Probabilistic Power Domains, Information Systems, and Locales

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Abstract

The probabilistic power domain construction of Jones and Plotkin [6, 7] is defined by a construction on dcpo's. We present alternative definitions in terms of information systems à la Vickers [12], and in terms of locales. On continuous domains, all three definitions coincide.

1 Introduction

To model probabilistic and randomized algorithms in the semantic framework of dcpo's and Scott continuous functions, Jones and Plotkin introduce in [6, 7] the *probabilistic power domain construction* $\mathcal{P}_D$. It forms a computational monad in the sense of [8] in the category of dcpo's and continuous functions and various of its subcategories of 'domains'. Every probabilistic powerdomain $\mathcal{P}_D X$ is equipped with a family of binary operations $+_p$ indexed by a real number p between 0 and 1 such that $A +_p B$ denotes the result of choosing A with probability p and B with probability $1 - p$.

Other applications of $\mathcal{P}_D$ were found in [1]. The probabilistic powerdomain of the upper power space [10] of a second countable locally compact Hausdorff space X can be used for an effective treatment of probability measures on X, and thus for the study of coloured fractals on X, where the colour is modeled by a probability distribution. For these applications, a description of $\mathcal{P}_D$ in terms of information systems would be useful.

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In [6, 7], it is shown that $\mathcal{P}_D$ preserves (ω-)continuity of dcpo's. On the other hand, it does not preserve algebraicity: even the power domain of the one-point domain is not algebraic. Thus, $\mathcal{P}_D$ cannot be described by the more conventional information systems, which are only suited for certain classes of algebraic domains, as for instance the information systems of [9] for the class of bounded complete algebraic domains (Scott Domains). The information systems of [3, 4] present bounded complete continuous domains, but this is still not sufficient, since $\mathcal{P}_D$ does not preserve bounded completeness as shown in [6, Section 4.5].

In [12], Vickers introduced a kind of information systems (*infosyses*) suitable to cover all continuous domains. In the paper at hand, we show that $\mathcal{P}_D$ can be described in terms of these infosyses. This was already conjectured by Vickers at the end of his paper. We also looked for a localic description of the power construction. Starting from the frame of opens of a continuous base domain, we show how the frame of opens of the power domain may be constructed in terms of generators and relations.

The paper is organized as follows: in Section 2, we sketch the theoretical background and introduce the probabilistic power domain construction $\mathcal{P}_D$. It can be applied to all topological systems producing a dcpo. In Section 3, we introduce the probabilistic power locale construction $\mathcal{P}_L$. It can be applied to all topological systems producing a locale. We show that for all topological systems, power domain and power locale have the same points.

In Section 4, we show how to construct a probabilistic power infosys $\mathcal{P}_I D$ for every infosys D. In Section 5, we prove that for continuous domains X, the power infosys of an infosys for X has the same opens as the power locale of X. The proof uses a hard lemma, which is postponed to Section 6. The two properties that power domain and power locale have the same points, and power infosys and power locale have the same opens imply that all three of $\mathcal{P}_D$, $\mathcal{P}_I$, and $\mathcal{P}_L$ agree on continuous domains.

2 The Background

In this section, we introduce the background of the theory in this paper. We assume the reader to be familiar with the basic notions of dcpo, continuous dcpo, continuous function between dcpo's, and the frames and locales of [5] or [11]. We denote the least element of a frame by 0 or F standing for 'false', and the greatest element by T for 'true', never 1. We use 'domain' as synonym for 'dcpo', in particular when speaking of continuous dcpo's.

In Subsection 2.1, we recall the topological systems of [11]. In Subsection 2.2, we review the definition and properties of infosyses from [12]. Then we present an infosys for the unit interval of the real line in Subsection 2.3. In Subsection 2.4, we introduce the construction $\mathcal{P}_D$ of [6, 7].

2.1 Topological Systems

In [11], Vickers introduced *topological systems* as a common generalization of topological spaces and locales. In analogy to his abbreviation 'infosys' for 'information system', we shall abbreviate 'topological system' by 'topsys'. A *topsys* is a pair $X = (\mathsf{pt}\,X, \Omega X)$ of a set of 'points' $\mathsf{pt}\,X$ and a frame of 'opens' ΩX together with a relation '$\models$' between points and opens, which respects the frame operations: for every index set I, $x \models \bigvee_{i \in I} u_i$ iff there is i in I with $x \models u_i$; and for every finite index set I, $x \models \bigwedge_{i \in I} u_i$ iff for all i in I, $x \models u_i$.

A topological space X induces a topsys, where $\mathsf{pt}\,X$ is the underlying set of X, ΩX is the frame of open sets of X, and $x \models u$ iff $x \in u$. This in particular applies to dcpo's with their Scott topology.

A locale X is defined by a frame ΩX. It becomes a topsys by taking $\mathsf{pt}\,X$ as the set of frame homomorphisms from ΩX to $\mathbf{2} = \{\mathsf{F}, \mathsf{T}\}$, with $x \models u$ iff $xu = \mathsf{T}$.

A continuous function $f : X \to Y$ between topsyses has two components, $\mathsf{pt}\,f : \mathsf{pt}\,X \to \mathsf{pt}\,Y$ and $\Omega f : \Omega Y \to \Omega X$, where Ωf is a frame homomorphism and $\mathsf{pt}\,f x \models v$ iff $x \models \Omega f v$ for all x in $\mathsf{pt}\,X$ and v in ΩY. In case of topological spaces, these are the usual continuous functions, and f is determined by $\mathsf{pt}\,f$. In case of locales, these are frame homomorphisms in the opposite direction, and f is determined by Ωf.

2.2 Vickers Information Systems (Infosyses)

In this subsection, we briefly review the theory of infosyses as given in [12].

An *infosys* is a pair $(D, <)$ of a set of *tokens* D and a binary relation '$<$' on D which is *transitive* — $a < b$ and $b < c$ implies $a < c$ — and *interpolative* — if $a < c$, then there is a token b with $a < b$ and $b < c$. In contrast to the more conventional preorders, reflexivity is not required.

We need several notions and notations: for a set $A \subseteq D$, $\uparrow A = \{b \in D \mid \exists a \in A : a < b\}$ is the upper set of A. We use the abbreviation $\sharp a$ for $\uparrow\{a\}$.[1] The operator '$\uparrow$' is monotonic — $A \subseteq B$ implies $\uparrow A \subseteq \uparrow B$ — and idempotent — $\uparrow(\uparrow A) = \uparrow A$. In posets, there is an additional property $A \subseteq \uparrow A$, which is a consequence of reflexivity and thus not true for general infosyses.

The set of upper bounds of a token set A is $\mathsf{ub}\,A = \{b \in D \mid \forall a \in A : a < b\}$.

A *point* of an infosys is a subset x of D with properties analogous to those of ideals: if $a < b$ and $b \in x$, then $a \in x$; and for $a_1, \ldots, a_n$ in x, there is b in x with $a_1, \ldots, a_n < b$. For the second condition, cases $n = 0$ (x is not empty) and $n = 2$ are enough. The points of an infosys D ordered by inclusion form a continuous dcpo $\mathsf{pt}\,D$. This is the continuous domain represented by the infosys D.

[1] Vickers has no such abbreviations.

Conversely, for every continuous dcpo X, there is an infosys I with $\mathsf{pt}\, I \cong X$, namely $I = (X, \ll)$, where '$\ll$' is the way-below relation on X. In general, there are many other infosyses I with $\mathsf{pt}\, I \cong X$.

If the dcpo X is even algebraic, then there is an infosys I with reflexive order, i.e., a poset, with $\mathsf{pt}\, I \cong X$, namely the basis of X.

An *open* of an infosys is a subset u of D with $\uparrow u = u$. Ordered by inclusion, the opens of D form a frame ΩD. The joins of this frame are simply given by union, whereas the meets are *not* given by intersection: $u \wedge v$ is $\uparrow(u \cap v)$. The mapping $\omega : \Omega D \to 2^{\mathsf{pt}\, D}$ with $\omega u = \{x \in \mathsf{pt}\, D \mid x \cap u \neq \emptyset\}$ provides an isomorphism between the frame ΩD and the frame $\Omega(\mathsf{pt}\, D)$ of Scott open sets of the continuous domain $\mathsf{pt}\, D$. Thus, the topsys induced by the dcpo $\mathsf{pt}\, D$ is isomorphic to $(\mathsf{pt}\, D,\, \Omega D)$ with $p \models u$ iff $p \cap u \neq \emptyset$. We call the latter topsys $[D]$. Two infosyses D and E are *equivalent* iff $[D] \cong [E]$. Equivalent infosyses may look quite different, for instance one may be finite and the other one infinite.

The frame ΩD of an infosys D can be presented by generators and relations: Generators are $\sharp a$ for a in D, and the relations are $\bigwedge_{a \in S} \sharp a = \bigvee_{b \in \mathsf{ub}\, S} \sharp b$ for every finite subset S of D. This one relation can be equivalently replaced by three:

(1) Monotonicity: If $a < b$, then $\sharp a \geq \sharp b$;

(2) All tokens: $\bigvee_{a \in D} \sharp a = \mathsf{T}$;

(3) Meets: $\sharp a \wedge \sharp b \leq \bigvee_{c > a, b} \sharp c$.

Here, (1) corresponds to '$\geq$' in the single relation scheme, (2) to '$\leq$' with empty S, and (3) to '$\leq$' with $S = \{a, b\}$.

2.3 The Unit Interval

The unit interval of the real line plays a major role in the theory of the probabilistic power construction. In this paper, we denote the usual order on real numbers by '$\sqsubseteq$' instead of '$\leq$', and accordingly its strict variant ($x \sqsubseteq y$ and $x \neq y$) by '$\sqsubset$' instead of '$<$'. This is done to avoid confusion with the infosys order introduced below.

By these conventions, the unit interval is $\mathbf{I} = \{r \in \mathbf{R} \mid 0 \sqsubseteq r \sqsubseteq 1\}$ ordered by '$\sqsubseteq$'. It forms a continuous domain, which can be described by many different information systems. As pointed out above, one of these is $(\mathbf{I}, <)$, where '$<$' is the way-below relation of $\mathbf{I}$, namely $x < y$ iff $x \sqsubset y$ or $x = y = 0$. Another one with less tokens is $\mathbf{Q}_0^1 = \{q \in \mathbf{Q} \mid 0 \sqsubseteq q \sqsubset 1\}$ with the same order '$<$'.

In the sequel, we list some arithmetic properties of this non-standard order. All these properties are thought of to be quantified over the non-negative reals ($r \sqsupseteq 0$). In contrast to '$\sqsubset$', $a < b$ does not imply $a + c < b + c$ (take $a = b = 0$ and $c \neq 0$). On the other hand, $a_1 < b_1$ and … and $a_n < b_n$ implies $\sum_{i=0}^{n} a_i < \sum_{i=0}^{n} b_i$ even in the case $n = 0$; and $a < b$ implies $a \cdot c < b \cdot c$ even if $c = 0$. Furthermore, if $a > b$, then there is some rational q with $a > q > b$, and there is some rational $r < 1$ with $r \cdot a > b$. The latter can be extended to any finite number of relations: if $a_1 > b_1$ and ... and $a_n > b_n$, then there is one rational $r < 1$ with $r \cdot a_i > b_i$ for all i.

The frame $\Omega \mathbf{Q}_0^1 \cong \Omega \mathbf{I}$ can be presented by generators $\sharp q$ for q in $\mathbf{Q}_0^1$ with the three relations at the end of Subsection 2.2. By special properties of $\mathbf{Q}_0^1$, namely linearity and existence of joins of tokens, these relations can be simplified to

(1) Rational zero: $\sharp 0 = \top$;

(2) Rational continuity: $\bigvee_{r>q} \sharp r = \sharp q$.

