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Ordered Equilogical Spaces and Equilogical D-Spaces

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Abstract

Equilogical spaces form a cartesian closed complete category that contains all $\mathcal{T}_0$ spaces. We first present the cartesian closed complete subcategory of *ordered equilogical spaces* that still contains all $\mathcal{T}_0$ spaces, but excludes certain pathological equilogical spaces. Then, we further restrict to the cartesian closed complete subcategory of *equilogical d-spaces* that still contains all dcpo's and all $\mathcal{T}_1$ spaces. Its main advantage is that equilogical d-spaces with least element admit a least fixed point operator.

1 Introduction

In December 1996, Dana Scott and his group proposed the category EQU of *equilogical spaces* [5,1,2], a cartesian closed complete category that contains TOP_0, the category of $\mathcal{T}_0$ topological spaces, as a full subcategory. Section 2 recalls the main definitions and basic facts.

Topological spaces carry a preorder, the *specialisation preorder*, that becomes an ordering for $\mathcal{T}_0$ spaces. In Section 3, we define a specialisation preorder for equilogical spaces, and a so-called *path ordering*. While both orderings coincide for those equilogical spaces that come from $\mathcal{T}_0$ topological spaces, they are quite different in general. In fact, there are some equilogical spaces with a rather peculiar structure very different from the structure of topological spaces; we shall present some examples in Section 3. These examples are ruled out by imposing some restrictions on equilogical spaces. The resulting subcategory is presented in Section 4. It is again cartesian closed and complete, and still contains TOP_0.

D-spaces are TOP_0-spaces which are dcpo's in their specialisation preorder with the additional property that all open sets are Scott open (these spaces are called *monotone convergence spaces* in [3, Def. II.3.9]). In Section 5, we shall present the cartesian closed complete category of equilogical d-spaces which intersects TOP_0 exactly in the class of topological d-spaces. The nice thing about this category is that equilogical d-spaces with $\perp$ enjoy a least fixed point theorem and a fixed point morphism, admitting the solution of recursive equations.

2 Equilogical Spaces

2.1 The Category EQU

In [1], Scott and his group present three equivalent categories whose objects are equivalence relations on $\mathcal{T}_0$ spaces, partial equivalence relations on algebraic lattices, and so-called modest sets over algebraic lattices. Officially, EQU is the first of these categories. Here, we prefer to work with the last category because its morphisms are functions (not equivalence classes of functions), and its function space construction is very simple. In a slightly sloppy way, we shall transfer the name EQU to this category. The reference for the following is Section 4 of [1].

An object of EQU, called *equilogical space*, is a triple (X, E, L) where X is a set (the carrier set), L an algebraic lattice, and E a function from X to the powerset of L such that all sets $E(x)$ are non-empty and pairwise disjoint. The elements of $E(x)$ are called *realisers* for x.

Let $\mathcal{X} = (X, E, L)$ and $\mathcal{Y} = (Y, E', M)$ be two equilogical spaces. A Scott continuous function $\varphi : L \to M$ is a *realiser* for a function $f : X \to Y$ if for all x in X and a in $E(x)$, φa is in $E(fx)$. A function $f : X \to Y$ is *realisable* if it has at least one realiser. In EQU, the morphisms from $\mathcal{X}$ to $\mathcal{Y}$ are the realisable functions from X to Y. All constant functions are realisable.

Often, we shall follow the custom known from topology to speak of an element or subset of an equilogical space if we mean an element or subset of its carrier set.

2.2 Product and Function Space

The *product* of a family $(\mathcal{X}_i)_{i\in I} = (X_i, E_i, L_i)_{i\in I}$ is $\prod_{i\in I} \mathcal{X}_i = (\prod_{i\in I} X_i, E, \prod_{i\in I} L_i)$ where $E((x_i)_{i\in I}) = \prod_{i\in I} E(x_i)$. For $\mathcal{X} = (X, E, L)$ and $\mathcal{Y} = (Y, E', L)$, the function space $[\mathcal{X} \to \mathcal{Y}]$ is $(F, E'', [L \to M])$ where F is the set of realisable functions from $\mathcal{X}$ to $\mathcal{Y}$, $[L \to M]$ is the algebraic lattice of Scott continuous functions from L to M, and $E(f)$ is the set of realisers of f.

With these constructions, EQU becomes a cartesian closed category with products. The necessary proofs are quite simple; projections are realised by projections, application by application, curried functions by curried realisers etc.

