Approximation of Metric Spaces by Partial Metric Spaces

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Abstract. Partial metrics are generalised metrics with non-zero self-distances. We slightly generalise Matthews' original definition of partial metrics, yielding a notion of *weak partial metric*. After considering weak partial metric spaces in general, we introduce a weak partial metric on the poset of formal balls of a metric space. This weak partial metric can be used to construct the completion of classical metric spaces from the domain-theoretic rounded ideal completion.

Key words: Generalised Metric Spaces, Domains, Scott Topology, Preorder

1. Introduction

Partial metrics were introduced by Matthews (1992) as a generalisation of metrics where self-distances are not necessarily zero. Later, O'Neill (1995) generalised partial metrics a bit further by admitting negative distances. Here, we stick to non-negative distances, but generalise Matthews' partial metrics in a different direction by omitting his axiom of small self-distances which is of minor importance in our opinion. The resulting notion of partial metric without small self-distances is called *weak partial metric*.

In Section 2, we study weak partial metric spaces. They induce a topology, which in general is not Hausdorff, so that it is worthwhile to consider its specialisation preorder. We derive some new results about the structure of weak partial metric spaces (and hence in particular of partial metric spaces): the weak partial metric is continuous w.r.t. its induced topology if the non-negative reals $\mathbf{R}_+$ are topologised by the Scott topology of $\mathbf{R}_+^{\mathrm{op}}$ (Theorem 2.13), and the specialisation preorder has the nice property that every directed subset contains an ascending sequence with the same upper bounds (Theorem 2.18). Thus, reasoning about directed sets can be replaced by reasoning about ascending sequences when the specialisation preorder is studied domain-theoretically.

In Section 3, we consider the poset $\mathbf{B}X$ of formal balls of a metric space X as introduced in (Edalat and Heckmann, 1997). It approximates the metric space X in the sense that X is homeomorphic to the subspace of maximal elements of $\mathbf{B}X$ in the relative Scott topology. Here, we define a weak partial metric p on $\mathbf{B}X$ which is universal in the sense that every weak partial metric space P can be isometrically embedded into the weak partial metric space $\mathbf{B}X$ for some metric space X. The weak partial metric on $\mathbf{B}X$ extends the metric on X, and induces the Scott topology on $\mathbf{B}X$, whence it is Scott continuous as a map from $\mathbf{B}X \times \mathbf{B}X$ to $\mathbf{R}_+^{\mathrm{op}}$. Thus, a domain-theoretic construction of the metric completion is obtained: From a metric space (X, d), construct the weak partial metric space $(\mathbf{B}X, p)$, use rounded ideal completion to obtain $(\mathcal{I}(\mathbf{B}X), \widehat{p})$, and restrict to the maximal elements which gives the completion $(\overline{X}, \overline{d})$.

2. Partial Metric Spaces

2.1. Definition and Basic Properties

Partial metric spaces were introduced by Matthews (1992). Here, we generalise his notion slightly by dropping one of his axioms, calling the result weak partial metric. We also distinguish between a partial pseudo-metric and a proper partial metric.

DEFINITION 2.1.
A *weak partial pseudo-metric* on a set P is a function $p : P \times P \to \mathbf{R}_+$ which is
– symmetric: $p(a, b) = p(b, a)$ for all a, b in P,
– and satisfies the modified triangle inequation:
$p(a, c) \leq p(a, b) + p(b, c) - p(b, b)$ for all a, b, c in P.
A *weak partial metric* is a weak partial pseudo-metric which in addition satisfies the $\mathcal{T}_0$-separation axiom: $p(a, a) = p(a, b) = p(b, b)$ implies $a = b$.

As the following proposition shows, the shift from metrics to weak partial metrics consists of dropping the property that self-distances are zero.

PROPOSITION 2.2.
A weak partial (pseudo-)metric p on P is a (pseudo-)metric iff $p(a, a) = 0$ for all a in P.

Proof. A pseudo-metric p has the property $p(a, a) = 0$. With this property, the modified triangle inequality becomes identical with the ordinary one, and the $\mathcal{T}_0$-separation axiom becomes $p(a, b) = 0 \Rightarrow a = b$. □

Partial metrics in the sense of Matthews satisfy an additional axiom, namely the following SSD property.

DEFINITION 2.3. A weak partial pseudo-metric has *small self-distances (SSD)* if $p(a, b) \geq p(a, a)$ for all a, b in P. A weak partial (pseudo-)metric with SSD is called partial (pseudo-)metric.

We introduced weak partial metrics since SSD is not required for many properties, and our 'partial metric' on $\mathbf{B}X$ does not satisfy it. On the other hand, a weak version of SSD can be derived from the remaining axioms:

PROPOSITION 2.4. *For all a, b in a weak partial pseudo-metric space (P, p), we have* $p(a, b) \geq \frac{1}{2}(p(a, a) + p(b, b))$.

Proof. By the triangle inequation with $c = a$, $p(a,a) \leq p(a,b) + p(b,a) - p(b,b)$ holds. With symmetry, the claimed inequality follows. □

The weak SSD property shows that our weak partial pseudo-metric spaces are not far from SSD: in any case, $p(a,b)$ is above at least one of $p(a,a)$ and $p(b,b)$ since it is above their mean value.

Later, we need the notion of isometric embedding.

