Quantitative Semantics, Topology, and Possibility Measures

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Abstract

Given a topological space X and a complete lattice L, we study the space of L-predicates $\mathsf{F}_L(X) = [X \to L^{\mathrm{op}}]^{\mathrm{op}}$, continuous maps from X to L^{op} in its Scott-topology. It yields a functor $\mathsf{F}_L(\cdot)$ from TOP-L, a full subcategory of TOP subsuming continuous domains, to SUP, the category of complete sup-lattices and maps preserving suprema. Elements of $\mathsf{F}_2(X)$ are continuous predicates (= closed sets), and elements of $\mathsf{F}_{[0,1]}(X)$ may be viewed as probabilistic predicates. Alternatively, one may consider the complete sup-lattice $\mathsf{P}_L(X) = \mathcal{O}(X) \multimap L$ of maps $\mu: \mathcal{O}(X) \to L$ preserving suprema (= possibility measures), which results in another functor $\mathsf{P}_L(\cdot)$ from TOP to SUP. We show that these functors are equivalent for two restrictions. First, we leave SUP unchanged and restrict TOP-L to CONT, the category of continuous domains in their Scott-topology; second, we fix TOP but restrict L to co-continuous lattices. Conversely, if $\mathsf{F}_L(X)$ and $\mathsf{P}_L(X)$ are isomorphic for all topological spaces X then L is indeed co-continuous. Further, if X is a sober space and $\mathsf{F}_L(X)$ and $\mathsf{P}_L(X)$ are isomorphic for all complete lattices L then X is a continuous domain and its topology is the Scott-topology. Possibility measures have extensions to the upper powerdomain κ_X, or to the full power set $\mathcal{P}(X)$, which are defined similarly to outer measures. Utilizing the notion of sup-semirings, we employ such extensions and the isomorphism $\mathsf{F}_L(X) \cong \mathsf{P}_L(X)$ to show that the sup-primes of $\mathsf{P}_L(X)$ are exactly scalar multiples of point valuations with sup-primes as scalars for an underlying sober space X. Combining this with classical results in the theory of continuous lattices, we restate the notion of cones in our setting and show that the space of possibility measures $\mathsf{P}_L(X)$ of a continuous domain X is the free L-module over X for the sup-semiring $L = [0, \infty]$.

Key words: Quantitative Predicates, Possibility Measures, Continuous Lattices
AMS classification codes: 06B35, 06B30, 68Q55

1 Introduction

The formal semantics of programming languages, be it in an axiomatic, operational, or denotational setting, was at its outset mainly concerned with studying languages endowed with discrete data types like booleans, integers, strings, and list structures [31]. In recent years, however, Domain Theory [2] seems to have reached a level of maturity, that allows more challenging data types. One prominent such development has been the extensive work on the semantics of (exact) real number computation [8] and integration [6].

The theory of integration, of course, is based upon the notion of measures (see for example [11]). Traditionally, a measure is defined on a σ-algebra of sets. The set-theoretic operations of such algebras, set complement, for example, cannot be easily reconciled with topological notions. The usual approach is to take the set of opens $\mathcal{O}(X)$ of a topological space and generate its σ-algebra (the Borel algebra) as the domain of definition for measures on X.

In [21], Claire Jones studied valuations $\mu: \mathcal{O}(X) \to [0, \infty]$, which satisfy the same modular law as do ordinary measures and map the empty set to 0. However, the requirement of countable additivity is replaced with that of Scott-continuity: for every directed family of opens $\mathcal{O}$ we have

$$\bigvee_{O \in \mathcal{O}} \mu(O) = \mu\left(\bigcup \mathcal{O}\right).$$

If D is a continuous domain, then the set of all continuous valuations on D is a continuous domain $\mathcal{V}_D$, where measures are ordered pointwise [21]. This construction naturally extends to a monadic functor, rendering a probabilistic version of powerdomains in the usual domain-theoretic framework [1].

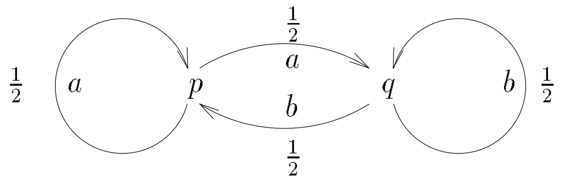
The theories of continuous valuations and measures on Borel algebras are not unrelated. For example, if X is a metric space, or locally compact Hausdorff, then every continuous valuation $\mu: \mathcal{O}(X) \to [0, \infty]$ can be extended uniquely to an *outer regular* Borel measure [23]. Unfortunately, it is not yet known, whether there are any *cartesian closed* categories of continuous domains, which are closed under the construction $\mathcal{V}_D$. Thus, one cannot yet tell a probabilistic story of higher-order semantic behavior. What is known is that $D \mapsto \mathcal{V}_D$ maps Lawson-compact spaces [2] to Lawson-compact ones [22]. Alas, the latter do not form a cartesian closed category of continuous domains, but Lawson-compactness is pretty close to being an FS-domain [2]. Hence, the cartesian closed category of FS-domains is a respectable candidate for being closed under $\mathcal{V}_D$. This is the subject of ongoing work.

There are other important areas of theoretical computer science, where semantics reflects quantitative information. Probabilistic frameworks for programming languages extended with parallel composition [30], stochastic process algebras in systems design and performance evaluation [10,15], or probabilistic versions of temporal logics (see for example [12]) have such a focus on quantitative behavior.

One natural underlying model for process algebra terms or temporal (branching time) logics are labeled transition systems (Q, Δ) over some set of action types Act, where Q is a set of states and $\Delta \subseteq Q \times \mathsf{Act} \times Q$ the set of labeled transitions modeling state changes (see e.g. [26,7]); for convenience we write $q \to^a q'$ instead of $(q, a, q') \in \Delta$. The probabilistic aspect can then be modeled along a variety of dimensions. We focus on *generative systems* for the purpose of illustration. For such models we additionally have a probability distribution assigned to every set

$$T_q = \{(q, a, q') \mid a \in \mathsf{Act},\ q \to^a q'\} \qquad (q \in Q).$$

All this information may be represented in the graph of the underlying labeled transition system. The figure below contains an example of such a probabilistic transition system, which models a fair coin state swap ($Q = \{p, q\}$ and $\mathsf{Act} = \{a, b\}$):



We may view such models, and related ones, as discrete random processes, where, given some initial state q_0, one probabilistically draws a transition $q_0 \to^{a_0} q_1$ from T_{q_0} according to the distribution to end up in state q_1, where one draws a transition $q_1 \to^{a_1} q_2$ from T_{q_1} and so on. For such a process there are at least two principal questions one might want to ask.

#1: Assuming a certain probability distribution on states at $t = 0$ and given some discrete point of time t_0, which could be ∞, what is the *probability* to be in state q at time t_0?

#2: Given a certain property ϕ of some specification language, suitable for the system at hand, and given a state q, what are the *possibilities* for ϕ to hold in state q at time t_0?

Fixing t_0 for the moment we expect that the answer to question #1 is going to be a probability distribution π on the set of states, i.e. a function $\pi\colon Q \to [0,1]$

such that

$$\sum_{q \in Q} \pi(q) = 1.$$

On the other hand, what should the type of meaning be for ϕ? Evidently, it has to assign to each state q some measure of possibility for ϕ. Assuming that this measure is finite, we may thus think of the meaning of ϕ as a function $[\![\phi]\!]: Q \to [0,1]$; but this is really all we may say about $[\![\phi]\!]$ in general. For example, if ϕ expresses the property that one may perform a *fair*, infinite sequence of a and b actions beginning in a particular state, then the meaning of ϕ should be 1 for states p and q in the figure above. Thus, such meanings, although established by probabilistic reasoning, cannot be expected to be probability distributions.

A similar scenario is that of an imperative programming language with a probabilistic choice [30], where the usual Floyd-Hoare logic [16], or Dijkstra's predicate transformers [5] turn into probabilistic predicate transformers. Probabilistic predicates [28] are functions of type $S \to [0,1]$, where S is the set of possible states of programs. The meaning of a program P is then a function

$$[\![P]\!]: (S \to [0,1]) \to (S \to [0,1])$$

reflecting changes of possibilities of being in a particular state [28,30,27].

The two points of view, *state-based* (question #1) versus *action-based* (question #2) semantics, each are of interest in their own right. But often one wants to combine such semantic insights. For example, a term of a stochastic process algebra might give rise to a steady-state probability distribution π and a function f which associates to each state a quantity reflecting the use of some resource. By means of *integrating* f with respect to π one obtains the average utilization of the resource being considered.

The main objective of this paper is to develop the notion of quantitative predicates in a *topological* setting that encompasses that of continuous domains, and to provide a dual view of such quantitative predicates as *possibility measures on the lattice of opens.*

Such an abstraction may be compared with the generalization of concrete domains $S \to \mathsf{values}_\perp$ to general function spaces of domains $[D \to E]$ [29,31]. One of the fundamental insights of our work is that, given a topological space X, the order dual of the space of meanings for possibilities $[X \to [0,1]]$, denoted as $\mathsf{F}_{[0,1]}(X)$, is isomorphic to a powerdomain of possibility measures $\mathcal{O}(X) \multimap [0,1]$, denoted as $\mathsf{P}_{[0,1]}(X)$. Elements of $\mathsf{P}_{[0,1]}(X)$ correspond to continuous valuations in the sense that '+' is being replaced by '$\vee$' in their modular law. This also shows a clear connection between possibility measures and continuous, fuzzy sets [33].

In fact, we prove this result not just for $[0,1]$ but for all co-continuous complete lattices L. It turns out that this isomorphism extends to a correspondence of functors as well. However, the actual duality at hand is

$$\mathsf{F}_L(X) = [X \to L^{\mathrm{op}}]^{\mathrm{op}} \cong \mathcal{O}(X){-\!\circ} L = \mathsf{P}_L(X)$$

amounting to $[X \to [0,1]]^{\mathrm{op}} \cong \mathcal{O}(X){-\!\circ}[0,1]$ in that special case, for $[0,1]$ and $[0,1]^{\mathrm{op}}$ are isomorphic.

If we make no assumptions about the complete lattice L at all, then we can still show this isomorphism for all continuous domains X in their Scott-topology. We further show that this isomorphism also 'classifies' co-continuous lattices and continuous domains as sober spaces.

Convention. For the rest of this paper we assume that continuous domains are always considered as topological spaces in their Scott-topology, unless stated otherwise.

