Observable Modules and Power Domain Constructions

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Abstract

An R-module M is observable iff all its elements can be distinguished by observing them by means of linear morphisms from M to R. We show that free observable R-modules can be explicitly described as the cores of the final power domains with characteristic semiring R. Then, the general theory is applied to the cases of the lower and the upper semiring. All lower modules are observable, whereas there are non-observable upper modules. Accordingly, all known lower power constructions coincide, whereas there are at least three different upper power constructions. We show that they coincide for continuous ground domains, but differ on more general domains.

1 Introduction

A power domain construction maps every domain $\mathbf{X}$ into a so-called power domain over $\mathbf{X}$ whose points represent sets of points of the ground domain. Power domain constructions were originally proposed to model the semantics of non-deterministic programming languages [Plo76, Smy78, HP79, Mai85]. Other motivations are the semantic representation of a set data type [Hec90b], or of relational data bases [BDW88, Gun89].

In 1976, Plotkin [Plo76] proposed the *convex* power domain construction. A short time later, Smyth [Smy78] introduced a simpler construction, the *upper* power construction. In [Smy83], a third power domain construction occurs, the *lower* construction. Recently, two further power constructions were defined [BDW88, Gun89, Gun90, Hec90b].

Given at least five different power domain constructions, the question arises what is the essence of these constructions. In [Hec90a, Hec91], we defined power domain constructions algebraically by axioms concerning existence and properties of the basic operations of empty set, singleton, binary union, and function extension.

The algebraic theory of power constructions of [Hec90a, Hec91], which is summarized in section 3, shows that every construction has a *characteristic semiring* reflecting its inherent logic. For every semiring R, there are an *initial* and a *final* R-construction. The initial

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construction is given by the construction of free R-modules, whereas the final construction maps ground domains $\mathbf{X}$ to the space of linear second order R-predicates over $\mathbf{X}$ (see Th. 3.2).

In the present paper, we introduce the method of *core formation* to construct a new power construction from a given one (section 4). The core of a power construction $\mathcal{P}$ is the least subconstruction of $\mathcal{P}$ that still has the same characteristic semiring as $\mathcal{P}$. The power domains of the core result from the original power domains by discarding all junk, i.e., all formal sets that are not limits of finite unions of formal singletons.

Initial R-constructions coincide with their core. This is not true in general for final constructions. In section 5, we introduce the notion of *observable* R-modules and show that the core of the final R-construction produces free observable R-modules (Th. 5.8). An R-module is observable iff its members can be compared by observing them by linear mappings from the module to R.

The general theory is then applied to the cases of the lower semiring $\mathbf{L} = \{0 < 1\}$ and the dual upper semiring $\mathbf{U} = \{1 < 0\}$. Although these two semirings look quite similar, the resulting power constructions behave completely different. In the case of the lower semiring, all modules are observable. The initial construction, the final construction, and its core yield the same power domains for all ground domains (section 6).

In the case of the upper semiring however, these three constructions differ, and there are non-observable modules (section 7). We show that the three constructions coincide for continuous ground domains at least (Sec. 7.5), and we present an example that exhibits the difference between the initial construction and the core of the final one (Sec. 7.6). The difference between the final construction and its core follows from an example that was already presented in [Hec92]. Thus, one has to be careful to avoid confusion when speaking about upper power domains and going beyond the class of continuous dcpo's.

At the end, we briefly consider two further semirings: the one of the Sandwich and mixed construction [BDW88, Gun89, Gun90, Hec90b], and the one of Plotkin's construction.

2 Theoretical Background

In this section, we introduce some notions and notations from domain theory and algebra. For $f : X \to Y$ and $A \subseteq X$ and $B \subseteq Y$, $f[A]$ is the image of A and $f^{-1}[B]$ the inverse image of B. fx means application of f to x, and accordingly, $g \circ f$ means $\lambda x.\, g\,(fx)$. The identity function is denoted by id. Gfx parses as $(Gf)\,x$, and $Fg \circ h$ as $(Fg) \circ h$.

2.1 Posets and Domains

A *poset* (partially ordered set) is a set P together with a reflexive, antisymmetric, and transitive relation '$\leq$'. We often identify the poset $\mathbf{P} = (P, \leq)$ with its carrier P.

For $A \subseteq \mathbf{P}$, let $\downarrow\! A$ be the set of all points *below* some point of A, and correspondingly $\uparrow\! A$ the set of all points *above* some point of A. We use the abbreviations $\downarrow\! x = \downarrow\!\{x\}$ and $\uparrow\! x = \uparrow\!\{x\}$. A set $A \subseteq \mathbf{P}$ is a *lower* set iff $\downarrow\! A = A$, and an *upper* set iff $\uparrow\! A = A$.

We refer to the standard notions of join or least upper bound denoted by '$\sqcup$', meet or greatest lower bound '$\sqcap$', directed set, monotonic and continuous function. A *domain* is a poset where every directed set has a join, also called *limit*. A domain need neither have a least element, nor be algebraic.[1] Continuous functions between domains are sometimes called *morphisms*. For domains $\mathbf{X}$ and $\mathbf{Y}$, $\mathbf{X} \times \mathbf{Y}$ denotes the domain of all pairs of points of $\mathbf{X}$ and $\mathbf{Y}$, and $[\mathbf{X} \to \mathbf{Y}]$ is the domain of all morphisms from $\mathbf{X}$ to $\mathbf{Y}$ ordered pointwise. The notation $f : [\mathbf{X} \to \mathbf{Y}]$ includes the continuity of f. A morphism $e : [\mathbf{X} \to \mathbf{Y}]$ is an *embedding* iff $ex \leq ex'$ implies $x \leq x'$.

2.2 Monoids, Semirings, and Modules

Monoids, semirings, and modules are well-known algebraic concepts. Here, we define variants of these notions where the carrier is a domain, and all operations are continuous.

Definition 2.1 (Monoid domains and additive maps)
A *monoid domain* (or simply monoid) $(M, +, 0)$ is a domain M together with an associative operation $+ : [M \times M \to M]$ and an element 0 of (the carrier of) M which is the neutral element of '+'. The monoid is *commutative* iff '+' is.
A map $f : [X \to Y]$ between two monoids is *additive* iff it is a *monoid homomorphism*, i.e., $f(0_X) = 0_Y$ and $f(a + b) = fa + fb$ hold.