DEFINITION 2.5. A function $f : (P,p) \to (P',p')$ between two weak partial pseudo-metric spaces is *isometric* iff $p'(fa, fb) = p(a,b)$ holds for all a, b in P. An *isometric embedding* is an injective isometric function.

PROPOSITION 2.6. *If (P,p) satisfies the $\mathcal{T}_0$-separation axiom, then every isometric function $f : (P,p) \to (P',p')$ is an isometric embedding.*

Proof. If $fa = fb$, then $p'(fa,fa) = p'(fa,fb) = p'(fb,fb)$, whence $p(a,a) = p(a,b) = p(b,b)$ by the isometry condition. With the $\mathcal{T}_0$-separation axiom, $a = b$ follows. □

2.2. AN ALTERNATIVE DESCRIPTION OF WEAK PARTIAL METRIC SPACES

In (Matthews, 1992), Matthews shows that a partial metric (with SSD) can be alternatively represented by a metric plus a weight function. This is also possible in our case replacing the metric by a pseudo-metric. The following definitions differ from those of Matthews by a factor of two. This will turn out to be more natural in the example of formal balls.

In a weak partial pseudo-metric space (P,p), define $\delta : P \times P \to \mathbf{R}_+$ by $\delta(a,b) = p(a,b) - \frac{1}{2}(p(a,a) + p(b,b))$. This difference is indeed non-negative because of the weak SSD property (Prop. 2.4). Function δ is a pseudo-metric: $\delta(a,a) = 0$ and $\delta(a,b) = \delta(b,a)$ are obvious, and $\delta(a,c) \leq \delta(a,b) + \delta(b,c)$ follows from the triangle inequality for p. In addition, we define a 'weight function' $w : P \to \mathbf{R}_+$ by $wa = \frac{1}{2}p(a,a)$.

Conversely, if δ is a pseudo-metric on a set P, and w is a function from P to $\mathbf{R}_+$, then $p : P \times P \to \mathbf{R}_+$ defined by $p(a,b) = \delta(a,b) + wa + wb$ is a weak partial pseudo-metric on P. It is easily seen that the two constructions are inverse to each other so that we obtain the first claim of the following proposition.

PROPOSITION 2.7.

1. *On every set P, there is a one-to-one correspondence between weak partial pseudo-metrics p on P and pairs (δ, w) of a pseudo-metric δ on P and a function $w : P \to \mathbf{R}_+$. It is given by $\delta(a,b) = p(a,b) - \frac{1}{2}(p(a,a) + p(b,b))$ and $wa = \frac{1}{2}p(a,a)$, and conversely $p(a,b) = \delta(a,b) + wa + wb$.*
2. *p is a weak partial metric iff (δ, w) has the property $\delta(a,b) = 0$ & $wa = wb \Rightarrow a = b$. (Note that even in this case, δ need not be a metric.)*

3. *p has property SSD iff $\delta(a, b) \geq |wa - wb|$ for all a, b in P.*
4. *In the case considered by Matthews, where p is a partial metric with SSD, δ is a proper metric.*
5. *A function $f : (P, p, \delta, w) \to (P', p', \delta', w')$ is isometric, i.e., $p'(fa, fb) = p(a, b)$ for all a, b in P, iff $\delta'(fa, fb) = \delta(a, b)$ and $w'(fa) = wa$ for all a, b in P.*

Proof.

1. was shown above.
2. The hypothesis of the $\mathcal{T}_0$-separation property, $p(a, a) = p(a, b) = p(b, b)$, translates into $2\,wa = \delta(a, b) + wa + wb = 2\,wb$, which is equivalent to $\delta(a, b) = 0$ and $wa = wb$. The space of formal balls will provide an example for a weak partial metric where δ is not a metric.
3. Property SSD, $p(a, b) \geq p(a, a)$, is equivalent to $\delta(a, b) + wa + wb \geq 2\,wa$, or $\delta(a, b) \geq wa - wb$. Because of symmetry, this is equivalent to $\delta(a, b) \geq \max(wa - wb, wb - wa) = |wa - wb|$.
4. Because of (3), $\delta(a, b) = 0$ implies $|wa - wb| = 0$, whence $wa = wb$. With (2), $a = b$ follows.
5. is obvious. □

2.3. (WEAK) PARTIAL METRIC TOPOLOGY

For partial metric spaces (with SSD), Matthews (1992) and O'Neill (1995) defined a topology. Their definition also works for weak partial pseudo-metric spaces. We first define open balls according to O'Neill.

DEFINITION 2.8. Let (P, p) be a weak partial pseudo-metric space. For a in P and $r > 0$, the *open ball* $O(a, r)$ around a with radius r is the set $\{b \in P \mid p(a, b) < p(a, a) + r\}$.

This definition has the advantage that a is a member of $O(a, r)$. This was not always true with the original definition of Matthews, $\{b \in P \mid p(a, b) < r\}$. Both definitions collapse to the usual one in the case of a proper metric where $p(a, a) = 0$.

The definition of open balls can be easily translated into the (δ, w) world:

PROPOSITION 2.9. $O(a, r) = \{b \in P \mid \delta(a, b) < wa - wb + r\}$.

Using open balls, we can define open sets as follows:

DEFINITION 2.10. A subset U of a weak partial pseudo-metric space is *open* if for all a in U there is $\epsilon > 0$ such that $O(a, \epsilon) \subseteq U$.