It is instructive to examine the case $L = \mathbf{2}$ of truth values (classical sets). In our topological setting of $[X \to \mathbf{2}^{\mathrm{op}}]^{\mathrm{op}}$ and $\mathcal{O}(X){-\!\circ}\mathbf{2}$ these spaces result in two different ways of spelling out the lifted, lower powerdomain of X: the lattice of all closed sets of X ordered by inclusion. The lattice $[X \to \mathbf{2}^{\mathrm{op}}]^{\mathrm{op}}$ views such sets as *characteristic functions*, whereas $\mathcal{O}(X){-\!\circ}\mathbf{2}$ thinks of such sets as principal ideals of opens under reverse inclusion ($\mu \mapsto \mu^{-1}(1)$). Note that the duality presented in this paper is therefore based on the equivalence of two representations as complete lattices. This is in contrast to, say, the usual Stone duality, where objects can be seen as spatial frames (certain complete lattices) or sober spaces (certain topological spaces).

We use the isomorphism between $\mathsf{F}_L(X)$ and $\mathsf{P}_L(X)$ to characterize the sup-primes in $\mathcal{O}(X){-\!\circ}L$ as sup-prime scalar multiples of point valuations, provided that X is a sober space and L a sup-semiring. The latter characterization allows us to prove that $\mathsf{P}_{[0,\infty]}(X)$, and therefore $\mathsf{F}_{[0,\infty]}(X)$, is the free left $[0,\infty]$-module of X, if X is a continuous domain. Left $[0,\infty]$-modules correspond to the cones in [21] in that one replaces sums in the cone and the reals by suprema. Given a map $h: L \to L'$ between two different targets of quantitative semantics, it induces functorial actions between $\mathsf{F}_L(\cdot)$ and $\mathsf{F}_{L'}(\cdot)$, respectively $\mathsf{P}_L(\cdot)$ and $\mathsf{P}_{L'}(\cdot)$. We state conditions on h under which these actions preserve scalar multiplication. The latter is indeed the case for the embedding $h: [0,1] \to$

$[0, \infty]$, although, the function spaces $\mathsf{F}_{[0,\infty]}(X)$ and $\mathsf{P}_{[0,\infty]}(X)$ behave differently for multiplication with the scalar ∞.

Finally, given a completely distributive lattice L we conclude with an analysis of how $\mathsf{P}_L(X)$ sits inside the larger space $[\mathcal{O}(X) \to L]$ using recent results on the theory of linear FS-lattices [20].

2 Distributions versus Measures

Given a topological space X and a complete lattice L, we think of L as a topological space in its Scott-topology. The function space $[X \to L]$ denotes all continuous maps $f: X \to L$ ordered in the pointwise order: $f \leq g$ if $f(x) \leq g(x)$ in L for all $x \in X$. Note that this corresponds to the usual function space $[D \to L]$ if D is a dcpo in its Scott-topology. The prototypical complete lattices we are thinking of are $[0, 1]$ and $[0, \infty]$. The elements of $[X \to [0, 1]]$ are *continuous* possibility distributions over X.

If we think of X as a finite set in the discrete topology then functions $f \in [X \to [0, 1]]$ have no restrictions other than being maps from X to $[0, 1]$. Such an f induces a map μ_f sending any subset Y of X to the supremum of all values $f(y)$, $y \in Y$.

Conversely, given a map $\mu: \mathcal{O}(X) \to [0, 1]$, μ is completely determined by its values $\mu(\{x\})$, $x \in X$, if μ preserves suprema. The values $\mu(\{x\})$, $x \in X$, thus render an element $f_\mu \in [X \to [0, 1]]$ (see e.g. [4]). We generalize this observation about sets to a general dual isomorphism between possibility distributions and possibility measures for arbitrary topological spaces X.

In what sense can we think of sup-maps $\mu: \mathcal{O}(X) \to [0, 1]$, or $\mu: \mathcal{O}(X) \to [0, \infty]$, as measures? Certainly, like continuous valuations, such maps are Scott-continuous and send the empty set to 0. The remaining property of a continuous valuation is its modular law

$$\mu(U \cup V) + \mu(U \cap V) = \mu(U) + \mu(V) \tag{1}$$

for all opens U and V. If we interpret addition in this equation in a fuzzy way as supremum, then, noting that μ is monotone, this modified modular law

$$\mu(U \cup V) \vee \mu(U \cap V) = \mu(U) \vee \mu(V) \tag{2}$$

reduces to a new modular law

$$\mu(U \cup V) = \mu(U) \vee \mu(V).$$

Since μ preserves suprema, we infer that it actually satisfies the modular law in (2). In summary, sup-maps $\mu: \mathcal{O}(X) \to [0,1]$ are characterized as strict Scott-continuous functions which satisfy the usual modular law, where addition '+' is interpreted as supremum '$\vee$'.

Definition 1 *Given two complete lattices M and N we let $M \multimap N$ denote the space of all maps $f: M \to N$ preserving suprema, ordered pointwise.*

Maps $\mu \in \mathcal{O}(X) \multimap [0,1]$ are then topological versions of the possibility measures in [4].

3 The Isomorphism for Co-Continuous L

As indicated earlier, we chose to prove $\mathsf{F}_L(X) \cong \mathsf{P}_L(X)$ in a very general setting. Namely, we assume that the range of quantitative meaning may be any complete lattice L such that L^{op} is continuous. In the next section we weaken this assumption to L being just a complete lattice but strengthen the assumptions about X such that it is a continuous domain in its Scott-topology.

Convention. For the remainder of this paper we talk about order, suprema, infima, lower and upper sets *with respect to L* whenever we consider the range of possibility functions or measures, unless stated otherwise.

So a Scott-open set in L^{op}, for example, will be referred to as a lower set inaccessible by filtered infima. While this puts some strain on reading this material, it actually reduces most of the head spin one would encounter if one switched between L and L^{op} and their relative terminology at all times.

Definition 2 *Let X be a topological space with lattice of opens $\mathcal{O}(X)$. For $x \in X$ we let $\mathcal{F}_x$ be the completely prime filter of all opens $O \in \mathcal{O}(X)$ containing x. For a complete lattice L we let $\mathsf{F}_L(X)$ be the partial order $[X \to L^{\mathrm{op}}]^{\mathrm{op}}$ of possibility distributions over L, ordered pointwise. Further, $\mathsf{P}_L(X)$ denotes the complete lattice $\mathcal{O}(X) \multimap L$ of possibility measures of X over L.*

Theorem 3 *Let X be a topological space and L a complete lattice such that L^{op} is continuous. Then the maps*

$$\begin{aligned} \Psi: \mathsf{F}_L(X) \to \mathsf{P}_L(X), \;\; \Psi\, f\, O &= \bigvee_{x \in O} f(x) \\ \Phi: \mathsf{P}_L(X) \to \mathsf{F}_L(X), \;\; \Phi\, \mu\, x &= \bigwedge_{O \in \mathcal{F}_x} \mu(O) \end{aligned} \tag{3}$$

are well-defined, monotone, and order-inverses of each other. In particular, the partial order $\mathsf{F}_L(X)$ is a complete lattice.

PROOF. Given $f \in [X \to L^{\mathrm{op}}]$, the function Ψf preserves all suprema by the general associativity of suprema:

$$\begin{aligned}\Psi f \left(\bigcup \mathcal{O}\right) &= \bigvee_{x \in \bigcup \mathcal{O}} f(x) \\ &= \bigvee_{O \in \mathcal{O}} \bigvee_{x \in O} f(x) \\ &= \bigvee_{O \in \mathcal{O}} \Psi f\, O.\end{aligned}$$

Therefore Ψ is well-defined as a set-theoretic map. To see that Ψ is actually monotone, note that the order in $\mathsf{F}_L(X)$ is the pointwise one, for we reverse the order twice. Thus, $f \le g$ in $\mathsf{F}_L(X)$ means $f(x) \le g(x)$ in L for all $x \in X$, which readily entails $\bigvee_{x \in O} f(x) \le \bigvee_{x \in O} g(x)$ for all $O \in \mathcal{O}(X)$, whence $\Psi f \le \Psi g$ in $\mathsf{P}_L(X)$.

Next, we show that Φ is well-defined and monotone. Given $\mu \in \mathsf{P}_L(X)$ we need to show that f, defined as $\Phi\,\mu$, is a continuous map between X and L^{op} in its Scott-topology. To this end let V be a lower set of L inaccessible by filtered infima. We are done if $f^{-1}(V)$ is open in X. But $x \in f^{-1}(V)$ means that $\bigwedge_{O \in \mathcal{F}_x} \mu(O) \in V$. Since $\mathcal{F}_x$ is a filter we see that the infimum is filtered and since V is inaccessible by filtered infima we conclude that there must be some $O \in \mathcal{F}_x$ such that $\mu(O) \in V$. Since $x \in O$ it suffices to show $O \subseteq f^{-1}(V)$. Given any $x' \in O$ we have $O \in \mathcal{F}_{x'}$ so $f(x') = \bigwedge_{O' \in \mathcal{F}_{x'}} \mu(O') \le \mu(O) \in V$ follows. But then $f(x') \in V$ since V is a lower set.

If $\mu \le \nu$ in $\mathsf{P}_L(X)$ then $\mu(O) \le \nu(O)$ for all $O \in \mathcal{O}(X)$ implies that the infimum $\bigwedge_{O \in \mathcal{F}_x} \mu(O)$ is less than or equal to $\bigwedge_{O \in \mathcal{F}_x} \nu(O)$, so $\Phi\,\mu \le \Phi\,\nu$ holds in $\mathsf{F}_L(X)$.

We still need to show that these two maps compose to the respective identities. First, let ν be the possibility measure $\Psi(\Phi\,\mu)$. For any $O \in \mathcal{O}(X)$ we have

$$\begin{aligned}\nu\, O &= \bigvee_{x \in O} \Psi\,\mu\, x \\ &= \bigvee_{x \in O} \bigwedge_{U \in \mathcal{F}_x} \mu\, U\end{aligned}$$

and for all $x \in O$ we obtain $\bigwedge_{U \in \mathcal{F}_x} \mu\, U \le \mu\, O$, which shows $\nu\, O \le \mu\, O$.