Definition 2.2 (Semirings)
A *semiring* (domain) $(R, +, 0, \cdot, 1)$ is a domain R with continuous operations such that $(R, +, 0)$ is a commutative monoid, $(R, \cdot, 1)$ is a monoid, and multiplication '$\cdot$' is additive in both arguments.

Semiring homomorphisms are continuous mappings that preserve the semiring operations.

Semirings are generalizations of both rings and distributive lattices. These in turn are generalizations of fields and Boolean algebras. Hence, both the notations $(R, +, 0, \cdot, 1)$ of the definition above and a logical notation $(R, \vee, \mathsf{F}, \wedge, \mathsf{T})$ seem to be adequate.

Assuming the logical interpretation, morphisms from a domain $\mathbf{X}$ to a semiring R are called *(R-)predicates*. Second order predicates are then morphisms in $[[\mathbf{X} \to R] \to R]$.

Definition 2.3 (Modules)
Let $R = (R, +, 0, \cdot, 1)$ be a semiring domain. $M = (M, +, 0, \cdot)$ is a *(left) R-module* iff $(M, +, 0)$ is a commutative monoid domain, and $\cdot : [R \times M \to M]$ is additive in both arguments and satisfies
$1_R \cdot A = A$ and $a \cdot (b \cdot C) = (a \cdot b) \cdot C$.

Let M_1 and M_2 be two R-modules. A morphism $f : [M_1 \to M_2]$ is *(left) (R-)linear* iff

$$f(A + B) = fA + fB \quad \text{and} \quad f(r \cdot A) = r \cdot fA$$

We speak of *right modules* and *right linear* morphisms if the semiring factor occurs to the right, i.e., $\cdot : [M \times R \to M]$. The axioms are analogous to the ones above.

[1] Usually, such 'domains' are called dcpo's. We refrain from doing so in order to avoid the term 'power dcpo construction'.

3 Power Domain Constructions

In this section, we present a short summary of the algebraic theory of power constructions as contained in [Hec90a, Hec91, Hec93].

3.1 Specification of Power Constructions

A *power (domain) construction* $\mathcal{P}$ maps *ground domains* $\mathbf{X}$ into *power domains* over $\mathbf{X}$. The power domains have to satisfy the following axioms:

Empty set: There is a distinguished element 0 in every power domain $\mathcal{P}\mathbf{X}$.

Binary union: There is a continuous operation $+ : [\mathcal{P}\mathbf{X} \times \mathcal{P}\mathbf{X} \to \mathcal{P}\mathbf{X}]$ in every power domain. '+' is commutative and associative, and 0 is its neutral element.

Singleton sets: There is a continuous mapping $\mathsf{s} : [\mathbf{X} \to \mathcal{P}\mathbf{X}]$, $x \mapsto \{\!|x|\!\}$ for every ground domain $\mathbf{X}$.

Extension of functions: For every two domains $\mathbf{X}$ and $\mathbf{Y}$, there is a higher order function $\mathsf{E} : [[\mathbf{X} \to \mathcal{P}\mathbf{Y}] \to [\mathcal{P}\mathbf{X} \to \mathcal{P}\mathbf{Y}]]$ mapping set-valued functions on ground domain elements into set-valued functions on sets. The intuitive meaning of $\mathsf{E} f A$ is $\bigcup_{a \in A} fa$. Extension has to satisfy the following axioms:

- $\mathsf{E}\, f\, A$ is additive in both A and f.
- $\mathsf{E}\, f \circ \mathsf{s} = f$ and $\mathsf{E}\, \mathsf{s} = \mathsf{id}$.
- For every two morphisms $f : [\mathbf{X} \to \mathcal{P}\mathbf{Y}]$ and $g : [\mathbf{Y} \to \mathcal{P}\mathbf{Z}]$,

  $$\mathsf{E}\, g\, (\mathsf{E}\, f\, A) = \mathsf{E}\, (\lambda a.\, \mathsf{E}\, g\, (fa))\, A$$

  holds for all A in $\mathcal{P}\mathbf{X}$, or: $\mathsf{E}\, g \circ \mathsf{E}\, f = \mathsf{E}\, (\mathsf{E}\, g \circ f)$.

$$\begin{tikzcd}
\mathcal{P}\mathbf{X} \arrow[r, "\mathsf{E}\,f"] & \mathcal{P}\mathbf{Y} \arrow[r, "\mathsf{E}\,g"] & \mathcal{P}\mathbf{Z} \\
\mathbf{X} \arrow[u, "\mathsf{s}"] \arrow[ur, "f"'] & \mathbf{Y} \arrow[u, "\mathsf{s}"] \arrow[ur, "g"'] &
\end{tikzcd}$$

The axioms of extension are just additivity plus the axioms of a Kleisli triple [Mog91]. Kleisli triples are an equivalent description for monads. In our case, the monad is given by $((\mathcal{P}, \mathsf{M}), \mathsf{s}, \mathsf{u})$ where $\mathsf{M} = \lambda f.\ \mathsf{E}(\mathsf{s} \circ f) : [[\mathbf{X} \to \mathbf{Y}] \to [\mathcal{P}\mathbf{X} \to \mathcal{P}\mathbf{Y}]]$, and $\mathsf{u} = \mathsf{E}\,\mathsf{id} : [\mathcal{P}(\mathcal{P}\mathbf{X}) \to \mathcal{P}\mathbf{X}]$. Note that $\mathcal{P}$ and M together form the underlying functor of the monad.

In the category of domains, E or M are continuous iff the considered monad is strong in the sense of [Mog91]. Thus, power domain constructions are strong monads where Moggi's morphism t is given by $\mathsf{t} = \lambda(a, B).\, \mathsf{M}\, (\lambda b.\, (a, b))\, B : [\mathbf{X} \times \mathcal{P}\mathbf{Y} \to \mathcal{P}(\mathbf{X} \times \mathbf{Y})]$.

3.2 Power Domains are Modules

The operations listed in the specification of power constructions allow the derivation of many other operations with useful algebraic properties. We already mentioned M, u, and t

that turn the power construction into a strong monad. From the remaining operations, we include the external product only; for other ones, we refer to [Hec90a, Hec91].