It is easy to prove that these open sets form a topology. It is slightly more difficult to show that the open balls themselves are open. Let b be in $O(a, r)$. Then $p(a, b) <$

$p(a,a)+r$. Let $\epsilon = p(a,a)+r-p(a,b) > 0$. We show $O(b,\epsilon) \subseteq O(a,r)$. For c in $O(b,\epsilon)$, $p(a,c) \le p(a,b)+p(b,c)-p(b,b) < p(a,b)+\epsilon = p(a,a)+r$ holds, whence c is in $O(a,r)$. Thus we obtain:

PROPOSITION 2.11.
The open sets of Def. 2.10 form a topology with basis the open balls $O(a,r)$.

As expected, isometric embeddings are topological embeddings.

PROPOSITION 2.12. *Isometric embeddings are topological subspace embeddings.*

Proof. Let $e : (P,p) \to (P',p')$ be an isometric embedding. First we show that e is continuous. Let a be in the inverse image set $e^{-}(O(b,s))$ for b in P' and $s > 0$. Then $p'(ea,b) < p'(b,b)+s$. Let $\epsilon > 0$ be the difference of the two sides. We claim $O(a,\epsilon) \subseteq e^{-}(O(b,s))$. For c in $O(a,\epsilon)$,

$$\begin{aligned} p'(ec,b) &\le p'(ec,ea)+p'(ea,b)-p'(ea,ea) \\ &= p(a,c)-p(a,a)+p'(ea,b) \\ &< \epsilon + p'(ea,b) \\ &= p'(b,b)+s \end{aligned}$$

holds, whence c is in $e^{-}(O(b,s))$.

Now let a in P and $r > 0$. Since $p(a,c) < p(a,a)+r$ and $p'(ea,ec) < p'(ea,ea)+r$ are equivalent, $O(a,r) = e^{-}(O(ea,r))$ holds. Thus every open set of P is the inverse image of some open set of P', whence e is a topological embedding. $\square$

For every metric space (X,d), the function $d : X \times X \to \mathbf{R}_+$ is continuous if X is given the metric topology, and $\mathbf{R}_+$ the usual Hausdorff topology. A direct generalisation to partial metric spaces is not possible; the topology of $\mathbf{R}_+$ has to be replaced by the Scott topology belonging to the opposite of the usual ordering. This gives a weaker result than in the metric case.

THEOREM 2.13. *For every weak partial pseudo-metric space (P,p), the function $p : P \times P \to \mathbf{R}_+^{\mathrm{op}}$ is continuous, where P is given the topology induced by p, and $\mathbf{R}_+^{\mathrm{op}}$ is endowed with the Scott topology.*

Proof. The Scott topology of $\mathbf{R}_+^{\mathrm{op}}$ has basic open sets $\{s \in \mathbf{R}_+ \mid s < r\}$ for $r > 0$. If $p(a,b) < r$, let $\epsilon = r - p(a,b) > 0$. Then $p(a',b') < r$ for all (a',b') in $O(a,\epsilon/2) \times O(b,\epsilon/2)$. For,

$$\begin{aligned} p(a',b') &\le p(a',a)+p(a,b)+p(b,b')-p(a,a)-p(b,b) \\ &< (p(a,a)+\epsilon/2)+(r-\epsilon)+(p(b,b)+\epsilon/2)-p(a,a)-p(b,b) \\ &= r. \end{aligned}$$ $\square$

2.4. THE INDUCED PREORDER

In contrast to the topology of a proper metric space, the partial metric topology is not Hausdorff in general. Thus it is worthwhile to study its specialisation preorder, which we denote by '$\sqsubseteq$'.

PROPOSITION 2.14. *For two elements a and b of a weak partial pseudo-metric space (P, p, δ, w), the following are equivalent:*

1. *$a \sqsubseteq b$ in the specialisation preorder of the topology induced by p.*
2. *For all $\epsilon > 0$, b is in $O(a, \epsilon)$.*
3. *$p(a, b) \leq p(a, a)$.*
4. *For all c in P, $p(a, c) \geq p(b, c)$.*
5. *$\delta(a, b) \leq wa - wb$.*
6. *For all c in P, $\delta(a, c) + wa \geq \delta(b, c) + wb$.*

Proof.

$1 \Rightarrow 4$: Every continuous function is monotonic w.r.t. the specialisation preorders of its domain and codomain. By monotonicity of $p : P \times P \to \mathbf{R}_+^{\mathrm{op}}$, $a \sqsubseteq b$ implies $p(a, c) \geq p(b, c)$ for all c in P.

$4 \Rightarrow 3$: Consider $c = a$.

$3 \Rightarrow 2$: If $p(a, b) \leq p(a, a)$, then $p(a, b) < p(a, a) + \epsilon$ for all $\epsilon > 0$, whence $b \in O(a, \epsilon)$.

$2 \Rightarrow 1$: By definition, $a \sqsubseteq b$ means that b is in every open set which contains a. If a is in an open set U, then $O(a, \epsilon) \subseteq U$ for some $\epsilon > 0$, whence b is in U.

The equivalences $3 \Leftrightarrow 5$ and $4 \Leftrightarrow 6$ are obtained by straightforward translations. □

Matthews (1992) defines this preorder ad-hoc by $a \sqsubseteq b$ iff $p(a, b) = p(a, a)$ (see part (3) of the proposition; '$\geq$' follows from SSD). He does not point out that it is the specialisation preorder of the induced topology. While this is done by O'Neill, both did not mention the equivalence with statement (4) of the proposition.