To establish the reverse inequality, we use the fact that the infimum $\bigwedge_{U \in \mathcal{F}_x} \mu\, U$

is filtered *and* that L^{op} is continuous. We write $l' \prec l$ for l and l' in L if l is way-below l' in the lattice L^{op}; recall that x is way-below y in a dcpo D (usually written as $x \ll y$) means that for all directed sets A with $y \leq \bigvee A$ we have $x \leq a$ for some $a \in A$. Since L^{op} is continuous it suffices to prove that $\nu\, O \prec l$ implies $\mu\, O \leq l$. Given $\nu\, O \prec l$ we have in particular that $\bigwedge_{U \in \mathcal{F}_x} \mu\, U \prec l$ for all $x \in O$. Since the latter infimum is filtered we know that there is some $U_x \in \mathcal{F}_x$ such that $\mu\, U_x \leq l$. But $x \in U_x$ entails $O \subseteq \bigcup_{x \in O} U_x$. Using that μ is monotone and preserves suprema we conclude $\mu\, O \leq \mu\,(\bigcup_{x \in O} U_x) \leq \bigvee_{x \in O} \mu(U_x) \leq l$; hence $\mu = \nu$.

Second, for $f \in \mathsf{F}_L(X)$ we define g to be $\Phi(\Psi\, f)$. Then for any $x \in X$ we have

$$
\begin{aligned}
g\, x &= \bigwedge_{O \in \mathcal{F}_x} \Phi\, f\, O \\
&= \bigwedge_{O \in \mathcal{F}_x} \bigvee_{y \in O} f\, y.
\end{aligned}
$$

This renders $g\, x \geq f\, x$ since for all $O \in \mathcal{F}_x$ we have $\bigvee_{y \in O} f\, y \geq f\, x$.

Again, we use the continuity of L^{op} to show the reverse inequality. So let $f\, x \prec l$ be given. We are done if $g\, x \leq l$. The set

$$V = \{a \in L \mid a \prec l\}$$

is Scott-open in L^{op} as L^{op} is continuous, so $f^{-1}(V)$ is open in X since $f\colon X \to L^{\mathrm{op}}$ is continuous. For all $y \in f^{-1}(V)$ we have $f\, y \in V$ and in particular $f\, y \leq l$. Thus, $\bigvee_{y \in f^{-1}(V)} f\, y \leq l$. But since $f\, x \prec l$ we have $x \in f^{-1}(V)$, i.e. $f^{-1}(V) \in \mathcal{F}_x$ and therefore $g\, x \leq \bigvee_{y \in f^{-1}(V)} f\, y$, where the latter is below l. $\square$

4 The Isomorphism for Continuous X

In this section we assume that $\mathcal{O}(X)$ is the Scott-topology of a continuous domain X and show the isomorphism above for *all* complete lattices L. We only need to adapt the former proof in two places, where we used the fact that L^{op} is continuous.

Theorem 4 *Let X be a continuous domain and $\mathcal{O}(X)$ its Scott-topology. If L is any complete lattice then the maps in (3) are well-defined, monotone and order-inverses of each other.*

PROOF. We focus only on the two reverse inequalities of the previous proof. First, to show $\mu \leq \nu$ it suffices to show $\mu(\Uparrow x) \leq \nu(\Uparrow x)$ for all $x \in X$ since both maps preserve suprema and the sets of the form $\Uparrow x = \{y \in X \mid x \ll y\}$ form a basis of the Scott-topology $\mathcal{O}(X)$. Interpolating in X we have $\Uparrow x = \bigcup_{y \in \Uparrow x} \Uparrow y$, so it suffices to show $\mu(\Uparrow y) \leq \nu(\Uparrow x)$ for all $y \in \Uparrow x$ since μ preserves suprema. But for any $z \ll y$ in X we have $\Uparrow y \subseteq \Uparrow z$ and so $\mu(\Uparrow y) \leq \bigwedge_{z \in \Downarrow y} \mu(\Uparrow z)$ for all $y \in \Uparrow x$ which gives us $\mu(\Uparrow y) \leq \nu(\Uparrow x)$ as desired (since X is continuous it suffices to consider only opens of the form $\Uparrow z$ in the infimum defining ν).

Second, since X is continuous we see that $g\, x$ equals $\bigwedge_{\Uparrow y \in \mathcal{F}_x} \bigvee_{z \in \Uparrow y} f\, z$. On the other hand, the (Scott-)continuity of $f\colon X \to L^{\mathrm{op}}$ together with the continuity of X entail $f(x) = f\left(\bigvee_{y \ll x} y\right) = \bigwedge_{y \ll x} f(y)$. Thus, it suffices to show

$$\bigvee_{z \in \Uparrow y} f\, z \leq f\, y$$

for all $y \ll x$, which is true since $f\colon X \to L$ is *antitone* and $y \ll z$ in X implies $y \leq z$ in X. $\Box$

5 Characterizing Continuity

It is not accidental that the isomorphism between possibility distributions and possibility measures reflects a certain tradeoff: assuming that the complete lattice L^{op} is continuous, we had no restrictions on the topological space X; conversely, assuming that $\mathcal{O}(X)$ is the Scott-topology of a continuous domain X, we had no restrictions on the complete lattice L in securing the isomorphism above.

In this section we demonstrate that the condition of L^{op} being continuous is as general as it could be. However, we can slightly weaken the assumption that $\mathcal{O}(X)$ be the Scott-topology of a continuous domain X.

Theorem 5 *Let L be a complete lattice. Then the following are equivalent:*

(1) The lattice L^{op} is continuous.
(2) The lattices $\mathsf{F}_L(X)$ and $\mathsf{P}_L(X)$ are isomorphic via Ψ and Φ for all topological spaces X.
(3) The map $\Phi \circ \Psi$ equals $\mathrm{id}_{\mathsf{F}_L(X)}$ for all topological spaces X.
(4) The map $\Phi \circ \Psi$ equals $\mathrm{id}_{\mathsf{F}_L(X)}$ for all complete lattices X, where $\mathcal{O}(X)$ is the Scott-topology.
(5) The map $\Phi \circ \Psi$ equals $\mathrm{id}_{[L^{\mathrm{op}} \to L^{\mathrm{op}}]^{\mathrm{op}}}$, where $\mathcal{O}(L^{\mathrm{op}})$ is the Scott-topology.
(6) We have $\Phi(\Psi\, \mathrm{id}_{L^{\mathrm{op}}}) = \mathrm{id}_{L^{\mathrm{op}}}$, where $\mathcal{O}(L^{\mathrm{op}})$ is the Scott-topology.

PROOF. We already showed that (1) implies (2). The implication chain

$$(2)\Rightarrow(3)\Rightarrow(4)\Rightarrow(5)\Rightarrow(6)$$

consists entirely of restrictions. To show (6)⇒(1) note that $\Phi(\Psi\,\mathrm{id}_{L^{\mathrm{op}}}) = \mathrm{id}_{L^{\mathrm{op}}}$ says

$$x = \bigwedge_{O\in\mathcal{F}_x} \bigvee_{y\in O} y$$

for all $x \in L^{\mathrm{op}}$. Given any $O \in \mathcal{F}_x$ and a filtered set $(y_i)_{i\in I}$ in L with $\bigwedge_{i\in I} y_i \leq x$ we have $\bigwedge_{i\in I} y_i \in O$ as O is a lower set in L. Since O is Scott-open in L^{op} we obtain some $y_{i_0} \in O$ which implies $y_{i_0} \leq \bigvee_{y\in O} y$. Therefore, all elements $\bigvee_{y\in O} y$ with $O \in \mathcal{F}_x$ are way-below x in L^{op}, and their supremum in L^{op} ($\bigwedge$ in L) is x. Since L^{op} is a complete lattice the set of elements way-below x in L^{op} is directed. $\square$

Thus, we cannot expect to extend this isomorphism to a larger class of complete lattices, unless we make restrictions on the class of topological spaces X. The resulting class of spaces turns out to be slightly larger than the one of continuous domains.

Definition 6 *Given a topological space X, we call it a* continuous space *if for all $O \in \mathcal{O}(X)$ we have*

$$O = \bigcup_{y\in O} (\uparrow y)^{o}, \tag{4}$$

where $\uparrow y$ is the set of all x with $x \geq y$ in the specialization order and A^{o} the interior of a set A.

Note that every continuous domain is a continuous space. This is also the case for any preorder in its Alexandroff topology. In fact, for sober spaces the notions of continuous space and continuous domain coincide:

Proposition 7 *For a topological space X the following are equivalent:*

(1) The specialization order on X makes X into a dcpo such that all opens of X are Scott-open, and X is a continuous space.
(2) X is a continuous sober space.
(3) X is a continuous domain and its topology is the Scott-topology.

Continuous spaces turn out to be exactly those topological spaces X which make $\mathsf{P}_L(X)$ and $\mathsf{F}_L(X)$ isomorphic for all complete lattices L.

Theorem 8 *Let X be a topological space. Then the following are equivalent:*

(1) The topological space X is continuous.
(2) The lattices $\mathsf{F}_L(X)$ and $\mathsf{P}_L(X)$ are isomorphic via Ψ and Φ for all complete lattices L.
(3) The map $\Psi \circ \Phi$ equals $\mathrm{id}_{\mathsf{P}_L(X)}$ for all complete lattices L.
(4) The map $\Psi \circ \Phi$ equals $\mathrm{id}_{\mathsf{P}_L(X)}$ for all frames L.
(5) The map $\Psi \circ \Phi$ equals $\mathrm{id}_{\mathcal{O}(X) \multimap \mathcal{O}(X)}$ for the frame $\mathcal{O}(X)$.
(6) We have $\Psi(\Phi\, \mathrm{id}_{\mathcal{O}(X)}) = \mathrm{id}_{\mathcal{O}(X)}$ for the frame $\mathcal{O}(X)$.

PROOF. The proof of (1)$\Rightarrow$(2) proceeds as in the previous section. One only has to replace $x \ll y$ by $y \in (\uparrow x)^o$; the set $\{y \in X \mid y \in (\uparrow x)^o\}$ is directed with respect to the specialization order since X is assumed to be a continuous space. As before, it remains to show (6)$\Rightarrow$(1). Now (6) reads as

$$\bigcup_{x \in U} \bigwedge_{V \in \mathcal{F}_x} V = U$$

for all $U \in \mathcal{O}(X)$. Since $\bigwedge$ is the infimum in $\mathcal{O}(X)$ we readily see that $\bigwedge_{V \in \mathcal{F}_x} V = (\uparrow x)^o$. Thus, we have $U = \bigcup_{x \in U} (\uparrow x)^o$ and X is a continuous space. $\square$

5.1 Example

Here is an example where the isomorphism above *does not* hold. Take X to be the space $[0, 1]$ in its Hausdorff/Interval-topology. Let L be the frame of opens of X. Note that the latter is a continuous lattice since X is locally compact. However, the isomorphism cannot hold since X would then be a continuous space, i.e. a continuous domain as X is sober. Therefore it is neither necessary nor sufficient to assume that L be continuous. Our results also imply that the lattice L^{op} is not continuous.