2.3 The Embedding of TOP$_0$

Given a topological space X, there exist algebraic lattices L with a topological embedding $e : X \hookrightarrow L$. For any such e and L, $(X, \bar{e}, L)$ with $\bar{e}(x) = \{e(x)\}$ forms an equilogical space that corresponds to the topological space X in the following sense:

Proposition 2.1 *Let X_1 and X_2 be topological spaces with topological embeddings $e_i : X_i \hookrightarrow L_i$ into algebraic lattices L_1 and L_2. Then $f : X_1 \to X_2$ is continuous if and only if $f : (X_1, \bar{e}_1, L_1) \to (X_2, \bar{e}_2, L_2)$ is realisable.*

This proposition shows that TOP$_0$ is equivalent to a full subcategory of EQU. It also shows that the choice of the lattice does not matter: In the case $X_1 = X_2 = X$, the identity from X_1 to X_2 and vice versa is continuous, hence realisable; so we get $(X, \bar{e}_1, L_1) \cong (X, \bar{e}_2, L_2)$.

The notion of isomorphism used here is the usual one: there are morphisms (realisable functions) in both directions which compose to the identities. Note that it is not required that their realisers compose to the identities on the lattices; two equilogical spaces may be isomorphic even if their lattices are very different.

2.4 Embeddings and Subspaces

Because of the concreteness of (the modest set representation of) EQU, it is immediate that a morphism is monic if and only if it is injective. This is the same situation as in TOP, but there, monics play a minor role compared with the stronger topological embeddings. A similar notion can be defined for EQU: A monomorphism $e : \mathcal{X} \to \mathcal{Y}$ is an *equilogical embedding* iff for all (set-theoretic) functions $f : \mathcal{Z} \to \mathcal{X}$, the realisability of $e \circ f : \mathcal{Z} \to \mathcal{Y}$ implies the realisability of f.

This notion is compatible with the corresponding topological notion: if two equilogical spaces $\mathcal{X}$ and $\mathcal{Y}$ come from $\mathcal{T}_0$ topological spaces X and Y, then $e : \mathcal{X} \to \mathcal{Y}$ is an equilogical embedding if and only if $e : X \to Y$ is a topological embedding.

Given an equilogical space $\mathcal{X} = (X, E, L)$ and a subset S of X, there is an equilogical space $(S, E|_S, L)$, called the *subspace* S of $\mathcal{X}$. In this case, the subset inclusion is an equilogical embedding, and conversely, if $e : \mathcal{X} \hookrightarrow \mathcal{Y}$ is any equilogical embedding, then $\mathcal{X}$ is isomorphic to the subspace $e(\mathcal{X})$ of $\mathcal{Y}$.

The *equaliser* of two realisable functions $f, g : \mathcal{X} \to \mathcal{Y}$ is the subspace $\{x \in \mathcal{X} \mid f(x) = g(x)\}$ of $\mathcal{X}$. In EQU, it happens that all subspace embeddings are equalisers of suitable pairs of morphisms, but this is no longer true in most of the subcategories of EQU considered later.

2.5 The Induced Topology

Let Ω be the equilogical Sierpinski space, i.e., $\Omega = (\{0,1\}, E, \{0,1\})$ with $0 \leq 1$ and $E(x) = \{x\}$ for $x = 0, 1$. This space comes from the topological Sierpinski space by using the identical embedding. It can be used to define a topology on (the carrier set of) an equilogical space $\mathcal{X}$ as follows: A subset O of $\mathcal{X}$ is *open* iff its characteristic function $\chi_O : \mathcal{X} \to \Omega$ is realisable. Arbitrary unions and finite intersections of opens are open, so we get indeed a topology on $\mathcal{X}$. We call the resulting topological space $T\mathcal{X}$.

If $f : \mathcal{X} \to \mathcal{Y}$ is realisable, then $f : T\mathcal{X} \to T\mathcal{Y}$ is continuous. The opposite implication does not hold in general, but it holds if $\mathcal{Y}$ comes from a topological space Y; in this case, $T\mathcal{Y}$ is Y.

3 Ordering Equilogical Spaces

In those branches of topology where non-Hausdorff spaces are studied, the notion of specialisation preorder '$\sqsubseteq_{\mathrm{S}}$' is useful. We shall generalise this notion to equilogical spaces, and compare it with the path ordering '$\sqsubseteq_{\mathrm{P}}$', a notion inspired by research in synthetic domain theory.