Extension depends on two ground domains, $\mathbf{X}$ and $\mathbf{Y}$. Particularly interesting instances of extension are obtained if one of $\mathbf{X}$ and $\mathbf{Y}$ is the one-point domain $\mathbf{1} = \{\diamond\}$. In case $\mathbf{X} = \mathbf{1}$, extension has functionality $\mathsf{E} : [[\mathbf{1} \to \mathcal{P}\mathbf{Y}] \to [\mathcal{P}\mathbf{1} \to \mathcal{P}\mathbf{Y}]]$. Rearranging the arguments leads to the morphism $\cdot : [\mathcal{P}\mathbf{1} \times \mathcal{P}\mathbf{Y} \to \mathcal{P}\mathbf{Y}]$ where $b \cdot S = \mathsf{E}\,(\lambda\diamond.\, S)\, b$. If we additionally choose $\mathbf{Y} = \mathbf{1}$, then '$\cdot$' becomes an inner operation of $\mathcal{P}\mathbf{1}$.

The power domain $\mathcal{P}\mathbf{1}$ represents the inherent logic of the power construction $\mathcal{P}$. It contains two specific values: 0 representing 'false' and $1 = \{\!|\diamond|\!\}$ representing 'true'. Binary union '+' stands for disjunction since $0+0=0$ and $0+1=1+0=1$. The operation '$\cdot$' may be interpreted as conjunction since $0\cdot 0 = 0\cdot 1 = 1\cdot 0 = 0$ and $1\cdot 1 = 1$. The axioms of power constructions moreover imply that $\mathcal{P}\mathbf{1}$ is a semiring and power domains are $\mathcal{P}\mathbf{1}$-modules.

Theorem 3.1 ([Hec90a, Hec91])
Let $\mathcal{P}$ be a power construction. Then $(\mathcal{P}\mathbf{1}, +, 0, \cdot, \{\!|\diamond|\!\})$ is a semiring domain, the *characteristic semiring* of $\mathcal{P}$. $(\mathcal{P}\mathbf{X}, +, 0, \cdot)$ is a left $\mathcal{P}\mathbf{1}$-module for all domains $\mathbf{X}$. For $f : [\mathbf{X} \to \mathcal{P}\mathbf{Y}]$, the result of extension $\mathsf{E}f : [\mathcal{P}\mathbf{X} \to \mathcal{P}\mathbf{Y}]$ is linear.

Power constructions with characteristic semiring R are called R-constructions. The result above connects our work with that of Main [Mai85] where power domains are introduced as free semiring modules. In our theory, power domains are not introduced as modules, but a posteriori happen to be modules, and these modules may be non-free. As a minor point, Main requires the singleton function s to be strict, whereas our s need not be strict.

3.3 Power Homomorphisms

Homomorphisms between algebraic structures are mappings preserving all operations of these structures. Power constructions may be considered algebraic structures on a higher level. Thus, it is also possible and useful to define corresponding homomorphisms.

A power homomorphism $H : \mathcal{P} \dot{\to} \mathcal{Q}$ between two power constructions $\mathcal{P}$ and $\mathcal{Q}$ is a family of morphisms $H = (H_\mathbf{X})_\mathbf{X} : [\mathcal{P}\mathbf{X} \to \mathcal{Q}\mathbf{X}]$ commuting over all power operations, i.e.,

- The empty set in $\mathcal{P}\mathbf{X}$ is mapped to the empty set in $\mathcal{Q}\mathbf{X}$: $H0 = 0$.
- The image of a union is the union of the images: $H(A+B) = HA + HB$.
- Singletons in $\mathcal{P}\mathbf{X}$ are mapped to singletons in $\mathcal{Q}\mathbf{X}$: $H\{\!|x|\!\}_\mathcal{P} = \{\!|x|\!\}_\mathcal{Q}$.
- Let $f : [\mathbf{X} \to \mathcal{P}\mathbf{Y}]$. Then $H \circ f : [\mathbf{X} \to \mathcal{Q}\mathbf{Y}]$, and $H(\mathsf{E}_\mathcal{P}\, f\, A) = \mathsf{E}_\mathcal{Q}\,(H \circ f)\,(HA)$ has to hold for all A in $\mathcal{P}\mathbf{X}$.

It is easily seen that power homomorphisms may be composed, and there are also identity power homomorphisms. Thus, we get the category PC of power constructions as objects and power homomorphisms as arrows.

If the two constructions $\mathcal{P}$ and $\mathcal{Q}$ share the same characteristic semiring, then one can define: A power homomorphism is linear iff all the functions $H_\mathbf{X}$ are linear. The category of R-constructions and R-linear power homomorphisms is denoted by $\mathsf{PC}(R)$.

3.4 Final R-Constructions

For every semiring R, the category $\mathsf{PC}(R)$ has an initial object $\mathcal{P}_R^i$ as well as a final object $\mathcal{P}_R^f$ [Hec90a, Hec91]. An R-construction $\mathcal{P}$ is *final* iff for every R-construction $\mathcal{Q}$ there is exactly one R-linear power homomorphism $\mathcal{Q}\dot{\to}\mathcal{P}$. Final R-constructions can be explicitly described by means of second order predicates. In the following theorem, $[[\mathbf{X} \to R] \stackrel{rlin}{\to} R]$ denotes the domain of right linear morphisms between the right R-modules $[\mathbf{X} \to R]$ and R. Categorically, it can be derived from the function domain $[[\mathbf{X} \to R] \to R]$ as an equalizer.

Theorem 3.2 ([Hec90a, Hec91])
Let R be a given semiring. The final R-construction $\mathcal{P} = \mathcal{P}_f^R$ is explicitly given by $\mathcal{P}\mathbf{X} = [[\mathbf{X} \to R] \stackrel{rlin}{\to} R]$. Its operations are defined by

- $0 = \lambda p.\, 0$
- $A + B = \lambda p.\, Ap + Bp$
- $\{\!|x|\!\} = \lambda p.\, p\, x$ for $x \in \mathbf{X}$.
- $\mathsf{E} f A = \lambda p.\, A\, (\lambda a_{\in \mathbf{X}}\, .\, f a p)$ for $f : [\mathbf{X} \to \mathcal{P}\mathbf{Y}]$ and $A \in \mathcal{P}\mathbf{X}$.
- $\mathsf{M} f A = \lambda p.\, A\, (p \circ f)$

The power domain $\mathcal{P}\mathbf{1} = [[\mathbf{1} \to R] \stackrel{rlin}{\to} R]$ is isomorphic to R by the isomorphism $\varphi : [R \to \mathcal{P}\mathbf{1}]$ with $\varphi(r) = \lambda p.\, r \cdot p\diamond$. The inverse of φ is $\psi(A) = A\,(\lambda\diamond.\, 1)$. The unique R-linear power homomorphism from another R-construction $\mathcal{Q}$ to $\mathcal{P}_f^R$ is given by the morphism $\mathcal{E} : [\mathcal{Q}\mathbf{X} \to [[\mathbf{X} \to R] \stackrel{rlin}{\to} R]]$ defined by $\mathcal{E} A = \lambda p.\, \mathsf{E}_{\mathcal{Q}}\, p\, A$.