The case $q = 0$ of rational continuity is redundant, since it follows from rational zero and $0 > 0$.

2.4 Probabilistic Power Domains

For given dcpo X, the *probabilistic power domain* $\mathcal{P}_D X$ is defined in [6, 7] as $\mathcal{P}_D X = [\Omega X \overset{mod}{\to} \mathbf{I}]$, the dcpo of *continuous evaluations* from ΩX to $\mathbf{I}$ with the pointwise order. A function μ from a frame to $\mathbf{I}$ is an evaluation if it satisfies the zero law $\mu(0) = 0$ and the modular law $\mu(a \vee b) + \mu(a \wedge b) = \mu a + \mu b$.

Since the definition of $\mathcal{P}_D X$ only refers to ΩX, it can be applied to any topsys X producing a dcpo and thus a topsys. Similarly, the probabilistic power locale construction $\mathcal{P}_L$ soon to be introduced only relies on ΩX and thus can be applied to any topsys. We shall prove (Theorem 3.2):

(1) For every topsys X, $\mathsf{pt}\,\mathcal{P}_D X$ and $\mathsf{pt}\,\mathcal{P}_L X$ are isomorphic posets (in fact dcpo's).

In Section 4, we shall introduce the probabilistic power infosys construction $\mathcal{P}_I$ mapping infosyses to infosyses. We shall prove (Theorem 5.2):

(2) For every infosys D, $\Omega \mathcal{P}_I D$ and $\Omega \mathcal{P}_L [D]$ are isomorphic frames.

Since the topsyses described by infosyses are continuous domains and thus localic (sober), (2) suffices to conclude:

(3) For every infosys D, $[\mathcal{P}_I D]$ and $\mathcal{P}_L [D]$ are isomorphic topsyses.

First, we can conclude from this that $\mathcal{P}_I$ preserves equivalence of infosyses, which would not be obvious a priori. Second, we can conclude that for every continuous domain X, $\mathcal{P}_L X$ is a continuous domain again, and thus a dcpo. The topsyses induced by dcpo's are spatial, whence (1) suffices to conclude

(4) For every continuous domain X, $\mathcal{P}_D X$ and $\mathcal{P}_L X$ are isomorphic topsyses (in fact continuous domains).

This is our main result: on continuous domains, $\mathcal{P}_D$, $\mathcal{P}_L$, and $\mathcal{P}_I$ coincide.

3 A Probabilistic Power Locale

In this section, we introduce the probabilistic power locale construction $\mathcal{P}_L$ and show that its power locales have the same points as Jones's power domains.

3.1 The Generators

In this subsection, we motivate our choice for the generators of $\Omega\mathcal{P}_L X$. Let us first analyze the probabilistic power domain construction $\mathcal{P}_D X = [\Omega X \stackrel{mod}{\to} \mathbf{I}]$ for dcpo's X. It can be dismantled into two consecutive steps: first mapping the dcpo X into its frame of opens ΩX, and then applying a construction $D : F \mapsto [F \stackrel{mod}{\to} \mathbf{I}]$ mapping frames into dcpo's. In investigating D, we may assume that it is applied to arbitrary frames, not just those frames arising from the Scott topology of a dcpo.

Analogously, we shall define a construction L mapping frames into locales, and then let $\mathcal{P}_L X = L(\Omega X)$. The locale $L(F)$ to a frame F should be close to the dcpo $D(F) = [F \stackrel{mod}{\to} \mathbf{I}]$. Ideally, $\Omega L(F)$ would be the frame of Scott open sets of $D(F)$, but this frame is difficult to axiomatize. Instead, we shall consider the pointwise topology on the function space $[F \to \mathbf{I}]$. Its subbasic opens are $F(u, v)$ for members u of F and opens v in $\Omega\mathbf{I}$, where $F(u, v) = \{\mu : F \to \mathbf{I} \mid \mu u \in v\}$. Instead of all opens in $\Omega\mathbf{I}$, it suffices to consider the basic opens $\sharp q$ for q in $\mathbf{Q}_0^1$. Thus, the frame $\Omega L(F)$ will have generators $\langle u, q\rangle$ with u in F and q in $\mathbf{Q}_0^1$, where the intended meaning of $\langle u, q\rangle$ is $\{\mu \mid \mu u > q\}$.

Now, our goal is to find suitable relations on these generators such that the posets $\mathsf{pt}\, D(F)$ and $\mathsf{pt}\, L(F)$ become isomorphic. Indeed, we shall prove a little bit more:

Theorem 3.1 For every frame F, there is a continuous function $f : D(F) \to L(F)$, whose points part $\mathsf{pt}\, f : \mathsf{pt}\, D(F) \to \mathsf{pt}\, L(F)$ is an isomorphism.

We shall prove this theorem by first comparing the dcpo D_0 of *all* functions from F to $\mathbf{I}$ with a certain locale L_0, and then gradually introducing more restrictions on the functions of D_0 as well as more relations on the frame of L_0.

3.2 Arbitrary Functions

For our fixed frame F, we define D_0 to be the dcpo of *all* (not even necessarily monotonic) functions from F to $\mathbf{I}$. For the corresponding locale L_0, we present the frame ΩL_0 by the generators $\langle u, q\rangle$ for u in F and q in $\mathbf{Q}_0^1$ with two relations which are directly motivated by the two relations on $\Omega\mathbf{Q}_0^1$ of Subsection 2.3:

- Rational zero: $\langle u, 0\rangle = \mathsf{T}$ for all u in F.
- Rational continuity: $\bigvee_{r>q} \langle u, r\rangle = \langle u, q\rangle$ for all u in F and q in $\mathbf{Q}_0^1$.

The second relation implies that $\langle u, r\rangle \leq \langle u, q\rangle$ whenever $r > q$. Notice that '>' here is meant to refer to (the dual of) the infosys relation of Subsection 2.3, which differs from the usual meaning of '<' by the additional relationship $0 < 0$.

Now, we define a continuous function $f : D_0 \to L_0$ by defining $\Omega f : \Omega L_0 \to \Omega D_0$ with

$$\Omega f\langle u, q\rangle \;=\; \{\mu : F \to \mathbf{I} \mid \mu u > q\}$$

and $\mathsf{pt}\, f : \mathsf{pt}\, D_0 \to \mathsf{pt}\, L_0 = \mathit{Frame}(\Omega L_0,\, \mathbf{2})$ with

$$\mathsf{pt}\, f \mu\, \langle u, q\rangle \;=\; [\mu u > q]$$

where $[\mu u > q] = \mathsf{T}$ if $\mu u > q$, and $= \mathsf{F}$ otherwise.

To show that this makes sense, we have to check several things. First, $\{\mu \mid \mu u > q\}$ is indeed Scott open in D_0.
Second, $\mathsf{pt}\, f$ and Ωf fit together, i.e., for all μ in $\mathsf{pt}\, D_0$ and U in ΩL_0, $\mu \models \Omega f U$ iff $\mathsf{pt}\, f\mu \models U$. It suffices to show this for generators, and $\mu \models \Omega f\langle u, q\rangle$ iff $\mu u > q$ iff $\mathsf{pt}\, f\mu \models \langle u, q\rangle$ holds indeed.
Third, Ωf and $\mathsf{pt}\, f\mu$ preserve the two relations: for rational zero, $\mu u > 0$ holds for every μ in D_0; for rational continuity, note that $\mu u > q$ iff there is a rational r with $\mu u > r > q$.

In the sequel, we show that $\mathsf{pt}\, f : \mathsf{pt}\, D_0 \to \mathsf{pt}\, L_0$ is an isomorphism. The proof becomes simpler if $\mathbf{I}$ is considered as the set of ideals $\mathsf{pt}\, \mathbf{Q}_0^1$ of the infosys $\mathbf{Q}_0^1$. The dcpo's $\mathbf{I}$ and $\mathsf{pt}\, \mathbf{Q}_0^1$ are isomorphic, and directed join in $\mathbf{I}$ corresponds to directed union in $\mathsf{pt}\, \mathbf{Q}_0^1$. The definition of $\mathsf{pt}\, f$ then becomes

$$\mathsf{pt}\, f\mu\, \langle u, q\rangle \;=\; [q \in \mu u].$$

We claim that the inverse of $\mathsf{pt}\, f$ is $\gamma : \mathsf{pt}\, L_0 \to \mathsf{pt}\, D_0$ with

$$\gamma p u = \{q \in \mathbf{Q}_0^1 \mid p\langle u, q\rangle = \mathsf{T}\}$$

for p in $\mathsf{pt}\, L_0$ and u in F. If r in $\gamma p u$ and $r > q$, then q is in $\gamma p u$ since $\langle u, r\rangle \leq \langle u, q\rangle$. If q is in $\gamma p u$, then there is $r > q$ with r in $\gamma p u$ by rational continuity. 0 is always in $\gamma p u$ by rational zero. These facts suffice to show that $\gamma p u$ is a point (a directed lower set) because $\mathbf{Q}_0^1$ is linearly ordered. Thus, $\gamma p u$ is in $\mathsf{pt}\, \mathbf{Q}_0^1$ as required.

Next, we show that γ is the inverse of $\mathsf{pt}\, f$. One direction is easy:

$$q \in \gamma(\mathsf{pt}\, f\mu)u \;\text{ iff }\; \mathsf{pt}\, f\mu\, \langle u, q\rangle = \mathsf{T} \;\text{ iff }\; q \in \mu u.$$

Conversely, we claim $\mathsf{pt}\, f(\gamma p) = p$ for all p in $\mathsf{pt}\, L_0$. It suffices to show that both sides coincide on generators. The statement $\mathsf{pt}\, f(\gamma p)\langle u, q\rangle = \mathsf{T}$ is by definition of $\mathsf{pt}\, f$ equivalent to $q \in \gamma p u$, which in turn is equivalent to $p\langle u, q\rangle = \mathsf{T}$.

3.3 Restricted Functions

We carry on by comparing various subdcpo's of D_0 defined by restrictions on the functions with sublocales of L_0 defined by additional relations. This neither affects the definitions of Ωf, $\mathsf{pt}\, f$, and γ, nor the proof that $\mathsf{pt}\, f$ and γ are inverse to each other. We only have to show in any case that Ωf and $\mathsf{pt}\, f\mu$ preserve the additional relations and that γp satisfies the restrictions. For all but modularity, these proofs are quite obvious and omitted.

Monotonicity: Restriction: If $u \le v$, then $\mu u \le \mu v$.
Relations: If $u \le v$, then $\langle u, q\rangle \le \langle v, q\rangle$.

Continuity: Restriction: If $\mathcal{U} \subseteq F$ is directed, then $\mu(\bigvee \mathcal{U}) = \bigvee_{u\in\mathcal{U}} \mu u$.
Relations: If $\mathcal{U} \subseteq F$ is directed, then $\langle \bigvee \mathcal{U}, q\rangle = \bigvee_{u\in\mathcal{U}} \langle u, q\rangle$.

Zero law: Restriction: $\mu(0) = 0$.
Relations: If $q \ne 0$, then $\langle 0, q\rangle = \mathsf{F}$.
By the rational zero relation, $\langle 0, 0\rangle = \mathsf{T}$ holds. All these relations look quite contradictory in presence of $\bigvee_{r>0}\langle 0, r\rangle = \langle 0, 0\rangle$, but remember $0 > 0$.