The specialisation preorder '$\sqsubseteq_{\mathrm{S}}$' on an equilogical space $\mathcal{X}$ is defined as the specialisation preorder of the induced topological space $T\mathcal{X}$, i.e., for two elements x and x' of $\mathcal{X}$, we have $x \sqsubseteq_{\mathrm{S}} x'$ if for all realisable functions $u : \mathcal{X} \to \Omega$, $ux \leq ux'$ holds. This relation is a preorder, i.e., reflexive and transitive, but not anti-symmetric in general. Since realisable functions are continuous, we get monotonicity: $x \sqsubseteq_{\mathrm{S}} x' \Rightarrow fx \sqsubseteq_{\mathrm{S}} fx'$ for realisable f.

The *path ordering* on $\mathcal{X}$ is defined by $x \sqsubseteq_{\mathrm{P}} x'$ iff the function mapping 0 to x and 1 to x' is a realisable function from Ω to $\mathcal{X}$. This is equivalent to the existence of a in $E(x)$ and a' in $E(x')$ such that $a \leq a'$. Again, realisable functions are monotonic. The relation $x \sqsubseteq_{\mathrm{P}} x'$ implies $x \sqsubseteq_{\mathrm{S}} x'$, but the converse is not true.

Note that, despite its name, the path ordering is not an ordering; it is trivially reflexive, but in general, it is neither transitive nor anti-symmetric, as the following examples show:

(i) $\mathcal{X}_1$ with $X = \{a, b\}$, $L = \{0 < 1 < 2\}$, and $E(a) = \{0, 2\}$ and $E(b) = \{1\}$. Then $a \sqsubseteq_{\mathrm{P}} b$, $b \sqsubseteq_{\mathrm{P}} a$, and same for '$\sqsubseteq_{\mathrm{S}}$'.

(ii) $\mathcal{X}_2$ with $X = \{a, b, c\}$, L the powerset of $\{0, 1, 2\}$ and $E(a) = \{\{0\}\}$, $E(b) = \{\{0, 1\}, \{1\}\}$, and $E(c) = \{\{1, 2\}\}$. Then $a \sqsubseteq_{\mathrm{P}} b$ and $b \sqsubseteq_{\mathrm{P}} c$, but not $a \sqsubseteq_{\mathrm{P}} c$. The specialisation preorder is the chain $a \sqsubseteq_{\mathrm{S}} b \sqsubseteq_{\mathrm{S}} c$ (including the relation $a \sqsubseteq_{\mathrm{S}} c$).

(iii) $\mathcal{X}_3$ with $X = \{a, b\}$, $L = \{0 < 1 < 2 < \cdots < \infty\}$ and $E(a) = L \setminus \{\infty\}$ and $E(b) = \{\infty\}$. Then $a \sqsubseteq_{\mathrm{S}} b$, $b \sqsubseteq_{\mathrm{S}} a$, $a \sqsubseteq_{\mathrm{P}} b$, but not $b \sqsubseteq_{\mathrm{P}} a$.

We consider these spaces as more or less pathological. In the next section, some restrictions will be imposed on equilogical spaces that exclude these pathological examples. Before doing so, we show how the two "orderings" are affected by the constructions of equilogical spaces considered here.

- If $\mathcal{X}$ comes from a $\mathcal{T}_0$ topological space X, then '$\sqsubseteq_{\mathrm{P}}$' and '$\sqsubseteq_{\mathrm{S}}$' coincide with each other and the specialisation preorder of X, which is anti-symmetric by the $\mathcal{T}_0$ property.
- If $e : \mathcal{X} \hookrightarrow \mathcal{Y}$ is an equilogical embedding, i.e., $\mathcal{X}$ is (isomorphic to) a subspace of $\mathcal{Y}$, then $x \sqsubseteq_{\mathrm{P}} x' \Leftrightarrow ex \sqsubseteq_{\mathrm{P}} ex'$ and $x \sqsubseteq_{\mathrm{S}} x' \Rightarrow ex \sqsubseteq_{\mathrm{S}} ex'$. The opposite implication does not hold in general; for instance, in $\mathcal{X}_2$, $a \sqsubseteq_{\mathrm{S}} c$ holds, but in its subspace $\{a, c\}$, $a \sqsubseteq_{\mathrm{S}} c$ does not hold.
- On products, '$\sqsubseteq_{\mathrm{P}}$' is componentwise, and $(x_i)_{i \in I} \sqsubseteq_{\mathrm{S}} (x'_i)_{i \in I}$ implies $x_i \sqsubseteq_{\mathrm{S}} x'_i$ for all i in I by the realisability of projection. We do not know whether the converse holds or not.
- For f and g in the function space $[\mathcal{X} \to \mathcal{Y}]$, realisability of $f \mapsto fx$ gives us the implication $f \sqsubseteq_{\mathrm{P}} g \Rightarrow fx \sqsubseteq_{\mathrm{P}} gx \ \forall x \in \mathcal{X}$, and likewise for '$\sqsubseteq_{\mathrm{S}}$'. Unfortunately, the opposite implication does not hold for '$\sqsubseteq_{\mathrm{P}}$'. For an example, consider $f, g : \Omega \to \mathcal{X}_2$ where $f(0) = a$, $f(1) = g(0) = b$, and $g(1) = c$. These functions are realisable, i.e., members of $[\Omega \to \mathcal{X}_2]$, and $fx \sqsubseteq_{\mathrm{P}} gx$ holds for all x in Ω, but $f \sqsubseteq_{\mathrm{P}} g$ does *not* hold.