3.5 Initial R-Constructions

An R-construction $\mathcal{P}$ is *initial* iff there is exactly one linear power homomorphism *to* every R-construction $\mathcal{Q}$. The initial R-construction is obtained by mapping every ground domain $\mathbf{X}$ into the *free R-module* over $\mathbf{X}$.

Definition 3.3 An R-module $\mathbf{F}$ with a morphism $\mathsf{s} : [\mathbf{X} \to \mathbf{F}]$ is *free* over $\mathbf{X}$ iff for every R-module M and every morphism $f : [\mathbf{X} \to M]$, there is a unique R-linear extension $\bar{f} : [\mathbf{F} \to M]$ of f. *Extension* of f means $\bar{f} \circ \mathsf{s} = f$.

By usual algebraic arguments, all free R-modules over $\mathbf{X}$ are isomorphic to each other. Thus, we sometimes denote *the* free R-module over $\mathbf{X}$ by $R \odot \mathbf{X}$. The free R-module over $\mathbf{1}$ is $(R,\, \lambda\diamond.\, 1)$. The initial R-construction can be built from free R-modules.

Theorem 3.4 ([Hec90a, Hec91]) Let R be a fixed semiring. The free R-module $R \odot \mathbf{X}$ exists for every domain $\mathbf{X}$. The power construction $\mathcal{P}_R^i$ defined by $\mathcal{P}_R^i\mathbf{X} = R \odot \mathbf{X}$ is the initial R-construction. The a priori given external product of the modules $\mathcal{P}_R^i\mathbf{X}$ coincides with the external product derived from the power operations.

4 Core Formation

In the previous section, we shortly repeated the general theory of power domain constructions as contained in [Hec90a, Hec91]. In this section, we present a method to derive a new construction from an existing one. The *core* of an R-construction $\mathcal{P}$ is the least sub-construction of $\mathcal{P}$ that still has characteristic semiring R. Definition and general properties of cores were published in the thesis [Hec90a], but not elsewhere.

4.1 R-$\mathbf{X}$-Modules

As technical means to simplify reasoning, we introduce R-$\mathbf{X}$-modules in this subsection. R-$\mathbf{X}$-modules are R-modules together with a map from $\mathbf{X}$.

Definition 4.1 Let R be a semiring and $\mathbf{X}$ a domain. An R-$\mathbf{X}$-*module* $\mathbf{M}$ is a pair $\mathbf{M} = (M, \eta)$ of an R-module domain M and a morphism $\eta : [\mathbf{X} \to M]$.

A morphism $f : (M, \eta) \to (M', \eta')$ is R-$\mathbf{X}$-*linear* iff $f : [M \to M']$ is R-linear and $f \circ \eta = \eta'$.

We already met examples for such R-$\mathbf{X}$-modules and R-$\mathbf{X}$-linear mappings. Free R-modules $\mathbf{F}$ over $\mathbf{X}$ with $\mathsf{s} : [\mathbf{X} \to \mathbf{F}]$ are just initial R-$\mathbf{X}$-modules, since $\bar{f}$ is the unique R-$\mathbf{X}$-linear morphism from $(\mathbf{F}, \mathsf{s})$ to (M, f).

In connection with power domain constructions, R-$\mathbf{X}$-modules occur in two situations. If $f : [\mathbf{X} \to \mathcal{P}\mathbf{Y}]$, then the extension $\mathsf{E}f$ is R-$\mathbf{X}$-linear between the R-$\mathbf{X}$-modules $(\mathcal{P}\mathbf{X}, \mathsf{s})$ and $(\mathcal{P}\mathbf{Y}, f)$ since $\mathsf{E}f \circ \mathsf{s} = f$. If $H : \mathcal{P} \dot{\to} \mathcal{Q}$ is a linear power homomorphism between two R-constructions, then for every ground domain $\mathbf{X}$, the instance $H_{\mathbf{X}}$ is an R-$\mathbf{X}$-linear mapping between the two R-$\mathbf{X}$-modules $(\mathcal{P}\mathbf{X}, \mathsf{s}_{\mathcal{P}})$ and $(\mathcal{Q}\mathbf{X}, \mathsf{s}_{\mathcal{Q}})$. Thus, the R-$\mathbf{X}$-modules with R-$\mathbf{X}$-linear mappings provide a common abstraction of extension and power homomorphisms.

4.2 The Core of R-$\mathbf{X}$-Modules

We first define the core of an R-$\mathbf{X}$-module as its least submodule. From this, the core of a power construction is derived later.

Definition 4.2 Let $\mathbf{M} = ((M, +, 0, \cdot), \eta)$ be an R-$\mathbf{X}$-module. A subset S of (the carrier of) M is called an R-$\mathbf{X}$-*submodule* of $\mathbf{M}$ iff

(1) ηx is in S for all $x \in \mathbf{X}$, i.e., $\eta[\mathbf{X}] \subseteq S$.

(2) 0 is in S.

(3) If a and b are in S, then so is $a + b$.

(4) If a is in S, then $r \cdot a$ is in S for all $r \in R$.

(5) S is *directed closed* in M, i.e., if D is a directed subset of S, then the limit of D w.r.t. M is in S.

R-submodules of an R-module are defined analogously by omitting condition (1).

By definition, S may be assumed to be an R-$\mathbf{X}$-module again, and the natural inclusion map $e : S \to M$ is an R-$\mathbf{X}$-linear morphism.

It is easily verified that the intersection of a family of R-$\mathbf{X}$-submodules is again an R-$\mathbf{X}$-submodule. Hence, the R-$\mathbf{X}$-submodules form a complete lattice, and there is a least R-$\mathbf{X}$-submodule for every given R-$\mathbf{X}$-module $\mathbf{M}$. We call it the *core* $\mathbf{M}^c$ of $\mathbf{M}$.

