X}$ for all x in $\mathbf{X}$,

- If $f : [\mathbf{X} \to \mathcal{Q}\mathbf{Y}]$ and A in $\mathcal{Q}\mathbf{X}$, then $ext\, f\, A$ is in $\mathcal{Q}\mathbf{Y}$,
- $\mathcal{Q}\mathbf{X}$ is closed w.r.t. lubs of directed sets.

In shorter terms, $\mathcal{Q}\mathbf{X}$ is closed w.r.t. all power operations of $\mathcal{P}$. $\mathcal{Q}$ is obviously a power construction since the validity of the power axioms for $\mathcal{Q}$ is inherited from $\mathcal{P}$.

One easily verifies that the intersection of a family of sub-constructions of a power construction $\mathcal{P}$ is again a sub-construction of $\mathcal{P}$, if we define $(\bigcap_{i\in I} \mathcal{Q}_i)\mathbf{X} = \bigcap_{i\in I}(\mathcal{Q}_i\mathbf{X})$. Hence, the sub-constructions of $\mathcal{P}$ form a complete lattice.

Let R be a semiring domain. R' is a *sub-semiring* of R iff R' is a subset of R containing 0 and 1, and being closed w.r.t. addition, multiplication, and lubs of directed sets. Because the operations in the characteristic semiring are derived from the power operations, the semiring of a sub-construction $\mathcal{Q}$ of $\mathcal{P}$ is a sub-semiring of the semiring of $\mathcal{P}$.

The following theorem presents a method to obtain the greatest sub-construction for a given sub-semiring.

Theorem 4.2 Let $\mathcal{P}$ be an R-construction, and let R' be a sub-semiring of R. Then the *existential restriction* of $\mathcal{P}$ to R' defined by

$$\mathcal{Q}\mathbf{X} = \mathcal{P}|_{R'}\mathbf{X} = \{A \in \mathcal{P}\mathbf{X} \mid \forall p : [\mathbf{X} \to R'] : \mathcal{E}\, A\, p \in R'\}$$

is the greatest sub-construction of $\mathcal{P}$ with semiring R'.

Proof: We first show $\mathcal{Q}$ is a sub-construction of $\mathcal{P}$.

- $\mathcal{E}\theta\, p = 0 \in R'$ implies $\theta \in \mathcal{Q}\mathbf{X}$.
- If A and B are in $\mathcal{Q}\mathbf{X}$, then for all $p : [\mathbf{X} \to R']$, $\mathcal{E}\, A\, p$ and $\mathcal{E}\, B\, p$ are in R', whence $\mathcal{E}\,(A \uplus B)\, p = \mathcal{E}\, A\, p + \mathcal{E}\, B\, p$ is in R'.
- For x in $\mathbf{X}$ and $p : [\mathbf{X} \to R']$, $\mathcal{E}\{\!|x|\!\}\, p = p\, x$ is in R'. Hence, $\{\!|x|\!\}$ is in R' for all x in $\mathbf{X}$.
- Let $f : [\mathbf{X} \to \mathcal{Q}\mathbf{Y}]$ and $A \in \mathcal{Q}\mathbf{X}$. We have to show $ext\, f\, A$ in $\mathcal{Q}\mathbf{Y}$.

  For all $p : [\mathbf{X} \to R']$, $\mathcal{E}\,(ext\, f\, A) = \mathcal{E}A(\lambda a.\, \mathcal{E}(f a)\, p)$ holds as indicated in section 3.2. $f a \in \mathcal{Q}\mathbf{Y}$ implies $\mathcal{E}(f a)\, p \in R'$. Thus, $(\lambda a.\, \mathcal{E}(f a)\, p) : [\mathbf{X} \to R']$, whence the value of the whole term is in R'.
- Let $(A_i)_{i\in I}$ be a directed family of members of $\mathcal{Q}\mathbf{X}$ with limit A. Then for all $p : [\mathbf{X} \to R']$, $\mathcal{E}(\bigsqcup_{i\in I} A_i)\, p = \bigsqcup_{i\in I}(\mathcal{E}A_i\, p) \in R'$ holds by continuity of $\mathcal{E}$.

Next, we show $\mathcal{Q}\mathbf{1} = R'$. For $p : [\mathbf{1} \to R]$ and $a \in R = \mathcal{P}\mathbf{1}$, we may simplify $\mathcal{E}\, a\, p = ext\,(\lambda\diamond.\, p\, \diamond)\, a = a \cdot p\, \diamond$. Hence, $\mathcal{Q}\mathbf{1} = \{a \in \mathcal{P}\mathbf{1} \mid \forall p : [\mathbf{1} \to R'] : \mathcal{E}\, a\, p \in R'\} = \{a \in R \mid \forall r \in R' : a \cdot r \in R'\}$. This set is a subset of R', since $a \in \mathcal{Q}\mathbf{1}$ and $1 \in R'$ implies $a = a \cdot 1 \in R'$. Conversely, if r' is in R', then for all r in R', $r' \cdot r$ is in R', whence r' is in $\mathcal{Q}\mathbf{1}$.

If $\mathcal{Q}'$ is an arbitrary sub-construction of $\mathcal{P}$ with $\mathcal{Q}'\mathbf{1} = R'$, then $\mathcal{Q}'\mathbf{X} \subseteq \mathcal{Q}\mathbf{X}$ holds for all ground domains $\mathbf{X}$ since existential quantification in $\mathcal{Q}'$ maps $\mathcal{Q}'\mathbf{1}$-predicates to $\mathcal{Q}'\mathbf{1}$. □

Because of its definition in terms of existential quantification, one might believe that the existential restriction of a final construction for R is a final construction for R'. However this is not true as pointed out in section 9.6. There are two reasons for this. First, two distinct second order predicates in $[[\mathbf{X} \to R] \stackrel{rlin}{\to} R]$ may produce equal results for predicates in $[\mathbf{X} \to R']$. They are then still different in the restriction of the final construction for R, but equal in $[[\mathbf{X} \to R'] \stackrel{rlin}{\to} R']$. Second, there may be additional members in $[[\mathbf{X} \to R'] \stackrel{rlin}{\to} R']$ that cannot be obtained by restricting predicates in $[[\mathbf{X} \to R] \stackrel{rlin}{\to} R]$.

Despite of this general result, we also meet examples for semirings R and R' where the existential restriction of the final construction for R is final for R' — see Th. 8.1.

5 The known power constructions and their semirings

The algebraic theory of power constructions covers the five known constructions mentioned in the introduction if the empty set is not artificially excluded. We shall see this in the remainder of the paper. The characteristic semiring of the lower power construction is the 'lower semiring' $\mathbf{L} = \{0 < 1\}$. In its logic, only positive answers 1 are durable whereas negative answers 0 may become positive if the computation proceeds. The logic of the 'upper semiring' $\mathbf{U} = \{1 < 0\}$ belonging to the upper power construction behaves conversely. The semiring of the convex power construction is $\mathbf{C} = \{0, 1\}$ where 0 and 1 are incomparable.

Sandwich and mixed power construction share the same characteristic semiring $\mathbf{B} = \{\perp, 0, 1\}$ where $\perp$ is below the incomparable values 0 and 1. Addition and multiplication in this semiring correspond to parallel disjunction and conjunction. The logic of this semiring was investigated in [13]. The value $\perp$ denotes a state of ignorance which may turn to 'true' or 'false' when the computation proceeds.

To obtain a better connection among these semirings, we additionally introduce the 'double semiring' $\mathbf{D} = \mathbf{L} \times \mathbf{U}$. It has four elements ordered as follows:

$$\begin{tikzcd} & (1,0) & \\ (0,0) \arrow[ur] & & (1,1) \arrow[ul] \\ & (0,1) \arrow[ul] \arrow[ur] & \end{tikzcd} \qquad \begin{tikzcd} & \top & \\ 0 \arrow[ur] & & 1 \arrow[ul] \\ & \perp \arrow[ul] \arrow[ur] & \end{tikzcd}$$

The picture to the left shows a representation of $\mathbf{D}$ in terms of pairs of members of the lower and upper semiring. The picture to the right shows a logical interpretation of $\mathbf{D}$. Again, the least element $\perp$ denotes a state of ignorance. In contrast, $\top$ denotes a state of inconsistency: a computation returning $\top$ subsumes both 0 and 1. The logic of the double semiring was investigated in [1].

The five semirings $\mathbf{L}$, $\mathbf{U}$, $\mathbf{D}$, $\mathbf{B}$, and $\mathbf{C}$ are related as follows: $\mathbf{C}$ is a sub-semiring of $\mathbf{B}$, which in turn is a sub-semiring of $\mathbf{D}$, which is the product of $\mathbf{L}$ and $\mathbf{U}$. In [7, 5], it is shown that we need not worry about linearity when considering these semirings.

Proposition 5.1 Let R be any of the semirings $\mathbf{L}$, $\mathbf{U}$, $\mathbf{C}$, $\mathbf{B}$, and $\mathbf{D}$. All additive maps between left (right) R-modules are left (right) R-linear.

Since $\mathbf{L} = \{0 < 1\}$ and $\mathbf{U} = \{1 < 0\}$, there is an order isomorphism between them that interchanges 0 and 1. This order isomorphism is interpreted as negation and denoted by overlining. By pointwise extension to functions, any $\mathbf{L}$-predicate $p : [\mathbf{X} \to \mathbf{L}]$ can be negated to a $\mathbf{U}$-predicate $\overline{p} : [\mathbf{X} \to \mathbf{U}]$ and vice versa.

Negation becomes an inner operation of $\mathbf{D} = \mathbf{L} \times \mathbf{U}$ by defining $\overline{(a, b)} = (\overline{b}, \overline{a})$. This operation maps $0 = (0, 0)$ to $1 = (1, 1)$ and vice versa, and maps $\bot = (0, 1)$ and $\top = (1, 0)$ to themselves. Hence, the sub-semirings $\mathbf{B}$ and $\mathbf{C}$ of $\mathbf{D}$ are also closed w.r.t. negation. For all instances of negation, i.e. $[\mathbf{L} \to \mathbf{U}]$, $[\mathbf{U} \to \mathbf{L}]$, and $[\mathbf{D} \to \mathbf{D}]$, the equations $\overline{a + b} = \overline{a} \cdot \overline{b}$ and $\overline{a \cdot b} = \overline{a} + \overline{b}$ are easily verified.