6 The Ambient Categories

We defined a space of possibility distributions $\mathsf{F}_L(X)$ and a space of possibility measures $\mathsf{P}_L(X)$ for every topological space X. These definitions extend to functors in a natural way.

Definition 9 *Given a complete lattice L, we extend $\mathsf{F}_L(\cdot)$ and $\mathsf{P}_L(\cdot)$ to functors. Let TOP-L be the full subcategory of TOP of all topological spaces X such that*

$\mathsf{F}_L(X)$ *is a complete lattice in the pointwise order. Given a morphism* $f\colon X \to Y$ *in TOP-L, we define* $\mathsf{F}_L(f)$ *as the lower adjoint of the map* $\tilde{f}\colon \mathsf{F}_L(Y) \to \mathsf{F}_L(X)$ *defined by* $\tilde{f}(g) = g \circ f$. *If* f *is any morphism in TOP, then* $\mathsf{P}_L(f)\colon \mathsf{P}_L(X) \to \mathsf{P}_L(Y)$ *is defined as*

$$\mathsf{P}_L(f)\,\mu\,O = \mu(f^{-1}(O))$$

for all $\mu \in \mathsf{P}_L(X)$ *and all* $O \in \mathcal{O}(Y)$.

Lemma 10 *(1) The map* $\mathsf{P}_L(\cdot)$ *is well-defined and results in a functor from TOP to SUP, the category of complete sup-lattices and maps preserving suprema.*

(2) On TOP-L, the map $\mathsf{F}_L(\cdot)$ *is well-defined and results in a functor from TOP-L to SUP.*

(3) The category TOP-L subsumes all dcpos and continuous spaces for all complete lattices L. *If* L^{op} *is continuous then TOP-L is just TOP.*

PROOF.

(1) Clearly, $\mathsf{P}_L(X)$ is a complete lattice, where the supremum operation is the pointwise one. Given a continuous function $f\colon X \to Y$ we see that $\mathsf{P}_L(f)$ is a well-defined map from $\mathcal{O}(Y)$ to L. For any $\mu \in \mathsf{P}_L(X)$ the map $\mathsf{P}_L(f)\,\mu$ preserves suprema since μ and f^{-1} preserve suprema. Thus, $\mathsf{P}_L(\cdot)$ is well-defined and clearly functorial.

(2) Given spaces X and Y in TOP-L, consider the function

$$\hat{f}\colon [Y \to L^{\mathrm{op}}] \to [X \to L^{\mathrm{op}}]$$

which maps $g \in [Y \to L^{\mathrm{op}}]$ to $g \circ f$. Since suprema in $[X \to L^{\mathrm{op}}]$ and $[Y \to L^{\mathrm{op}}]$ are pointwise over L^{op} we see that $\hat{f}$ preserves suprema. In other words, $\tilde{f}\colon \mathsf{F}_L(Y) \to \mathsf{F}_L(X)$ with $\tilde{f}(g) = g \circ f$ is a map preserving infima. But $\mathsf{F}_L(Y)$ is a complete lattice, so $\tilde{f}$ has a lower adjoint $\mathsf{F}_L(f)\colon \mathsf{F}_L(X) \to \mathsf{F}_L(Y)$ preserving suprema. Since the process of taking lower adjoints is a contravariant functor and since we composed that with the contravariant functor $f \mapsto \hat{f}$ in TOP, we see that $\mathsf{F}_L(\cdot)$ is a well-defined functor from TOP-L into SUP.

(3) If D is a dcpo and L a complete lattice, then $[D \to L]$ is a complete lattice. The rest follows from Theorems 5 and 8. □

7 Naturality of the Isomorphisms

We showed that the objects $\mathsf{F}_L(X)$ and $\mathsf{P}_L(X)$ are isomorphic whenever X is a continuous space or L^{op} a continuous lattice. These results extend to morphisms as well. Given a complete lattice L, we parameterize the maps Φ and Ψ with X. Given a continuous function $f\colon X \to Y$ we need to show that $\mathsf{F}_L(f)$ and $\mathsf{P}_L(f)$ correspond via the isomorphisms for X and Y. Thus, we need to show the commuting diagrams

$$\begin{aligned}
&\mathsf{F}_L(f) \circ \Phi_X = \Phi_Y \circ \mathsf{P}_L(f)\\
&\mathsf{P}_L(f) \circ \Psi_X = \Psi_Y \circ \mathsf{F}_L(f).
\end{aligned}$$

Since Φ and Ψ are inverses of each other for X and Y, it suffices to show

$$\mathsf{F}_L(f) = \Phi_Y \circ \mathsf{P}_L(f) \circ \Psi_X.$$

Since lower adjoints are unique, we are done if $\Phi_Y \circ \mathsf{P}_L(f) \circ \Psi_X$ is a lower adjoint of $\tilde{f}$, i.e. we need to show

$$\begin{aligned}
&\tilde{f} \circ (\Phi_Y \circ \mathsf{P}_L(f) \circ \Psi_X) \geq \mathrm{id}_{\mathsf{F}_L(X)}\\
&(\Phi_Y \circ \mathsf{P}_L(f) \circ \Psi_X) \circ \tilde{f} \leq \mathrm{id}_{\mathsf{F}_L(Y)}
\end{aligned}$$

Using the fact that Φ_X and Ψ_X are inverses of each other, we may rewrite the first inequality as

$$\tilde{f} \circ \Phi_Y \circ \mathsf{P}_L(f) \geq \Phi_X$$

or equivalently

$$\tilde{f}(\Phi_Y(\mathsf{P}_L(f)\,\mu)) \geq \Phi_X\,\mu$$

for all $\mu \in \mathsf{P}_L(X)$. Recalling the definitions of $\tilde{f}$ and $\mathsf{P}_L(f)$ the latter is equivalent to

$$\Phi_Y(\mu \circ f^{-1}) \circ f \geq \Phi_X\,\mu.$$

Since this is an inequality in $\mathsf{F}_L(X)$ this is equivalent to

$$\Phi_Y(\mu \circ f^{-1})(f\,x) \geq \Phi_X\,\mu\,x$$

for all $x \in L$, where '$\geq$' is taken in L. But this always holds since

$$\begin{aligned}\Phi_Y(\mu \circ f^{-1})(f\,x) &= \bigwedge_{V \in \mathcal{F}_{f(x)}} \mu(f^{-1}(V)) \\ &= \bigwedge\{\mu(f^{-1}(V)) \mid f^{-1}(V) \in \mathcal{F}_x\} \\ &\geq \bigwedge_{U \in \mathcal{F}_x} \mu\,U \\ &= \Phi_X\,\mu\,x.\end{aligned}$$

Unwinding the meaning of the second inequality $(\Phi_Y \circ \mathsf{P}_L(f) \circ \Psi_X) \circ \tilde{f} \leq \mathrm{id}_{\mathsf{F}_L(Y)}$ we may similarly rewrite this as $\mathsf{P}_L(f) \circ \Psi_X \circ \tilde{f} \leq \Psi_Y$, which means $\mathsf{P}_L(f)(\Psi_X(\tilde{f}(g))) \leq \Psi_Y\,g$ for all $g \in \mathsf{F}_L(Y)$. Applying the definitions of $\tilde{f}$ and $\mathsf{P}_L(f)$ this rewrites to $\Psi_X(g \circ f) \circ f^{-1} \leq \Psi_Y\,g$, which is an inequality in $\mathsf{P}_L(Y)$. Thus, it is equivalent to

$$\Psi_X(g \circ f)(f^{-1}(V)) \leq \Psi_X\,g\,V$$

for all $V \in \mathcal{O}(Y)$, where the '$\leq$' refers to the order in L. The latter inequality is satisfied since

$$\begin{aligned}\Psi_X(g \circ f)(f^{-1}(V)) &= \bigvee_{x \in f^{-1}(V)} g(f(x)) \\ &= \bigvee\{g(f(x)) \mid f(x) \in V\} \\ &\leq \bigvee_{y \in V} g(y) \\ &= \Psi_Y\,g\,V.\end{aligned}$$

Theorem 11 *Let L be a complete lattice such that L^{op} is continuous. Then the functors $\mathsf{F}_L(\cdot)$ and $\mathsf{P}_L(\cdot)$ are naturally isomorphic via the natural isomorphisms Φ and Ψ. If L' is any complete lattice, then we get a similar equivalence of functors if we restrict $\mathsf{F}_{L'}(\cdot)$ and $\mathsf{P}_{L'}(\cdot)$ to the class of continuous spaces.*

8 Functoriality in L

So far we studied $\mathsf{F}_L(\cdot)$ and $\mathsf{P}_L(\cdot)$ for arbitrary, but fixed, complete lattices L. This was motivated by a desire for mathematical generality and by the fact that we want to compare such functors for different choices of L. The most prominent candidates for L are $[0,1]$ and $[0,\infty]$. Exploiting the uniform definition of $\mathsf{F}_L(\cdot)$ and $\mathsf{P}_L(\cdot)$ in L, such comparisons are now relatively straightforward.

First, consider a map $h: L \to L'$ preserving *suprema* between two complete lattices L and L'. Then we get a functorial action $\mathcal{P}_h: \mathsf{P}_L(X) \to \mathsf{P}_{L'}(X)$ for any topological space X such that $\mathcal{P}_h\,\mu = h \circ \mu$.

Second, if a map $h: L \to L'$ preserves *filtered infima*, it is just a continuous map $h: L^{\mathrm{op}} \to L'^{\mathrm{op}}$, where we endow L and L' with their Scott-topology. So given any continuous map $f: X \to L^{\mathrm{op}}$ we see that the map

$$\mathcal{F}_h\, f = h \circ f: X \to L'^{\mathrm{op}}$$

is therefore continuous as well. Thus, $\mathcal{F}_h$ is a well-defined functorial action.