In the sequel, we sometimes need the following lemma about cores.

Lemma 4.3 Let $\mathbf{M} = (M, \eta)$ be an R-$\mathbf{X}$-module, M' an R-module, and S a submodule of M'. If an R-linear morphism $F : [M \to M']$ maps all points ηx into S, then F maps all points of $\mathbf{M}^c$ into S.

Proof: Let $S' = \{x \in \mathbf{M} \mid Fx \in S\}$. It is easy to verify that S' is an R-$\mathbf{X}$-submodule of $\mathbf{M}$. Hence, $\mathbf{M}^c \subseteq S'$ follows. □

From the lemma, we conclude that core formation is functorial.

Proposition 4.4 Every R-$\mathbf{X}$-linear morphism $F : \mathbf{M}_1 \to \mathbf{M}_2$ induces an R-$\mathbf{X}$-linear morphism $F^c : \mathbf{M}_1^c \to \mathbf{M}_2^c$ by restriction and corestriction.

Proof: $\mathbf{M}_2^c$ is an R-$\mathbf{X}$-submodule of $\mathbf{M}_2$. F maps all points ηx into $\eta' x \in \mathbf{M}_2^c$. By Lemma 4.3, it maps all points in $\mathbf{M}_1^c$ into the core of $\mathbf{M}_2$. □

4.3 Reduced R-X-Modules

An R-$\mathbf{X}$-module is *reduced* iff it coincides with its core. Equivalently, an R-$\mathbf{X}$-module is reduced iff it does not allow proper R-$\mathbf{X}$-submodules. Reduced R-$\mathbf{X}$-modules enjoy particularly nice properties.

Proposition 4.5 Let $\mathbf{M} = (M, \mathsf{s})$ be a reduced R-$\mathbf{X}$-module.

(1) There is at most one R-$\mathbf{X}$-linear morphism from $\mathbf{M}$ to an R-$\mathbf{X}$-module $\mathbf{M}'$.

(2) Let M' be an R-module and $F_1, F_2 : [M \to M']$ be R-linear morphisms. Then $F_1 = F_2$ is equivalent to $F_1 \circ \mathsf{s} = F_2 \circ \mathsf{s}$.

Proof:

(1) Let $F_1, F_2 : \mathbf{M} \to \mathbf{M}'$ be two R-$\mathbf{X}$-linear morphisms. The equalizing set $E = \{x \in \mathbf{M} \mid F_1 x = F_2 x\}$ is an R-$\mathbf{X}$-submodule of $\mathbf{M}$. Since $\mathbf{M}$ is reduced, $E = \mathbf{M}$ follows.

(2) $F_1 \circ \mathsf{s} = F_2 \circ \mathsf{s}$ implies that F_1 and F_2 are R-$\mathbf{X}$-linear from $\mathbf{M} = (M, \mathsf{s})$ to $(M', F_1 \circ \mathsf{s})$. By (1), they are equal. □

Next, we show that every free R-module over $\mathbf{X}$ is reduced.

Proposition 4.6 Every initial R-$\mathbf{X}$-module is reduced.

Proof: Let $\mathbf{F}$ be an initial R-$\mathbf{X}$-module. By initiality, there is a unique R-$\mathbf{X}$-linear morphism $F : \mathbf{F} \to \mathbf{F}^c$. The inclusion map $E : \mathbf{F}^c \to \mathbf{F}$ is R-$\mathbf{X}$-linear, too. $E \circ F = \mathsf{id} : \mathbf{F} \to \mathbf{F}$ follows from initiality, and $F \circ E = \mathsf{id} : \mathbf{F}^c \to \mathbf{F}^c$ from Prop. 4.5, (1). □

4.4 The Core of a Power Construction

Let $\mathcal{P}$ be a power construction with characteristic semiring R. Then the power domains $\mathcal{P}\mathbf{X}$ together with the singleton mappings $\mathsf{s} : [\mathbf{X} \to \mathcal{P}\mathbf{X}]$ form R-$\mathbf{X}$-modules. We denote the core of $(\mathcal{P}\mathbf{X}, \mathsf{s})$ by $(\mathcal{P}^c\mathbf{X}, \mathsf{s})$ and show that $\mathcal{P}^c$ is a power domain construction with the same characteristic semiring as $\mathcal{P}$.

The operations 0, '+', and $\{\!|.|\!\}$ obviously cut down to operations for $\mathcal{P}^c$ because $\mathcal{P}^c\mathbf{X}$ was defined to allow these operations together with R-multiplication. We only have to provide an extension satisfying the power axioms.

For every $f : [\mathbf{X} \to \mathcal{P}^c\mathbf{Y}]$, the linear extension $\mathsf{E}_\mathcal{P} f$ maps singletons $\{\!|x|\!\}$ of $\mathcal{P}^c\mathbf{X}$ into $fx \in \mathcal{P}^c\mathbf{Y}$. Hence by Lemma 4.3, $\mathsf{E}_\mathcal{P} f$ maps all members of $\mathcal{P}^c\mathbf{X}$ to $\mathcal{P}^c\mathbf{Y}$. Thus, the extension of $\mathcal{P}$ cuts down to the extension of $\mathcal{P}^c$. It satisfies the power axioms because $\mathsf{E}_\mathcal{P}$ did. Since the power operations for $\mathcal{P}^c$ are derived from those of $\mathcal{P}$ by restriction and corestriction, the embeddings $E : \mathcal{P}^c\mathbf{X} \to \mathcal{P}\mathbf{X}$ form a power homomorphism.

$\mathcal{P}^c\mathbf{1}$ equals $\mathcal{P}\mathbf{1}$ since every member b of $\mathcal{P}\mathbf{1}$ equals $b \cdot \{\!|\diamond|\!\}$, whence $b \in \mathcal{P}^c\mathbf{1}$. Thus, $\mathcal{P}^c$ has the same characteristic semiring as $\mathcal{P}$.