6 The lower power construction

The lower power construction has characteristic semiring $\mathbf{L} = \{0 < 1\}$ where $1 + 1 = 1$, whereas the upper power construction has the dual semiring $\mathbf{U} = \{1 < 0\}$. In this section and the next one, we investigate the final constructions with these semirings. Their representation in terms of second order predicates may be translated first into terms of open sets, then into topological terms of Scott closed sets and Scott compact upper sets. This shows our final constructions to be equivalent with the well known classical constructions.

6.1 From predicates to open sets

According to Th. 3.5, the final construction for semiring $\mathbf{L}$ is given in predicative form by $\mathcal{L}_p\mathbf{X} = [[\mathbf{X} \to \mathbf{L}] \overset{rlin}{\to} \mathbf{L}]$, and the final $\mathbf{U}$-construction by $\mathcal{U}_p\mathbf{X} = [[\mathbf{X} \to \mathbf{U}] \overset{rlin}{\to} \mathbf{U}]$. By Prop. 5.1, we obtain the simpler descriptions
$\mathcal{L}_p\mathbf{X} = [[\mathbf{X} \to \mathbf{L}] \overset{add}{\to} \mathbf{L}]$ and $\mathcal{U}_p\mathbf{X} = [[\mathbf{X} \to \mathbf{U}] \overset{add}{\to} \mathbf{U}]$

It is well known that the domain $\Omega\mathbf{X}$ of open sets of $\mathbf{X}$ ordered by inclusion is isomorphic to the function space $[\mathbf{X} \to \mathbf{2}]$. The isomorphism is given by the following table:

$\Omega\mathbf{X}$	$x \in O$	$x \notin O$	$\subseteq$	$\cup$	$\cap$	$\emptyset$	$\mathbf{X}$
$[\mathbf{X} \to \mathbf{2}]$	$p\,x = \top$	$p\,x = \bot$	$\leq$	$\sqcup$	$\sqcap$	$\lambda x.\,\bot$	$\lambda x.\,\top$

Thus, $\Omega(\Omega\mathbf{X})$ is isomorphic to $[[\mathbf{X} \to \mathbf{2}] \to \mathbf{2}]$ by means of $\varphi P = \{O \mid P p_O = \top\}$ where

$$p_O = \lambda x. \begin{cases} \top & \text{if } x \in O \\ \bot & \text{otherwise} \end{cases} \qquad \text{and its inverse} \qquad \mathcal{E}\mathcal{O} = \lambda p. \begin{cases} \top & \text{if } p^{-1}[\top] \in \mathcal{O} \\ \bot & \text{otherwise} \end{cases}$$

Hence, both $\mathcal{L}_p\mathbf{X}$ and $\mathcal{U}_p\mathbf{X}$ correspond to subsets of $\Omega(\Omega\mathbf{X})$. For $\mathcal{L}_p\mathbf{X}$, $P(\lambda x.\,\bot) = \bot$ has to hold which translates into $\emptyset \notin \mathcal{G}$ where $\mathcal{G}$ is the open set of open sets corresponding to P. In addition, $P(p \sqcup q) = P\,p \sqcup P\,q$ has to hold, or equivalently $P(p \sqcup q) = \top$ iff $P\,p = \top$ or $P\,q = \top$. This translates into $O \cup O' \in \mathcal{G}$ iff $O \in \mathcal{G}$ or $O' \in \mathcal{G}$. The implication from right to left always holds since $\mathcal{G}$ is an upper set because it is open. Hence, only the implication from left to right matters. In analogy to a topological notion, we call open sets with these properties *grills*.

The translation for $\mathcal{U}_p\mathbf{X}$ is just dual. $P(\lambda x.\,\top) = \top$ in $[[\mathbf{X} \to \mathbf{U}] \to \mathbf{U}]$ corresponds to $\mathbf{X} \in \mathcal{F}$ in $\Omega(\Omega\mathbf{X})$ where $\mathcal{F}$ is the set of open sets corresponding to P. In addition, $P(p \sqcap q) = P\,p \sqcap P\,q$ has to hold, or equivalently $P(p \sqcap q) = \top$ iff $P\,p = \top$ and $P\,q = \top$. This translates into $O \cap O' \in \mathcal{F}$ iff $O \in \mathcal{F}$ and $O' \in \mathcal{F}$. Here, only the implication from right to left matters. Sets with these properties are called *open filters* in [17].

In the remainder of this section, we proceed by the investigation of $\mathcal{L}$. $\mathcal{U}$ is considered in section 7.

6.2 The lower power construction in terms of grills

An *open grill* of $\mathbf{X}$ is an open set $\mathcal{G}$ in $\Omega\mathbf{X}$ satisfying the two grill properties:

(1) $\emptyset$ is not in $\mathcal{G}$,

(2) Let O and O' be open sets in $\mathbf{X}$. If $O \cup O'$ is in $\mathcal{G}$, then at least one of O and O' is in $\mathcal{G}$.

Let $\mathcal{L}_\Gamma\mathbf{X}$ be the poset of open grills of $\mathbf{X}$ ordered by inclusion.

Theorem 6.1 $\mathcal{L}_p\mathbf{X}$ and $\mathcal{L}_\Gamma\mathbf{X}$ are isomorphic for all ground domains $\mathbf{X}$. The power operations for $\mathcal{L}_\Gamma\mathbf{X}$ are given by the following table:

$\mathcal{L}\mathbf{X}$	$[[\mathbf{X} \to \mathbf{L}] \stackrel{add}{\to} \mathbf{L}]$	$\mathcal{L}_\Gamma\mathbf{X}$
$A \leq B$	$\forall p : A\,p \leq B\,p$	$A \subseteq B$
$\bigsqcup\mathcal{D}$	$\lambda p.\, \bigsqcup_{D\in\mathcal{D}} Dp$	$\bigcup\mathcal{D}$
θ	$\lambda p.\, 0$	$\emptyset$
$A \uplus B$	$\lambda p.\, Ap + Bp$	$A \cup B$
$\{\!\mid x \mid\!\}$	$\lambda p.\, p\,x$	$\mathcal{O}(x) = \{O \mid x \in O\}$
$ext\ f\ A$	$\lambda p.\, A\,(\lambda x.\, fxp)$	$\{O \mid \{x \mid O \in fx\} \in A\}$

Proof: Isomorphism and order are already known. One easily verifies that arbitrary unions of open grills are open grills again. Hence, $\bigcup\mathcal{D}$ is the lub of the directed set $\mathcal{D}$.

$\theta = \varphi(\lambda p.\, 0) = \{O \mid (\lambda p.\, \perp)\, p_O = \top\} = \emptyset$

$A \uplus B = \varphi(\lambda p. \mathcal{E}A\,p + \mathcal{E}B\,p) = \{O \mid \mathcal{E}A\,p_O \sqcup \mathcal{E}B\,p_O = \top\} = \{O \mid O \in A \text{ or } O \in B\} = A \cup B$

$\{\!\mid x \mid\!\} = \varphi(\lambda p.\, px) = \{O \mid p_O x = \top\} = \{O \mid x \in O\} = \mathcal{O}(x)$

$$\begin{aligned} ext\ f\ A &= \varphi(\lambda p.\, \mathcal{E}A\,(\lambda x.\, \mathcal{E}(fx)p)) \\ &= \{O \mid \mathcal{E}A\,(\lambda x.\, \mathcal{E}(fx)p_O) = \top\} \\ &= \{O \mid (\lambda x.\, \mathcal{E}(fx)p_O)^{-1}[\top] \in A\} \\ &= \{O \mid \{x \mid \mathcal{E}(fx)p_O = \top\} \in A\} \\ &= \{O \mid \{x \mid O \in fx\} \in A\} \qquad \square \end{aligned}$$

Summarizing, we see that the lower power domain in terms of open grills is quite unhandy, and the realization of the power operations, in particular of extension, is quite complex. Fortunately, we need neither show the continuity of *ext f* nor the validity of the power axioms for $\mathcal{L}_\Gamma$ since the isomorphism gives this for free.

6.3 The lower power construction in terms of closed sets

In this section, we show that the common lower power construction in terms of closed sets is isomorphic to $\mathcal{L}_\Gamma\mathbf{X}$.

Proposition 6.2

$\mathcal{L}_\Gamma \mathbf{X}$ is isomorphic to the poset $\mathcal{L}_C \mathbf{X}$ of closed sets of $\mathbf{X}$ ordered by inclusion.

Proof: Given a closed set C, let $\mathcal{G}(C)$ be the set of all open sets of $\mathbf{X}$ that meet C. $\mathcal{G}(C)$ is easily shown to be an open grill. Obviously, $C \subseteq C'$ implies $\mathcal{G}(C) \subseteq \mathcal{G}(C')$. Conversely, assume $\mathcal{G}(C) \subseteq \mathcal{G}(C')$ holds, and let c be a point in C. Then every open environment of c is in $\mathcal{G}(C')$, i.e. meets C'. Thus, c is in $\mathsf{cl}\, C' = C'$ by Prop. 2.7. Summarizing, $C \subseteq C'$ holds iff $\mathcal{G}(C) \subseteq \mathcal{G}(C')$.

Finally, we have to show that the mapping $\mathcal{G}(.)$ is surjective. Let $\mathcal{G}$ be an open grill. Let U be the union of all open sets of $\mathbf{X}$ that are *not* in $\mathcal{G}$, and let C be its complement. U is open as union of open sets, whence C is closed. We claim $\mathcal{G} = \mathcal{G}(C)$.