Modularity: Restriction: For all u and v, $\mu u + \mu v = \mu(u \vee v) + \mu(u \wedge v)$.
To find the corresponding relations, we have to check when $\mu u + \mu v > q$ holds. It is the case iff there are rational numbers r and s with $\mu u > r$, $\mu v > s$, and $r + s = q$. This suggests the following
Relations: For all rational numbers q with $0 \sqsubseteq q \sqsubset 2$,

$$\bigvee\{\langle u, r\rangle \wedge \langle v, s\rangle \mid r, s \in \mathbf{Q}_0^1,\ r+s=q\} \ = $$
$$\bigvee\{\langle u \vee v, r\rangle \wedge \langle u \wedge v, s\rangle \mid r, s \in \mathbf{Q}_0^1,\ r+s=q\}.$$

Ωf preserves the relations by the reasoning above. To show that γp is modular, we perform the following computation, where q always ranges over the rationals with $0 \sqsubseteq q \sqsubset 2$:

$$\begin{array}{lll} p\,(\bigvee\{\langle u, r\rangle \wedge \langle v, s\rangle \mid r, s \in \mathbf{Q}_0^1,\ r+s=q\}) = \mathsf{T} & & \\ \quad \text{iff} \quad \bigvee\{p\langle u, r\rangle \wedge p\langle v, s\rangle \mid r, s \in \mathbf{Q}_0^1,\ r+s=q\} = \mathsf{T} & & (p \text{ is homomorphism}) \\ \quad \text{iff} \quad \exists r, s \in \mathbf{Q}_0^1 : r+s=q,\ p\langle u, r\rangle = \mathsf{T},\ p\langle v, s\rangle = \mathsf{T} & & (p \text{ maps to } \{\mathsf{F}, \mathsf{T}\}) \\ \quad \text{iff} \quad \exists r, s \in \mathbf{Q}_0^1 : r+s=q,\ \gamma p u > r,\ \gamma p v > s & & \\ \quad \text{iff} \quad \gamma p u + \gamma p v > q. & & \end{array}$$

An analogous computation may be performed for $s = \gamma p(u \vee v) + \gamma p(u \wedge v)$, and the relations assure that $s > q$ iff $\gamma p u + \gamma p v > q$, whence $s = \gamma p u + \gamma p v$.

This completes the proof of Theorem 3.1. Summarizing, we obtain:

Theorem 3.2 For every topsys X, let $\mathcal{P}_D X$ be the dcpo $[\Omega X \overset{mod}{\to} \mathbf{I}]$, and $\mathcal{P}_L X$ be the locale, whose frame of opens $\Omega \mathcal{P}_L X$ is presented by:

Generators: $\langle u, q\rangle$ with u in ΩX and q in $\mathbf{Q}_0^1$,

Relations: Rational zero: $\langle u, 0\rangle = \mathsf{T}$ for all u in ΩX;

Rational continuity: $\bigvee_{r>q}\langle u, r\rangle = \langle u, q\rangle$ for all u in ΩX and q in $\mathbf{Q}_0^1$;

Continuity: If $\mathcal{U} \subseteq \Omega X$ is directed, then $\langle \bigvee \mathcal{U}, q\rangle = \bigvee_{u\in\mathcal{U}}\langle u, q\rangle$;

Zero law: If $q \ne 0$, then $\langle 0, q\rangle = 0$;

Modularity: For all rational q with $0 \sqsubseteq q \sqsubset 2$, and all u, v in ΩX,

$$\bigvee\{\langle u, r\rangle \wedge \langle v, s\rangle \mid r, s \in \mathbf{Q}_0^1,\ r+s=q\} \ =$$
$$\bigvee\{\langle u \vee v, r\rangle \wedge \langle u \wedge v, s\rangle \mid r, s \in \mathbf{Q}_0^1,\ r+s=q\}.$$

Then there is a continuous function $f : \mathcal{P}_D X \to \mathcal{P}_L X$ defined by

$$\Omega f\langle u, q\rangle \ = \ \{\mu \in [\Omega X \overset{mod}{\to} \mathbf{I}] \mid \mu u > q\}$$

such that $\mathsf{pt}\, f : \mathsf{pt}\, \mathcal{P}_D X \to \mathsf{pt}\, \mathcal{P}_L X$ is a poset isomorphism. □

4 The Power Construction on Infosyses

In this section, we define the probabilistic construction $\mathcal{P}_I$ on infosyses. Subsection 4.1 deals with the tokens of the power infosyses, and Subsection 4.2 with their order.

4.1 The Tokens of the Power Infosys

In this subsection, we define the tokens of the power infosys $\mathcal{P}_I D$ of some given infosys D. We then investigate two different ways to interpret such power tokens as functions from the frame ΩD to $\mathbf{I}$. These functions will be used in the next subsection to define the order on the power tokens.

The power tokens may be thought of as formal convex combinations $\sum_{i=1}^{n} r_i \cdot a_i$ with a_i in D and r_i in $\mathbf{Q}_0^1$ such that $\sum_{i=1}^{n} r_i < 1$. This is formalized in the following definition, which was already conjectured in [12].

Definition 4.1 The tokens of $\mathcal{P}_I D$ are finite bags A of pairs (r, a) from $\mathbf{Q}_0^1 \times D$ with $\sum\{\!|r \mid (r, a) \in A|\!\} < 1$.

Bags or *multi-sets* are similar to sets, but elements may occur more than once in them. Notationally, we use $\{\!|.|\!\}$ for bags in contrast to $\{.\}$ for sets, but keep on using '$\emptyset$', '$\in$', and '$\subseteq$' as for sets. The bag union of two bags is denoted by '+', e.g., $\{\!|1|\!\} + \{\!|1, 2|\!\} = \{\!|1, 1, 2|\!\}$. Bags will be manipulated by means of *bag abstractions*, whose meaning is analogous to that of set abstractions, but keeping different occurrences of the same element apart. Thus, for instance $\sum\{\!|r \mid (r, a) \in \{\!|(0.2, b), (0.2, b), (0.2, c)|\!\}|\!\} = \sum\{\!|0.2, 0.2, 0.2|\!\} = 0.6$.

The power infosys of a finite infosys need not be finite because of the rational numbers involved. On the other hand, countability of infosyses is preserved by the power infosys operation.

In the sequel, we need some notions and notations concerning power tokens.

Definition 4.2 For power tokens A in $\mathcal{P}_I D$, let $\mathsf{set}\, A = \{a \mid (r, a) \in A\}$ be the *set* of D-tokens occurring in A, and $\mathsf{num}\, A = \{\!|r \mid (r, a) \in A|\!\}$ be the *bag* of numbers in A.

For power tokens A in $\mathcal{P}_I D$ and opens u in ΩD, let $A_p u = \sum\{\!|r \mid (r,a) \in A,\ a \in u|\!\}$ and $A_o u = \sum\{\!|r \mid (r,a) \in A,\ \sharp a \subseteq u|\!\}$.

We use the indices p and o to indicate that in the first case, the token a is considered as a *point* of the open u, whereas in the latter case, it codes for the *open* $\sharp a$. If the infosys D is reflexive, i.e., in the algebraic case, the conditions $a \in u$ and $\sharp a \subseteq u$ are equivalent, whence $A_p = A_o$ for all power tokens A. This is not valid in the general case.

In the sequel, we look at the elementary properties of the two functional interpretations $A_p, A_o : \Omega D \to \mathbf{Q}_0^1 \subseteq \mathbf{I}$.

Proposition 4.3 For all A in $\mathcal{P}_I D$:

(1) For all u in ΩD: $0 < A_p u \sqsubseteq A_o u \sqsubseteq \sum \mathsf{num}\, A < 1$.

(2) A_p is continuous, and A_o is monotonic.

(3) A_p satisfies the zero law: $A_p \emptyset = 0$.

(4) For all u in ΩD: $\emptyset_p u = \emptyset_o u = 0$.

Unfortunately, neither A_p nor A_u can be expected to be modular. Since joins in ΩD are unions, '$a \in u \vee v$ iff $a \in u$ or $a \in v$' holds, but since meets are in general not intersections, '$a \in u \wedge v$ iff $a \in u$ and $a \in v$' does not hold in general. Dually, '$\sharp a \subseteq u \wedge v$ iff $\sharp a \subseteq u$ and $\sharp a \subseteq v$' holds by the very nature of meets, but since $\sharp a$ is not a single token, '$\sharp a \subseteq u \vee v$ iff $\sharp a \subseteq u$ or $\sharp a \subseteq v$' does not hold in general. Fortunately, A_p and A_o satisfy some kind of modularity in co-operation.

Proposition 4.4
For all A in $\mathcal{P}_I D$ and u, v in ΩD, $A_p u + A_p v \sqsubseteq A_p(u \vee v) + A_o(u \wedge v) \sqsubseteq A_o u + A_o v$.

Proof: For a in D and u in ΩD, let $[a \in u]$ be 1 or 0 depending on if a is in u or not. Then $A_p u = \sum\{\!|r \cdot [a \in u] \mid (r,a) \in A|\!\}$, and $A_o u = \sum\{\!|r \cdot [\sharp a \subseteq u] \mid (r,a) \in A|\!\}$ with an analogous notation. Thus, the statement of the proposition can be derived from the more basic statement

$$[a \in u] + [a \in v] \sqsubseteq [a \in u \vee v] + [\sharp a \subseteq u \wedge v] \sqsubseteq [\sharp a \subseteq u] + [\sharp a \subseteq v],$$

which can be shown by case analysis. □

4.2 The Order of the Power Infosys

So far, we only defined the tokens of $\mathcal{P}_I D$. For the order, we shall establish several equivalent definitions in this subsection.

Definition 4.5 For two functions f and g from ΩD to $\mathbf{I}$, we define $f < g$ iff for all u in ΩD, $fu < gu$ holds. For two tokens A and B in $\mathcal{P}_I D$, we define $A < B$ iff $A_o < B_p$ holds.

So far, the token order is not quite effective because of the potentially infinite quantification over all opens. We shall soon derive effective characterizations.

The definition above has the virtue that transitivity can be shown easily: from $A < B$ and $B < C$, $A_o u < B_p u \sqsubseteq B_o u < C_p u$ for all opens u, whence $A < C$. Also, $\emptyset$ is readily seen to be the least power token: $\emptyset_o u = 0 < B_p u$ holds for all opens u. For interpolation however, some hard work is needed. Before doing it, we derive two effective characterizations.

Proposition 4.6 For a token A in $\mathcal{P}_I D$ and a monotonic function $\mu : \Omega D \to \mathbf{I}$, the following three statements are equivalent:

(1) $A_o < \mu$, i.e., for all u in ΩD, $A_o u < \mu u$ holds.

(2) $A_o u < \mu u$ holds for all u in $\mathcal{U}(A) = \{\uparrow \mathsf{set}\, S \mid S \subseteq A\}$.

(3) For all subbags S of A, $\sum \mathsf{num}\, S < \mu(\uparrow \mathsf{set}\, S)$ holds.

In (3), one may also quantify 'for all non-empty subbags'.

Notice how the universal quantification of (1) is reduced to the finite quantifications in (2) and (3).

Proof:

$(1) \Rightarrow (2)$: Trivial; $\mathcal{U}(A)$ is a subset of ΩD.

$(2) \Rightarrow (3)$: Let $u = \uparrow \mathsf{set}\, S$, which is in $\mathcal{U}(A)$. If (r, a) is in S, then a is in $\mathsf{set}\, S$, whence $\sharp a \subseteq u$. Thus,

$\sum \mathsf{num}\, S \;=\; \sum \{\!| r \mid (r, a) \in S |\!\} \;\sqsubseteq\; \sum \{\!| r \mid (r, a) \in A,\, \sharp a \subseteq u |\!\} \;=\; A_o u \;<\; \mu u.$

$(3) \Rightarrow (1)$: For given u, let $S = \{\!| (r, a) \in A \mid \sharp a \subseteq u |\!\}$. Then $\uparrow \mathsf{set}\, S \subseteq u$, whence with monotonicity of μ, $A_o u = \sum \mathsf{num}\, S < \mu(\uparrow \mathsf{set}\, S) \sqsubseteq \mu u$.