Let $\mathcal{P}$ and $\mathcal{Q}$ be two power constructions with the same characteristic semiring. For every linear power homomorphism $H : \mathcal{P} \dot\to \mathcal{Q}$, the instance $H_\mathbf{X} : (\mathcal{P}\mathbf{X}, \mathsf{s}_\mathcal{P}) \to (\mathcal{Q}\mathbf{X}, \mathsf{s}_\mathcal{Q})$ is R-$\mathbf{X}$-linear. By Lemma 4.4, H cuts down to a power homomorphism $H^c : \mathcal{P}^c \dot\to \mathcal{Q}^c$.

All results of this section are collected in the following theorem:

Theorem 4.7 For every power construction $\mathcal{P}$, the core $\mathcal{P}^c$ is the least sub-construction of $\mathcal{P}$ with the same characteristic semiring. The embeddings of the cores $\mathcal{P}^c\mathbf{X}$ into the power domains $\mathcal{P}\mathbf{X}$ provide a linear power homomorphism $E : \mathcal{P}^c \dot\to \mathcal{P}$.

For every linear power homomorphism $H : \mathcal{P} \dot\to \mathcal{Q}$, there is a linear homomorphism $H^c : \mathcal{P}^c \dot\to \mathcal{Q}^c$ with $E \circ H^c = H^c \circ E$.

5 Observable Modules

An R-module is *observable* iff all its points can be compared by observing them using linear mappings to R (cf. formal definition below). In this section, we present the theory of observable R-modules. The ultimate goal is to show that the core of the final R-construction creates free observable R-modules. All results about observable modules are new and not even part of the thesis [Hec90a].

Definition 5.1 An R-module M is *observable* iff for all points $a, b \in M$ the relation $a \leq b$ is implied by $pa \leq pb$ for all $p : [M \stackrel{lin}{\to} R]$.

In the definition, $[M \stackrel{lin}{\to} R]$ denotes the domain of all continuous linear mappings from M to R where R is turned into an R-module in the obvious way.

Examples for non-observable R-modules are easily found in the case of $R = \{0, 1\}$ where 0 and 1 are incomparable and $1 + 1 = 1$ holds. The domain $M = \{0 < m\}$ with $m + m = m$ is an R-module. It is not observable since the only linear map from M to R is $\lambda m.\, 0$, which cannot distinguish between 0 and m. On the other hand, examples for observable modules are abundant and given by the following propositions.

Proposition 5.2 R is an observable R-module.

Proof: The identity is among the morphisms $p : [R \stackrel{lin}{\to} R]$. □

Proposition 5.3

Every submodule of an observable R-module is observable.

Proof: Let M' be a submodule of the observable module M, i.e., there is a linear embedding $e : [M' \to M]$. Let $a, b \in M'$ and assume $pa \leq pb$ for all $p : [M' \stackrel{lin}{\to} R]$. Let $q : [M \stackrel{lin}{\to} R]$. Then $q \circ e : [M' \stackrel{lin}{\to} R]$, whence $q(ea) \leq q(eb)$. By observability of M, $ea \leq eb$ follows, whence $a \leq b$ because e is an embedding. □

Proposition 5.4 Every product of observable R-modules is observable.

Proof: Let $M = \prod_{i \in I} M_i$, and let $a = (a_i)_{i \in I}$ and $b = (b_i)_{i \in I}$ in M such that $pa \le pb$ for all $p : [M \stackrel{lin}{\to} R]$. If $q : [M_k \stackrel{lin}{\to} R]$, then $q \circ \pi_k : [M \stackrel{lin}{\to} R]$, whence $qa_k \le qb_k$. By observability of M_k, $a_k \le b_k$ follows. Thus, we obtain $a \le b$. □

By the propositions above, the observable R-modules form a complete subcategory of all R-modules.

Proposition 5.5 For every domain $\mathbf{X}$, the function domain $[\mathbf{X} \to R]$ is an observable R-module.

Proof: We apply the three propositions above. The module $[\mathbf{X} \to R]$ of continuous functions forms a submodule of the module of all functions from $\mathbf{X}$ to R. This module is a product $\prod_{x \in \mathbf{X}} R$ of the observable module R. □

The properties of observability collected so far allow to prove an equivalent criterion.

Theorem 5.6

For an R-module M, the following statements are equivalent:

(1) M is observable;

(2) M can be embedded into $\widehat{M} = [[M \stackrel{lin}{\to} R] \to R]$;

(3) There is a set I such that M can be embedded into $\prod_{i \in I} R$.

In (2), the embedding e is given by $e\, m = \widehat{m} = \lambda p.\, p\, m$.

Proof:

$(2) \Rightarrow (3)$: Let $I = [M \stackrel{lin}{\to} R]$. The domain $\widehat{M}$ of continuous functions from I to R can be embedded into the domain $\prod_{i \in I} R$ of all functions from I to R.

$(3) \Rightarrow (1)$: By Prop. 5.2, 5.4 and 5.3.

$(1) \Rightarrow (2)$: Let $e\, m = \widehat{m} = \lambda p.\, p\, m$ for all m in M. e is a morphism from M to $\widehat{M}$. It is easily shown to be linear using the fact that $e\, m$ is applied to linear maps only. $ea \le eb$ is equivalent to $pa \le pb$ for all $p : [M \stackrel{lin}{\to} R]$. Hence, e is an embedding by observability of M. □

The module $\widehat{M}$ in Th. 5.6 has a similar structure as the final power domain over M. Indeed, there is some relationship:

Proposition 5.7

For all R-modules M, there is a linear morphism $\sigma : [\mathcal{P}^f_R M \to \widehat{M}]$ such that $\sigma \{\!|m|\!\} = \widehat{m}$.

Proof: Remember $\mathcal{P}^f_R M = [[M \to R] \stackrel{rlin}{\to} R]$ and $\widehat{M} = [[M \stackrel{lin}{\to} R] \to R]$. Let i_1 be the inclusion of $[M \stackrel{lin}{\to} R]$ into $[M \to R]$, and i_2 the inclusion of $[[M \to R] \stackrel{rlin}{\to} R]$ into $[[M \to R] \to R]$. Then we can define $\sigma P = i_2\, (\lambda p.\, P\, (i_1 p))$, i.e., σP results from P by restricting the argument domain and extending the result domain. The morphism σ is obviously linear. It maps $\{\!|m|\!\}$ into $\widehat{m}$ since both are described by $\lambda p.\, p\, m$. □

Now, we have collected enough facts to prove the announced theorem.

Theorem 5.8 The core of the final R-construction creates free observable R-modules: for every domain $\mathbf{X}$, the power domain $(\mathcal{P}^f_R)^c \mathbf{X}$ is a free observable R-module over $\mathbf{X}$.