The set $\mathcal{S} = \{O \text{ open} \mid O \notin \mathcal{G}\}$ is directed: It is not empty since $\emptyset$ is in it, and $O, O' \notin \mathcal{G}$ implies $O \cup O' \notin \mathcal{G}$. If U, the union of the directed set $\mathcal{S}$, were in $\mathcal{G}$, then one of the members of $\mathcal{S}$ would be in $\mathcal{G}$ as $\mathcal{G}$ is open. Thus, U is not in $\mathcal{G}$.

If an open set O meets C, then O is not a subset of U. Thus, $O \cup U$ is a proper superset of U. Hence, it is in $\mathcal{G}$ since U is the union of all open sets not in $\mathcal{G}$. $O \cup U \in \mathcal{G}$ and $U \notin \mathcal{G}$ imply $O \in \mathcal{G}$.

If O does not meet C, then O is a subset of U. If O were in $\mathcal{G}$, then U were in $\mathcal{G}$, too, as $\mathcal{G}$ is an upper set. The last two paragraphs together show $\mathcal{G}(C) = \mathcal{G}$. $\square$

After establishing this isomorphism, we translate the power operations into terms of closed sets.

Theorem 6.3 The final $\mathbf{L}$-construction $[[\mathbf{X} \to \mathbf{L}] \stackrel{rlin}{\to} \mathbf{L}]$ is isomorphic to

(1) $\{C \subseteq \mathbf{X} \mid C \text{ is Scott closed}\}$ ordered by inclusion '$\subseteq$',

(2) $\bigsqcup_{i \in I} A_i = \mathsf{cl} \bigcup_{i \in I} A_i$ where 'cl' denotes Scott closure,

(3) $\theta = \emptyset$,

(4) $A \uplus B = A \cup B$,

(5) $\{\!|x|\!\} = \downarrow x$,

(6) $\mathit{ext}\, f\, A = \bigsqcup f[A] = \mathsf{cl} \bigcup f[A] = \mathsf{cl} \bigcup_{a \in A} f a$.

Proof:

(1) The isomorphism is already known (Prop. 6.2).

(2) Because $\mathsf{cl} \bigcup_{i \in I} A_i$ is the least closed superset of $\bigcup_{i \in I} A_i$.

(3) $\mathcal{G}(\emptyset) = \{O \mid O \cap \emptyset \neq \emptyset\} = \emptyset = \theta_\Gamma$.

(4) An open set meets $A \cup B$ iff it meets A or meets B. Hence, $\mathcal{G}(A \cup B) = \mathcal{G}(A) \cup \mathcal{G}(B) = \mathcal{G}(A) \uplus_\Gamma \mathcal{G}(B)$.

(5) An open set meets $\downarrow x$ iff it contains x. Hence, $\mathcal{G}(\downarrow x) = \{O \mid x \in O\} = \{\!|x|\!\}_\Gamma$.

(6) By Prop. 2.6, an open set meets $\mathsf{cl}\, S$ iff it meets S. Hence,

$$
\begin{aligned}
\mathcal{G}(\mathsf{cl} \bigcup f[A]) &= \{O \mid O \cap \bigcup f[A] \neq \emptyset\} \\
&= \{O \mid \exists a \in A : O \cap fa \neq \emptyset\} \\
&= \{O \mid \exists a \in A : O \in \mathcal{G}(fa)\} \\
&= \{O \mid A \cap \{a \mid O \in \mathcal{G}(fa)\} \neq \emptyset\} \\
&= \{O \mid \{a \mid O \in \mathcal{G}(fa)\} \in \mathcal{G}(A)\} \\
&= ext_{\Gamma}\, (\mathcal{G}(.) \circ f)\, (\mathcal{G}(A))
\end{aligned}
$$

Here, we have to make sure that $\{a \mid O \in \mathcal{G}(fa)\}$ is open. It is the inverse image by f of the open set $\{C' \in \mathcal{L}_C \mathbf{Y} \mid O \in \mathcal{G}(C')\} = \{C' \mid O \cap C' \neq \emptyset\}$.

These equations show that $\mathcal{G}(.)$ becomes a power isomorphism if the operations for closed sets are chosen as in the theorem. □

7 Upper power constructions

The upper power construction as introduced by [16] has characteristic semiring $\mathbf{U} = \{1 < 0\}$ with $1 + 1 = 1$. Although this semiring looks as simple as the lower semiring $\mathbf{L} = \{0 < 1\}$, the situation here is much more complex. The theory is considerably harder than in the lower case, and nevertheless produces weaker results.

7.1 The upper construction in terms of open filters

In section 6.1, we already saw that the final construction $\mathcal{U}_p\mathbf{X} = [[\mathbf{X} \to \mathbf{U}] \overset{add}{\to} \mathbf{U}]$ is isomorphic to the set of open filters of $\mathbf{X}$.

Definition 7.1 An open filter in a domain $\mathbf{X}$ is an open set $\mathcal{F}$ of open sets of $\mathbf{X}$ with

(1) $\mathbf{X} \in \mathcal{F}$

(2) If O_1 and O_2 are in $\mathcal{F}$, then so is their intersection $O_1 \cap O_2$.

The poset of all open filters of $\mathbf{X}$ ordered by inclusion '$\subseteq$' is denoted by $\mathcal{U}_\Phi\mathbf{X}$.

Theorem 7.2 $\mathcal{U}_p\mathbf{X}$ and $\mathcal{U}_\Phi\mathbf{X}$ are isomorphic for all ground domains $\mathbf{X}$. The power operations for $\mathcal{U}_\Phi\mathbf{X}$ are given by the following table:

$\mathcal{U}\mathbf{X}$	$[[\mathbf{X} \to \mathbf{U}] \overset{add}{\to} \mathbf{U}]$	$\mathcal{U}_\Phi\mathbf{X}$
$A \leq B$	$\forall p : A\, p \leq B\, p$	$A \subseteq B$
$\bigsqcup \mathcal{D}$	$\lambda p.\, \bigsqcup_{D \in \mathcal{D}} Dp$	$\bigcup \mathcal{D}$
θ	$\lambda p.\, 0$	$\Omega\mathbf{X}$
$A \uplus B$	$\lambda p.\, Ap + Bp$	$A \cap B$
$\{\!\mid x \mid\!\}$	$\lambda p.\, p\, x$	$\mathcal{O}(x) = \{O \mid x \in O\}$
$ext\, f\, A$	$\lambda p.\, A\, (\lambda x.\, fxp)$	$\{O \mid \{x \mid O \in fx\} \in A\}$

Proof: Isomorphism and order are already known. One easily verifies that arbitrary unions of open filters are open filters again. Hence, $\bigcup \mathcal{D}$ is the lub of the directed set $\mathcal{D}$.

$\theta = \varphi\,(\lambda p.\, 0) = \{O \mid (\lambda p.\, \top)\, p_O = \top\} = \Omega\mathbf{X}$.

$A \uplus B = \varphi\,(\lambda p. \mathcal{E} A\, p + \mathcal{E} B\, p) = \{O \mid \mathcal{E} A\, p_O \sqcap \mathcal{E} B\, p_O = \top\} = \{O \mid O \in A \text{ and } O \in B\} = A \cap B$

The formulae for the operations $\{\!\mid . \mid\!\}$ and ext and their proofs look exactly as those in Th. 6.1.□

7.2 The upper power construction in terms of compact upper sets

As the upper power domain in terms of open filters is quite unhandy, we look for an representation in terms of subsets of $\mathbf{X}$. Following [17], we use compact upper sets to this end. Unfortunately, this approach does not work out for all domains. The class of allowed domains however is quite large.

For an arbitrary domain $\mathbf{X}$, let $\mathcal{U}_K\mathbf{X}$ be the set of all compact upper sets of $\mathbf{X}$. For every compact set K, the set of open environments of K is an open filter: it obviously contains intersections and certainly $\mathbf{X}$, and it is open because K is compact (see Prop. 2.9).

Thus, there is a mapping $\mathcal{O}(.) : \mathcal{U}_K\mathbf{X} \to \mathcal{U}_\Phi\mathbf{X}$. By Prop. 2.10, for every two compact upper sets K and K', $K \supseteq K'$ is equivalent to $\mathcal{O}(K) \subseteq \mathcal{O}(K')$. Since we ordered $\mathcal{U}_\Phi\mathbf{X}$ by '$\subseteq$', we have to order $\mathcal{U}_K\mathbf{X}$ by '$\supseteq$'. Then we obtain that $\mathcal{O}(.) : \mathcal{U}_K\mathbf{X} \to \mathcal{U}_\Phi\mathbf{X}$ has the property $K \leq K'$ iff $\mathcal{O}(K) \leq \mathcal{O}(K')$.

Hence, all what is needed further is the surjectivity of $\mathcal{O}(.)$. In contrast to the corresponding mapping $\mathcal{G}(.)$ of the lower power construction, there are domains where $\mathcal{O}(.)$ is not surjective. In [17], Smyth points out that surjectivity of $\mathcal{O}(.)$ is equivalent to the topological property of sobriety. He cites [9] for a proof of this fact. The class of sober domains is however large; it contains for instance all continuous domains (see [5] for a proof). An example for a non-sober domain is given in [11].

Sobriety allows to prove a topological property that is useful to analyze $\mathcal{U}_K\mathbf{X}$.

Lemma 7.3 Let $\mathbf{X}$ be a sober domain. Then for every open set O in $\mathbf{X}$, the set $\mathcal{K}(O)$ of compact upper subsets of O is open.

Proof: Since $\mathbf{X}$ is sober, $\mathcal{U}_K\mathbf{X}$ and $\mathcal{U}_\Phi\mathbf{X}$ are isomorphic. Hence, $\mathcal{U}_K\mathbf{X}$ is a domain, and the isomorphism is continuous as all order isomorphisms are.