4 Ordered Equilogical Spaces

In the previous section, we saw that equilogical spaces provide some pathologies that do not occur in TOP_0: path ordering not transitive, path ordering different from specialisation preorder, one or both of them not anti-symmetric, the order on functions not pointwise. Now, we introduce a restriction on equilogical spaces that rules out these pathologies. We admit that this restriction looks a bit ad hoc. It emerged from experimenting with several properties and comparing their consequences and also the difficulty to prove these consequences. Another problem is the name "ordered space". In topology, it means a (Hausdorff) space with an extra ordering that is different from the specialisation preorder in general. Here we mean a space where specialisation preorder and path "order" are well-behaved.

Definition 4.1 An equilogical space (X, E, L) is *directly ordered* if for all $x \sqsubseteq_{\mathrm{P}} x'$ and all $A \subseteq E(x) \cup E(y)$ that meet both $E(x)$ and $E(x')$, the join $\bigvee A$ is in $E(x')$. An equilogical space is *ordered* if it is isomorphic to a directly ordered space.

The distinction between ordered and directly ordered comes from the unfortunate fact that of two isomorphic spaces, one may be directly ordered, but the other one not. We do not know how to characterise ordered spaces in a way that is invariant under isomorphism. In the sequel, our results will be formulated for ordered spaces, but proved by considering directly ordered spaces.

The defining property of directly ordered spaces can be split into two properties:

(i) All sets $E(x)$ are closed under non-empty joins.

(ii) If $x \sqsubseteq_{\mathrm{P}} x'$, then $a \in E(x)$ and $b \in E(x')$ implies $a \vee b \in E(x')$.

Here, (i) holds because of $x \sqsubseteq_{\mathrm{P}} x$, and (ii) is the defining property for $A = \{a, b\}$. Conversely, (i) and (ii) imply the defining property since $\bigvee A = \bigvee(A \cap E(x)) \vee \bigvee(A \cap E(x'))$.

By property (i), all sets $E(x)$ have a greatest element $\hat{E}(x) := \bigvee E(x) \in E(x)$. The two orderings can be characterised in terms of this greatest element:

Proposition 4.2 *For x and x' in a directly ordered space, $x \sqsubseteq_{\mathrm{P}} x'$, $x \sqsubseteq_{\mathrm{S}} x'$, and $\hat{E}(x) \leq \hat{E}(x')$ are equivalent.*

Proof. $\hat{E}(x) \leq \hat{E}(x')$ obviously implies $x \sqsubseteq_{\mathrm{P}} x'$. Conversely, if $x \sqsubseteq_{\mathrm{P}} x'$, then $\hat{E}(x) \vee \hat{E}(x')$ is in $E(x')$ by (ii), hence $\hat{E}(x) \leq \hat{E}(x) \vee \hat{E}(x') \leq \hat{E}(x')$ follows. The implication $x \sqsubseteq_{\mathrm{P}} x' \Rightarrow x \sqsubseteq_{\mathrm{S}} x'$ holds in all equilogical spaces. For the opposite direction, assume $x \not\sqsubseteq_{\mathrm{P}} x'$. Let $u : X \to \Omega$ be the function that maps all $y \sqsubseteq_{\mathrm{P}} x$ to 0, and all other y to 1. Similarly, let $v : L \to \Omega$ be the function that maps all $b \leq \hat{E}(x)$ to 0, and all other b to 1. Function v is Scott continuous and realises u because if a is in $E(y)$, then $a \leq \hat{E}(x)$ iff $y \sqsubseteq_{\mathrm{P}} x$. Hence, $u^{-1}(\{1\})$ is an open set of the induced topology that contains x, but not x', and so we conclude $x \not\sqsubseteq_{\mathrm{S}} x'$. $\square$