Proof: We abbreviate $\mathcal{P}^f_R\mathbf{X}$ by $\mathbf{P}$. Its core $\mathbf{P}^c$ is observable by Prop. 5.5 and 5.3, since $\mathbf{P}^c$ is embedded into $\mathbf{P}$, which in turn is embedded into $[[\mathbf{X} \to R] \to R]$.

Let M be an observable R-module and $f : [\mathbf{X} \to M]$. By Th. 5.6, M can be embedded into $\widehat{M}$. Applying Prop. 5.7, we get a linear mapping $\widehat{f} = \sigma \circ \mathsf{M}f : \mathbf{P} = \mathcal{P}^f_R\mathbf{X} \to \mathcal{P}^f_R M \to \widehat{M}$. Because of $\widehat{f}\{\!|x|\!\} = \sigma(\mathsf{M}f\{\!|x|\!\}) = \sigma\{\!|fx|\!\} = \widehat{fx}$, it maps singletons of $\mathbf{P}$ into the submodule M of $\widehat{M}$. By Lemma 4.3, it can be restricted to a linear morphism $\bar{f} : [\mathbf{P}^c \to M]$. This morphism extends f since $\bar{f}\{\!|x|\!\} = fx$. It is the unique linear extension of f by Prop. 4.5. □

6 The Lower Power Construction

After having introduced the general theory of power domain constructions, we want to consider now power constructions for certain particular semirings. The semiring with the simplest non-trivial power constructions is the *lower semiring* $\mathbf{L} = \{0 < 1\}$ where $1 + 1 = 1$. We shall show that all $\mathbf{L}$-modules are observable and that initial and final $\mathbf{L}$-constructions coincide for all ground domains.

6.1 The Final Lower Construction

According to Th. 3.2, the final $\mathbf{L}$-construction is given in predicative form by $\mathcal{L}_f\mathbf{X} = [[\mathbf{X} \to \mathbf{L}] \overset{rlin}{\to} \mathbf{L}]$. Since $\mathbf{L}$ consists of 0 and 1 only, every additive morphism between $\mathbf{L}$-modules is left and right linear. Thus, we may also write $\mathcal{L}_f\mathbf{X} = [[\mathbf{X} \to \mathbf{L}] \overset{add}{\to} \mathbf{L}]$.

In [Hec90a, Hec93], we established an order isomorphism between these additive second order predicates and the Scott closed subsets of $\mathbf{X}$ ordered by inclusion. A subset of a domain $\mathbf{X}$ is called *(Scott) closed* iff it is a lower set closed w.r.t. joins of directed subsets. Lower cones $\downarrow x$ are obviously closed. Arbitrary intersections and finite unions of closed sets are closed. Hence, every set A has a least closed superset, the *closure* $\mathsf{cl}\, A$.

The above mentioned isomorphism is given by

$$\begin{array}{lll} \text{Scott closed set } C & \mapsto & \lambda p.\, \text{if } (\exists a \in C : pa = 1) \text{ then } 1 \text{ else } 0 \\ \text{Predicate } P & \mapsto & \widehat{P}^{-1}[\{0\}] \text{ where } \widehat{P} = \bigsqcup\{p : [\mathbf{X} \to \mathbf{L}] \mid Pp = 0\} \\ & & = \bigcap\{p^{-1}[\{0\}] \mid Pp = 0\} \end{array}$$

Note that the set $\{p \mid Pp = 0\}$ is directed due to the additivity of P. Hence, the join of this set exists.

In [Hec90a, Hec93], we also derived formulae for the power operations in terms of Scott closed sets.

Theorem 6.1

The final $\mathbf{L}$-construction $[[\mathbf{X} \to \mathbf{L}] \overset{rlin}{\to} \mathbf{L}]$ is isomorphic to

(1) $\{C \subseteq \mathbf{X} \mid C \text{ is Scott closed}\}$ ordered by inclusion '$\subseteq$',

(2) $\bigsqcup_{i \in I} A_i = \mathsf{cl} \bigcup_{i \in I} A_i$ where 'cl' denotes Scott closure,

(3) $0 = \emptyset$,

(4) $A + B = A \cup B$,

(5) $\{\!|x|\!\} = \downarrow x$,

(6) $\mathsf{E} f A = \sqcup f[A] = \mathsf{cl} \bigcup f[A] = \mathsf{cl} \bigcup_{a \in A} fa$,

(7) $\mathsf{M} f A = \mathsf{cl}\, f[A] = \mathsf{cl}\, \{fa \mid a \in A\}$.

Now, we show that the final $\mathbf{L}$-construction is reduced. If C is an arbitrary closed set, then $\mathcal{D} = \{{\downarrow} F \mid F$ is a finite subset of $C\}$ is directed and has union (and thus limit) C. If F is finite, then ${\downarrow} F$ is a finite union of sets ${\downarrow}\{x\} = \{\!|x|\!\}$. Hence, every member of $\mathcal{L}_f \mathbf{X}$ is a limit of finite unions of formal singletons. Thus, $\mathcal{L}_f \mathbf{X}$ is reduced for all $\mathbf{X}$, and the final $\mathbf{L}$-construction coincides with its core.

6.2 The Initial Lower Construction

Our next goal is to prove that the final $\mathbf{L}$-construction is also initial. Thus, we are in the fortunate situation that we can speak of *the* lower power construction $\mathcal{L}$. The power domains $\mathcal{L}\mathbf{X}$ can be described by three different but equivalent methods: as free complete lattices over $\mathbf{X}$, as sets of additive second order predicates, and concretely as sets of Scott closed subsets of $\mathbf{X}$.

Theorem 6.2 The final $\mathbf{L}$-construction is also initial.

The theorem could be proved by showing that the lower power domains in terms of closed sets are free $\mathbf{L}$-$\mathbf{X}$-modules. This is done in [Hec90a]. First notice that $\mathbf{L}$-module domains and $\mathbf{L}$-linear morphisms are the same as complete lattices and mappings preserving all joins. This is because addition in $\mathbf{L}$-modules is binary join, since $a + b = a + 1 \cdot b \geq a + 0 \cdot b = a + 0 = a$ and analogously $a + b \geq b$, and $a \leq u$ and $b \leq u$ imply $a + b \leq u + u = 1 \cdot u + 1 \cdot u = (1 + 1) \cdot u = 1 \cdot u = u$.