If a map $h: L \to L'$ preserves suprema *and* filtered infima, we have two well-defined functorial actions $\mathcal{P}_h$ and $\mathcal{F}_h$. One readily sees that they correspond via the isomorphisms Φ and Ψ:

$$\begin{aligned}\Psi(\mathcal{F}_h\, f)\, O &= \bigvee_{x \in O} h(f\, x)\\ &= h\left(\bigvee_{x \in O} f\, x\right)\\ &= \mathcal{P}_h(\Psi\, f\, O)\end{aligned}$$

since h preserves suprema and

$$\begin{aligned}\Phi(\mathcal{P}_h\, \mu)\, x &= \bigwedge_{O \in \mathcal{F}_x} h(\mu\, O)\\ &= h\left(\bigwedge_{O \in \mathcal{F}_x} \mu\, O\right)\\ &= \mathcal{F}_h(\Phi\, \mu\, x)\end{aligned}$$

since h preserves filtered infima. Note that this combined condition on h means that $h: L^{\mathrm{op}} \to L'^{\mathrm{op}}$ is Lawson continuous [9, Theorem 1.8].

8.1 Embedding $[0,1]$ *into* $[0,\infty]$

As an example let us take the embedding of $[0,1]$ into $[0,\infty]$. The map $e: [0,1] \to [0,\infty]$ with $e(r) = r$ preserves all suprema and non-empty infima. Thus, it has an upper adjoint $p: [0,\infty] \to [0,1]$ mapping s to $s \wedge 1$. The latter preserves suprema and infima, so both p and e have functorial actions for $\mathcal{F}$ and $\mathcal{P}$ corresponding via Φ and Ψ. Since these actions are monotone and functorial we get $\mathcal{P}_p \circ \mathcal{P}_e = \mathrm{id}$, $\mathcal{P}_e \circ \mathcal{P}_p \leq \mathrm{id}$ and likewise for $\mathcal{F}$.

For a fixed $r \in [0,1]$ we define $l_r: [0,1] \to [0,1]$ as left multiplication: $l_r(s) = r \cdot s$. This function preserves suprema and non-empty meets, so $\mathcal{P}_{l_r}$ is a well-defined function on $\mathsf{P}_{[0,1]}(X)$ for all topological spaces X. Using $\mathcal{P}_{l_r}$ we may define a multiplicative action of $[0,1]$ on $\mathsf{P}_{[0,1]}(X)$ by

$$r * \mu = \mathcal{P}_{l_r}\, \mu$$

which is the possibility measure sending $O \in \mathcal{O}(X)$ to $r \cdot (\mu\, O)$. Since $\mathcal{F}_{l_r}$ and $\mathcal{P}_{l_r}$ correspond via our isomorphisms, we know that the multiplicative action of $[0,1]$ on $\mathsf{F}_{[0,1]}(X)$ defined by $\mathcal{F}_{l_r}$ is also pointwise: $(r * f)\, x = r \cdot (f\, x)$.

However, things become less trivial if we try to figure out what to do with $r = \infty$ in $[0,\infty]$. How should we extend the multiplication in $[0,\infty)$ to $[0,\infty]$? If l_∞ is to preserve suprema, we need $\infty \cdot 0 = 0$. If l_∞ is to preserve filtered infima, then $\infty \cdot 0$ must be ∞.

In the theory of cones for *valuations* [21] one chooses $\infty \cdot 0 = 0$, so $\mathcal{P}_{l_\infty}$ is well-defined, whereas $\mathcal{F}_{l_\infty}$ is not, i.e. interpreting $\mathcal{P}_{l_\infty}$ via the isomorphism does not result in pointwise scalar multiplications. For example, let X be the ordinal $\omega + 1$ in its Scott-topology. The function $f: \omega + 1 \to [0,\infty]^{\mathrm{op}}$ with $f(n) = \frac{1}{n}$ and $f(\omega) = 0$ is continuous. Pointwise multiplication with ∞ yields a function $g: \omega + 1 \to [0,\infty]^{\mathrm{op}}$ with $g(n) = \infty$ and $g(\omega) = 0$, so g is *not* continuous.

This mismatch does not occur for any $r \in [0,\infty)$, for $s \mapsto r \cdot s: [0,\infty] \to [0,\infty]$ preserves suprema and filtered infima. It is therefore no surprise that only finite scalars are used in the approximation of valuations via *simple valuations*, or possibility measures via *simple possibility measures* (see Section 12).

9 Extensions of Possibility Measures

Let L^{op} be a continuous lattice and $\mu \in \mathsf{P}_L(X)$. We know that f defined as $\Phi\, \mu: X \to L^{\mathrm{op}}$ is continuous. If S is any subset of X we may define

$$\mu_1\, S = \bigvee_{x \in S} f\, x$$
$$\mu_2\, S = \bigwedge_{O \in \mathcal{F}_S} \mu\, O$$

where $\mathcal{F}_S$ is the filter of all opens O containing S. Note that the definition of μ_2 resembles that of an *outer measure* [11].

Proposition 12 *Let L^{op} be a continuous lattice and $\mu \in \mathsf{P}_{L}(X)$. For the maps μ_1 and μ_2 defined above we have:*

(1) μ_1 and μ_2 are equal,
(2) let μ' be the map μ_1 or μ_2; then μ' extends μ since $\mu' O = \mu O$ for all $O \in \mathcal{O}(X)$,
(3) $\mu'(\uparrow x) = (\Phi\, \mu)\, x$ for all $x \in X$,
(4) and $\mu' \colon \mathcal{P}(X) \to L$ preserves suprema.

PROOF. We only prove the first item $\mu_1 = \mu_2$ for purpose of illustration. The relationship $\mu_1 \leq \mu_2$ always holds; for $x \in S$ implies $\mathcal{F}_S \subseteq \mathcal{F}_x$, and thus,

$$\begin{aligned} f\,x &= \bigwedge_{O \in \mathcal{F}_x} \mu\, O \\ &\leq \bigwedge_{O \in \mathcal{F}_S} \mu\, O \\ &= \mu_2\, O. \end{aligned}$$

To prove $\mu_2 \leq \mu_1$ let l be way-below $\mu_1\, S$ in L^{op}: $\mu_1\, S \prec l$. Then $f\,x \prec l$ for all $x \in S$ follows; so there exists some $O_x \in \mathcal{O}(X)$ with $\mu\, O_x \prec l$ and $O = \bigcup_{x \in S} O_x$ contains S. Therefore, $\mu_2\, S \leq \mu\, O = \bigvee_{x \in S} \mu\, O_x \leq l$. $\square$

If we restrict μ' to the upper powerdomain κ_X, the dcpo of compact saturated subsets of X ordered by *reverse* inclusion [2], then it preserves all natural operations of that domain (we allow $\emptyset$ to be in κ_X):

Proposition 13 *Let X be a sober space and L a co-continuous complete lattice. If we restrict μ' to the upper powerdomain κ_X of X then $\mu' \colon \kappa_X \to L^{\mathrm{op}}$ is Scott-continuous and preserves finite infima.*

PROOF. Since we reversed the order in the domain and range of μ', the map $\mu' \colon \kappa_X \to L^{\mathrm{op}}$ preserves finite infima, for $\mu' \colon \mathcal{P}(X) \to L$ preserves suprema and finite unions of compact saturated sets are compact saturated. To see that $\mu' \colon \kappa_X \to L^{\mathrm{op}}$ is Scott-continuous, it suffices to show that

$$\mu'\left(\bigcap_{i \in I} K_i\right) = \bigwedge_{i \in I} \mu'(K_i)$$

for all filtered families $(K_i)_{i \in I}$ of compact saturated sets, where the infimum is the one in L. Since X is sober we may invoke the Hofmann-Mislove Theorem

[17], which guarantees

$$\mathcal{F}_{(\bigcap_{i\in I} K_i)} = \bigcup_{i\in I} \mathcal{F}_{K_i}$$

from which we obtain the desired Scott-continuity. $\square$

It is not the case that these maps preserve binary suprema in general. For example, let $\mathbf{2}\times\mathbf{2}$ be the product of the Sierpinski space and let $\mu = \eta_{(1,0)} \vee \eta_{(0,1)}$ where the map η_x sends all $O \in \mathcal{F}_x$ to 1 and all $O' \notin \mathcal{F}_x$ to 0. Then $\mu'\{(1,1)\} = 0$ but $\mu'\{(1,1),(1,0)\}$ and $\mu'\{(1,1),(0,1)\}$ both equal 1.

10 Sup-Semirings

We require that L induces an action on the space $\mathsf{P}_L(X)$ for any topological space X. Given $l \in L$, $\mu \in \mathsf{P}_L(X)$, and $O \in \mathcal{O}(X)$, we want $l * \mu$ to be an element of $\mathsf{P}_L(X)$ which 'multiplies' the values l and $\mu(O)$ in L. This motivates the concept of a sup-semiring and its corresponding modules.

Definition 14 *We call structures* $(L; *, 1)$ sup-semirings *provided that*

(1) L is a complete lattice,
(2) and $*\colon L \times L \to L$ is associative, preserves suprema in each coordinate separately, and 1 is an element of L which is a left and right unit of $*$.

Note that one could alternatively define sup-semirings as structures

$$(D; +, 0, *, 1),$$

where D is a dcpo with bottom 0, $(D; +, 0)$ is a commutative, idempotent monoid such that $+\colon D \times D \to D$ is continuous with respect to the Scott-topologies of $D \times D$ and D, $(D; *, 1)$ is a monoid such that $*\colon D \times D \to D$ is continuous as well, $l * 0 = 0 * l = 0$, $l * (n_1 + n_2) = (l * n_1) + (l * n_2)$, and $(n_1 + n_2) * l = (n_1 * l) + (n_2 * l)$ for all $l, n_1, n_2 \in L$. Canonical examples of sup-semirings are

- $\mathbf{2}$ and $[0,1]$, where $*$ is the infimum operation,
- $[0,1]$ and $[0,\infty]$, where $*$ is multiplication; in particular, 1 need not be the top element of the lattice,
- $[0,\infty]$, where $*$ is the infimum operation and 1 is now ∞.

Similarly to sup-semirings L, we can define left L-modules in two equivalent ways. We prefer to work with the shorter version based on complete lattices.

Definition 15 *Given a sup-semiring $(L; *_L, 1_L)$, we define a* left L-module *$(M; *_M)$ to be*

(1) a complete lattice M,
(2) and a map $*_M: L \times M \to M$, which preserves suprema in each coordinate separately, such that $(l_1 *_L l_2) *_M m = l_1 *_M (l_2 *_M m)$ and $1_L *_M m = m$ for all $l_1, l_2 \in L$ and $m \in M$.

In any left L-module we have that $0_L *_M m$ equals 0_M for all $m \in M$, since $*_M$ preserves suprema, and 0_L and 0_M are least elements (empty suprema).