Let $\mathcal{D}$ be a directed set in $\mathcal{U}_K\mathbf{X}$. Then $\mathcal{D}$ has a limit K, and $\bigcap\mathcal{D} \supseteq K$ holds because '$\supseteq$' is the order in $\mathcal{U}_K\mathbf{X}$. By continuity of $\mathcal{O}(.)$, $\mathcal{O}(K) = \bigcup_{A\in\mathcal{K}} \mathcal{O}(A)$ follows. Let O be an open set in $\mathbf{X}$ with $\bigcap\mathcal{D} \in \mathcal{K}(O)$. Then $K \subseteq \bigcap\mathcal{D} \subseteq O$ whence $O \in \mathcal{O}(K)$. Thus, there is A in $\mathcal{D}$ with $O \in \mathcal{O}(A)$ i.e. $A \in \mathcal{K}(O)$. □

We now are able to translate the power operations from $\mathcal{U}_\Phi$ to $\mathcal{U}_K$.

Theorem 7.4 If $\mathbf{X}$ is sober, the upper power domain $\mathcal{U}_\Phi\mathbf{X}$ is isomorphic to $\mathcal{U}_K\mathbf{X}$. The power domains and operations are given by

(1) $\mathcal{U}_K\mathbf{X}$ is the set of all compact upper sets of $\mathbf{X}$.

(2) $K \leq K'$ iff $K \supseteq K'$

(3) $\bigsqcup\mathcal{D} = \bigcap\mathcal{D}$ for directed sets $\mathcal{D}$ in $\mathcal{U}_K\mathbf{X}$.

(4) $\theta = \emptyset$

(5) $A \uplus B = A \cup B$

(6) $\{\!|x|\!\} = \uparrow x$ for all $x \in \mathbf{X}$.

(7) If both $\mathbf{X}$ and $\mathbf{Y}$ are sober, and $f : [\mathbf{X} \to \mathcal{U}_K\mathbf{Y}]$ is continuous and A is in $\mathcal{U}_K\mathbf{X}$, then $ext\ f\ A = \bigcup_{a\in A} fa = \bigcup f[A]$.

All these operations are well defined and continuous.

Proof: (1) and (2) are the definition of $\mathcal{U}_K\mathbf{X}$.

To prove the statement about $\bigsqcup\mathcal{D}$, we have to show that $\bigcap\mathcal{D}$ is a compact upper set. By set theory, it is then the least upper bound (w.r.t. '$\supseteq$') of $\mathcal{D}$.

Let $\mathcal{O}$ be an open cover of $\bigcap\mathcal{D}$, i.e. $\bigcap\mathcal{D} \subseteq \bigcup\mathcal{O}$. By Lemma 7.3, there is some A in $\mathcal{D}$ with $A \subseteq \bigcup\mathcal{O}$. As A is compact, there is a finite subset $\mathcal{F}$ of $\mathcal{O}$ with $\bigcap\mathcal{D} \subseteq A \subseteq \bigcup\mathcal{F}$.

To prove (4) through (7), we show that $\mathcal{O}(.)$ operates as a power isomorphism. By Prop. 2.9, the results of the operations in $\mathcal{U}_K$ are compact upper sets again.

(4) $\mathcal{O}(\emptyset) = \Omega\mathbf{X} = \theta_\Phi$

(5) If O is an open set in $\mathcal{O}(A \cup B)$, then $O \supseteq A \cup B \supseteq A, B$ holds, whence O is in $\mathcal{O}(A) \cap \mathcal{O}(B)$. Conversely, if O is in the intersection, then $O \supseteq A$ and $O \supseteq B$ implies $O \supseteq A \cup B$.

(6) Since open sets are upper, $\mathcal{O}(\uparrow x) = \mathcal{O}(x) = \{\!|x|\!\}_\Phi$ holds.

(7) Let $f : [\mathbf{X} \to \mathcal{U}_K\mathbf{Y}]$ be continuous and A in $\mathcal{U}_K\mathbf{X}$.

$$\begin{array}{rcl}
\mathcal{O}(\bigcup f[A]) & = & \{O \in \Omega\mathbf{Y} \mid \bigcup_{a \in A} fa \subseteq O\} \\
& = & \{O \in \Omega\mathbf{Y} \mid \forall a \in A : fa \subseteq O\} \\
& = & \{O \in \Omega\mathbf{Y} \mid A \subseteq \{x \in \mathbf{X} \mid fx \subseteq O\}\} \\
& = & \{O \in \Omega\mathbf{Y} \mid \{x \in \mathbf{X} \mid O \in \mathcal{O}(fx)\} \in \mathcal{O}(A)\} \\
& = & ext_\Phi\,(\mathcal{O}(.) \circ f)\,(\mathcal{O}(A))
\end{array}$$

Here, the set $\{x \in \mathbf{X} \mid fx \subseteq O\}$ is open as the inverse image of the open set $\mathcal{K}(O)$ by the continuous function f (see Lemma 7.3). □

A direct topological proof of the compactness of $\bigcup f[A]$ is also possible, but would be more tedious. The same remark is valid for a direct proof of the continuity of $ext\, f : \mathcal{U}_K\mathbf{X} \to \mathcal{U}_K\mathbf{Y}$. Both proofs are unnecessary because one may use that $ext\, f : \mathcal{U}_\Phi\mathbf{X} \to \mathcal{U}_\Phi\mathbf{Y}$ is well-defined and continuous. These facts are in turn inherited from the well-definedness and continuity of the operations in the final power construction defined in terms of functions of higher order.

8 The sandwich power construction

The sandwich power construction $\mathcal{S}$ was defined in [2, 3, 4] for algebraic ground domains only. In this section, we show that $\mathcal{S}$ may be extended to all domains as the final **B**-construction, or equivalently the existential restriction of the final **D**-construction to the sub-semiring **B** of **D**.

8.1 $\mathcal{S}$ — the existential restriction of $\mathcal{D}$ to B

By Th. 4.1, we know that the final construction for semiring $\mathbf{D} = \mathbf{L} \times \mathbf{U}$ is the product of the final constructions for semirings **L** and **U**.

$$[[\mathbf{X} \to \mathbf{D}] \overset{add}{\to} \mathbf{D}] \;=\; [[\mathbf{X} \to \mathbf{L}] \overset{add}{\to} \mathbf{L}] \times [[\mathbf{X} \to \mathbf{U}] \overset{add}{\to} \mathbf{U}]$$

Although the equality is only an isomorphism, we do not write down the isomorphisms explicitly for simplification. Instead, we directly apply pairs of functions to pairs of predicates subsuming an equality $(P^L, P^U)(p^L, p^U) = (P^L p^L, P^U p^U)$.

We denote the final $\mathbf{D}$-construction by $\mathcal{D}$. Since $\mathbf{B}$ is a sub-semiring of $\mathbf{D}$, Th. 4.2 delivers us the sub-construction $\mathcal{S} = \mathcal{D}|_{\mathbf{B}}$ with $\mathcal{S}\mathbf{X} = \{P \in \mathcal{D}\mathbf{X} \mid \forall p \in [\mathbf{X} \to \mathbf{B}] : Pp \in \mathbf{B}\}$.

In Th. 8.7 below, we shall see that $\mathcal{S}$ is a generalization of the *sandwich power construction* defined in [2] for algebraic ground domains and investigated further in [3, 4]. In anticipation of the theorem, we chose the abbreviation $\mathcal{S}$ and call the domain $\mathcal{S}\mathbf{X}$ *sandwich power domain* and its elements *sandwiches*. Consequently, the condition restricting $\mathcal{D}\mathbf{X}$ to $\mathcal{S}\mathbf{X}$ is called *sandwich condition* or shorter *condition S*.

If R' is a sub-semiring of some semiring R, then generally, the existential restriction of the final R-construction to R' is completely different from the final R'-construction. In the case of $\mathbf{B}$ and $\mathbf{D}$ however, these two constructions happen to coincide.

Theorem 8.1 The final $\mathbf{B}$-construction is isomorphic to the existential restriction of $\mathcal{D}$ to $\mathbf{B}$: $\mathcal{S}\mathbf{X} \cong [[\mathbf{X} \to \mathbf{B}] \overset{add}{\to} \mathbf{B}]$.

Proof: We have to establish an isomorphism between $[[\mathbf{X} \to \mathbf{B}] \overset{add}{\to} \mathbf{B}]$ and $\mathcal{S}\mathbf{X} = \{P \in [[\mathbf{X} \to \mathbf{D}] \overset{add}{\to} \mathbf{D}] \mid \forall p : [\mathbf{X} \to \mathbf{B}] : Pp \in \mathbf{B}\}$. An obvious choice is the restriction and co-restriction $\mathcal{R}$ of functions in $\mathcal{S}\mathbf{X}$ to arguments in $[\mathbf{X} \to \mathbf{B}]$ and results in $\mathbf{B}$. Since the power operations of $\mathcal{S}$ are inherited from those of $\mathcal{D}$, restriction $\mathcal{R}$ coincides with existential quantification in $\mathcal{S}$, whence it is a power homomorphism as indicated in section 3.5. We only have to show that all its instances are domain isomorphisms, then it is a power isomorphism.

Instead of $\mathcal{S}\mathbf{X} \cong [[\mathbf{X} \to \mathbf{B}] \overset{add}{\to} \mathbf{B}]$, we show the more general domain isomorphism $[[\mathbf{X} \to \mathbf{D}] \overset{add}{\to} \mathbf{D}] \cong [[\mathbf{X} \to \mathbf{B}] \overset{add}{\to} \mathbf{D}]$. Using $\mathbf{D} = \mathbf{L} \times \mathbf{U}$, we may also show $[[\mathbf{X} \to \mathbf{D}] \overset{add}{\to} \mathbf{L}] \cong [[\mathbf{X} \to \mathbf{B}] \overset{add}{\to} \mathbf{L}]$ and the corresponding isomorphism involving $\mathbf{U}$ instead of $\mathbf{L}$.

(1) For every $P : [[\mathbf{X} \to \mathbf{D}] \overset{add}{\to} \mathbf{L}]$, $P(f^L, f^U) = P(f^L, \underline{1})$ for all $(f^L, f^U) : [\mathbf{X} \to \mathbf{D}]$, where $\underline{1} = \lambda x.\, 1$.