As already mentioned in the proof above, continuous functions are monotonic. Hence we obtain:

PROPOSITION 2.15. *For every weak partial pseudo-metric space (P, p), $p : P \times P \to \mathbf{R}_+^{\mathrm{op}}$ and $w : P \to \mathbf{R}_+^{\mathrm{op}}$ are monotonic.*

Using the characterisation of the specialisation preorder, we can show that the $\mathcal{T}_0$-separation property is named appropriately.

PROPOSITION 2.16. *For a weak partial pseudo-metric on a set P, the following statements are equivalent:*

1. *p is a weak partial metric, i.e., $p(a, a) = p(a, b) = p(b, b)$ implies $a = b$.*

2. *The induced topology has the $\mathcal{T}_0$-property.*
3. *The induced preorder is a partial order, i.e.,* $a \sqsubseteq b$ *and* $b \sqsubseteq a$ *implies* $a = b$.

Proof. Statements (2) and (3) are equivalent for general topological spaces.

Clearly, $p(a,a) = p(a,b) = p(b,b)$ implies $a \sqsubseteq b$ and $b \sqsubseteq a$ by part (3) of Prop. 2.14. Conversely, $a \sqsubseteq b$ and $b \sqsubseteq a$ means $p(a,b) \leq p(a,a)$ and $p(a,b) \leq p(b,b)$. Together with the weak SSD property, i.e., $p(a,b) \geq \frac{1}{2}(p(a,a) + p(b,b))$ (Prop. 2.4), $p(a,b) = p(a,a) = p(b,b)$ follows. Thus, (1) and (3) are equivalent. □

PROPOSITION 2.17. *In every weak partial metric space* (P,p)*, elements* a *with* $p(a,a) = 0$*, or equivalently* $wa = 0$*, are maximal.*

Proof. Let $p(a,a) = 0$ and $a \sqsubseteq b$. From monotonicity of $p : P \times P \to \mathbf{R}_+^{\mathrm{op}}$, $p(b,b) \leq p(a,b) \leq p(a,a) = 0$ follows, whence $p(a,a) = p(a,b) = p(b,b) = 0$. Thus a and b are equal by the $\mathcal{T}_0$-separation property. □

Finally, we prove the theorem about directed sets which was announced in the introduction.

THEOREM 2.18. *Let* (P,p) *be a weak partial pseudo-metric space with the induced preorder. Every directed subset of* P *contains an ascending sequence with the same upper bounds.*

Proof. Let D be a directed subset of P, and let $s = \inf\{p(d,d) \mid d \in D\}$. For all n in N, there is a_n in D such that $p(a_n,a_n) \leq s + 1/n$. Let $b_1 = a_1$, and using directedness, let b_{n+1} be an upper bound of b_n and a_{n+1} in D. Then $b_n \sqsubseteq b_{n+1}$ and $p(b_n,b_n) \leq p(a_n,a_n) \leq s + 1/n$ for all n in N. The elements b_n form the desired ascending sequence. We have to show that every upper bound c of all b_n is above every element d of D.

Let e_n be an upper bound of b_n and d in D. Then $c \sqsupseteq b_n \sqsubseteq e_n \sqsupseteq d$. For every n in N, $p(c,d) \leq p(c,e_n) + p(e_n,d) - p(e_n,e_n)$ follows from the triangle inequation. Because of monotonicity of p, $c \sqsupseteq b_n \sqsubseteq e_n$ implies $p(c,e_n) \leq p(b_n,b_n) \leq s + 1/n$, and $e_n \sqsupseteq d$ implies $p(e_n,d) \leq p(d,d)$. Since e_n is in D, $p(e_n,e_n) \geq s$ holds. Taking everything together, $p(c,d) \leq (s + 1/n) + p(d,d) - s = p(d,d) + 1/n$ holds for all n in N. Thus, $p(c,d) \leq p(d,d)$ holds, whence $d \sqsubseteq c$ as required. □

2.5. Forcing Small Self-Distances

Here, we show that every topology which can be obtained from a weak partial metric can also be obtained from a partial metric (a weak partial metric with small self-distances). Thus, dropping the SSD axiom does not increase the 'expressiveness' of partial metrics. It does allow for simpler definitions in certain situations, as we shall see in the case of formal balls.

Let p be a weak partial pseudo-metric on a set P. Define a function p' by $p'(a,b) = \max(p(a,b), p(a,a), p(b,b))$. We claim that p' is a partial pseudo-metric which induces the same topology as p.

Note that $p'(a, a) = p(a, a)$ for all a in P. Clearly, p' is symmetric, and $p'(a, b) \geq p(a, a) = p'(a, a)$, i.e., p' has small self-distances. For the triangle inequation, we have to show

$$\begin{aligned}&\max(p(a, c), p(a, a), p(c, c)) + p(b, b) \leq \\ &\qquad \max(p(a, b), p(a, a), p(b, b)) + \max(p(b, c), p(b, b), p(c, c)).\end{aligned}$$

Let R be the right hand side. First, $p(a, c) + p(b, b) \leq p(a, b) + p(b, c) \leq R$ follows from the triangle inequation for p. Second, $p(a, a) + p(b, b) \leq R$ and $p(c, c) + p(b, b) \leq R$ hold. Together, $\max(p(a, c), p(a, a), p(c, c)) + p(b, b) \leq R$ follows.