By the disjointness condition of equilogical spaces, $\hat{E}(x) = \hat{E}(x')$ implies $x = x'$. Thus, '$\sqsubseteq_{\mathrm{P}}$ / $\sqsubseteq_{\mathrm{S}}$' is anti-symmetric in directly ordered spaces. Since both orderings are defined via morphisms and thus compatible with isomorphisms, we can extend these results to all ordered spaces:

Theorem 4.3 *In all ordered spaces, '$\sqsubseteq_{\mathrm{P}}$' and '$\sqsubseteq_{\mathrm{S}}$' are anti-symmetric and coincide. (Hence, the index can be omitted.)*

Before we come to the closure properties of the class of ordered spaces, we present an auxiliary lemma.

Lemma 4.4 *Let $f, g : \mathcal{X} \to \mathcal{Y}$ be realisable where $\mathcal{Y}$ is directly ordered. If $fx \sqsubseteq gx$ for all x in $\mathcal{X}$, then $\varphi \in E(f)$ and $\psi \in E(g)$ implies $\varphi \vee \psi \in E(g)$; therefore, $f \sqsubseteq g$ follows.*

Proof. First, $\varphi \vee \psi$ is Scott continuous as the (pointwise) join of two Scott continuous functions. For any x in $\mathcal{X}$ and a in $E(x)$, φa is in $E(fx)$ and ψa in $E(gx)$. Since $fx \sqsubseteq_{\mathrm{P}} gx$ by hypothesis, property (ii) implies $\varphi a \vee \psi a$ in $E(gx)$. Thus, $\varphi \vee \psi$ realises g. Since $E(f)$ and $E(g)$ are not empty, the addition $f \sqsubseteq_{\mathrm{P}} g$ follows. $\square$

The class of ordered spaces has all the closure properties we like:

Theorem 4.5 *The class of ordered equilogical spaces contains all spaces coming from $\mathcal{T}_0$ topological spaces, and is closed under subspace, product, and ex-*

ponentiation, i.e., if $\mathcal{Y}$ is ordered, then so is $[\mathcal{X} \to \mathcal{Y}]$ for any $\mathcal{X}$. Moreover, the order on realisable functions is pointwise.

Proof. The theorem is best proved by considering (i) and (ii) independently. Property (ii) for exponentiation essentially is Lemma 4.4 which gives also the fact about the pointwise ordering. $\square$

The class of ordered spaces is not closed under the quotients of EQU; in fact, any equilogical space whatsoever is the quotient of a space coming from a $\mathcal{T}_0$ topological space.

Theorem 4.3 shows that the three examples of the previous section are not ordered spaces, in particular, they are not topological (isomorphic to a space coming from a $\mathcal{T}_0$ topological space).

Theorem 4.6 *All finite ordered spaces are topological.*

Proof. Finite means finite carrier, a property preserved by isomorphism. Thus, we may assume a directly ordered space $\mathcal{X} = (X, E, L)$ with finite X. Let $\mathcal{Y} = (X, E', L)$ where $E'(x) = \{\hat{E}(x)\}$. We show $\mathcal{X} \cong \mathcal{Y}$. The set-theoretic identity from $\mathcal{Y}$ to $\mathcal{X}$ is realised by the identity on L. The realiser for the opposite direction is given by $\varphi a = \bigwedge\{\hat{E}(x) \mid \hat{E}(x) \geq a\}$. This function is Scott continuous by finiteness of X. It realises the identity since $a \in E(x)$ implies $\varphi a = \hat{E}(x)$; for '$\geq$', property (ii) is needed. By the disjointness condition of equilogical spaces, $\hat{E} : X \to L$ is injective. Hence, we topologise X as a subspace of L with the Scott topology. The equilogical space $\mathcal{Y}$ comes from this topological space X. $\square$

Apart from the restrictions leading to ordered spaces, other restrictions may be imposed, e.g., closure of $E(x)$ under binary meet or order-convexity of $E(x)$. The resulting classes satisfy closure theorems analogous to Theorem 4.5. The smallest such class is the class of Ω-*embedded spaces* introduced in [4]. A space $\mathcal{X}$ is Ω-embedded if it can be embedded into a function space $[\mathcal{Y} \to \Omega]$ for some space $\mathcal{Y}$. From Theorem 4.5, it is clear that all Ω-embedded spaces are ordered; we do not know about the converse.