Proof: In $\mathbf{U}$, 1 is least, whence $P(f^L, f^U) \geq P(f^L, \underline{1})$ by monotonicity. By additivity of P, $P(f^L, \underline{1}) = P(f^L, f^U) + P(f^L, \underline{1}) \geq P(f^L, f^U)$ since $1 = x + 1$ in $\mathbf{U}$ and $x \geq 0$ in $\mathbf{L}$.

There is an obvious continuous mapping from $[[\mathbf{X} \to \mathbf{D}] \overset{add}{\to} \mathbf{L}]$ to $[[\mathbf{X} \to \mathbf{B}] \overset{add}{\to} \mathbf{L}]$, namely restriction $\mathcal{R}$ to arguments in $[\mathbf{X} \to \mathbf{B}]$. $\mathcal{R}P \leq \mathcal{R}P'$ implies $P \leq P'$ by statement (1) since $(f^L, \underline{1})$ creates results $(0, 1) = \perp$ and $(1, 1) = 1$ only, i.e. maps from $\mathbf{X}$ to $\mathbf{B}$.

To show the surjectivity of $\mathcal{R}$, let Q be in $[[\mathbf{X} \to \mathbf{B}] \overset{add}{\to} \mathbf{L}]$. Then we define $P : [[\mathbf{X} \to \mathbf{D}] \to \mathbf{L}]$ by $P(f^L, f^U) = Q(f^L, \underline{1})$. If we show the additivity of P, then statement (1) implies $\mathcal{R}P = Q$.

$P(f + g) = Pf + Pg$ holds by additivity of Q because $(f^L + g^L, \underline{1}) = (f^L, \underline{1}) + (g^L, \underline{1})$. $P\underline{0} = Q(\underline{0}, \underline{1}) \leq Q(\underline{0}, \underline{0}) = 0$ holds because $1 \leq 0$ in $\mathbf{U}$. Since 0 is least in $\mathbf{L}$, $P\underline{0} = 0$ follows.

Now, $[[\mathbf{X} \to \mathbf{D}] \overset{add}{\to} \mathbf{L}] \cong [[\mathbf{X} \to \mathbf{B}] \overset{add}{\to} \mathbf{L}]$ has been proved. The analogous statement involving $\mathbf{U}$ is proved following the same lines using

(2) For every $P : [[\mathbf{X} \to \mathbf{D}] \overset{add}{\to} \mathbf{U}]$, $P(f^L, f^U) = P(\underline{0}, f^U)$ for all $(f^L, f^U) : [\mathbf{X} \to \mathbf{D}]$. □

8.2 The sandwich power construction in topological terms

For lower and upper semiring, we know — at least for sober ground domains — an explicit representation of the final power construction in topological terms. These representations may be used to derive a topological representation for $\mathcal{S}$.

The definition states $\mathcal{S}\mathbf{X} = \{A \in \mathcal{D}\mathbf{X} \mid \forall p \in [\mathbf{X} \to \mathbf{B}] : A\, p \in \mathbf{B}\}$. $\mathcal{D}\mathbf{X}$ may be represented as $\mathcal{L}_p\mathbf{X} \times \mathcal{U}_p\mathbf{X}$. For $A = (A^L,\, A^U)$, the sandwich condition may be transformed as follows:

$$\begin{array}{lll} \forall\, p \in [\mathbf{X} \to \mathbf{B}] : A\, p \in \mathbf{B} & \text{iff} & \forall\, p \in [\mathbf{X} \to \mathbf{D}] : ((\forall\, x \in \mathbf{X} : p\, x \in \mathbf{B}) \Rightarrow A\, p \in \mathbf{B}) \\ & \text{iff} & \forall\, p \in [\mathbf{X} \to \mathbf{D}] : (A\, p = \top \Rightarrow \exists x \in \mathbf{X} : p\, x = \top) \end{array}$$

This formula may be interpreted such that $\mathcal{S}\mathbf{X}$ consists of all consistent second order predicates of $\mathbf{D}$. A consistent second order predicate does not create inconsistencies by itself. If it results in an inconsistency ($Ap = \top$), then its argument already was inconsistent ($p\, x = \top$ for some x).

By splitting the pairs into components, we obtain further:

$$\begin{array}{ll} \text{iff} & \forall\, p^L \in [\mathbf{X} \to \mathbf{L}],\, p^U \in [\mathbf{X} \to \mathbf{U}] : \\ & \qquad (A^L\, p^L = \top_{\mathbf{L}} \text{ and } A^U\, p^U = \top_{\mathbf{U}} \Rightarrow \exists x \in \mathbf{X} : p^L\, x = \top_{\mathbf{L}} \text{ and } p^U\, x = \top_{\mathbf{U}}) \end{array}$$

By the next translation step, we want to represent the final lower power construction in terms of open grills and the final upper power construction in terms of open filters. Let $\mathcal{G}$ be the open grill belonging to A^L and $\mathcal{O}$ the open filter belonging to A^U. To complete the translation to set notation, we represent the predicates p^L and p^U by open sets O^L and O^U. Then $p^L\, x = \top$ means $x \in O^L$, and same for O^U. Similarly, $A^L p^L = \top$ means $O^L \in \mathcal{G}$, and $A^U p^U = \top$ becomes $O^U \in \mathcal{O}$.

Hence, the chain of equivalences above continues by

$$\begin{array}{lll} (\mathcal{G},\, \mathcal{O}) \in \mathcal{S}\mathbf{X} & \text{iff} & \forall\, O^L,\, O^U \in \Omega\mathbf{X} : \\ & & \qquad (O^L \in \mathcal{G} \text{ and } O^U \in \mathcal{O} \Rightarrow \exists x \in \mathbf{X} : x \in O^L \text{ and } x \in O^U) \\ & \text{iff} & \forall\, O^L \in \mathcal{G},\, O^U \in \mathcal{O} : O^L \cap O^U \neq \emptyset \end{array}$$

For sober ground domain $\mathbf{X}$, one can go one step further and translate the open filters into compact upper sets K. The translation of open grills into closed sets C is always possible. $O^L \in \mathcal{G}$ becomes $C \cap O^L \neq \emptyset$, and $O^U \in \mathcal{O}$ becomes $K \subseteq O^U$.

Hence, the restriction translates into: for all open sets O^L and O^U, if C meets O^L and $K \subseteq O^U$ then O^L meets O^U. For fixed C and O^U, the following holds:

Every open set meeting C meets O^U
  iff every open environment of every point of C meets O^U
  iff every point of C is in the closure of O^U by Prop. 2.7
  iff $C \subseteq \mathsf{cl}\, O^U$.

Hence, one obtains

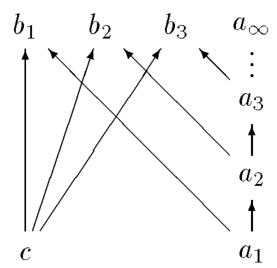
Theorem 8.2 The sandwich power domain $\mathcal{S}\mathbf{X}$ over a sober ground domain $\mathbf{X}$ is isomorphic to the set of all pairs $(C,\, K)$ of a closed set C and a compact upper set K such that for all open sets O with $K \subseteq O$ the inclusion $C \subseteq \mathsf{cl}\, O$ holds.

Two remarks seem to be appropriate. First, the condition '$K \subseteq O$ implies $C \subseteq \mathsf{cl}\, O$' looks quite strange, and it is not obvious how it could have been found without considering the second order predicates. Second, if we had defined a power domain construction directly as in the theorem above, we would have been forced to verify that each power operation respects the topological criterion. This would have been a non-trivial task, in particular for the extension functional.

For a special class of ground domains, the restriction '$K \subseteq O$ open implies $C \subseteq \mathsf{cl}\, O$' may be drastically simplified:

Theorem 8.3 If $\mathbf{X}$ is an M-domain, then a pair (C, K) of a closed set and a compact upper set is a sandwich iff $C \subseteq {\downarrow} K$.

We do not prove the theorem here. A proof is contained in [5]. Instead, we provide an example that shows that the theorem cannot be generalized to all algebraic ground domains. Let $\mathbf{X} = \{a_1, a_2, a_3, \ldots, a_\infty, b_1, b_2, b_3, \ldots, c\}$. There is no point b_∞. The a-points form an ascending sequence: $a_1 < a_2 < \cdots < a_\infty$, whereas the b-points are incomparable. Every a-point is below the corresponding b-point: $a_n < b_n$. The remaining point c is below all b-points, but not below any a-point, not even below a_∞.



This domain is algebraic, but not an M-domain. Let $C = {\downarrow} c = \{c\}$ and let $K = {\uparrow} a_\infty = \{a_\infty\}$. C and K satisfy the sandwich condition although $C \subseteq {\downarrow} K$ does not hold.

8.3 $\mathcal{S}$ for algebraic ground domain

Next, we turn to the algebraic case. If $\mathbf{X}$ is algebraic, then both $\mathcal{L}\mathbf{X}$ and $\mathcal{U}\mathbf{X}$ are algebraic. Their bases are given by the sets of all ${\downarrow} F$ and ${\uparrow} F$ respectively for finite subsets F of $\mathbf{X}^0$. Thus, $\mathcal{D}\mathbf{X}$ is algebraic, and its base is $\{({\downarrow} E, {\uparrow} F) \mid E, F \subseteq_f \mathbf{X}^0\}$. These pairs are also isolated in $\mathcal{S}\mathbf{X}$ provided they satisfy the sandwich condition because $\mathcal{S}\mathbf{X}$ is a sub-domain of $\mathcal{D}\mathbf{X}$. Every point in $\mathcal{D}\mathbf{X}$ is a directed limit of such pairs. Since all pairs below a sandwich are sandwiches again, every point of $\mathcal{S}\mathbf{X}$ is the limit of a directed set of isolated sandwiches. Thus, we obtain

Proposition 8.4
The sandwich power domain over an algebraic ground domain is algebraic. Its base is the set of all sandwiches $({\downarrow} E, {\uparrow} F)$ where E and F are finite subsets of $\mathbf{X}^0$.