Let us compare left L-modules to the modules used in the theory of valuations: continuous cones are dcpos D with the structure of a commutative monoid $(D; +, 0)$ and a continuous action $(\alpha, d) \mapsto \alpha * d: [0, \infty] \times D \to D$ of $[0, \infty]$ on D which interacts with that monoid structure in the expected way (see for example [21,24] for the axioms). Our setting requires that we replace the addition $+$ on D and the addition on $[0, \infty]$, respectively $[0, 1]$, by suprema. To do so we need to add the axiom $d + d = d$ to the commutative monoid and think of $+$ on reals as the maximum operation. This forces $+$ to be the lattice-theoretic supremum.

11 Sup-Primes in $\mathsf{P}_L(X)$

For continuous domains X we show that $\mathsf{P}_L(X)$, and therefore $\mathsf{F}_L(X)$, is a free sup-cone over X for L being $[0, \infty]$. For that we need to realize a canonical basis for $\mathsf{P}_L(X)$. Since $\mathcal{O}(X)$ is the Scott-topology of a continuous domain we know, by Stone duality [2], that $\mathcal{O}(X)$ is completely distributive. So $\mathsf{P}_L(X)$ will be completely distributive if L is such as well. Note that this holds for $[0, 1]$ and $[0, \infty]$. Completely distributive lattices have a very natural basis: finite suprema of sup-primes. This is so since every element in a completely distributive lattice is the supremum of sup-primes way-below it [9, Chapter 1, Theorem 3.15] (recall that p is a sup-prime in a lattice L if for all finite sets $F \subseteq L$ the relation $p \leq \vee F$ in L implies $p \leq x$ for some $x \in F$; for $F = \emptyset$ this implies that p cannot be 0_L).

Thus, it is important to characterize sup-primes in $\mathsf{P}_L(X)$. We do this under the weaker assumptions that X be a sober space and L be co-continuous.

Definition 16 *Let $(L; *_L, 1)$ be a sup-semiring. We make $\mathsf{P}_L(X)$ into a left*

L-module via:

$$r * \mu = (O \mapsto r *_L \mu(O)).$$

For $x \in X$ we define the point valuation $\eta_x : \mathcal{O}(X) \to L$ by $\eta_x O = 1_L$ if $O \in \mathcal{F}_x$, and $\eta_x O = 0_L$ otherwise.

Clearly $(\mathsf{P}_L(X), *)$ as defined above is a left L-module. For L being $[0,1]$ or $[0,\infty]$, point valuations are continuous valuations; but they are also possibility measures for all L.

Lemma 17 *Let X be a topological space, $x \in X$ and L a non-trivial sup-semiring. Then the point valuation η_x is a possibility measure over L, so we have a map $\eta^L_X : X \to \mathsf{P}_L(X)$ sending x to η_x. Moreover, this map is Scott-continuous and injective if $\mathcal{O}(X)$ is the Scott-topology on a dcpo X.*

PROOF. Given open sets $U, V \in \mathcal{O}(X)$ we have $\eta_x(U \cup V) = 1_L$ if and only if $x \in U$ or $x \in V$; the latter is equivalent to $\eta_x(U) \vee \eta_x(V) = 1_L$. Thus, $\eta_x \in \mathcal{O}(X) \multimap L = \mathsf{P}_L(X)$. The rest follows as in [21,24]. $\square$

Example 18 *Let L be $[0,1]$ or $[0,\infty]$. Further, take any topological space X with two incomparable elements x and y with respect to the specialization order on X. Suppose that we define the sum of the point valuations η_x and η_y in the same way as for continuous valuations*

$$(\eta_x + \eta_y)\, O = \eta_x(O) + \eta_y(O)$$

for all $O \in \mathcal{O}(X)$; then $\eta_x + \eta_y$ is not *a possibility measure; it is certainly Scott-continuous, but it fails to preserve binary suprema: since x and y are incomparable we can find open sets O and O' in X such that $x \in O \setminus O'$ and $y \in O' \setminus O$. Then $(\eta_x + \eta_y)(O \cup O') = \eta_x(O \cup O') + \eta_y(O \cup O') = 1 + 1 = 2$, whereas $(\eta_x + \eta_y)(O) \vee (\eta_x + \eta_y)(O') = (1+0) \vee (0+1) = 1$.*

Theorem 19 *Let X be a sober space and L a non-trivial co-continuous sup-semiring. Then $\mu \in \mathsf{P}_L(X)$ is a sup-prime in $\mathsf{P}_L(X)$ if and only if $\mu = r * \eta_x$ for some sup-prime $r \in L$ and some $x \in X$.*

PROOF. First, we show that all scalar multiples of point valuations are sup-primes if the scalar is a sup-prime in L. So let $x \in X$ and consider a sup-prime r in L. Then $r \neq 0_L$ implies $r * \eta_x \neq 0_{\mathsf{P}_L(X)}$. So suppose that $r * \eta_x \leq \mu \vee \nu$ in $\mathsf{P}_L(X)$. Assume that $r * \eta_x \not\leq \mu$ and $r * \eta_x \not\leq \nu$. Thus, there must be some $O \in \mathcal{O}(X)$ such that $r *_L \eta_x(O) \not\leq \mu(O)$. This implies $r *_L \eta_x(O) = r *_L 1_L = r$, and so $x \in O$, for $r *_L 0_L = 0_L$. Likewise, there must exist some $O' \in \mathcal{O}(X)$

such that $r *_L \eta_x(O') \not\leq \nu(O')$ and $r *_L \eta_x(O') = r$ follows. Thus, x is contained in the open set $O \cap O'$ and we compute

$$\begin{aligned} r &= r *_L \eta_x(O \cap O') \\ &\leq (\mu \vee \nu)(O \cap O') \\ &= \mu(O \cap O') \vee \nu(O \cap O') \\ &\leq \mu(O) \vee \nu(O') \end{aligned}$$

which contradicts $r \not\leq \mu(O)$ and $r \not\leq \nu(O')$, for r is a sup-prime in L.

Second, let π be a sup-prime in $\mathsf{P}_L(X)$. Consider

$$\mathcal{F} = \{O \in \mathcal{O}(X) \mid \pi(O) > 0_L\}.$$

We claim that $\mathcal{F}$ is a completely prime filter in $\mathcal{O}(X)$. Since π is monotone we see that $\mathcal{F}$ is an upper set in $\mathcal{O}(X)$.

In the previous section (Proposition 12) we saw that π extends to a map $\pi' \colon \mathcal{P}(X) \to L$ since $\mathsf{F}_L(X)$ and $\mathsf{P}_L(X)$ are isomorphic by Theorem 3. Given any subset $S \subseteq X$ we therefore obtain a possibility measure $\pi_S \in \mathsf{P}_L(X)$ defined by

$$\pi_S(O) = \pi'(O \cap S),$$

for $\pi' \colon \mathcal{P}(X) \to L$ preserves suprema. We use such measures to show that $\mathcal{F}$ is a filter. For let U and V be in $\mathcal{F}$. Using elementary set identities we obtain

$$\pi = \pi_{X \setminus (U \cap V)} \vee \pi_{U \cap V}.$$

Now, if $U \cap V$ is not in $\mathcal{F}$ we obtain $\pi(U \cap V) = 0_L$ and so $\pi_{U \cap V}(O) = \pi'(O \cap (U \cap V)) \leq \pi(U \cap V) = 0_L$ for all $O \in \mathcal{O}(X)$ since π' extends π. Therefore π equals $\pi_{X \setminus (U \cap V)}$ and the latter is just

$$\pi_{X \setminus U} \vee \pi_{X \setminus V}$$

since π' preserves suprema. Since π is a sup-prime in $\mathsf{P}_L(X)$ we may assume $\pi = \pi_{X \setminus U}$ without loss of generality. But then $\pi U = 0_L$ is immediate and a contradiction. Hence $U \cap V \in \mathcal{F}$.

If $\bigcup \mathcal{O}$ is in $\mathcal{F}$ for some $\mathcal{O} \subseteq \mathcal{O}(X)$ then

$$\bigvee_{O \in \mathcal{O}} \pi(O) = \pi\left(\bigcup \mathcal{O}\right) > 0_L$$

shows that $O \in \mathcal{F}$ for some $O \in \mathcal{O}$. Thus, $\mathcal{F}$ is a completely prime filter. Since X is sober we know that there exists a unique $x \in X$ such that $\mathcal{F} = \mathcal{F}_x$. Thus, $\pi(U) = 0_L$ for all U not containing x. Yet, for any open $O \in \mathcal{F}_x$ we have $\pi(O) > 0_L$. As before we may reason that

$$\pi = \pi_{X \setminus O} \vee \pi_O.$$

Since π is a sup-prime in $\mathsf{P}_L(X)$ we must have $\pi = \pi_{X\setminus O}$ or $\pi = \pi_O$. The former is impossible since $\pi(O) > 0_L$. Thus, $\pi = \pi_O$ renders $\pi(O) = \pi_O(O) = \pi_O(X) = \pi(X)$. Thus,

$$\pi = \pi(X) * \eta_x.$$

To see that $\pi(X)$ is a sup-prime in L, let $\pi(X) \leq l \vee k$ in L. Then $\pi = \pi(X) * \eta_x \leq l * \eta_x \vee k * \eta_x$ follows. Since π is a sup-prime in $\mathsf{P}_L(X)$ we obtain $\pi \leq l * \eta_x$ without loss of generality. Thus, $\pi(X) \leq l *_L \eta_x(X) = l *_L 1_L = l$. $\square$

One can easily adapt the proof above to show the corresponding result, where we replace 'sup-prime' with 'sup-irreducible'. It should be clear that these results also characterize 'inf-primes' and 'inf-irreducible' elements in $[X \to L^{\mathrm{op}}]$ whenever the isomorphism $\mathsf{F}_L(X) \cong \mathsf{P}_L(X)$ holds.

12 Continuity of $\mathsf{P}_L(X)$

In this section we assume that $\mathcal{O}(X)$ is the Scott-topology of a continuous domain X and that L is a completely distributive lattice. Using Stone duality [2], we know that $\mathcal{O}(X)$ is a completely distributive lattice as well. Thus, $\mathsf{P}_L(X)$ is completely distributive, too [19]; in particular, every element in $\mathsf{P}_L(X)$ is the supremum of sup-primes way-below it [9, Chapter 1, Theorem 3.15]. Since $l, l' \ll l''$ implies $l \vee l' \ll l''$ in any complete lattice we see that the set of finite suprema of sup-primes in $\mathsf{P}_L(X)$ is a basis for $\mathsf{P}_L(X)$. Continuous domains are sober spaces [2], so we may invoke the main result of the previous section to characterize this basis.