The empty subbag need not be included in (3), since $\sum \mathsf{num}\, \emptyset = 0 < s$ always holds.□

The proposition applies in particular to the case $\mu = B_p$ needed for $A < B$:

Corollary 4.7

For two tokens A and B in $\mathcal{P}_I D$, $B > A$ iff for all $S \subseteq A$, $B_p(\uparrow \mathsf{set}\, S) > \sum \mathsf{num}\, S$.

The corollary shows that the order on $\mathcal{P}_I D$ is decidable, if the order on D is decidable: $A < B$ can be checked by performing a finite number of comparisons of rational numbers

$\sum \mathsf{num}\, S = \sum \{\!| r \mid (r, a) \in S |\!\}$ and

$B_p(\uparrow \mathsf{set}\, S) = \sum \{\!| s \mid (s, b) \in B,\, \exists (r, a) \in S : b > a |\!\}$,

which can be computed with a finite number of comparisons in D and additions of rationals.

The intuitions of Vickers in [12] suggest a quite different definition of the order on the tokens. A token A is below a token B iff every element (r, a) can be split into some (r_i, a) with $\sum r_i = r$, then grown by enlarging both the rational number and the ground token, then recombined again to obtain some (not necessarily all) of the elements of B. A formalization of these intuitions is provided by Lemma 4.13 of [6], which handles the more concrete problem of comparing convex combinations of point evaluations. This leads to the following theorem:

Theorem 4.8 Let $A = \{\!|(r_1, a_1), \ldots, (r_n, a_n)|\!\}$ and $B = \{\!|(s_1, b_1), \ldots, (s_m, b_m)|\!\}$ be two power tokens with a fixed order to enumerate their elements, and let $I = \{1, \ldots, n\}$ and $J = \{1, \ldots, m\}$. Then the following are equivalent:

(1) $A < B$, i.e., for all opens u, $A_o u < B_p u$;

(2) there are numbers t_{ij} in $\mathbf{I}$ for every i in I and j in J, such that $t_{ij} \neq 0$ implies $a_i < b_j$, $\sum_{j \in J} t_{ij} = r_i$ for every i in I, and $\sum_{i \in I} t_{ij} < s_j$ for every j in J.

(3) as (2), but the t_{ij} are in $\mathbf{Q}_0^1$, and $\sum_{j \in J} t_{ij} > r_i$.

Proof:

$(1) \Rightarrow (2)$: This is essentially the proof of the Splitting Lemma 4.10 of [6] or Lemma 9.2 of [7]. It applies the Max-Flow Min-Cut Theorem 5.1 of [2]. In our case, it is applied to a graph with nodes $\perp$ (source), $1, \ldots, n, 1', \ldots, m'$, and $\top$ (sink), and edges from $\perp$ to i with capacities r_i, from i to j' with capacities 1 if $a_i < b_j$ and 0 otherwise, and from j' to $\top$ with capacities $\rho \cdot s_j$, where ρ is a previously chosen rational number with $\sum \mathsf{num}\, S < \rho \cdot B_p(\uparrow\mathsf{set}\, S)$ for all subbags S of A. The remainder of the proof is in analogy to [6, 7] and thus omitted.

$(2) \Rightarrow (3)$: Choose a (rational) $0 \neq \rho < 1$ with still $\sum_{i \in I} t_{ij} < \rho \cdot s_j$ for all j in J. For every i in I and j in J, choose a rational t'_{ij} with $t_{ij} < t'_{ij} < t_{ij} \cdot \rho^{-1}$. The numbers t'_{ij} do the job for (3).

$(3) \Rightarrow (1)$: We use Cor. 4.7 to prove $A < B$. Let S be a subbag of A, I' a corresponding subset of I, and $J' = \{j \in J \mid b_j \in \uparrow\mathsf{set}\, S\}$. We start with

$$\sum \mathsf{num}\, S = \sum_{i \in I'} r_i < \sum_{i \in I'} \sum_{j \in J} t_{ij}$$

We only need to sum those t_{ij} with $t_{ij} \neq 0$. For these, $a_i < b_j$ holds. Thus it suffices to consider those j with $b_j \in \uparrow\mathsf{set}\, S$, and we may continue

$$\cdots \sqsubseteq \sum_{i \in I'} \sum_{j \in J'} t_{ij} \sqsubseteq \sum_{j \in J'} \sum_{i \in I} t_{ij} < \sum_{j \in J'} s_j = B_p(\uparrow\mathsf{set}\, S)$$

□

The last statement of Theorem 4.8 enables us to prove interpolation. Let A and B as in the theorem with $A < B$, and let t_{ij} be the numbers of its last statement. For every i in I and j in J with $a_i < b_j$, we choose a ground token c_{ij} with $a_i < c_{ij} < b_j$, and construct the bag C of pairs (t_{ij}, c_{ij}). Then $A < C < B$ holds, as is easily verified using parts (2) or (3) of Theorem 4.8.

5 Comparing Power Infosys and Power Locale

In this section, we show — as announced in the beginning — that the frames of opens of the power infosys and the power locale are isomorphic.

5.1 The Two Frames

In this subsection, we present the two frames to be compared. Let D be some fixed infosys. The frame F_1 is $\Omega\mathcal{P}_I D$, which can be presented as shown in Section 2.2. The frame F_2 is $\Omega\mathcal{P}_L[D]$. It is defined in terms of $\Omega[D]$, which is isomorphic to ΩD. Thus, we arrive at the following presentations:

- Generators for F_1: $\sharp A$ for A in $\mathcal{P}_I D$.
- Relations:
  (1) Monotonicity: If $A < B$, then $\sharp A \geq \sharp B$;
  (2) All tokens: $\bigvee_{A \in \mathcal{P}_I D} \sharp A = \mathsf{T}$;
  (3) Meets: $\sharp A \wedge \sharp B \leq \bigvee_{C > A,B} \sharp C$.
  With monotonicity, one obtains '=' in (3). There is also the special case $A = B$ of this, which yields
  (4) $\sharp A = \bigvee_{C > A} \sharp C$.
- Generators for F_2: $\langle u, q\rangle$ with u in ΩD and q in $\mathbf{Q}_0^1$,
- Relations:
  (1) $\langle u, 0\rangle = \mathsf{T}$;
  (2) $\bigvee_{r>q} \langle u, r\rangle = \langle u, q\rangle$;
  (3) If $\mathcal{U} \subseteq \Omega X$ is directed, then $\langle \bigvee \mathcal{U}, q\rangle = \bigvee_{u \in \mathcal{U}} \langle u, q\rangle$;
  (4) If $q \neq 0$, then $\langle 0, q\rangle = 0$;
  (5) For all rational q with $0 \sqsubseteq q \sqsubset 2$, $\bigvee\{\langle u, r\rangle \wedge \langle v, s\rangle \mid r, s \in \mathbf{Q}_0^1,\ r + s = q\} = \bigvee\{\langle u \vee v, r\rangle \wedge \langle u \wedge v, s\rangle \mid r, s \in \mathbf{Q}_0^1,\ r + s = q\}$.

In case of F_2, relation (1) makes the cases $q = 0$ of (2), (3), and (5) redundant; we may assume $q \neq 0$ in (2) through (5).

5.2 A Homomorphism from F_2 to F_1

In view of the 'intended meaning' of the generators $\langle u, q\rangle$, we define $\varphi : F_2 \to F_1$ by

$$\varphi\langle u, q\rangle = \bigvee\{\sharp A \mid A_p u > q\}$$

We have to show that this definition preserves the relations of F_2.

(1) $\varphi\langle u, 0\rangle = \bigvee\{\sharp A \mid A_p u > 0\} = \bigvee\{\sharp A \mid A \in \mathcal{P}_I D\} = \mathsf{T}$ by relation (2) of F_1.

(2) $\bigvee_{r>q} \varphi\langle u, r\rangle = \bigvee_{r>q} \bigvee\{\sharp A \mid A_p u > r\} = \bigvee\{\sharp A \mid \exists r > q : A_p u > r\}$. By transitivity and interpolation in $\mathbf{Q}_0^1$, the latter equals $\bigvee\{\sharp A \mid A_p u > q\} = \varphi\langle u, q\rangle$.

(3) Here, we need continuity of A_p (Prop. 4.3 (2)). Let $(u_i)_{i\in I}$ be a directed family in ΩD.

$$\begin{aligned}
\varphi\langle \bigvee_{i\in I} u_i, q\rangle &= \bigvee\{\sharp A \mid A_p(\bigvee_{i\in I} u_i) > q\} \\
&= \bigvee\{\sharp A \mid \exists i \in I : A_p u_i > q\} \\
&= \bigvee_{i\in I} \bigvee\{\sharp A \mid A_p u_i > q\} \\
&= \bigvee_{i\in I} \varphi\langle u_i, q\rangle
\end{aligned}$$

(4) By Prop. 4.3 (3), $A_p 0 = 0$ holds. Thus, $\varphi\langle 0, q\rangle = \bigvee\{\sharp A \mid A_p 0 > q\}$ is an empty join if $q \neq 0$.

(5) For modularity, we compute

$$\begin{aligned}
\bigvee_{r+s=q} \varphi\langle u, r\rangle \wedge \varphi\langle v, s\rangle &= \bigvee_{r+s=q}(\bigvee\{\sharp A \mid A_p u > r\}) \wedge (\bigvee\{\sharp B \mid B_p v > s\}) \\
&= \bigvee\{\sharp A \wedge \sharp B \mid r+s=q,\ A_p u > r,\ B_p v > s\}
\end{aligned}$$

With the '=' version of relation (3) of F_1, we obtain

$$\bigvee\{\sharp C \mid C > A, B,\ A_p u + B_p v > q\}.$$

Applying relation (4) of F_1 yields

$$\bigvee\{\sharp D \mid D > C > A, B,\ A_p u + B_p v > q\}.$$

In this situation, $A_p u + B_p v > q$ implies $C_p u + C_p v > q$. Conversely, if $D > C$ and $C_p u + C_p v > q$, then we can interpolate a new C' between D and C and let $A = B = C$. Thus, the join above equals

$$x = \bigvee\{\sharp D \mid D > C,\ C_p u + C_p v > q\}.$$

Analogously, the other side of the modularity relation becomes

$$y = \bigvee\{\sharp D \mid D > C,\ C_p(u \vee v) + C_p(u \wedge v) > q\}.$$

We have to show $x = y$. For this, we use Prop. 4.4. It states

$$C_p u + C_p v \sqsubseteq C_p(u \vee v) + C_o(u \wedge v) \sqsubseteq C_o u + C_o v.$$

For $x \leq y$, we interpolate $D > C$ to $D > C' > C$. Then $C'_p > C_o \sqsupseteq C_p$, whence

$$C'_p(u \vee v) + C'_p(u \wedge v) \sqsupseteq C_p(u \vee v) + C_o(u \wedge v) \sqsupseteq C_p u + C_p v > q.$$

The other relation $x \geq y$ is shown analogously.

Now, we have shown that φ preserves all relations of F_2. Thus, it extends to a frame homomorphism $\varphi : F_2 \to F_1$.

5.3 A Homomorphism from F_1 to F_2

To establish a frame homomorphism $\psi : F_1 \to F_2$, we have to specify $\psi(\sharp A)$ for A in $\mathcal{P}_I D$. Here, we refer to Prop. 4.6, which says $A_o < \mu$ iff for all subbags S of A, $\sum \mathsf{num}\, S < \mu(\Uparrow \mathsf{set}\, S)$. This motivates the following definition:

$$\psi(\sharp A) = \bigwedge_{S \subseteq A} \gamma S \quad \text{where} \quad \gamma S = \langle \Uparrow \mathsf{set}\, S, \textstyle\sum \mathsf{num}\, S\rangle,$$

which involves a finite meet by finiteness of A. We have to prove that ψ preserves the relations of F_1.