The sandwich criterion simplifies drastically for such isolated pairs:

Lemma 8.5 Let $\mathbf{X}$ be a domain. If E and F are finite sets of isolated points of $\mathbf{X}$, then $({\downarrow} E, {\uparrow} F)$ satisfies the sandwich condition iff $E \subseteq {\downarrow}{\uparrow} F$.

Proof: $E \subseteq {\downarrow}{\uparrow} F$ obviously implies condition S. For the opposite, note that ${\uparrow} F$ is open since F consists of isolated points. Thus, the sandwich condition implies $E \subseteq {\downarrow} E \subseteq \mathsf{cl}\, {\uparrow} F$. Since E consists of isolated points, Prop. 2.8 yields $E \subseteq {\downarrow}{\uparrow} F$. $\square$

The representation of the base of $\mathcal{S}\mathbf{X}$ may even be further simplified choosing suitable sets E and F.

Lemma 8.6 Let $\mathbf{X}$ be algebraic, and let E and F be finite subsets of $\mathbf{X}^0$ with $E \subseteq \downarrow\uparrow F$. Then there is a finite subset F' of $\mathbf{X}^0$ with $\uparrow F = \uparrow F'$ and $E \subseteq \downarrow F'$.

Proof: Since $E \subseteq \downarrow\uparrow F$, for every $e \in E$ there is some point $x_e \in \mathbf{X}$ and some point $f_e \in F$ such that $e \leq x_e \geq f_e$. By Prop. 2.5, the points x_e may be assumed to be in the base $\mathbf{X}^0$. With $E' = \{x_e \mid e \in E\}$, we define $F' = E' \cup F$. E' is a finite subset of $\mathbf{X}^0$, whence F' also is. All points e in E are below x_e in F', whence $E \subseteq \downarrow F'$ follows. $\uparrow F \subseteq \uparrow F'$ immediately follows from $F \subseteq F'$. For the opposite inclusion, x_e is above f_e for all e in E, whence $E' \subseteq \uparrow F$ whence $F' \subseteq \uparrow F$. $\square$

Summarizing, we obtain the following theorem:

Theorem 8.7 For algebraic ground domain $\mathbf{X}$, our sandwich power domain over $\mathbf{X}$ is algebraic and coincides with the sandwich power domain of [2, 3, 4]. Its base is the set of all pairs $(\downarrow E, \uparrow F)$ with $E \subseteq \downarrow\uparrow F$, or equivalently the set of all pairs $(\downarrow E, \uparrow F)$ with $E \subseteq \downarrow F$, where in both cases E and F are finite subsets of $\mathbf{X}^0$.

Proof: For the comparison with the sandwich power domain in [2, 3, 4] notice that the authors of these papers write the sandwiches the other way round, i.e. the lower set to the right. Correcting this and translating notation, the paper [3] defines the sandwich power domain to be the ideal completion of all pairs (E, F) of finite subsets of $\mathbf{X}^0$ such that there is a finite subset G of $\mathbf{X}^0$ with $E \subseteq \downarrow G$ and $G \subseteq \uparrow F$. This directly implies $E \subseteq \downarrow\uparrow F$, and conversely, G may be chosen as the set E' in the proof of Lemma 8.6. These pairs are pre-ordered by $(E, F) \preceq (E', F')$ iff $\downarrow E \subseteq \downarrow E'$ and $\uparrow F \supseteq \uparrow F'$. Hence, the poset of equivalence classes of this pre-ordered set is just our base as presented in the theorem. $\square$

9 Mixed and convex power domain

Up to now, we were able to describe lower, upper, and sandwich power domains in terms of second order predicates. We now look for predicative descriptions of mixed and convex power domains. Indeed, such descriptions exist. In case of algebraic ground domain, both the mixed and the convex power construction are sub-constructions of $\mathcal{D}$. The mixed power domain is characterized by the *mix condition* M. There is also a *dual mix condition* $\overline{M}$, and the Plotkin power domain consists of all members of $\mathcal{D}\mathbf{X}$ that satisfy both M and $\overline{M}$.

9.1 Lower and upper implication for D-predicates

The definition of condition M and its dual, condition $\overline{M}$, is prepared by investigating the logic of $\mathbf{D}$ more closely. Because of $\mathbf{D} = \mathbf{L} \times \mathbf{U}$, all $\mathbf{D}$-predicates a may be written as pairs (a^L, a^U).

In addition to the logical operations of disjunction '+', conjunction '$\cdot$', and negation '$^-$' (see section 5), we introduce a kind of difference for $\mathbf{D}$-predicates: $a - b = a \cdot \overline{b}$. It is mainly used as an notational abbreviation.

The following relations are easily verified:

Proposition 9.1 For all **D**-predicates a and b:

$$
\begin{array}{lllll}
(1) & (a+b)^L = a^L + b^L & \text{and} & (a+b)^U = a^U + b^U \\
(2) & (a \cdot b)^L = a^L \cdot b^L & \text{and} & (a \cdot b)^U = a^U \cdot b^U \\
(3) & \overline{a}^L = \overline{a^U} & \text{and} & \overline{a}^U = \overline{a^L} \\
(4) & (a-b)^L = a^L \cdot \overline{b^U} & \text{and} & (a-b)^U = a^U \cdot \overline{b^L}
\end{array}
$$

The next proposition claims the equivalence of various conditions. They are coined as lower and upper implication.

Proposition 9.2 For **D**-predicates, the following equivalences hold:

(1) $a^L \le b^L$ iff $a + b \le b$. In this case, we say that a and b are in the relation of *lower implication* '$\overset{L}{\mapsto}$'.

(2) $a^U \ge b^U$ iff $a + b \ge b$. In this case, a and b are in the relation of *upper implication* '$\overset{U}{\mapsto}$'.

Proof:

(1) By part (1) of Prop. 9.1, $a + b \le b$ holds iff $a^L + b^L \le b^L$ and $a^U + b^U \le b^U$. Since the inequality involving 'U' is a tautology, it can be dropped. Hence, $a+b \le b$ iff $a^L+b^L \le b^L$. This inequality is equivalent to $a^L \le b^L$.

(2) Similarly. □

Lower and upper implication enjoy some properties that are needed in the next section.

Proposition 9.3 Let $X = L$ or U in the following.

(1) The relation '$\overset{X}{\mapsto}$' is reflexive and transitive.

(2) If $a \overset{X}{\mapsto} a'$ and $b \overset{X}{\mapsto} b'$, then $a + b \overset{X}{\mapsto} a' + b'$.

(3) $(a + b) - (a' + b') \overset{X}{\mapsto} (a - a') + (b - b')$.

(4) If P is an additive second order predicate, then $a \overset{X}{\mapsto} b$ implies $Pa \overset{X}{\mapsto} Pb$.

(5) If $(a_i)_{i\in I}$ and $(b_i)_{i\in I}$ are directed families of **D**-predicates with $a_i \overset{X}{\mapsto} b_i$ for all $i \in I$, then $\bigsqcup_{i\in I} a_i \overset{X}{\mapsto} \bigsqcup_{i\in I} b_i$.

Proof: We show the statements for 'L'; the proofs for 'U' are similar.

(1) Immediate by definition.

(2) $a^L \le a'^L$ and $b^L \le b'^L$ implies $(a+b)^L = a^L \sqcup b^L \le a'^L \sqcup b'^L = (a'+b')^L$.

(3) $((a + b) - (a' + b'))^L = (a^L + b^L) \cdot \overline{a'^U + b'^U} = (a^L \cdot \overline{a'^U} \cdot \overline{b'^U}) + (b^L \cdot \overline{a'^U} \cdot \overline{b'^U}) \le$
$(a^L \cdot \overline{a'^U}) + (b^L \cdot \overline{b'^U}) = ((a - a') + (b - b'))^L$.
Here, '$\le$' holds since $p \cdot q \le p$ holds for **L**-predicates p and q.

(4) $a \overset{L}{\mapsto} b$ implies $a + b \le b$, whence $Pa + Pb = P(a + b) \le Pb$, i.e. $Pa \overset{L}{\mapsto} Pb$.

(5) We use the equivalence $a \overset{L}{\mapsto} b$ iff $a^L \le b^L$. If $(a_i)_{i\in I}$ is directed, then $(a_i^L)_{i\in I}$ is directed, too. $a_i^L \le b_i^L$ implies $(\bigsqcup_{i\in I} a_i)^L = \bigsqcup_{i\in I} a_i^L \le \bigsqcup_{i\in I} b_i^L = (\bigsqcup_{i\in I} b_i)^L$. □

9.2 The conditions M and $\overline{M}$

After the preliminaries of the previous section, we are now able to define the conditions M and $\overline{M}$ in terms of second order predicates:

Definition 9.4 Let P be in $[[\mathbf{X} \to \mathbf{D}] \stackrel{add}{\to} \mathbf{D}]$.
P satisfies condition M iff $Pp - Pq \stackrel{L}{\mapsto} P\,(p-q)$ for all predicates $p,\,q : [\mathbf{X} \to \mathbf{D}]$.
P satisfies condition $\overline{M}$ iff $Pp - Pq \stackrel{U}{\mapsto} P\,(p-q)$ for all predicates $p,\,q : [\mathbf{X} \to \mathbf{D}]$.

We now show that the power operations preserve the conditions. Thus, we get two subconstructions $\mathcal{M}$ and $\overline{\mathcal{M}}$ of $\mathcal{D}$. Using the generic Prop. 9.3, the proofs for M and $\overline{M}$ are completely analogous. We formulate them for M.