So far, we have shown that p' is a partial pseudo-metric (with SSD). Next, we claim $O_p(a, \epsilon/2) \subseteq O_{p'}(a, \epsilon) \subseteq O_p(a, \epsilon)$ for all a in P and $\epsilon > 0$.

For the first inclusion, assume $p(a, b) < p(a, a) + \epsilon/2$. Then $p(a, b) < p(a, a) + \epsilon$ follows, and $p(a, a) < p(a, a) + \epsilon$ is obvious. Using the triangle inequation,

$$\begin{aligned}p(b, b) &\leq p(b, a) + p(a, b) - p(a, a) \\ &< 2(p(a, a) + \epsilon/2) - p(a, a) \\ &= p(a, a) + \epsilon\end{aligned}$$

holds as well. Taking all three statements together, $p'(a, b) < p(a, a) + \epsilon = p'(a, a) + \epsilon$ follows.

The second inclusion is much simpler: if $p'(a, b) < p'(a, a) + \epsilon$, then $p(a, b) \leq p'(a, b) < p'(a, a) + \epsilon = p(a, a) + \epsilon$ holds.

From the two inclusions, the equality of the two topologies follows: If U is p-open, then for a in U, there is $\epsilon > 0$ such that $O_p(a, \epsilon) \subseteq U$, whence $O_{p'}(a, \epsilon) \subseteq O_p(a, \epsilon) \subseteq U$. If U is p'-open, then for a in U, there is $\epsilon > 0$ such that $O_{p'}(a, \epsilon) \subseteq U$, whence $O_p(a, \epsilon/2) \subseteq O_{p'}(a, \epsilon) \subseteq U$. Because of the equality of the induced topologies and Prop. 2.16, p' is a partial metric iff p is a weak partial metric.

Summarising, we obtain:

THEOREM 2.19. *If p is a weak partial (pseudo-)metric on P, then p' with $p'(a, b) =$ $\max(p(a, b), p(a, a), p(b, b))$ is a partial (pseudo-)metric (with small self-distances) on P which induces the same topology and thus the same preorder as p.*

3. The Weak Partial Metric Space of Formal Balls

3.1. The Poset of Formal Balls

In (Edalat and Heckmann, 1997), a poset $\mathbf{B}X$ of formal balls was defined for every metric space (X, d). We repeat the key results of that paper here.

A *formal ball* in a metric space (X, d) is a pair (x, r) of an element x of X and a number r in $\mathbf{R}_+$. Formal balls are ordered by $(x, r) \sqsubseteq (y, s)$ iff $d(x, y) \leq r - s$.

They form a poset, but not a dcpo in general. They do form a dcpo if and only if X is complete. Nevertheless, a way-below relation '$\ll$' can be defined as usual; $a \ll b$ holds iff for all directed sets D having a least upper bound (lub) above b, some element of D is above a. For every metric space X, $\mathbf{B}X$ is a continuous poset in the sense that for every b in $\mathbf{B}X$, the set $\Downarrow b = \{a \in \mathbf{B}X \mid a \ll b\}$ is directed with lub b.

The maximal elements of $\mathbf{B}X$ are the formal balls $(x, 0)$. Hence, $i : X \to \mathbf{B}X$ with $ix = (x, 0)$ is a one-to-one-correspondence between the points of X and the maximal elements of $\mathbf{B}X$. Moreover, this function is a topological embedding w.r.t. the Scott topology of $\mathbf{B}X$. Thus, X with its metric topology is homeomorphic to $Max\,(\mathbf{B}X)$ with the relative Scott topology.

In (Edalat and Heckmann, 1997), it was also shown that the completion $\overline{X}$ of a metric space (X, d) can be obtained as follows: form the continuous poset $\mathbf{B}X$ of formal balls of X, and construct the rounded ideal completion $\mathcal{I}(\mathbf{B}X)$ of $\mathbf{B}X$ which is a continuous dcpo. Since $\mathcal{I}(\mathbf{B}X)$ is isomorphic to the continuous dcpo $\mathbf{B}\overline{X}$, $Max\,(\mathcal{I}(\mathbf{B}X))$ with the relative Scott topology is homeomorphic to $\overline{X}$ with the metric topology. Thus, $\overline{X}$ is obtained as a topological space without the completed metric $\overline{d}$. In the sequel, we show how to obtain $\overline{d}$ as well by defining a weak partial metric on $\mathbf{B}X$.

3.2. A Weak Partial Metric on Formal Balls

Using the (δ, w)-approach, a natural weak partial metric on formal balls can be defined by $\delta((x, r), (y, s)) = d(x, y)$ and $w(x, r) = r$. Clearly, δ is a pseudo-metric so that $p : \mathbf{B}X \times \mathbf{B}X \to \mathbf{R}_+$ defined by $p\,((x, r), (y, s)) = d(x, y) + r + s$ is a weak partial pseudo-metric by Prop. 2.7 (1). Since $\delta((x, r), (y, s)) = 0$ implies $x = y$ and $w(x, r) = w(y, s)$ means $r = s$, the condition in Prop. 2.7 (2) is satisfied and p is a weak partial metric.