For monotonicity, we have to show that $A < B$ implies $\psi(\sharp A) \geq \psi(\sharp B)$, or $\bigwedge_{S \subseteq A} \gamma S \geq \bigwedge_{T \subseteq B} \gamma T$. To prove this, it is sufficient to show that for each $S \subseteq A$, there is $T \subseteq B$ with $\gamma S \geq \gamma T$. For $S \subseteq A$, let $T = \{\!|(r, b) \in B \mid b \in {\uparrow}\mathsf{set}\, S|\!\}$. Then $\mathsf{set}\, T \subseteq {\uparrow}\mathsf{set}\, S$, whence ${\uparrow}\mathsf{set}\, T \leq {\uparrow}\mathsf{set}\, S$. Moreover, $\sum \mathsf{num}\, T = B_p({\uparrow}\mathsf{set}\, S) > \sum \mathsf{num}\, S$ holds by $B > A$ and Cor. 4.7. Since the generators of F_2 are monotonic in the first and anti-monotonic in the second argument, these facts imply $\gamma T \leq \gamma S$.

For the all-tokens relation, we have to show $\bigvee\{\psi(\sharp A) \mid A \in \mathcal{P}_I D\} = \mathsf{T}$. This equality holds, since $\psi(\sharp\emptyset) = \gamma\emptyset = \langle\emptyset, 0\rangle = \mathsf{T}$ by relation (1) of F_2.

Finally, for the meets relation, we have to show $\psi(\sharp A) \wedge \psi(\sharp B) \leq \bigvee_{C > A,B} \psi(\sharp C)$. This turns out to be difficult and is postponed until later. For the moment, we assume that it has been shown, so that $\psi : F_1 \to F_2$ is a well defined frame homomorphism.

5.4 The Homomorphisms are Inverse

Now, we show that the two frame homomorphisms $\varphi : F_2 \to F_1$ and $\psi : F_1 \to F_2$ are inverse to each other. It suffices to apply φ and ψ to generators.

First, we show $\varphi(\psi(\sharp A)) = \sharp A$ for all A in $\mathcal{P}_I D$.

$$\begin{array}{rcl} \varphi(\psi(\sharp A)) & = & \bigwedge_{S \subseteq A} \varphi\langle {\uparrow}\mathsf{set}\, S, \sum \mathsf{num}\, S\rangle \\ & = & \bigwedge_{S \subseteq A} \bigvee\{\sharp B \mid B_p({\uparrow}\mathsf{set}\, S) > \sum \mathsf{num}\, S\} \end{array}$$

We have to show that this equals $\sharp A$. For this, we do not use the presentation of F_1 by generators and relations, but the concrete form of F_1 as consisting of the 'open' subsets of $\mathcal{P}_I D$, with join being union, meet being intersection followed by '$\uparrow$', and $\sharp A = \{B \mid B > A\}$.

First, let D be in the meet above. Then $D > C$ for some C such that for all $S \subseteq A$, $C > B^S$ for some B^S with $B^S_p({\uparrow}\mathsf{set}\, S) > \sum \mathsf{num}\, S$. By $C > B^S$, for all $S \subseteq A$, $C_p({\uparrow}\mathsf{set}\, S) > \sum \mathsf{num}\, S$ holds. By Cor. 4.7, this just means $C > A$. Thus, $D > C > A$, whence D is in $\sharp A$.

Conversely, let D be in $\sharp A$. By interpolation, let $D > C > B > A$. By Cor. 4.7, $B > A$ means $B_p(\mathsf{set}\, S) > \sum \mathsf{num}\, S$ for all $S \subseteq A$. By $C > B$, C is in $\bigvee\{\sharp B \mid B_p({\uparrow}\mathsf{set}\, S) > \sum \mathsf{num}\, S\}$ for all $S \subseteq A$. Thus, C is in $\bigcap_{S \subseteq A} \ldots$, and because of $D > C$, D is in $\bigwedge_{S \subseteq A} \cdots$.

Next, we compute $\psi(\varphi\langle u, q\rangle) = \psi(\bigvee\{\sharp A \mid A_p u > q\}) = \bigvee\{\bigwedge_{S \subseteq A} \gamma S \mid A_p u > q\}$.

First, we show that this is below $\langle u, q\rangle$. For this, it suffices to show that for every A with $A_p u > q$, there is some $S \subseteq A$ with $\gamma S \leq \langle u, q\rangle$. Given A with $A_p u > q$, let $S = \{\!|(r, a) \in A \mid a \in u|\!\}$. Then $\mathsf{set}\, S \subseteq u$, whence ${\uparrow}\mathsf{set}\, S \leq u$; and $\sum \mathsf{num}\, S = A_p u > q$. By monotonicity of the generators of F_2 in the first argument and anti-monotonicity in the second, $\gamma S = \langle {\uparrow}\mathsf{set}\, S, \sum \mathsf{num}\, S\rangle \leq \langle u, q\rangle$ holds.

The final thing to show is $\psi(\varphi\langle u, q\rangle) \geq \langle u, q\rangle$. Again, this turns out to be difficult, and we postpone it until later. Except for the two postponed statements, the proof of $F_1 \cong F_2$ is now completed.

5.5 The Missing Relations

Let us now analyze the two relations, whose proofs were postponed. The meet relation is

$$\psi(\sharp A) \wedge \psi(\sharp B) \leq \bigvee_{C>A,B} \psi(\sharp C)$$

or

$$\bigwedge_{R\subseteq A} \gamma R \wedge \bigwedge_{S\subseteq B} \gamma S \leq \bigvee\{ \bigwedge_{T\subseteq C} \gamma T \mid C > A, B\}.$$

By Cor. 4.7, the condition $C > A$ may be replaced by $C_p(\Uparrow\mathsf{set}\, R) > \sum \mathsf{num}\, R$ for all $R \subseteq A$, and analogously for $C > B$.

After renaming the bound variables, the missing part of the inverse relations is

$$\langle u, q\rangle \leq \bigvee\{ \bigwedge_{T\subseteq C} \gamma T \mid C_p u > q\}.$$

Written this way, the two postponed relations reveal a common structure. On the left, there is a finite meet of generators $\langle u_i, q_i\rangle$, and the join on the right is quantified over those C with $C_p u_i > q_i$ for all i. Thus, both relations may be derived from a more general *critical lemma*:

Lemma 5.1 For every infosys D, for every finite index set I and all families $(u_i)_{i\in I}$ in ΩD and $(q_i)_{i\in I}$ in $\mathbf{Q}_0^1$,

$$\bigwedge_{i\in I}\langle u_i, q_i\rangle \leq \bigvee\{ \bigwedge_{S\subseteq C} \langle\Uparrow\mathsf{set}\, S, \sum \mathsf{num}\, S\rangle \mid C \in \mathcal{P}_I D,\ C_p u_i > q_i\ \forall i \in I\}$$

holds in $F_2 = \Omega\mathcal{P}_L[D]$.

Because of the complexity of our proof of the critical lemma, we devote the whole next section to it. Summarizing the results of this section, we have shown — modulo a proof of the critical lemma — the following theorem:

Theorem 5.2 For every infosys D, the frames $\Omega\mathcal{P}_I D$ and $\Omega\mathcal{P}_L[D]$ are isomorphic.

6 Proof of the Critical Lemma

In this section, the critical lemma will be proved. The very basic idea of the proof is taken from the proof of Lemma 5.3 in [6] or Lemma 8.3 in [7], which state a vaguely similar property involving points and concrete open sets. The added difficulty in our proof is due to the fact that we have to work pointless, because the spatiality of the locale of F_2 is a priori unknown. (A posteriori, it follows from Theorem 5.2.)

6.1 Outline of the Proof

The above mentioned proof of Jones employs so-called *crescents*, which are set differences of two concrete open sets. To mimic this, we set up a new frame F_3 in Subsection 6.3 with generators $\langle u, v, q\rangle$, whose intended spatial meaning is the set of all μ with $\mu(u \setminus v) > q$, or $\mu u - \mu(u \wedge v) > q$. With an appropriate choice of relations for F_3, the obvious assignment $\langle u, q\rangle \mapsto \langle u, 0, q\rangle$ becomes a frame homomorphism $\alpha : F_2 \to F_3$.

Let the critical lemma be $L \leq R$ in F_2. The added possibilities of F_3 allow proving $\alpha L \leq \alpha R$ in F_3 in vague analogy to Jones's proof of the Lemmas mentioned above (Subsection 6.4). After having managed this, we only have to show that α is an order embedding; then $\alpha L \leq \alpha R$ in F_3 implies $L \leq R$ in F_2.

By the Corollary in paragraph II, 2.6 of [5, page 53], F_2 can be embedded by a frame homomorphism η into a complete Boolean algebra G. Given η and G, we are able to define a (non-trivial) frame homomorphism $\beta : F_3 \to G$ with $\beta \circ \alpha = \eta$ (Subsection 6.6). Then α is an embedding, since η is an embedding.

The remaining subsections 6.2 and 6.5 introduce auxiliary notation to master the complexities of the proofs.

6.2 Real Valued Functions I

Before we present F_3 and prove the α-image of the critical lemma in F_3, we introduce some pieces of auxiliary notation, which allows for replacing complex join-and-meet expressions by simple arithmetically looking expressions with familiar laws.

Let F be an arbitrary frame and X the locale with $\Omega X = F$. Then continuous functions $f : X \to \mathbf{R}_0^\infty$ with values in the positive reals (including 0 and ∞) correspond to frame homomorphisms $\Omega f : \Omega \mathbf{R}_0^\infty \to F$. Analogously to the infosys $\mathbf{Q}_0^1$ for $\mathbf{I}$, an infosys for $\mathbf{R}_0^\infty$ is given by the positive rationals $\mathbf{Q}_0^+$ (including 0, but not ∞), with order '$<$' which is the usual order '$\sqsubset$' plus the one additional relationship $0 < 0$. Like $\Omega \mathbf{Q}_0^1$, the frame $\Omega \mathbf{R}_0^\infty \cong \Omega \mathbf{Q}_0^+$ can be presented by generators $\sharp q$ for q in $\mathbf{Q}_0^+$ and the two relations of rational zero and rational continuity. Thus, frame homomorphisms from $\Omega \mathbf{R}_0^\infty$ to F correspond to functions defined on the generators $\sharp q$ satisfying the two relations.

To obtain a real notational benefit, we now forget about X, write f for Ωf, and q for $\sharp q$. Thus, we define:

Definition 6.1 For every frame F, let $\mathcal{R}F$ be the set of all functions f from $\mathbf{Q}_0^+$ to F satisfying the two properties

(1) $f0 = \mathsf{T}$;

(2) $\bigvee_{r>q} fr = fq$ (and thus $r > q$ implies $fr \leq fq$).

A function f is *bounded* iff there is q with $fq = \mathsf{F}$.

We now define several operations of $\mathcal{R}F$. For every operation, satisfaction of the two conditions for $\mathcal{R}F$ has to be shown. We shall omit most of these proofs.