- $\theta = \lambda p.\,0$, whence $\theta p - \theta q = 0 - 0 = 0 = \theta(p-q)$. By reflexivity of lower implication (Prop. 9.3 (1)), θ satisfies M.
- $\{\!|x|\!\} = \lambda p.\,p\,x$, whence $\{\!|x|\!\}\,p - \{\!|x|\!\}\,q = p\,x - q\,x = (p-q)\,x = \{\!|x|\!\}\,(p-q)$.
  $p\,x - q\,x = (p-q)\,x$ holds since all logical operations are defined pointwise on predicates.
- For A, B in $\mathcal{M}\mathbf{X}$,

$$\begin{array}{rcll} (A \uplus B)\,p - (A \uplus B)\,q & = & (A\,p + B\,p) - (A\,q + B\,q) & \\ & \stackrel{L}{\mapsto} & (A\,p - A\,q) + (B\,p - B\,q) & \text{by Prop. 9.3 (3)} \\ & \stackrel{L}{\mapsto} & A\,(p-q) + B\,(p-q) & \\ & & \quad \text{since } A,\, B \text{ in } \mathcal{M}\mathbf{X} \text{ by Prop. 9.3 (2)} & \\ & = & (A \uplus B)\,(p-q) & \end{array}$$

- For $f : [\mathbf{X} \to \mathcal{M}\mathbf{Y}]$ and A in $\mathcal{M}\mathbf{X}$,

$$\begin{array}{rcll} ext\,f\,A\,p - ext\,f\,A\,q & = & A\,(\lambda x.\,fxp) - A\,(\lambda x.\,fxq) & \\ & \stackrel{L}{\mapsto} & A\,(\lambda x.\,fxp - fxq) & \text{since } A \text{ in } \mathcal{M}\mathbf{X} \\ & \stackrel{L}{\mapsto} & A\,(\lambda x.\,fx(p-q)) & \\ & & \quad \text{since } fx \text{ in } \mathcal{M}\mathbf{X} \text{ by Prop. 9.3 (4); } A \text{ is additive} & \end{array}$$

- If $(A_i)_{i\in I}$ is a directed family in $\mathcal{M}\mathbf{X}$, then both $(A_i p - A_i q)_{i\in I}$ and $(A_i\,(p-q))_{i\in I}$ are directed families with $A_i p - A_i q \stackrel{L}{\mapsto} A_i\,(p-q)$ for all $i \in I$. By Prop. 9.3 (5) $A\,p - A\,q \stackrel{L}{\mapsto} A\,(p-q)$ follows where $A = \bigsqcup_{i\in I} A_i$.

9.3 $\mathcal{M}$ — the mixed power domain construction

In the sequel, we want to translate the mix condition into topological terms. This is done in analogy to the sandwich power construction. The first step leads to pairs of open grills and open filters, and the second step to pairs of closed sets and compact upper sets. In the course of this translation, we also prove that condition M implies condition S, i.e. the mixed power domains are subsets of the sandwich power domains.

Let $p = (p^L, p^U)$ and $q = (q^L, q^U)$ be two predicates. For $A = (A^L, A^U)$, the mix condition may then be transformed using the facts collected in Prop. 9.1.

$$
\begin{array}{lll}
A\,p - A\,q \stackrel{L}{\mapsto} A\,(p-q) & \text{iff} & (A\,p - A\,q)^L \le (A\,(p-q))^L \\
 & \text{iff} & (A\,p - A\,q)^L = 1 \Rightarrow (A\,(p-q))^L = 1 \\
 & \text{iff} & A^L p^L \cdot \overline{A^U q^U} = 1 \Rightarrow A^L(p^L \cdot \overline{q^U}) = 1 \\
 & \text{iff} & A^L p^L = 1 \text{ and } A^U q^U = 0 \Rightarrow A^L(p^L \cdot \overline{q^U}) = 1 \\
 & \text{iff} & A^L p^L = \top \text{ and } A^U q^U = \top \Rightarrow A^L(p^L \sqcap q^U) = \top
\end{array}
$$

In the very last line, we replaced **L** and **U** by their common carrier domain **2**.

We now translate the predicates to open sets. p^L becomes O^L and q^U becomes O^U. Then $p^L \sqcap q^U$ corresponds to $O^L \cap O^U$. The lower second order predicate A^L is translated into an open grill $\mathcal{G}$, and the upper one into an open filter $\mathcal{O}$. We remember $A^L p^L = \top$ iff $O^L \in \mathcal{G}$, and $A^U p^U = \top$ iff $O^U \in \mathcal{O}$.

Thus, we obtain $\qquad (\mathcal{G}, \mathcal{O}) \in \mathcal{M}\mathbf{X}$ iff $\forall\, O^L \in \mathcal{G},\, O^U \in \mathcal{O} : O^L \cap O^U \in \mathcal{G}$

An open grill does not contain $\emptyset$. Hence, $O^L \cap O^U \in \mathcal{G}$ implies $O^L \cap O^U \neq \emptyset$ — the conclusion of the sandwich condition. Thus, $\mathcal{M}\mathbf{X} \subseteq \mathcal{S}\mathbf{X}$ holds.

For sober ground domain $\mathbf{X}$, one can translate the open filters into compact upper sets. $O \in \mathcal{G}$ then becomes $C \cap O \neq \emptyset$, and $O' \in \mathcal{O}$ becomes $K \subseteq O'$. Hence, the mix condition becomes: for all open sets O and O', if C meets O and $K \subseteq O'$ then $C \cap O \cap O' \neq \emptyset$. For fixed C and O', every open set meeting C meets $C \cap O'$ iff $C \subseteq \mathsf{cl}\,(C \cap O')$ (cf. the transformation of condition S). Hence, one obtains

Theorem 9.5 The mixed power domain $\mathcal{M}\mathbf{X}$ over a sober ground domain $\mathbf{X}$ is isomorphic to the set of all pairs (C, K) of a closed set C and a compact upper set K such that for all open sets O with $K \subseteq O$ the inclusion $C \subseteq \mathsf{cl}\,(C \cap O)$ holds.

Similar to the sandwich condition, the mix condition may be simplified in case of M-domains. The result is $C \subseteq \downarrow(C \cap K)$. A proof may be found in [5].

9.4 $\mathcal{C}$ — the convex power domain construction

As indicated above, we claim $\mathcal{C}\mathbf{X} = \mathcal{M}\mathbf{X} \cap \overline{\mathcal{M}}\mathbf{X}$. To derive a topological description of $\mathcal{C}\mathbf{X}$, we have to transform condition $\overline{M}$.

$$
\begin{array}{lll}
A\,p - A\,q \stackrel{U}{\mapsto} A\,(p-q) & \text{iff} & (A\,p - A\,q)^U = 1 \Rightarrow (A\,(p-q))^U = 1 \\
 & \text{iff} & A^U p^U \cdot \overline{A^L q^L} = 1 \Rightarrow A^U(p^U \cdot \overline{q^L}) = 1 \\
 & \text{iff} & A^U p^U = \bot \text{ and } A^L q^L = \bot \Rightarrow A^U(p^U \sqcup q^L) = \bot
\end{array}
$$

The transformation of condition $\overline{M}$ proceeds by translating the predicates to open sets:

$$
\begin{array}{lll}
(\mathcal{G}, \mathcal{O}) \in \overline{\mathcal{M}}\mathbf{X} & \text{iff} & O^L \notin \mathcal{G} \text{ and } O^U \notin \mathcal{O} \Rightarrow O^L \cup O^U \notin \mathcal{O} \\
 & \text{iff} & O^L \cup O^U \in \mathcal{O} \Rightarrow O^L \in \mathcal{G} \text{ or } O^U \in \mathcal{O}
\end{array}
$$

For sober ground domain $\mathbf{X}$, we translate the open filters into compact upper sets.

$$
(C, K) \in \overline{\mathcal{M}}\mathbf{X} \quad \text{iff} \quad K \subseteq O^L \cup O^U \Rightarrow C \cap O^L \neq \emptyset \text{ or } K \subseteq O^U
$$

Let C' be the complement of C. Then the condition above is equivalent to '$K \subseteq C' \cup O \Rightarrow K \subseteq O$'. To simplify further, note that $K \subseteq C' \cup O$ is equivalent to $C \cap K \subseteq O$. Thus, we obtain $(C, K) \in \overline{\mathcal{M}}\mathbf{X}$ iff $C \cap K \subseteq O$ implies $K \subseteq O$ for all open sets O. By Lemma 2.10, this is equivalent to $K \subseteq \uparrow(C \cap K)$.

Theorem 9.6 Our convex power domain $\mathcal{C}\mathbf{X}$ over a sober ground domain $\mathbf{X}$ is isomorphic to the set of all pairs (C, K) of a closed set C and a compact upper set K such that (C, K) is in $\mathcal{M}\mathbf{X}$ and $K \subseteq \uparrow(C \cap K)$ holds.

9.5 The case of an algebraic ground domain

So far, we have only claimed, but not proven, that our mixed construction $\mathcal{M}$ generalizes the one of [3, 4], which is defined for algebraic ground domains only. Thus, we consider now the case of an algebraic ground domain $\mathbf{X}$.

Lemma 9.7 Let $\mathcal{R}$ be any sub-construction of $\mathcal{D}$. Then every pair in $(\mathcal{D}\mathbf{X})^0 \cap \mathcal{R}\mathbf{X}$ is isolated in $\mathcal{R}\mathbf{X}$.

Proof: Because $\mathcal{R}\mathbf{X}$ is closed w.r.t. directed limits of $\mathcal{D}\mathbf{X}$. □

Lemma 9.8 Let $\mathcal{R}$ be one of $\mathcal{M}$ or $\mathcal{C}$. Let P be a member of $\mathcal{R}\mathbf{X}$, and let A be an isolated point of $\mathcal{D}\mathbf{X}$ below P. Then there is an isolated point B in $\mathcal{D}\mathbf{X}$ that lies within $\mathcal{R}\mathbf{X}$ and is between A and P.

$$A \in (\mathcal{D}\mathbf{X})^0,\ P \in \mathcal{R}\mathbf{X},\ A \leq P \Rightarrow \exists B \in (\mathcal{D}\mathbf{X})^0 \cap \mathcal{R}\mathbf{X} : A \leq B \leq P.$$

Before we are going to prove this lemma, we show that the two lemmata imply algebraicity. Let P be in $\mathcal{R}\mathbf{X}$. Then let $\mathcal{A} = \{A \in (\mathcal{D}\mathbf{X})^0 \mid A \leq P\}$ and $\mathcal{B} = \{B \in (\mathcal{D}\mathbf{X})^0 \cap \mathcal{R}\mathbf{X} \mid B \leq P\}$. Since $\mathcal{D}\mathbf{X}$ is algebraic, $\mathcal{A}$ is directed with lub P. Obviously, $\mathcal{B} \subseteq \mathcal{A}$ holds, and Lemma 9.8 implies $\mathcal{A} \subseteq \downarrow\mathcal{B}$. From these facts one can show that $\mathcal{B}$ is directed because $\mathcal{A}$ is, and both sets have the same lub. Lemma 9.7 states that $\mathcal{B}$ is a set of isolated points in $\mathcal{R}\mathbf{X}$.