The partial metric p' with small self-distances which belongs to p by Theorem 2.19 is given by

$$\begin{aligned} p'((x, r), (y, s)) &= \max(d(x, y) + r + s,\ 2r,\ 2s) \\ &= \max(d(x, y),\ r - s,\ s - r) + r + s \\ &= \max(d(x, y),\ |r - s|) + r + s. \end{aligned}$$

In the sequel, we prefer to work with p since it is simpler and allows for the characterisation of the way-below relation of $\mathbf{B}X$ by $a \ll b$ iff $p\,(a, b) < p\,(a, a)$ (see below). A corresponding characterisation involving p' is impossible since $p'(a, b) < p'(a, a)$ never holds because of the SSD property. Moreover, p has the additional advantage to admit the following representation theorem.

THEOREM 3.1. *For every weak partial metric space (P, π), there is a metric space (X, d) such that (P, π) can be isometrically embedded into $(\mathbf{B}X, p)$ (where p is defined as above).*

Proof. Let (δ, w) be the alternative description of π as given by Prop. 2.7 (1). Define a relation '$\sim$' on P by $a \sim b$ iff $\delta(a, b) = 0$. This relation obviously is reflexive and symmetric. Transitivity follows from the triangle inequation for δ: if $a \sim b \sim c$, then $\delta(a, c) \leq \delta(a, b) + \delta(b, c) = 0$. Relation '$\sim$' is a δ-congruence since $a \sim a'$ and $b \sim b'$ implies $\delta(a, b) \leq \delta(a, a') + \delta(a', b') + \delta(b', b) = \delta(a', b')$ and vice versa. Thus, we can define $d : X \times X \to \mathbf{R}_+$ on the set X of '$\sim$'-classes by $d([a], [b]) = \delta(a, b)$, and (X, d) becomes a metric space. Let $e : P \to \mathbf{B}X$ be defined by $ea = ([a], wa)$. Then $p(ea, eb) = d([a], [b]) + wa + wb = \delta(a, b) + wa + wb = \pi(a, b)$. Therefore, e is isometric. It is an isometric embedding by Prop. 2.6 since π is a weak partial metric (has the $\mathcal{T}_0$-property). □

If we start with $(P, \pi) = (\mathbf{B}X, p)$, then we have $(x, r) \sim (y, s)$ iff $d(x, y) = 0$ iff $x = y$. Thus, the metric space X constructed in the proof above is (isometric to) the original metric space X; the class of (x, r) is x. Therefore, the embedding e constructed in the proof is the identity. This confirms the choice of the weak partial metric on formal balls and of the correspondence between p and (δ, w).

3.3. The Induced Order and Topology

By Prop. 2.14 (5), the partial order induced by p is $(x, r) \sqsubseteq (y, s)$ iff $d(x, y) \leq r - s$. This is exactly the order of (Edalat and Heckmann, 1997). Or, saying it the other way round, the order of (Edalat and Heckmann, 1997) can be characterised by $a \sqsubseteq b$ iff $p(a, b) \leq p(a, a)$.

As shown in (Edalat and Heckmann, 1997), the way-below relation '$\ll$' of $\mathbf{B}X$ can be characterised by $(x, r) \ll (y, s)$ iff $d(x, y) < r - s$. Thus, we obtain $a \ll b$ iff $p(a, b) < p(a, a)$ for a, b in $\mathbf{B}X$. It is also shown that $\mathbf{B}X$ is a continuous poset; its Scott topology is generated by the basic open sets $\Uparrow a$ with a in $\mathbf{B}X$ as usual.

THEOREM 3.2. *The topology induced by the weak partial metric p on formal balls is the Scott topology of the induced preorder.*

Proof. For (x, r) in $\mathbf{B}X$ and $\epsilon > 0$, (y, s) is in $O((x, r), \epsilon)$ iff $d(x, y) + r + s < 2r + \epsilon$, iff $d(x, y) < r + \epsilon - s$, iff $(x, r + \epsilon) \ll (y, s)$ using the characterisation of '$\ll$'. Hence, $O((x, r), \epsilon) = \Uparrow(x, r + \epsilon)$ is Scott open.

Conversely, consider $\Uparrow(x, r)$ for (x, r) in $\mathbf{B}X$. If $r = 0$, then $\Uparrow(x, r) = \emptyset$ is p-open. Otherwise, $\Uparrow(x, r) = O((x, r/2), r/2)$ holds by the equality derived in the first part of the proof. Thus, $\Uparrow(x, r)$ is p-open. □

Since $p((x, 0), (y, 0)) = d(x, y)$, the function $i : X \to \mathbf{B}X$ with $ix = (x, 0)$ is an isometric embedding. By Prop. 2.12, we directly obtain the result of (Edalat and Heckmann, 1997) that X with the metric topology is homeomorphic to $\{(x, 0) \in \mathbf{B}X \mid x \in X\}$ with the relative Scott topology.

3.4. Completion via Partial Metrics

For a metric space (X, d), we first construct the weak partial metric space $(P, p) = (\mathbf{B}X, p)$ of formal balls. By Theorem 3.2 and Theorem 2.13, $p : P \times P \to \mathbf{R}_+^{\mathrm{op}}$ is Scott continuous. Let $\widehat{P}$ be the *rounded ideal completion* of P which consists of the rounded ideals of P ordered by subset inclusion. Here, *ideals* are directed lower sets, and a set I is *rounded* if for all a in I, there is b in I such that $a \ll b$.