The members of $\mathcal{R}F$ are ordered pointwise: $f \le g$ iff for all q, $fq \le gq$ holds. Obviously, if $f \le g$ and g is bounded, then f is bounded.
Non-empty joins may be defined pointwise: $(\bigvee_{i \in I} f_i)q = \bigvee_{i \in I}(f_i q)$.
The empty join 0 or F is given by $0(0) = \mathsf{T}$ and $0(q) = \mathsf{F}$ for $q \ne 0$. It is not given pointwise, since $0(0)$ is T instead of F.
Finite meets (including the empty meet T) are given pointwise: $(\bigwedge_{i \in I} f_i)q = \bigwedge_{i \in I}(f_i q)$. With these joins and meets, $\mathcal{R}F$ becomes a frame.
We define a special member 1 of $\mathcal{R}F$ by $1(q) = \mathsf{T}$ for $q \sqsubset 1$ and F otherwise. In spatial intuitions, this corresponds to the function with constant real value 1. Thus, functions from the locale of F to $\mathbf{I}$ exactly correspond to those f in $\mathcal{R}F$ with $f \le 1$, or equivalently $f1 = \mathsf{F}$.

For two reals x and y, $x + y > q$ holds iff $x > r$ and $y > s$ for some rationals r and s with $r + s = q$. Thus, we define

$$(f+g)q \;=\; \bigvee_{r,s:\, r+s=q} fr \wedge gs.$$

(This is the same kind of expression as used in the modularity relation.) It is easy to show that $(\mathcal{R}F, +, 0)$ forms a commutative monoid.
Because they are given pointwise, addition preserves non-empty joins: $f + \bigvee_{i \in I} g_i = \bigvee_{i \in I}(f + g_i)$. In particular, it is continuous and thus monotonic. By monotonicity, $f = f + 0 \le f + g$ always holds, whence $f \vee g \le f + g$. Addition does not preserve the empty join, since $f + 0 = f$ holds instead of $f + 0 = 0$.
Because of their pointwise nature, finite joins and non-empty meets of bounded functions are bounded. Also the sum of two bounded functions is bounded: if $fr_0 = \mathsf{F}$ and $gs_0 = \mathsf{F}$, then $(f+g)(r_0 + s_0) = \mathsf{F}$, since $r + s = r_0 + s_0$ implies $r \sqsupseteq r_0$ or $s \sqsupseteq s_0$.

The benefit of the new notations becomes obvious when we consider the frame F_2. For every u in ΩD, the assignment $q \mapsto \langle u, q\rangle$ for $q \sqsubset 1$ and $q \mapsto \mathsf{F}$ for $q \sqsupseteq 1$ becomes a function in $\mathcal{R}F_2$ by the relations of rational zero and rational continuity. We call this function $[u]$. Abstracting out the q's in the generators, we reach at a formal 'presentation' of $\mathcal{R}F_2$ with 'generators' $[u]$ for u in ΩD and 'relations'

- Bounded by one: $[u] \le 1$;
- Continuity: for directed families $(u_i)_{i \in I}$, $[\bigvee_{i \in I} u_i] = \bigvee_{i \in I}[u_i]$;
- Zero law: $[0] = 0$;
- Modularity: $[u \vee v] + [u \wedge v] = [u] + [v]$.

We do not claim that this 'presentation' presents anything directly, although it could be probably achieved by some more work, but we consider it as notational shorthand for our presentation of F_2. Intuitively and informally, $[u]$ can be best understood as the size or area of the 'region' u; the relations above then get a quite appealing interpretation.

6.3 The Frame F_3

We now present the frame F_3 by giving a 'presentation' of $\mathcal{R}F_3$: it has generators $[u, v]$ for every u, v in ΩD, whose intuitive meaning is the size or area of the 'region' $u \setminus v$, and relations

(1) Bounded by one: $[u, v] \le 1$;

(RC) Restricted Continuity: for directed families $(u_i)_{i \in I}$ and v with $v \le u_i$ for all i, $[\bigvee_{i \in I} u_i, v] = \bigvee_{i \in I} [u_i, v]$;

(0) Zero law: $[0, 0] = 0$;

($\wedge$) Meet law: $[u, v] = [u, u \wedge v]$;

($\vee$) Join law: $[u, v] = [u \vee v, v]$;

(S) Split law: if $u \ge v \ge w$, then $[u, w] = [u, v] + [v, w]$.

Again, this should not be understood as an actual presentation of anything, but as shorthand for a presentation of F_3, which results by adding q's. Thus, the generators for F_3 are $\langle u, v, q \rangle = [u, v](q)$, and the split law for instance becomes: if $u \ge v \ge w$, then for all q, $\langle u, w, q \rangle = \bigvee_{r,s:\, r+s=q} \langle u, v, r \rangle \wedge \langle v, w, s \rangle$.

A frame homomorphism $\alpha : F_2 \to F_3$ is specified by $\alpha\langle u, q \rangle = \langle u, 0, q \rangle$, or in shorthand $\alpha[u] = [u, 0]$. (Actually, it should be $\alpha \circ [u] = [u, 0]$, but we drop '$\circ$'.) Preservation of the relations of F_2 is immediate except for modularity. In the sequel, we shall take the relations for $\mathcal{R}F_3$ as axioms and derive a host of conclusions, including modularity, needed for the proof of the α-image of the critical lemma.

(C) Full Continuity: for directed families $(u_i)_{i \in I}$ and arbitrary v, $[\bigvee_{i \in I} u_i, v] = \bigvee_{i \in I} [u_i, v]$.

Proof: Applying the join law on both sides yields $[\bigvee_{i \in I} u_i \vee v, v] = \bigvee_{i \in I} [u_i \vee v, v]$, which holds by restricted continuity (RC). □

(M1) Monotonicity: if $u \le u'$, then $[u, v] \le [u', v]$.

Proof: Directly from full continuity (C), or from the split law like (M2) below. □

(M2) Anti-Monotonicity: if $v \le v'$, then $[u, v] \ge [u, v']$.

Proof: $[u, v] \overset{\wedge}{=} [u, u \wedge v] \overset{S}{=} [u, u \wedge v'] + [u \wedge v', u \wedge v] \ge [u, u \wedge v'] \overset{\wedge}{=} [u, v']$. □

($\vee'$) Extended join law: if $v' \le v$, then $[u \vee v', v] = [u, v]$.

Proof: $[u, v] \overset{M1}{\le} [u \vee v', v] \overset{M1}{\le} [u \vee v, v] \overset{\vee}{=} [u, v]$. □

($\wedge'$) Extended meet law: if $u' \ge u$, then $[u, u' \wedge v] = [u, v]$.

Proof: $[u, v] \overset{M2}{\le} [u, u' \wedge v] \overset{M2}{\le} [u, u \wedge v] \overset{\wedge}{=} [u, v]$. □

(E) Extinction: if $u \le v$, then $[u, v] = 0$.

Proof: $[u, v] = [0 \vee u, v] \overset{\vee'}{=} [0, v] \overset{\wedge}{=} [0, 0] \overset{0}{=} 0$. □

The next property is the α-image of modularity.

(Mod) $[u \vee v, 0] + [u \wedge v, 0] = [u, 0] + [v, 0]$.

Proof:

$$\begin{array}{rcl} [u \vee v, 0] + [u \wedge v, 0] & \overset{S}{=} & [u \vee v, v] + [v, 0] + [u \wedge v, 0] \\ & \overset{\vee,\wedge}{=} & [u, u \wedge v] + [u \wedge v, 0] + [v, 0] \\ & \overset{S}{=} & [u, 0] + [v, 0] \qquad \square \end{array}$$

The following properties are auxiliary lemmas needed in the proof of the α-image of the critical lemma. The first statement shows how to partition $[u, v]$ into parts inside and outside some w.

(1) $[u, v] = [u \wedge w, v] + [u, v \vee w]$.

Proof: By $(\vee)$ and split, $[u, v] = [u \vee v, v] = [u \vee v, (u \wedge w) \vee v] + [(u \wedge w) \vee v, v]$. By $(\vee)$, the second summand equals $[u \wedge w, v]$. By $(\vee')$, the first becomes $[u, (u \wedge w) \vee v]$. By two applications of $(\wedge)$, this equals $[u, w \vee v]$. $\square$

Next, we generalize (1) to a finite number of w's.

(2) If W is a finite set of opens, then $[u, v] = \sum_{T \subseteq W} [u \wedge \bigwedge T, v \vee \bigvee (W \setminus T)]$.

Proof: The proof is performed by induction on $|W|$. For $W = \emptyset$, the sum just equals $[u, v]$. For $W \neq \emptyset$, let w be a member of W and $W' = W \setminus \{w\}$. By (1), $[u, v]$ is $[u \wedge w, v] + [u, v \vee w]$. By induction hypothesis, we obtain

$$[u \wedge w, v] = \sum_{T \subseteq W'} [u \wedge w \wedge \bigwedge T, v \vee \bigvee (W' \setminus T)]$$

Replacing T by $T \cup \{w\}$, this becomes

$$\sum_{T:\, w \in T \subseteq W} [u \wedge \bigwedge T, v \vee \bigvee (W \setminus T)]$$

The other summand is treated similarly:

$$\begin{array}{rcl} [u, v \vee w] & = & \sum_{T \subseteq W'} [u \wedge \bigwedge T, v \vee w \vee \bigvee (W' \setminus T)] \\ & = & \sum_{T:\, w \notin T \subseteq W} [u \wedge \bigwedge T, v \vee \bigvee (W \setminus T)] \qquad \square \end{array}$$

For the actual proof of the critical lemma, we need a slight variant of (2):

(3) Let U be a finite set of opens. For all u in U: $[u, 0] = \sum_{T:\, u \in T \subseteq U} [\bigwedge T, \bigvee (U \setminus T)]$.

Proof: Apply (2) to $[u, 0]$ with $W = U \setminus \{u\}$. $\square$

The next statement shows how to transform a join into a sum.

(4) $[u \vee u', v] = [u, v] + [u', u \vee v]$.

Proof: Applying (1) with $w = u$, we obtain $[u \vee u', v] = [(u \vee u') \wedge u, v] + [u \vee u', v \vee u]$. The first summand is $[u, v]$, and with $(\vee')$, the second becomes $[u', u \vee v]$. $\square$

The next step is the generalization of (4) to arbitrary finite joins.

(5) $[\bigvee_{i=1}^{n} u_i, v] = \sum_{i=1}^{n} [u_i, v \vee \bigvee_{j=1}^{i-1} u_j]$

Proof: For $n = 0$, the equation is $[0, v] = 0$, which is true by (E). For $n > 0$, we compute:

$$\begin{aligned} &\textstyle\sum_{i=1}^{n-1}[u_i, v \vee \bigvee_{j=1}^{i-1} u_j] + [u_n, v \vee \bigvee_{j=1}^{n-1} u_j] \\ &\overset{IH}{=} [\textstyle\bigvee_{i=1}^{n-1} u_i, v] + [u_n, v \vee \bigvee_{j=1}^{n-1} u_j] \\ &\overset{4}{=} [\textstyle\bigvee_{i=1}^{n} u_i, v] \end{aligned}$$ □

The next statement is another form of modularity.

(7) If $u_1 \geq v_1$ and $u_2 \geq v_2$, then $[u_1, v_1] + [u_2, v_2] = [u_1 \vee u_2, v_1 \vee v_2] + [u_1 \wedge u_2, v_1 \wedge v_2]$.