5 Equilogical D-Spaces

In this section, we further restrict the ordered spaces so that they obtain a well-behaved dcpo structure. If there is in addition a least element, we get a realisable least fixed point operator.

5.1 Definition of D-Spaces

We start with some notation. For an equilogical space (X, E, L), E is extended to subsets S of X by defining $E(S) = \bigcup_{x \in S} E(x)$. A subset A of L *realises* a subset S of X if $A \subseteq E(S)$ and $A \cap E(x) \neq \emptyset$ for all x in S.

Lemma 5.1 *Let (X, E, L) be directly ordered. If $A \subseteq L$ realises $S \subseteq X$ and $\bigvee A$ is in $E(y)$, then y is the least upper bound $\bigsqcup S$ of S.*

Proof. For every x in S, there is a in $A \cap E(x)$. Since $a \leq \bigvee A \in E(y)$, $x \sqsubseteq y$ follows. Let z be an upper bound of S. Each $a \in A$ is in $E(x)$ for some x in S, and $x \sqsubseteq z$ implies $a \leq \hat{E}(z)$. Thus, $\bigvee A \leq \hat{E}(z)$, and as $\bigvee A$ is in $E(y)$, $y \sqsubseteq z$ follows. □

Theorem 5.2 *For a directly ordered space (X, E, L), the following are equivalent:*

(i) *$E(X)$ is closed under directed joins.*

(ii) *If a directed subset A of L realises a directed subset D of X, then $\bigvee A$ is in $E(x)$ for some x in X.*

(iii) *If an arbitrary subset A of L realises a directed subset D of X, then $\bigvee A$ is in $E(x)$ for some x in X.*

(iv) *All directed subsets D of X have a least upper bound $\bigsqcup D$, and $\bigvee A$ is in $E(\bigsqcup D)$ for all directed sets $A \subseteq L$ realising D.*

(v) *All directed subsets D of X have a least upper bound $\bigsqcup D$, and $\bigvee A$ is in $E(\bigsqcup D)$ for all sets $A \subseteq L$ realising D.*

Definition: *Directly ordered spaces satisfying these conditions are called* direct d-spaces. *An equilogical space is a* d-space *if it is isomorphic to a direct d-space.*

Proof.

(i) $\Rightarrow$ (ii): If A realises D, then $A \subseteq E(X)$. By (i), $\bigvee A$ is in $E(x)$ for some x in X.

(ii) $\Rightarrow$ (iii): Let $\mathcal{F}$ be the set of non-empty finite subsets of A. For all F in $\mathcal{F}$, we define $x_F \in D$ and $b_F \in E(x_F)$ such that $G \subseteq F \Rightarrow b_G \leq b_F$. This is done by induction over the size of F.

For $F = \{a\}$, let $b_F = a$. As $A \subseteq E(D)$, there is some x_F in D such that $b_F \in E(x_F)$. For F in $\mathcal{F}$ with more than one element, let $x_F \in D$ be an upper bound of all x_G with $G \subset F$. Since A realises D, there is some a_F in $A \cap E(x_F)$. Now let $b_F = a_F \vee \bigvee_{G \subset F} b_G$. By iterating property (ii) of directly ordered spaces, b_F is in $E(x_F)$.

By this construction, we obtained a *directed* set $B = \{b_F \mid F \in \mathcal{F}\}$ realising the directed set $D' = \{x_F \mid F \in \mathcal{F}\}$. Condition (ii) of the theorem implies that $\bigvee B$ is in $E(x)$ for some x in X.

The final step is to verify $\bigvee A = \bigvee B$. Here, '$\leq$' holds since $a = b_{\{a\}}$ for all a in A, and '$\geq$' since all b_F are (finite) joins of elements of A.

(iii) $\Rightarrow$ (v): Any directed $D \subseteq X$ is realised by $A = \{\hat{E}(d) \mid d \in D\}$. By (iii), $\bigvee A$ is in $E(x)$ for some x, and by Lemma 5.1, x is the least upper bound of D. The second part of (v) also follows from Lemma 5.1.

(v) $\Rightarrow$ (iv) is immediate.

(iv) $\Rightarrow$ (i): Given a directed subset A of $E(X)$, every a in A is in $E(x_a)$ for some

x_a in X. The set $D = \{x_a \mid a \in A\}$ is directed, and so, $\bigvee A$ is in $E(\bigsqcup D) \subseteq E(X)$.

□

5.2 Properties of D-Spaces

We now present the main properties of equilogical d-spaces.