The rounded ideal completion $(\widehat{P}, \subseteq)$ is a continuous dcpo, the function $e : P \to \widehat{P}$ with $ea = \Downarrow a$ is a continuous order embedding, and the set $\{ea \mid a \in P\}$ is a basis of $\widehat{P}$.

On $\widehat{P}$, a function $\widehat{p}$ is defined by $\widehat{p}(I, J) = \inf\{p(a, b) \mid a \in I, b \in J\}$. Here, 'inf' refers to the usual ordering on $\mathbf{R}_+$; thus, it is the least upper bound in $\mathbf{R}_+^{\mathrm{op}}$. Hence, $\widehat{p} : \widehat{P} \times \widehat{P} \to \mathbf{R}_+^{\mathrm{op}}$ is Scott continuous. Using Scott continuity of p and continuity of the poset $P = \mathbf{B}X$, we obtain

$$\widehat{p}(ea, eb) = \inf\{p(a', b') \mid a' \ll a,\ b' \ll b\} = p(\bigsqcup_{a' \ll a} a', \bigsqcup_{b' \ll b} b') = p(a, b).$$

Since $\{ea \mid a \in P\}$ is a basis of $\widehat{P}$ and $\widehat{p}$ is Scott continuous, all equalities and inequalities satisfied by p carry over to $\widehat{p}$. Therefore, $\widehat{p}$ is a weak partial pseudometric on $\widehat{P}$. Analogously, we define $\widehat{w} : \widehat{P} \to \mathbf{R}_+$ by $\widehat{w}I = \inf\{wa \mid a \in I\} = \inf\{r \mid (x, r) \in I\}$. Again, $\widehat{w}$ is Scott continuous, $\widehat{w} \circ e$ is w, and the equality $\widehat{p}(a, a) = 2\,\widehat{w}a$ holds.

In the sequel, we show several properties of $(\widehat{P}, \widehat{p})$.

PROPOSITION 3.3. *The preorder induced by $\widehat{p}$ is set inclusion.*

Proof. If $I \subseteq J$, then $\{p(a, b) \mid a \in I,\ b \in I\} \subseteq \{p(a, b) \mid a \in I,\ b \in J\}$, whence $\widehat{p}(I, I) \geq \widehat{p}(I, J)$. Conversely, assume $\widehat{p}(I, J) \leq \widehat{p}(I, I)$, and let a be in I. By roundedness, there is b in I such that $a \ll b$, whence $p(a, b) < p(a, a)$. Let $\epsilon = p(a, a) - p(a, b) > 0$. Since $\widehat{p}(I, J) < \widehat{p}(I, I) + \epsilon$, there are c in I and d in J such that $p(c, d) < \widehat{p}(I, I) + \epsilon$. Let u be an upper bound of b and c in I. Together, we have $a \ll b \sqsubseteq u \sqsupseteq c$. The triangle inequation gives us

$$\begin{aligned} p(a, d) \leq\ & p(a, b) + p(b, u) + p(u, c) + p(c, d) \\ & - p(b, b) - p(u, u) - p(c, c). \end{aligned}$$

The relations $b \sqsubseteq u$ and $c \sqsubseteq u$ imply $p(b, u) - p(b, b) \leq 0$ and $p(c, u) - p(c, c) \leq 0$. Thus we obtain

$$p(a, d) \leq p(a, b) + p(c, d) - p(u, u).$$

Since u is in I, $p(c, d) < \widehat{p}(I, I) + \epsilon \leq p(u, u) + \epsilon$ holds, whence

$$p(a, d) \leq p(a, b) + \epsilon = p(a, a).$$

Thus $a \sqsubseteq d$, and since d is in the lower set J, a is in J as well. □

PROPOSITION 3.4. *$(\widehat{P}, \widehat{p})$ is a weak partial metric space.*

Proof. The induced preorder '$\subseteq$' is a partial order. Hence, $\widehat{p}$ is a weak partial metric (has the $\mathcal{T}_0$-property) by Prop. 2.16. □

PROPOSITION 3.5. *An element I of $\widehat{P}$ is maximal in $\widehat{P}$ iff $\widehat{w}I = 0$ holds.*

Proof. Elements I with $\widehat{w}I = 0$ are maximal by Prop. 2.17. Conversely, let I be maximal, and let $s = \widehat{w}I = \inf\{r \mid (x, r) \in I\}$. It is easily verified that $J = \{(x, r - s) \mid (x, r) \in I\}$ is a rounded ideal. Since $(x, r) \sqsubseteq (x, r - s)$ and J is lower, $I \subseteq J$ holds, whence $I = J$ follows by maximality of I. Thus, $\widehat{w}I = \widehat{w}J = \widehat{w}I - s = 0$. □

Let $\overline{X}$ be the subset of maximal elements of $\widehat{P}$, and let $\overline{d}$ be the restriction of $\widehat{p}$ to $\overline{X}$. Since $\overline{d}$ is a weak partial metric with $\overline{d}(I, I) = 2\,\widehat{w}I = 0$ for all I in $\overline{X}$, $\overline{d}$ is a proper metric.