Proof: By (1) with $w = v_2$, $[u_1, v_1] = [u_1 \wedge v_2, v_1] + [u_1, v_1 \vee v_2]$.
By (1) with $w = u_1$, $[u_2, v_2] = [u_1 \wedge u_2, v_2] + [u_2, u_1 \vee v_2]$.
We show that the right hand side can be partitioned into the same four summands.
By (4), $[u_1 \vee u_2, v_1 \vee v_2] = [u_1, v_1 \vee v_2] + [u_2, u_1 \vee v_1 \vee v_2]$ holds. By $u_1 \geq v_1$, the second summand simplifies to $[u_2, u_1 \vee v_2]$.
By (1) with $w = v_2$, $[u_1 \wedge u_2, v_1 \wedge v_2] = [u_1 \wedge u_2 \wedge v_2, v_1 \wedge v_2] + [u_1 \wedge u_2, (v_1 \wedge v_2) \vee v_2]$.
By $u_2 \geq v_2$, the first summand simplifies to $[u_1 \wedge v_2, v_1 \wedge v_2]$, which by $(\wedge')$ equals $[u_1 \wedge v_2, v_1]$. □

Our next goal is to show that the sum of the areas of a finite number of disjoint regions is bounded by the area of a region that covers them all. To formalize disjointness in our framework, consider proper sets. The intersection of $A \setminus B$ and $A' \setminus B'$ is $(A \cap A') \setminus (B \cup B')$; it is empty iff $A \cap A' \subseteq B \cup B'$.

(8) Let $u_1, \ldots, u_n$ and $v_1, \ldots, v_n$ be opens with $u_i \wedge u_j \leq v_i \vee v_j$ for $i \neq j$ (disjointness condition). Then $\sum_{i=1}^{n}[u_i, v_i] \leq [\bigvee_{i=1}^{n} u_i, 0]$.

Proof: Applying $(\wedge)$, we may replace v_i by $u_i \wedge v_i$. The disjointness condition then still holds. Thus, we may assume without restriction $u_i \geq v_i$ for all i.

Before proving the statement by induction on n, we consider the effect of one application of (7) from left to right to two summands of $\sum[u_i, v_i]$. By simple computations, it can be checked that such a rewriting step keeps the number of summands and the join $\bigvee u_i$ invariant, and preserves the conditions $u_i \geq v_i$ and the disjointness condition.

For $n = 0$, $0 \leq [0, 0]$ holds. For the case of $n + 1$, consider $[u_0, v_0] + [u_1, v_1] + \cdots + [u_n, v_n]$. We rewrite this expression n times by (7), going from left to right:

$$\begin{aligned} &[u_0, v_0] + [u_1, v_1] + \cdots + [u_n, v_n] \\ &= [u_0 \wedge u_1, v_0 \wedge v_1] + [u_0 \vee u_1, v_0 \vee v_1] + [u_2, v_2] + \cdots + [u_n, v_n] \\ &= [u_0 \wedge u_1, v_0 \wedge v_1] + [(u_0 \vee u_1) \wedge u_2, (v_0 \vee v_1) \wedge v_2] \\ &\qquad + [u_0 \vee u_1 \vee u_2, v_0 \vee v_1 \vee v_2] + [u_3, v_3] + \cdots + [u_n, v_n] \\ &= \cdots \end{aligned}$$

The final outcome is $\sum_{i=1}^{n}[u_i', v_i'] + [u, v]$, where $u_i' = (\bigvee_{j=0}^{i-1} u_j) \wedge u_i$, $v_i' = (\bigvee_{j=0}^{i-1} v_j) \wedge v_i$, $u = \bigvee_{i=0}^{n} u_i$, and $v = \bigvee_{i=0}^{n} v_i$. As indicated above, the rewriting steps preserve the disjointness condition. Thus, the induction hypothesis may be used to show that the sum is bounded by $[u,v] + [\bigvee_{i=1}^{n} u'_i, 0]$. Using the disjointness condition, we may compute:

$$u'_i \;=\; \bigvee_{j=0}^{i-1} (u_j \wedge u_i) \;\le\; \bigvee_{j=0}^{i-1} (v_j \vee v_i) \;\le\; v$$

Thus,

$$[u,v] + [\bigvee_{i=1}^{n} u'_i, 0] \;\overset{M1}{\le}\; [u,v] + [v,0] \;\overset{S}{=}\; [u,0]$$

which is the required result. □

Next, we proof that summands as occurring in (3) satisfy the disjointness condition of (8).

(9) Let U be a finite set of opens, and let T vary over the subsets of U. The families $u^T = \bigwedge T$ and $v^T = \bigvee (U \setminus T)$ satisfy the disjointness condition of (8).

Proof: If $T \neq T'$, then without restriction, there is u in T with u not in T'. Then $u^T \wedge u^{T'} \le \bigwedge T \le u \le \bigvee (U \setminus T') \le v^T \vee v^{T'}$. □

For the last auxiliary statement, opens have to be considered as token sets, to which set difference and subset relation can be applied.

(10) $[u,v] = \bigvee_{F \subseteq_{\mathit{fin}} u \setminus v} [\uparrow F, v]$; this join is directed.

Proof: The join is directed by monotonicity of '$\uparrow$'. Thus, continuity (C) can be applied, and we have to show $[u,v] = [\bigvee_{F \subseteq_{\mathit{fin}} u \setminus v} \uparrow F, v]$. The relation '$\ge$' directly follows from monotonicity (M1). For '$\le$', we show $u \le v \vee \bigvee_{F \subseteq_{\mathit{fin}} u \setminus v} \uparrow F$, and then apply (M1) and ($\vee$). Let a be in u. If a is in v, we are done. Otherwise, let $a > b \in u$. Token b is not in v, since a is not in v. Thus, a is in $\uparrow\{b\}$ where $\{b\} \subseteq u \setminus v$. □

For the application of the statements above, it is useful to note how statements involving sums may be turned into statements involving meets. If $f_1, \ldots, f_n$ are functions in $\mathcal{R}F$ for some frame F and we know $\sum_{i=1}^{n} f_i \le g$, then $\bigwedge_{i=1}^{n} f_i q_i \le g(\sum_{i=1}^{n} q_i)$ follows, since the meet on the left is just one join component of the expanded form of $(\sum_i f_i)(\sum_i q_i)$.

6.4 The Critical Lemma in the Frame F_3

With the auxiliary statements collected in the previous subsection, we are now able to prove the α-image of the critical lemma. The statement to be shown is:

$$\bigwedge_{i \in I} \langle u_i, 0, q_i \rangle \le \bigvee \{ \bigwedge_{S \subseteq A} \langle \uparrow \mathsf{set}\, S, 0, \textstyle\sum \mathsf{num}\, S \rangle \mid A \in \mathcal{P}_I D,\; A_p u_i > q_i \;\forall i \in I \}$$

for finite families $u_1, \ldots, u_n$ and $q_1, \ldots, q_n$.

For $n = 0$, the statement holds, since $A = \emptyset$ is then involved in the join on the right, and $\bigwedge_{S \subseteq \emptyset} \langle \uparrow \mathsf{set}\, S, 0, \sum \mathsf{num}\, S \rangle = \mathsf{T}$.

For $n \neq 0$, our strategy is to break up the left hand side into a huge join, and then show that every component of this join is below the right hand side. Without restriction, we may assume that the u_i are pairwise different; for the left, this is since $\langle u, 0, q\rangle \wedge \langle u, 0, q'\rangle = \langle u, 0, q \sqcup q'\rangle$, and on the right, since $A_p u > q$ and $A_p u > q'$ iff $A_p u > q \sqcup q'$. Thus, we may assume in the sequel that the index set I coincides with $\{u_1, \ldots, u_n\}$.

By rational continuity, $\langle u_i, 0, q_i\rangle$ is join of $\langle u_i, 0, q_i'\rangle$ with $q_i' > q_i$. Applying (3) to $\langle u_i, 0, q_i'\rangle$, we see that this generator is a join of components $\bigwedge_{T:\, i \in T \subseteq I} \langle u^T, v^T, q_i^T\rangle$, where $u^T = \bigwedge_{j \in T} u_j$ and $v^T = \bigvee_{j \in I \setminus T} u_j$, and the numbers $(q_i^T)_{T:\, i \in T \subseteq I}$ are some numbers with $\sum_{T:\, i \in T \subseteq I} q_i^T = q_i' > q_i$.

By the step just performed, the left hand side is a meet of joins of meets. Applying distributivity, it can be rewritten into a join of more meets. For a fixed $T \subseteq I$, the meet of all $\langle u^T, v^T, q_i^T\rangle$ with i in T can be contracted into $\langle u^T, v^T, q^T\rangle$, where $q^T = max_{i \in T}\, q_i^T$. Thus, we reach at a join of components $\bigwedge_{T:\, \emptyset \neq T \subseteq I} \langle u^T, v^T, q^T\rangle$ with u^T and v^T as above, and some numbers q^T such that for all i in I, $\sum_{T:\, i \in T \subseteq I} q^T > q_i$. If we index something over T in the sequel, this should be always understood as indexed over $\{T \mid \emptyset \neq T \subseteq I\}$.

There are two kinds of components $\bigwedge_T \langle u^T, v^T, q^T\rangle$: those with $q := \sum_T q^T < 1$ and those with $q \geq 1$. We treat the latter first. The component $\bigwedge_T \langle u^T, v^T, q^T\rangle$ is just one join component of $(\sum_T [u^T, v^T])(q)$. By (9), the opens u^T and v^T satisfy the disjointness condition of (8), whence $(\sum_T [u^T, v^T])(q)$ is below $[\bigvee_T u^T, 0](q)$. If $q \geq 1$, the latter is F by the 'below 1' condition of $\mathcal{R}F_3$. Thus, components $\bigwedge_T \langle u^T, v^T, q^T\rangle$ with $\sum_T q^T \geq 1$ equal F and deserve no further attention.

Using (10), the remaining components $\bigwedge_T \langle u^T, v^T, q^T\rangle$ can be written as a join of $\bigwedge_T \langle \uparrow F^T, v^T, q^T\rangle$, where F^T is some finite subset of $u^T \setminus v^T$. Let $F^T = \{a_1^T, \ldots, a_{m^T}^T\}$. Using (5), we can write $\bigwedge_T \langle \uparrow F^T, v^T, q^T\rangle$ as a join of components

$$\bigwedge_T \bigwedge_{j=1}^{m^T} \langle \sharp a_j^T, \bigvee_{k=1}^{j-1} \sharp a_k^T \vee v^T, r_j^T\rangle \quad \text{for some numbers } r_j^T \text{ with } \textstyle\sum_{j=1}^{m^T} r_j^T = q^T.$$

For every such component z, we shall now construct a power token A with two properties: it is in the set on the right of the critical lemma, and the join component z is below $\bigwedge_{S \subseteq A} \langle \uparrow \mathsf{set}\, S, 0, \sum \mathsf{num}\, S\rangle$. We define $A = \sum_T \{\!|(r_1^T, a_1^T), \ldots, (r_{m^T}^T, a_{m^T}^T)|\!\}$. This is a legal power token since $\sum_T \sum_j r_j^T = \sum_T q^T < 1$.

First, we verify $A_p u_i > q_i$ for every i in I. We may compute $A_p u_i = \sum \{\!| r_j^T \mid a_j^T \in u_i |\!\}$. We know $a_j^T \in F^T \subseteq u^T = \bigwedge_{l \in T} u_l \leq u_i$ if i is in T. Thus, i in T implies a_j^T in u_i, whence $A_p u_i \geq \sum \{\!| r_j^T \mid i \in T |\!\} = \sum_{T \ni i} q^T > q_i$.