Proposition 5.3 *Let f be a realisable function from a d-space $\mathcal{X}$ to an ordered space $\mathcal{Y}$. If D is a directed subset of $\mathcal{X}$, then $f(D)$ is a directed subset of $\mathcal{Y}$, and $f(\bigsqcup D)$ is the least upper bound of $f(D)$.*

Proof. First, replace $\mathcal{X}$ by an isomorphic direct d-space, and $\mathcal{Y}$ by an isomorphic directly ordered space.—$f(D)$ is directed because f is monotonic. Let φ be a realiser of f. For every d in D, $\hat{E}(d)$ is in $E(d)$, hence $\varphi(\hat{E}(d))$ in $E(fd)$. Thus, $A = \{\hat{E}(d) \mid d \in D\}$ is a directed set realising D, and $\varphi(A)$ realises $f(D)$. Since $\mathcal{X}$ is a direct d-space, $\bigvee A$ is in $E(\bigsqcup D)$, hence $\varphi(\bigvee A)$ in $E(f(\bigsqcup D))$. By Scott continuity of φ, $\bigvee \varphi(A) \in E(f(\bigsqcup D))$ follows. Since $\varphi(A)$ realises $f(D)$, $f(\bigsqcup D)$ is the least upper bound of $f(D)$ by Lemma 5.1. □

From the proposition, we may immediately conclude:

Theorem 5.4 *All d-spaces are dcpo's with '$\sqsubseteq$', and all realisable functions between d-spaces are Scott continuous.*

Recall that the open sets of $T\mathcal{X}$ are defined via realisable maps $u : \mathcal{X} \to \Omega$. As Ω obviously is a (direct) d-space, we obtain:

Theorem 5.5 *If $\mathcal{X}$ is an equilogical d-space, all open sets of the induced topological space $T\mathcal{X}$ are Scott open.*

5.3 Closure Properties of the Class of D-Spaces

First, we consider subspaces of d-spaces.

Proposition 5.6 *If $\mathcal{X}$ is a d-space, and $Y \subseteq \mathcal{X}$ is closed under least upper bounds of directed sets, then the subspace Y of $\mathcal{X}$ is a d-space.*

Theorem 5.7 *If $\mathcal{X}$ is a d-space, $\mathcal{Y}$ an ordered space, and $f, g : \mathcal{X} \to \mathcal{Y}$ are realisable functions, then the equaliser $\{x \in \mathcal{X} \mid fx = gx\}$ is a d-space.*

Proof. If D is a directed subset of the equaliser, then $f(\bigsqcup D) = \bigsqcup f(D) = \bigsqcup g(D) = g(\bigsqcup D)$ holds by Proposition 5.3, whence $\bigsqcup D$ is in the equaliser. By Proposition 5.6, the equaliser is a d-space. □

Now we consider directed sets of functions.

Lemma 5.8 *If $\mathcal{X} = (X, E, L)$ is arbitrary, $\mathcal{Y} = (Y, E', M)$ is a direct d-space, and $\mathcal{F}$ is a directed set in $[\mathcal{X} \to \mathcal{Y}]$ realised by $\Phi \subseteq [L \to M]$, then the pointwise least upper bound $g = x \mapsto \bigsqcup_{f \in \mathcal{F}} fx$ is well-defined and realised by $\bigvee \Phi$.*

Proof. Since $\mathcal{F}$ is directed, the sets $\mathcal{F}(x) = \{fx \mid f \in \mathcal{F}\}$ are directed for all x in X. Thus, $g = x \mapsto \bigsqcup \mathcal{F}(x)$ is a well-defined function. For all x in X and a in $E(x)$, $\Phi(a) = \{\varphi a \mid \varphi \in \Phi\}$ realises $\mathcal{F}(x)$. Since $\mathcal{Y}$ is a direct d-space, $(\bigvee \Phi)(a) = \bigvee(\Phi(a))$ is in $E'(gx)$, and therefore, $\bigvee \Phi$ realises g. □

From the lemma, we immediately obtain:

Theorem 5.9 *If $\mathcal{X}$ is arbitrary and $\mathcal{Y}$ is a d-space, the pointwise least upper bound of a directed set of realisable functions from $\mathcal{X}$ to $\mathcal{Y}$ is well-defined and realisable again.*

Now we can formulate the closure properties of the class of d-spaces:

Theorem 5.10 *The class of equilogical d-spaces is closed under product, (internal) equalisers, and exponentiation.*