Theorem 20 *Let $\mathcal{O}(X)$ be the Scott-topology of a continuous domain and let $(L, *_L, 1_L)$ be a completely distributive sup-semiring. Then*

$$\{r_1 * \eta_{x_1} \vee r_2 * \eta_{x_2} \vee \ldots \vee r_k * \eta_{x_k} \mid \\ k \geq 1;\ r_i \textit{ sup-prime in } L,\ x_i \in X\ (i = 1, \ldots, k)\}$$

is a basis of $\mathsf{P}_L(X)$. If $L = [0, \infty]$ one may also assume all r_i to be finite.

PROOF. Using [9, Chapter 1, Theorem 3.15], this is immediate from Theorem 19. For $L = [0, \infty]$ we may restrict the r_i's to finite values since the action $\mu \mapsto r * \mu$ is Scott-continuous. $\square$

In case of $L = [0, \infty]$, we call basis elements of the form

$$r_1 * \eta_{x_1} \vee r_2 * \eta_{x_2} \vee \ldots \vee r_k * \eta_{x_k}$$

with $0 < r_i < \infty$ $(i = 1, \ldots, k)$ *simple possibility measures*. They look like simple valuations [21,24] only that we change sums to suprema. Unlike simple valuations, possibility measures are not uniquely determined by their representation via point valuations. For example, take X to be the Sierpinski space $\{0 < 1\}$. Then $\eta_1 = \eta_0 \vee \eta_1$. Suprema of point valuations can be seen to model unsharpness of underlying data in a fuzzy way.

Let us recall that for continuous X and completely distributive lattices L our isomorphic functors restrict to completely distributive lattices in their range. In that case, the functors $\mathsf{P}_L(\cdot)$ and $\mathsf{F}_L(\cdot)$ are locally continuous.

Proposition 21 *Let L be a completely distributive lattice. If we restrict the functors $\mathsf{P}_L(\cdot)$ and $\mathsf{F}_L(\cdot)$ to CONT, the category of continuous domains and Scott-continuous maps, then they are locally continuous and map into CD, the category of completely distributive lattices and maps preserving suprema.*

PROOF. We only show that $\mathsf{P}_L(\cdot)$, and therefore $\mathsf{F}_L(\cdot)$ as well, is locally continuous. Given continuous domains D and E the inequality $f \leq g$ in $[D \to E]$ implies $f^{-1}(O) \subseteq g^{-1}(O)$ for all $O \in \sigma(E)$. Thus, $\mathsf{P}_L(\cdot)$ is a monotone functor. If $(f_i)_{i \in I}$ is a directed family in $[D \to E]$, we compute for $\mu \in \mathsf{P}_L(D)$ and $O \in \sigma(E)$

$$\begin{aligned}
(\mathsf{P}_L(\bigvee_{i \in I} f_i)\,\mu)\,O &= (\mu \circ (\bigvee_{i \in I} f_i)^{-1})\,O \\
&= \mu((\bigvee_{i \in I} f_i)^{-1}\,O) \\
&= \mu(\bigcup_{i \in I} f_i^{-1}(O)) \quad \text{as } O \text{ is Scott-open} \\
&= \bigvee_{i \in I} (\mu \circ f_i^{-1})\,O \quad \text{as } \mu \text{ preserves (directed) suprema} \\
&= \bigvee_{i \in I} ((\mathsf{P}_L(f_i)\,\mu)\,O \\
&= ((\bigvee_{i \in I} \mathsf{P}_L(f_i))\,\mu)\,O. \qquad \square
\end{aligned}$$

The last proof did not require that D or E be continuous. Note also that the local continuity of these functors allows us to solve domain equations with the standard machinery of locally continuous functors [2].

13 Initial Algebras

We consider the category of left L-modules. Morphisms between left L-modules $(A; *_A)$ and $(B; *_B)$ are continuous functions $f\colon A \to B$ such that

$$f(a +_A a') = f(a) +_B f(a')$$
$$f(\alpha *_A a) = \alpha *_B f(a)$$

for all $\alpha \in L$, where $+$ is interpreted as the binary supremum. Assuming that these topological spaces are dcpos, one may apply Freyd's General Adjoint Functor Theorem to secure the existence of free algebras [2,25]. In [2,25] it has been shown that such an initial algebra is a continuous domain if the underlying dcpo D is a continuous domain to begin with. Thus, we obtain initial, or free, left L-modules over a dcpo D. However, one often would like to have concrete representations of such initial algebras, which validate and strengthen our semantic intuitions. In this section we reveal that $\mathsf{P}_{[0,\infty]}(D)$ is the free left $[0,\infty]$-module over D if D is a continuous dcpo. Since D is continuous this gives us another representation of this free left module as $\mathsf{F}_{[0,\infty]}(D)$, with the caveat for scalar multiplication with ∞ raised in Section 8.2.

Lemma 22 *The dcpo $\mathsf{P}_{[0,\infty]}(X)$ with $\alpha * \mu = (O \mapsto \alpha \cdot \mu(O))$ is a $[0,\infty]$-module, where $[0,\infty]$ is the multiplicative sup-semiring.*

PROOF. Most of this is immediate. The continuity of the operations follows from the general associativity of suprema and from the fact that multiplication in $[0,\infty]$ is continuous. The equations are easily shown using the fact that suprema in $\mathsf{P}_{[0,\infty]}(X)$ are pointwise, and noting that scalar multiplication preserves suprema. $\square$

We already have a Scott-continuous map $\eta_D^{[0,\infty]}\colon D \to \mathsf{P}_{[0,\infty]}(D)$ which associates to each $x \in D$ its point valuation η_x. So let A be any left $[0,\infty]$-module. If $f\colon D \to A$ is a Scott-continuous function, we need to show the existence of a unique Scott-continuous homomorphism of left $[0,\infty]$-modules

$$\hat{f}\colon \mathsf{P}_{[0,\infty]}(D) \to A$$

such that $f = \hat{f} \circ \eta_D^{[0,\infty]}$. Since A is a left $[0,\infty]$-module, it is also a complete lattice. Thus, the function

$$\hat{f}(\mu) = \bigvee\{\alpha * f(x) \mid \alpha * \eta_x \ll \mu\}$$

is well-defined since $\alpha*\eta_x = \beta*\eta_y$ implies $\alpha = \beta$ and $x = y$ (the opens separate the points). Since $\nu \ll \mu \leq \mu'$ implies $\nu \ll \mu'$ we conclude that $\hat{f}$ is monotone. Since $\mathsf{P}_{[0,1]}(D)$ is continuous (Theorem 20), the way-below relation on $\mathsf{P}_{[0,\infty]}(D)$ satisfies the interpolation property [9,2]. Using that fact one readily sees that $\hat{f}$ is Scott-continuous.

Next, we verify one half of the statement that $\hat{f}$ is a module homomorphism. For that we need to establish that scalar actions preserve and reflect the way-below relation in left $[0,\infty]$-modules.

Lemma 23 *Let A be a left $[0,\infty]$-module and $\beta \neq 0,\infty$. Then we have $a \ll b$ in A if and only if $\beta * a \ll \beta * b$ in A.*

PROOF. The proof works with the scalar action of $\frac{1}{\beta}$, using that scalar actions are Scott-continuous; note that continuous order-embeddings reflect the way-below relation. $\square$

Lemma 24 *For the map $\hat{f}$ above we have*

$$\hat{f}(\alpha * \mu) = \alpha * \hat{f}(\mu)$$

for all $\alpha \in [0,\infty]$ and $\mu \in \mathsf{P}_{[0,\infty]}(D)$.

PROOF. Recall that $\mu \mapsto \alpha * \mu$ preserves *all* suprema since $\mathsf{P}_{[0,\infty]}(X)$ is a left $[0,\infty]$-module. If $\alpha = 0$, both sides equal 0_A due to $0 * a = 0_A$. For $0 < \alpha < \infty$, we compute

$$\begin{aligned}
\alpha * \hat{f}(\mu) &= \alpha * \bigvee\{\gamma * f(x) \mid \gamma * \eta_x \ll \mu\} \\
&= \bigvee\{\alpha * (\gamma * f(x)) \mid \gamma * \eta_x \ll \mu\} \\
&= \bigvee\{\alpha * (\gamma * f(x)) \mid \alpha * (\gamma * \eta_x) \ll \alpha * \mu\} \quad \text{by Lemma 23} \\
&= \bigvee\{(\alpha \cdot \gamma) * f(x) \mid (\alpha \cdot \gamma) * \eta_x \ll \alpha * \mu\} \\
&= \bigvee\{\beta * f(x) \mid \beta * \eta_x \ll \alpha * \mu\} \quad \text{since } \beta = \alpha \cdot \frac{\beta}{\alpha} \\
&= \hat{f}(\alpha * \mu)
\end{aligned}$$

The case $\alpha = \infty$ follows from $\alpha < \infty$ by the continuity of the operations. $\square$

We may use the property of $\hat{f}$ above to show that $\hat{f} \circ \eta_D^{[0,\infty]} = f$. For that we need to identify certain elements way-below η_x in $\mathsf{P}_{[0,\infty]}(D)$.

Lemma 25 *Let $\alpha \in [0,\infty]$, $y \in D$, and $\mu \in \mathsf{P}_{[0,\infty]}(D)$ such that*

$$\alpha < \mu(\{d \in D \mid y \ll d\}).$$

Then $\alpha * \eta_y \ll \mu$.