Second, we have to show

$$\bigwedge_T \bigwedge_{j=1}^{m^T} \langle \sharp a_j^T, \bigvee_{k=1}^{j-1} \sharp a_k^T \vee v^T, r_j^T\rangle \leq \bigwedge_{S \subseteq A} \langle \uparrow \mathsf{set}\, S, 0, \textstyle\sum \mathsf{num}\, S\rangle$$

Let S be a fixed subbag of A. For every T, let S^T be the part of S consisting of pairs (r_j^T, a_j^T) only. Then we obtain with $(M2)$:

$$\bigwedge_T \bigwedge_{j=1}^{m^T} \langle \sharp a_j^T, \bigvee_{k=1}^{j-1} \sharp a_k^T \vee v^T, r_j^T \rangle \le \bigwedge_T \bigwedge_{j \in S^T} \langle \sharp a_j^T, \bigvee_{k \in S^T,\, k<j} \sharp a_k^T \vee v^T, r_j^T \rangle$$

By (5), the latter is below $\bigwedge_T \langle \uparrow\mathsf{set}\, S^T, v^T, \sum \mathsf{num}\, S^T \rangle$. It remains to show that this is below $\langle \uparrow\mathsf{set}\, S, 0, \sum \mathsf{num}\, S \rangle$. This is true by (8), since the opens $\uparrow\mathsf{set}\, S^T \subseteq u^T$ and v^T satisfy the disjointness condition by (9), and $\bigvee_T \uparrow\mathsf{set}\, S^T = \uparrow\mathsf{set}\, S$ and $\sum_T \sum \mathsf{num}\, S^T = \sum \mathsf{num}\, S$ hold.

This concludes the proof of the critical lemma in F_3.

6.5 Real Valued Functions II

As announced in Subsection 6.1, we want to construct a frame homomorphism $\beta : F_3 \to G$ for complete Boolean algebras with $\eta : F_2 \to G$ such that $\beta \circ \alpha = \eta$. Before we do so, we investigate which additional properties the 'real valued functions' $\mathcal{R}G$ have if G is not just a frame, but a complete Boolean algebra. We shall show that the cBa structure allows defining a partial operation of subtraction on $\mathcal{R}G$.

Given two positive reals x and y with $x \sqsupseteq y$ and a positive rational q (all possibly being 0), when is $x - y > q$? If $q = 0$, the answer is always, because our order even satisfies $0 < 0$. For $q \neq 0$, $x - y > q$ iff there is $r > q$ with $x > r \sqsupseteq q + y$. With $s = r - q$, this is equivalent to: there is $s \neq 0$ with $x > q + s$ and $y \sqsubseteq s$, i.e., not $y > s$. In this existential statement, the case $s = 0$ may be included without harm since 'not $y > 0$' is always false.

This derivation motivates the following definition of subtraction in $\mathcal{R}G$ for a complete Boolean algebra G:

Definition 6.2 For f, g in $\mathcal{R}G$ with $f \geq g$, let

$$(f-g)(q) = \begin{cases} \top & \text{if } q = 0 \\ \bigvee_{s \in \mathbf{Q}_0^+} f(q+s) \wedge \neg gs & \text{if } q \neq 0 \end{cases}$$

The first thing to verify is that $f - g$ is a legal member of $\mathcal{R}G$. The condition $(f-g)(0) = \top$ is part of the definition, and $\bigvee_{q' > q} (f-g)(q') = (f-g)(q)$ holds for $q \neq 0$, since the argument q occurs positively in the definition of $f - g$.

In the sequel, we state and prove some properties of subtraction.

S0: '$-$' preserves non-empty joins in its first argument (but not in the second); thus, in particular it is continuous there.

Proof: Non-empty joins are given pointwise, and f occurs positively in the definition. □

S1: For all f, $f - 0 = f$.

Proof: For $q \neq 0$, $(f-0)(q) = \bigvee_s f(q+s) \wedge \neg 0(s) = \bigvee_{s \neq 0} f(q+s) = fq$ by rational continuity. □

S2: For all f, $f - f = 0$.

Proof: For $q \neq 0$, $(f-f)(q) = \bigvee_s f(q+s) \wedge \neg fs$. By $q + s \sqsupset s$, this is below $\bigvee_s fs \wedge \neg fs = \mathsf{F}$. □

The next statement prepares the following mixed associativity law.

S3: If $g \geq h$ and h is bounded, then $(f+g) - h \geq f$.

Proof: For $q \neq 0$, we obtain

$$((f+g)-h)(q) = \bigvee_{s,q_1,q_2:\, q_1+q_2=q+s} fq_1 \wedge gq_2 \wedge \neg hs$$

We have to show that this is above $fq = \bigvee_{r>q} fr$. For fixed $r > q$, consider fr. Fixing $q_1 = r$, we obtain

$$((f+g)-h)(q) \geq fr \wedge \bigvee_{s,q_2:\, r+q_2=q+s} gq_2 \wedge \neg hs$$

We show that the big join in this expression equals T. With $d = r - q$, it becomes $\bigvee_{q_2} gq_2 \wedge \neg h(q_2+d)$. Using $g \geq h$ and restricting to multiples of d, the latter join is above $\bigvee_{n \in \mathbf{N}_0} h(n \cdot d) \wedge \neg h((n+1) \cdot d)$. Since h is bounded and $d \sqsupset 0$, there is some m with $h((m+1) \cdot d) = \mathsf{F}$. Thus, the join equals $(\mathsf{T} \wedge \neg hd) \vee (hd \wedge \neg h(2d)) \vee \cdots \vee (h(m \cdot d) \wedge \neg \mathsf{F})$, which can be contracted to T. □

The next statement is the important mixed associativity law.

S4: If $g \geq h$ and h is bounded, then $f + (g-h) = (f+g) - h$.

Proof: For $q \neq 0$, we have $(f + (g-h))(q) = \bigvee_{q_1+q_2=q} fq_1 \wedge (g-h)(q_2)$. Here, we must single out the case $q_2 = 0$, i.e., $q_1 = q$. It yields a single join component fq. Thus,

$$Lq = fq \vee \bigvee_{q_1,q_2,s:\, q_2 \neq 0,\, q_1+q_2=q} fq_1 \wedge g(q_2+s) \wedge \neg hs$$

The right hand side yields

$$Rq = \bigvee_{s,q_1',q_2':\, q_1'+q_2'=q+s} fq_1' \wedge gq_2' \wedge \neg hs$$

For $Lq \leq Rq$, $fq \leq Rq$ holds by (S3). The remaining part of Lq is below Rq, as can be seen by letting $q_1' = q_1$ and $q_2' = q_2 + s$. For $Rq \leq Lq$, we have to differentiate two cases. The R-components with $q_2' \sqsupset s$ are below Lq, as can be seen by letting $q_1 = q_1'$ and $q_2 = q_2' - s$. If $q_2' \sqsubseteq s$, then $q_1' \sqsupseteq q$, whence $fq_1' \wedge gq_2' \wedge \neg hs \leq fq_1' \leq fq \leq Lq$. □

The next two statements show that subtraction inverts addition.

S5: If g is bounded, then $(f+g) - g = f$.

Proof: By (S4), $(f+g) - g = f + (g-g)$, which is f by (S2). □

S6: If $f \geq g$ and g is bounded, then $(f-g) + g = f$, and thus $f \geq f - g$.

Proof: By (S4) and commutativity of addition, $(f-g)+g=(f+g)-g$, which equals f by (S5). $\Box$

S7: If $f \geq g \geq h$ and g is bounded, then $(f-g)+(g-h)=f-h$.

Proof: Since g is bounded, h is bounded, too, and by (S6), $g-h$ is also bounded. By (S4), $(f-g)+(g-h)=((f-g)+g)-h$, which by (S6) equals $f-h$. $\Box$

S8: Let $f \geq f'$, $g \geq g'$, and f', g' be bounded. Then $f-f'=g-g'$ iff $f+g'=f'+g$.

Proof: Add / subtract f' and g' and apply (S4), (S5), and (S6). $\Box$

6.6 A Frame Homomorphism from F_3 to G

Now we come to the final step of the proof of the critical lemma: given a frame embedding η from F_2 to a complete Boolean algebra G, define a frame homomorphism $\beta : F_3 \to G$ with $\beta \circ \alpha = \eta$. According to the spatial intuition that $\langle u, v, q\rangle$ be the set of evaluations μ with $\mu(u \setminus v) > q$, or $\mu u - \mu(u \wedge v) > q$, we define $\beta\langle u, v, q\rangle = (\eta \circ [u] - \eta \circ [u \wedge v])q$ using the difference operator of the previous subsection.

In the sequel, we shall not explicitly write down η, and abstract out the q-argument. Thus, we obtain the concise definition $\beta[u, v] = [u]-[u \wedge v]$. We have to show $\beta \circ \alpha = \eta$ and the preservation of the relations of F_3 by β. In doing so, we may use all properties of subtraction, since all the functions $[u]$ are bounded: $[u](1) = \mathsf{F}$.

For the composition, we compute $\beta(\alpha[u]) = \beta[u, 0] = [u] - [u \wedge 0]$. By the zero law of F_2, $[u \wedge 0] = [0] = 0$, and by (S1), $[u] - 0 = [u]$ holds. Since this actually stands for $\eta \circ [u]$, we have shown $\beta \circ \alpha = \eta$.

For restricted continuity, let $(u_i)_{i \in I}$ be a directed family of opens, and v an open with $u_i \geq v$ for all i. We have to show $\beta[\bigvee_{i \in I} u_i, v] = \bigvee_{i \in I} \beta[u_i, v]$, or

$$[\bigvee_{i \in I} u_i] - [\bigvee_{i \in I} u_i \wedge v] = \bigvee_{i \in I}([u_i] - [u_i \wedge v]).$$

By the condition $u_i \geq v$, this simplifies to $[\bigvee_{i \in I} u_i] - [v] = \bigvee_{i \in I}([u_i] - [v])$, which is a valid statement by the continuity relation in F_2 and the continuity of subtraction in its first argument.

The zero relation $[0, 0] = 0$ becomes $[0] - [0 \wedge 0] = 0$, which is true since $[0] = 0$ by the zero relation of F_2, and $0 - 0 = 0$ by (S1) or (S2).

The meet relation $[u, v] = [u, u \wedge v]$ becomes $[u] - [u \wedge v] = [u] - [u \wedge (u \wedge v)]$, which is obviously true.

The join relation $[u, v] = [u \vee v, v]$ translates into $[u] - [u \wedge v] = [u \vee v] - [v]$. By (S8), this is equivalent to $[u] + [v] = [u \vee v] + [u \wedge v]$, which is just the modularity relation of F_2.

Finally, the split relation states $[u, w] = [u, v] + [v, w]$ if $u \geq v \geq w$. By β, this becomes $[u] - [w] = ([u] - [v]) + ([v] - [w])$ — a valid statement by (S7).

7 Conclusion and Future Work

In [6, 7], a probabilistic power construction $\mathcal{P}_D$ is defined for dcpo's. In this paper, we define a construction $\mathcal{P}_I$ for infosyses in the sense of Vickers [12], and a construction $\mathcal{P}_L$ for locales, and prove that $\mathcal{P}_D$, $\mathcal{P}_I$, and $\mathcal{P}_L$ are equivalent when restricted to continuous domains. In particular, the infosys construction is effective: given a countable infosys D with decidable order, the power infosys $\mathcal{P}_I D$ is again countable with decidable order.

The dcpo construction $\mathcal{P}_D$ is part of a monad on dcpo's, as shown in [6]. One might also wish to make $\mathcal{P}_I$ and $\mathcal{P}_L$ into monads with operations equivalent to those of Jones. This is a non-trivial task, since Jones's definition of the multiplication of the monad $\mathcal{P}_D$ involves Lebesgue integration (of Scott continuous functions w.r.t. continuous evaluations). For $\mathcal{P}_L$, we were already successful in defining the monad operations, but this a topic for a different paper.

Another interesting problem is to work out the theory of '$\mathbf{R}_0^\infty$-frames' that may hide behind the auxiliary notations of Subsection 6.2. With a proper axiomatization of $\mathbf{R}_0^\infty$-frames or of $\mathbf{I}$-frames, one would obtain a theory of locales that does not rely on the Sierpinski space $\mathbf{2}$ as usual, but on the positive reals $\mathbf{R}_0^\infty$ or the unit interval $\mathbf{I}$.

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