Proof of the Lemma:
We have to show the claim for each $\mathcal{R}$ separately. Generally, $A = (\downarrow E, \uparrow F)$ holds where E and F are finite subsets of $\mathbf{X}^0$, and $P = (C, K)$ where C is closed, K is a compact upper set, and $E \subseteq C$ and $K \subseteq \uparrow F$ hold because of $A \leq P$. Two finite subsets E' and F' of $\mathbf{X}^0$ are to be found that satisfy the conditions of $\mathcal{R}$ and lie between A and P, i.e. $E \subseteq \downarrow E'$, $E' \subseteq C$, and $K \subseteq \uparrow F' \subseteq \uparrow F$ have to hold.

$\mathcal{M}$: $\uparrow F$ is open by Prop. 2.4, whence we obtain $E \subseteq C \subseteq \mathsf{cl}\,(C \cap \uparrow F)$ by the mix property of (C, K). Thus, $E \subseteq \downarrow(C \cap \uparrow F)$ follows by Prop. 2.8. Hence, for all e in E, there is g_e in C and f_e in F such that $e \leq g_e \geq f_e$. By Prop. 2.5, g_e may be assumed to be isolated.
Let $E' = \{g_e \mid e \in E\} \subseteq_f \mathbf{X}^0$. $e \leq g_e \in E'$ for all e in E implies $E \subseteq \downarrow E' \subseteq C$. $g_e \geq f_e$ for all e in E implies $E' \subseteq \uparrow F$, whence $(\downarrow E', \uparrow F)$ is a mix because $\uparrow F \subseteq O$ implies $E' = E' \cap \uparrow F \subseteq \mathsf{cl}\,(\downarrow E' \cap O)$.

$\mathcal{C}$: $K \subseteq \uparrow F$ and condition $\overline{M}$ imply $K \subseteq \uparrow(C \cap K) \subseteq \uparrow(C \cap \uparrow F) = \uparrow(C \cap F)$. The last equality holds since C is lower. Let $F' = C \cap F$. Then $K \subseteq \uparrow F' \subseteq \uparrow F$ holds as required.

By defining E' as in '$\mathcal{M}$', $E \subseteq \downarrow E' \subseteq C$ holds. $E' \subseteq \uparrow F'$ holds since $g_e \geq f_e$, i.e. $f_e \in F'$.

Now let $G = E' \cup F'$. We claim that $(\downarrow G, \uparrow G)$ is the desired pair. $E \subseteq \downarrow E' \subseteq \downarrow G$ holds, and $G \subseteq C$ since $F' \subseteq C$. $E' \subseteq \uparrow F'$ implies $\uparrow G = \uparrow F'$, whence $K \subseteq \uparrow G \subseteq \uparrow F$.

$(\downarrow G, \uparrow G)$ is in $\mathcal{C}\mathbf{X}$ since $G \subseteq \downarrow G \cap \uparrow G$ whence conditions M and $\overline{M}$ follow. □

The proof above not only shows the algebraicity of $\mathcal{M}\mathbf{X}$ and $\mathcal{C}\mathbf{X}$ in case of algebraic $\mathbf{X}$, but also provides nice representations for the bases of these power domains. For $\mathcal{M}$, Lemma 9.8 characterizes the basic mixes by $E \subseteq \uparrow F$. This is $E \geq^\sharp F$ in Gunter's notation, whence we see that our mixed power construction generalizes Gunter's [3, 4].

The base of $\mathcal{C}\mathbf{X}$ is the set of all pairs $(\downarrow F, \uparrow F)$ where F is a finite subset of $\mathbf{X}^0$. The intersection of $\downarrow F$ and $\uparrow F$ is the convex hull $\updownarrow F$ of F. It suffices to recover $\downarrow F$ and $\uparrow F$ since $\downarrow F = \downarrow\updownarrow F$ and $\uparrow F = \uparrow\updownarrow F$. The ordering of these convex sets is given by $\updownarrow F \leq \updownarrow F'$ iff $\updownarrow F \subseteq \downarrow\updownarrow F'$ and $\updownarrow F' \subseteq \uparrow\updownarrow F$. This is the Egli-Milner ordering. Hence, $\mathcal{C}\mathbf{X}$ equals Plotkin's power domain for algebraic ground domains.

9.6 Other C-constructions

The power construction $\mathcal{C}$ that we derived as a sub-construction of $\mathcal{M}$ and ultimately of $\mathcal{D} = \mathcal{L} \times \mathcal{U}$ has characteristic semiring $\mathbf{C} = \{0, 1\}$. It *does not* coincide with the existential restriction of $\mathcal{D}$ to $\mathbf{C}$; this is a much larger sub-construction of $\mathcal{D}$ than $\mathcal{C}$.

The final $\mathbf{C}$-construction is not among the sub-constructions of $\mathcal{D}$. In both $\mathcal{L}$ and $\mathcal{U}$, $\{\!|x|\!\} \leq \{\!|y|\!\}$ implies $x \leq y$. This property carries over to their product $\mathcal{D}$, and is inherited by all sub-constructions of $\mathcal{D}$. On the other hand, a domain with least element only admits two predicates $[\mathbf{X} \to \mathbf{C}]$ because the two elements of $\mathbf{C}$ are unrelated. The two predicates are $\lambda x.\, 0$ and $\lambda x.\, 1$. Every additive second order predicate must map $\lambda x.\, 0$ to 0; it only has the choice to map $\lambda x.\, 1$. Thus, $[[\mathbf{X} \to \mathbf{C}] \stackrel{add}{\to} \mathbf{C}]$ has at most two elements, and $\{\!|x|\!\} = \{\!|x'|\!\}$ usually holds in it even for different points x and x'.

10 A note on initiality

In [7] and [5], the existence of an initial R-construction is shown for every semiring R. The initial R-construction maps every ground domain $\mathbf{X}$ to the free R-module domain over $\mathbf{X}$. Initial power constructions were proposed and investigated in [8, 10, 14].

Initial and final $\mathbf{L}$-constructions coincide (for all ground domains). The coincidence of our constructions $\mathcal{U}$, $\mathcal{M}$, and $\mathcal{C}$ defined predicatively with the initial constructions for $\mathbf{U}$, $\mathbf{B}$, and $\mathbf{C}$ respectively could however be shown for the case of continuous ground domains only. In all three cases, the coincidence does not hold for arbitrary domains. Thus, the predicative and the initial power constructions have to be carefully distinguished if non-continuous domains are considered.

11 Conclusion

The method to define power domains by second order predicates provides explicit representations for power domains over *all* ground domains. Using these representations in terms of second order predicates, it is possible to implement power domain constructions as polymorphic abstract data types in a functional language if only the semiring operations are provided. To realize power constructions with semiring $\mathbf{B}$ for instance, parallel disjunction is needed.

All five power domain constructions mentioned in the introduction may be characterized in terms of second order predicates:

$$
\begin{array}{lrcl}
\text{Lower construction:} & \mathcal{L}\mathbf{X} & = & [[\mathbf{X} \to \mathbf{L}] \stackrel{add}{\to} \mathbf{L}] \\
\text{Upper construction:} & \mathcal{U}\mathbf{X} & = & [[\mathbf{X} \to \mathbf{U}] \stackrel{add}{\to} \mathbf{U}] \\
\text{Sandwich construction:} & \mathcal{S}\mathbf{X} & = & \{P : [[\mathbf{X} \to \mathbf{D}] \stackrel{add}{\to} \mathbf{D}] \mid p : [\mathbf{X} \to \mathbf{B}] \Rightarrow Pp \in \mathbf{B}\} \\
 & & = & [[\mathbf{X} \to \mathbf{B}] \stackrel{add}{\to} \mathbf{B}] \\
\text{Mixed construction:} & \mathcal{M}\mathbf{X} & = & \{P : [[\mathbf{X} \to \mathbf{D}] \stackrel{add}{\to} \mathbf{D}] \mid Pp \wedge \neg Pq \geq \mathsf{T} \Rightarrow P(p \wedge \neg q) \geq \mathsf{T}\} \\
 & & = & \{P : [[\mathbf{X} \to \mathbf{B}] \stackrel{add}{\to} \mathbf{B}] \mid Pp \wedge \neg Pq = \mathsf{T} \Rightarrow P(p \wedge \neg q) = \mathsf{T}\} \\
\text{Convex construction:} & \mathcal{C}\mathbf{X} & = & \{P : [[\mathbf{X} \to \mathbf{B}] \stackrel{add}{\to} \mathbf{B}] \mid \\
 & & & \quad (Pp \wedge \neg Pq = \mathsf{T} \Rightarrow P(p \wedge \neg q) = \mathsf{T}) \text{ and} \\
 & & & \quad (Pp \wedge \neg Pq \neq \mathsf{F} \Rightarrow P(p \wedge \neg q) \neq \mathsf{F})\}
\end{array}
$$

For all these constructions $\mathcal{P}$, the power domains $\mathcal{P}\mathbf{X}$ are isomorphic to function spaces $[[\mathbf{X} \to R] \stackrel{res}{\to} R]$ where R is a semiring and '*res*' a logical restriction on the second order predicates. The respective operations of empty set, singleton, binary union, and functional extension may be uniformly described by λ-expressions (see Th. 3.5).

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