PROOF. In the proof of Theorem 20, we saw that μ is the supremum of scalar point valuations $\beta * \eta_x$ way-below μ. Thus, $\mu(\{d \in D \mid y \ll d\})$ equals $\bigvee\{\beta \mid y \ll x,\ \beta * \eta_x \ll \mu\}$. By our assumption we get

$$\alpha < \bigvee\{\beta \mid y \ll x,\ \beta * \eta_x \ll \mu\}$$

and all suprema in $[0,\infty]$ are directed. Thus, there is some $\beta * \eta_x \ll \mu$ such that $\alpha \leq \beta$ and $y \ll x$. Since Scott-open sets are upper sets it is immediate that $\alpha * \eta_y \leq \beta * \eta_x$. So $\alpha * \eta_y \ll \mu$ follows. $\square$

Lemma 26 *For the maps $\eta_D^{[0,\infty]}$ and $\hat{f}$ above we have $f = \hat{f} \circ \eta_D^{[0,\infty]}$.*

PROOF. By Lemma 25 we have $\alpha * \eta_y \ll \eta_x$ for all $\alpha < 1$ and $y \ll x$. Thus, we compute

$$\begin{aligned}
(\hat{f} \circ \eta_D^{[0,\infty]})\, x &\geq \bigvee\{\alpha * f(y) \mid \alpha < 1, y \ll x\} \\
&= \left(\bigvee\{\alpha \mid \alpha < 1\}\right) * \left(\bigvee\{f(y) \mid y \ll x\}\right) \\
&= 1 * f(x) \quad \text{as } f \text{ is Scott-continuous} \\
&= f(x).
\end{aligned}$$

Conversely, let $\alpha * \eta_y \ll \eta_x$. Clearly, $\alpha \leq 1$ follows. But we also get $y \leq x$; otherwise, y would be in the open set $D \setminus \downarrow x$ which does not contain x, contradicting $\alpha * \eta_y \leq \eta_x$. Therefore, $\alpha * f(y) \leq 1 * f(x) = f(x)$ as f and $*$ are monotone. This implies $(\hat{f} \circ \eta_D^{[0,\infty]})\, x \leq f(x)$. $\square$

Lemma 27 *The map $\hat{f}$ is the unique module homomorphism with $\hat{f} \circ \eta_D^{[0,\infty]} = f$.*

PROOF. It remains to verify uniqueness of $\hat{f}$ and $\hat{f}(\mu+\nu) = \hat{f}(\mu) + \hat{f}(\nu)$. The latter is just saying that $\hat{f}$ preserves binary suprema (and therefore *all* suprema). Since $\hat{f}$ is monotone, it suffices to prove $\hat{f}(\mu \vee \nu) \leq \hat{f}(\mu) \vee \hat{f}(\nu)$. The left hand side equals the supremum of all $\alpha * f(x)$, where $\alpha * \eta_x \ll \mu \vee \nu$. Since all such elements $\alpha * \eta_x$ are sup-primes in $\mathsf{P}_{[0,\infty]}(D)$ (Theorem 19), we may assume that $\alpha * \eta_x \leq \mu$ without loss of generality. Since $\hat{f}$ is monotone we obtain

$$\begin{aligned} \alpha * f(x) &= \alpha * \hat{f}(\eta_x) \text{ as } \hat{f} \circ \eta_D = f \\ &= \hat{f}(\alpha * \eta_x) \text{ as } \alpha * \hat{f}(\zeta) = \hat{f}(\alpha * \zeta) \text{ for all } \zeta \\ &\leq \hat{f}(\mu) \\ &\leq \hat{f}(\mu) \vee \hat{f}(\nu). \end{aligned}$$

Thus, $\hat{f}(\mu \vee \nu) = \bigvee\{\alpha * f(x) \mid \alpha * \eta_x \ll \mu \vee \nu\} \leq \hat{f}(\mu) \vee \hat{f}(\nu)$.

As for uniqueness, let $g\colon \mathsf{P}_{[0,\infty]}(D) \to A$ be any module homomorphism such that $g \circ \eta_D^{[0,\infty]} = f$. Given $\mu \in \mathsf{P}_{[0,\infty]}(D)$ we know by Theorem 20 that μ is the directed supremum of all simple valuations ν way-below it. Such a valuation ν is of the form $(\alpha_1 * \eta_{x_1}) \vee (\alpha_2 * \eta_{x_2}) \vee \ldots \vee (\alpha_k * \eta_{x_k})$ for some $k \geq 1$. In particular, all $\alpha_i * \eta_{x_i}$ are way-below μ $(1 \leq i \leq k)$. Thus,

$$\mu = \bigvee\{\alpha * \eta_x \mid \alpha * \eta_x \ll \mu\},$$

together with the fact that g preserves suprema and scalar multiplication, shows that $g = \hat{f}$, noting that $g \circ \eta_D^{[0,\infty]} = f$. $\square$

Theorem 28 *Let D be a continuous domain. Then $\mathsf{P}_{[0,\infty]}(D)$ is the initial left $[0,\infty]$-module over D.*

Incidentally, given a Scott-continuous function $f \in [D \to E]$, we readily see that $\mathsf{P}_{[0,\infty]}(f)$ is the unique module homomorphism $h\colon \mathsf{P}_{[0,\infty]}(D) \to \mathsf{P}_{[0,\infty]}(E)$ such that $\eta_E^{[0,\infty]} \circ f = h \circ \eta_D^{[0,\infty]}$.

14 Embedding $\mathsf{P}_L(X)$ into $[\mathcal{O}(X) \to L]$

The space of possibility measures $\mathsf{P}_L(X)$ sits in the larger space $[\mathcal{O}(X) \to L]$ of all Scott-continuous maps $\mu\colon \mathcal{O}(X) \to L$. The inclusion

$$E_X\colon \mathsf{P}_L(X) \to [\mathcal{O}(X) \to L]$$

with $E_X(\mu) = \mu$ preserves suprema as these are defined pointwise in both spaces. Thus, there exists an upper adjoint

$$U_X : [\mathcal{O}(X) \to L] \to \mathsf{P}_L(X),$$

where $U_X(\mu)$ computes the supremum of all maps $\nu : \mathcal{O}(X) \to L$ preserving suprema such that $\nu \leq \mu$. Incidentally, if we adopt the same definition of $\Phi : \mathsf{P}_L(X) \to \mathsf{F}_L(X)$ to the larger source space $[\mathcal{O}(X) \to L]$ then $\Phi\,\mu$ is still a continuous function from X to L^{op}. Thus, $\Psi(\Phi\,\mu)$ is well-defined and one can indeed show that the latter is just $U_X(\mu)$, provided that L^{op} is continuous or X a continuous domain.

Assuming that $\mathcal{O}(X)$ is the Scott-topology of a continuous domain X and L is completely distributive, it is the case that U_X preserves suprema as well [20]. Therefore, $E_X \circ U_X$ is a projection on $[\mathcal{O}(X) \to L]$ preserving suprema. This is important as it describes the way-below and way-way-below relations [2] on $\mathsf{P}_L(X)$ as being induced by the ones in $[\mathcal{O}(X) \to L]$ (recall that x is way-way-below y in L if for all sets $S \subseteq L$ the relation $y \leq \bigvee S$ implies the existence of some $s \in S$ such that $x \leq s$).

Proposition 29 *Let $\mathcal{O}(X)$ be the Scott-topology of a continuous domain X and L a completely distributive lattice. For $\mu, \nu \in \mathsf{P}_L(X)$ we have $\mu \ll \nu$ in $\mathsf{P}_L(X)$ if and only if $\mu \ll \nu$ in $[\mathcal{O}(X) \to L]$. Further, μ is way-way-below ν in $\mathsf{P}_L(X)$ if and only if μ is way-way-below ν in $[\mathcal{O}(X) \to L]$.*

We obtain a similar result for $\mathsf{P}_{[0,1]}(X)$ sitting inside $\mathsf{P}_{[0,\infty]}(X)$, for $\mathcal{P}_e \circ \mathcal{P}_p$ is a projection preserving suprema on $\mathsf{P}_{[0,\infty]}(X)$ with image $\mathsf{P}_{[0,1]}(X)$ (see Section 8.1).

One could also use the results in [20] to show that simple possibility measures are a basis in $\mathsf{P}_L(X)$ if L is $[0,1]$ or $[0,\infty]$, for every possibility measure μ is the supremum of step functions [2] $O \searrow l$, which map all O' with $O \ll O'$ in $\mathcal{O}(X)$ to l and all other elements to 0_L. The map $U_X(O \searrow l)$ then equals $V \nearrow l$, which maps all $U \subseteq V$ to 0_L and all $U \not\subseteq V$ to l, where

$$V = \bigcup \{W \in \mathcal{O}(X) \mid O \not\ll W\}$$

[20]. But the latter is just

$$\bigvee_{x \in X \setminus V} l * \eta_x$$

so $U_X(O \searrow l)$ is the supremum of simple possibility measures. Since U_X preserves suprema (the set of step functions way-below μ is *not* directed in general) this shows that simple possibility measures form indeed a basis; the embedding of $\mathsf{P}_L(X)$ into $[\mathcal{O}(X) \to L]$ reflects the way-below relation, so each $l * \eta_x$ with $x \in X \setminus V$ is way-below μ in $\mathsf{P}_L(X)$.

15 Related Work

The first author initially studied valuations in a topological setting [14]. In loc. cit. one finds an investigation of several spaces of valuations, their universal properties and a simple new definition of the integral of a real-valued function with respect to a valuation. In [13], real-valued valuations were replaced by three-valued 'abstract' valuations. The space of abstract valuations provides a novel representation of the Plotkin powerdomain of a continuous domain and the Vietoris hyperspace of a Hausdorff space.

Theorem 28 already had a proof in [18]; however, it was phrased using the algebraically cumbersome notion of *sup-cones* and its proof had to rely on 'hand-made' order-theoretic techniques since in loc. cit. it was not known that every sup-prime in $\mathsf{P}_L(X)$ is a scalar multiple of a point valuation (Theorem 19).

We should also mention Jan Rutten's work on generalized metric spaces [3] and Philipp Sünderhauf's work on quantitative $\mathcal{V}$-powerdomains [32].

16 Outlook

This paper presented a new junction between the world of topological spaces and complete lattices. Since we may view any complete lattice L as a topological space in its Scott-topology we obtain another representation of the order dual of the function space $[L^{\mathrm{op}} \to L^{\mathrm{op}}]$; the lattice $\mathcal{O}(L^{\mathrm{op}}) \multimap L$. This should open up the possibility of proving new results in the theory of continuous lattices, notably in the area of *bi*-continuous lattices [19] (lattices L where L and L^{op} are continuous).

The framework of sup-semirings employed in this article does have its shortcomings. For example, if one wants to transfer the initiality proof for $\mathsf{P}_{[0,\infty]}(X)$, Theorem 28, to $\mathsf{P}_{[0,1]}(X)$ one notices that this argument breaks down in two places: the proofs of Lemma 23 and 24 needed fractions $\frac{\alpha}{\beta}$ which might not be defined in $[0,1]$. From a rigorous mathematical perspective this defect warrants a careful analysis of such arguments in order to find suitable abstractions thereof. In particular, the universal property could be expressible and provable without having to resort to the additional structure of module operations. Intuitively, one would expect $\mathsf{P}_L(X)$ to be universal over the *pair* $X \times L$ where the universality should impose continuity in X and the lattice operation of suprema in L.

Acknowledgement

Klaus Keimel and Philipp Sünderhauf made valuable suggestions when the second author presented part of this material at the Comprox II meeting in Darmstadt, Germany, on September 25, 1996.

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