Proof. By internal equalisers, we mean equalisers of $f, g : \mathcal{X} \to \mathcal{Y}$ where both $\mathcal{X}$ and $\mathcal{Y}$ are d-spaces. Such equalisers are d-spaces again by Theorem 5.7. Closure under exponentiation is essentially Lemma 5.8. □

Since Ω is a (direct) d-space, the function space $[\mathcal{X} \to \Omega]$ is a d-space for any equilogical $\mathcal{X}$. By definition, the elements of this function space are in one-to-one correspondence with the open sets of (the induced topology of) $\mathcal{X}$. Hence, we obtain an equilogical d-space $\Omega\mathcal{X} \cong [\mathcal{X} \to \Omega]$ whose elements are the opens of $\mathcal{X}$. By Theorem 5.5, all open sets of (the induced topology of) $\Omega\mathcal{X}$ are Scott open.

5.4 Topological D-Spaces

Now we answer the question which $\mathcal{T}_0$ topological spaces become equilogical d-spaces when embedded into EQU: they are exactly the *topological d-spaces*, i.e., those $\mathcal{T}_0$ spaces which are dcpo's in their specialisation preorder with the additional property that all open sets are Scott open.

By Theorems 5.2 and 5.5, the induced topological space $T\mathcal{X}$ of any equilogical d-space $\mathcal{X}$ is a topological d-space. If $\mathcal{X}$ comes from a $\mathcal{T}_0$ topological space X, then $T\mathcal{X}$ is X. Therefore, we see that if $\mathcal{X}$ is an equilogical d-space coming from a $\mathcal{T}_0$ topological space X, then X must be a topological d-space.

For the opposite direction, let X be a topological d-space and $e : X \hookrightarrow L$ an embedding of X into an algebraic lattice. Then $\mathcal{X} = (X, \bar{e}, L)$ with $\bar{e}(x) = \{e(x)\}$ is ordered by Theorem 4.5, and its ordering is the specialisation preorder of X. Therefore, X with the order of $\mathcal{X}$ is a dcpo. Since $e : X \hookrightarrow L$ is an embedding, it is continuous. Hence, for any (Scott) open set V of L, $e^{-1}(V)$ is an open set of X, and therefore Scott open because X is a topological d-space. Together, we see that $e : X \to L$ is Scott continuous, and thus, $e(\bigsqcup D) = \bigvee e(D)$ holds for all directed $D \subseteq X$. This shows that $(X, \bar{e}, L)$ is a direct d-space.

Summarising, we obtain:

Theorem 5.11 *Let $\mathcal{X}$ be an equilogical space coming from a topological space X. Then $\mathcal{X}$ is an equilogical d-space if and only if X is a topological d-space.*

Clearly, all dcpo's with their Scott topology are topological d-spaces, and also all $\mathcal{T}_1$ topological spaces. Hence, we can find all dcpo's and all $\mathcal{T}_1$ spaces, in particular the space $\mathbb{R}$ of real numbers, inside the class of equilogical d-spaces.

5.5 Least Fixed Points

Since equilogical d-spaces are dcpo's in their induced ordering, and realisable functions are Scott continuous by Theorem 5.4, we immediately obtain:

Theorem 5.12 *If $\mathcal{X}$ is an equilogical d-space with a least element $\bot$, then every realisable function $f : \mathcal{X} \to \mathcal{X}$ has a least fixed point given by $\bigsqcup_{n\in\mathbb{N}} f^n(\bot)$.*

With a little more work, we can get the following stronger result:

Theorem 5.13 *If $\mathcal{X}$ is an equilogical d-space with a least element $\bot$, then $F = f \mapsto \bigsqcup_{n\in\mathbb{N}} f^n(\bot)$ is a realisable function from $[\mathcal{X} \to \mathcal{X}]$ to $\mathcal{X}$. For every f in $[\mathcal{X} \to \mathcal{X}]$, $F(f)$ is the least fixed-point of f.*

Proof. We only need to show that F is realisable. To this end, we define a family $(F_n)_{n\in\mathbb{N}}$ of functions from $[\mathcal{X} \to \mathcal{X}]$ to $\mathcal{X}$ by $F_0(f) = \bot$ and $F_{n+1}(f) = f(F_n(f))$. Function F_0 is realisable as a constant function. If F_n is realisable, then so is F_{n+1} because it can be obtained as a composition of realisable functions: $f \mapsto (f, f) \mapsto (f, F_n(f)) \mapsto f(F_n(f))$, the last step being application. The functions F_n form an ascending sequence of realisable functions, whence their least upper bound F is realisable by Theorem 5.9. □

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