For x in X, let $\eta x = e(ix) = \Downarrow(x, 0)$ in $\widehat{P}$. Since $\widehat{w}(\Downarrow(x, 0)) = w(x, 0) = 0$, η in fact maps into $\overline{X}$. It is an isometric embedding $\eta : X \hookrightarrow \overline{X}$ because e and i are isometric. Once we have shown that $(\overline{X}, \overline{d})$ is complete and the η-image of X is dense in $\overline{X}$, we can conclude that $(\overline{X}, \overline{d})$ is the metric completion of (X, d).

PROPOSITION 3.6. *The η-image of X is dense in $\overline{X}$.*

Proof. Let I be in $\overline{X}$, i.e., I in $\widehat{P}$ with $\widehat{w}I = 0$. For every n in N, there is (x_n, r_n) in I with $r_n < 2^{-n}$. We claim $\eta x_n \to I$.
For every n, $(x_n, 2^{-n}) \ll (x_n, 0)$ holds, whence $(x_n, 2^{-n})$ is in ηx_n. Thus,

$$\overline{d}(\eta x_n, I) \leq p\left((x_n, 2^{-n}), (x_n, r_n)\right) \leq 2^{-n} + r_n < 2 \cdot 2^{-n}. \quad \square$$

PROPOSITION 3.7. *$(\overline{X}, \overline{d})$ is complete.*

Proof. Let $(I_n)_{n \in \mathrm{N}}$ be a Cauchy sequence in $(\overline{X}, \overline{d})$. Let $n_0 = 0$. For every k in N, there is $n_k > n_{k-1}$ in N such that $\overline{d}(I_i, I_j) \leq 2^{-(k+1)}$ for all $i, j \geq n_k$. For every k in N, let $J_k = \{(x, r + 2^{-k}) \mid (x, r) \in I_{n_k}\}$. It is easy to show that J_k is a rounded ideal. Obviously, $\widehat{p}(I, J_k) = \widehat{p}(I, I_{n_k}) + 2^{-k}$ holds for arbitrary I in $\widehat{P}$. Using this property twice, we obtain $\widehat{p}(J_k, J_l) = \overline{d}(I_{n_k}, I_{n_l}) + 2^{-k} + 2^{-l}$, and thus

$$\begin{aligned}
\widehat{p}(J_k, J_{k+1}) &= \overline{d}(I_{n_k}, I_{n_{k+1}}) + 2^{-k} + 2^{-(k+1)} \\
&\leq 2^{-(k+1)} + 2^{-k} + 2^{-(k+1)} \\
&= 2 \cdot 2^{-k} \\
&= \overline{d}(I_{n_k}, I_{n_k}) + 2 \cdot 2^{-k} \\
&= \widehat{p}(J_k, J_k).
\end{aligned}$$

By Prop. 3.3, $J_k \subseteq J_{k+1}$ follows. Thus, $J = \bigcup_{k \in \mathrm{N}} J_k$ is a rounded ideal; it is the least upper bound of the ascending sequence $(J_k)_{k \in \mathrm{N}}$ in $\widehat{P}$. By continuity of $\widehat{w} : \widehat{P} \to \mathbf{R}_+^{\mathrm{op}}$, $\widehat{w}J = \inf_{k \in \mathrm{N}} \widehat{w}J_k = \inf_{k \in \mathrm{N}} 2^{-k} = 0$ holds, whence J is in $\overline{X}$. Using monotonicity of $\widehat{p} : \widehat{P} \times \widehat{P} \to \mathbf{R}_+^{\mathrm{op}}$,

$$\overline{d}(I_{n_k}, J) \leq \widehat{p}(I_{n_k}, J_k) = \overline{d}(I_{n_k}, I_{n_k}) + 2^{-k} = 2^{-k}$$

holds, which proves $I_{n_k} \to J$ as $k \to \infty$. Since $(I_n)_{n \in \mathbf{N}}$ is a Cauchy sequence, $I_n \to J$ follows. □

Summarising, we obtain the following theorem:

THEOREM 3.8. *The completion* $(\overline{X}, \overline{d})$ *of a metric space* (X, d) *is given by*

$$\overline{X} = \mathit{Max}\,(\mathcal{I}(\mathbf{B}X)) = \{I \in \mathcal{I}(\mathbf{B}X) \mid \inf\{r \mid (x, r) \in I\} = 0\}$$

$$\overline{d}(I, J) = \inf\{d(x, y) + r + s \mid (x, r) \in I,\, (y, s) \in J\}.$$

4. Conclusion and Future Work

We have generalised and extended the work of Matthews and O'Neill on partial metric spaces, and applied our results to the weak partial metric space $\mathbf{B}X$ on formal balls which approximate the points of a metric space X. Using $\mathbf{B}X$, a domain-theoretic construction of the metric completion $(\overline{X}, \overline{d})$ of a metric space (X, d) was established.

Some more examples of weak partial metric spaces should be studied. The work of Matthews and O'Neill on convergence of sequences in partial metric spaces should be extended. Also, O'Neill's partial metric completion which uses Cauchy sequences should be compared with rounded ideal completion.

Criteria should be found which ensure that a weak partial metric space with its induced partial order and topology is a continuous (or algebraic) dcpo with its Scott topology. Conversely, the question appears which continuous dcpo's are partially metrisable.

The special kind of relationship which exists between $\mathbf{B}X$ and X should be characterised in more abstract terms, so that it can be seen which features of $\mathbf{B}X$ are essential to obtain the metric completion from the rounded ideal completion.

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