If a morphism f from $\mathbf{X}$ to some complete lattice M is given, then its linear continuous extension $\bar{f}$ is defined by $\bar{f}(C) = \sqcup f[C]$ for Scott closed sets C. The verification of the continuity of $\bar{f}$ is tedious.

In this paper, we present a novel proof that applies Th. 5.8. We show that all $\mathbf{L}$-modules are observable, whence (the core of) the final $\mathbf{L}$-construction must be isomorphic to the initial $\mathbf{L}$-construction.

Proposition 6.3 All $\mathbf{L}$-modules are observable.

Proof: Let M be an $\mathbf{L}$-module, i.e., a complete lattice. Let a and b be two points of M and assume $a \not\leq b$. Let p be the function from M to $\mathbf{L} = \{0 < 1\}$ that maps ${\downarrow} b$ to 0 and all other points to 1. By $a \not\leq b$, we obtain $pa = 1$ and $pb = 0$, whence $pa \not\leq pb$.

The join $\sqcup_{i \in I} x_i$ is in ${\downarrow} b$ iff $x_i \in {\downarrow} b$ for all $i \in I$. Hence, $p(\sqcup_{i \in I} x_i) = 0$ iff $px_i = 0$ for all i iff $\sqcup_{i \in I} px_i = 0$. Thus, p preserves all joins, i.e., is an $\mathbf{L}$-linear morphism. □

7 Upper Power Constructions

Upper power constructions have characteristic semiring $\mathbf{U} = \{1 < 0\}$ with $1 + 1 = 1$. Although this semiring looks as simple as the lower semiring $\mathbf{L} = \{0 < 1\}$, the situation here is much more complex: the final $\mathbf{U}$-construction is not reduced, there are non-observable $\mathbf{U}$-modules, the initial and the core of the final $\mathbf{U}$-construction do not coincide. Thus, *the* upper power construction does not exist, because the final **U**-construction $\mathcal{U}_f$, its core $\mathcal{U}_c$, and the initial **U**-construction $\mathcal{U}_i$ all differ. At least, they all coincide on continuous ground domains as will be shown below. We also present examples that exhibit their difference on more general domains. Before we proceed, we introduce some topological notions, and consider the class of continuous domains and some of its generalizations.

7.1 Open, Compact, and Strongly Compact Sets

In section 6.1, we defined *closed sets* as lower sets that are closed w.r.t. directed joins. The complements of the closed sets are called *open*. Finite intersections and arbitrary unions of open sets are open. The union of all open subsets of a set A is its *interior* $\mathsf{int}\, A$. The importance of open sets comes from the following well-known theorem:

Theorem 7.1

A function $f : \mathbf{X} \to \mathbf{Y}$ is continuous, i.e., preserves directed joins, iff the inverse image $f^{-1}[O]$ of every open set O of $\mathbf{Y}$ is open in $\mathbf{X}$.

In the sequel, we often use open sets to compare upper sets or single points, as made possible by the following two propositions. In these propositions, the set of all open supersets of a set A is denoted by $\mathcal{O}(A)$. We abbreviate $\mathcal{O}(\{x\})$ by $\mathcal{O}(x)$.

Proposition 7.2

For two subsets A, B of $\mathbf{X}$, $\mathcal{O}(A) \subseteq \mathcal{O}(B)$ holds iff $\uparrow A \supseteq \uparrow B$.

Proof: Let $\uparrow A \supseteq \uparrow B$. If $A \subseteq O$ then $B \subseteq \uparrow A \subseteq O$ since open sets are upper sets.
Let $\mathcal{O}(A) \subseteq \mathcal{O}(B)$, and let b be a point of B and assume $b \notin \uparrow A$. Then there is no point of A below b, i.e., A does not meet the closed set $\downarrow b$. Hence, its complement $\mathbf{X} \setminus \downarrow b$ is in $\mathcal{O}(A) \subseteq \mathcal{O}(B)$. Because of $b \in B$, $b \in \mathbf{X} \setminus \downarrow b$ follows contradicting reflexivity of '$\leq$'. □

Proposition 7.3 For two points a, b of $\mathbf{X}$, $\mathcal{O}(a) \subseteq \mathcal{O}(B)$ holds iff $a \leq b$.

Proof:
This is a consequence of Prop. 7.2 since $\uparrow a \supseteq \uparrow b$ is equivalent to $a \leq b$. □

A subset K of a domain $\mathbf{X}$ is called *(Scott) compact* iff whenever K is covered by a family $(O_i)_{i \in I}$ of open sets, i.e., $K \subseteq \bigcup_{i \in I} O_i$, there is a finite subset F of I such that $K \subseteq \bigcup_{i \in F} O_i$. A subset S is called *strongly compact* iff whenever $S \subseteq O$ for an open set O, there is a finite set E such that $S \subseteq \uparrow E \subseteq O$.

The basic properties of strongly compact sets are analogous to those of compact sets [Hec90a, Hec92]. Every finite set is strongly compact, and every strongly compact set is compact. A set A is (strongly) compact iff $\uparrow A$ is so. Finite unions and images by continuous functions of (strongly) compact sets are (strongly) compact again.

7.2 Continuity, Multi-Continuity, and Sobriety

A point a in a domain $\mathbf{X}$ is *way-below* a point b iff for all directed sets $D \subseteq \mathbf{X}$ with $b \leq \bigsqcup D$, there is an element d in D such that $a \leq d$. $\mathbf{X}$ is *continuous* iff every point x of $\mathbf{X}$ is the join of a directed set of points that all are way-below x. In [Hec90a, Hec92], we proved a purely topological criterion for domain continuity:

Theorem 7.4 A domain $\mathbf{X}$ is continuous iff for all x in $\mathbf{X}$ and open sets O with $x \in O$, there are a point y and an open set O' such that $x \in O' \subseteq \uparrow y \subseteq O$.

This criterion suggests an obvious generalization of continuity:

Definition 7.5 A domain $\mathbf{X}$ is *multi-continuous* iff for all x in $\mathbf{X}$ and open sets O with $x \in O$, there are a finite set E and an open set O' such that $x \in O' \subseteq \uparrow E \subseteq O$.
Sets $\uparrow E$ where E is finite are called *finitary.*

The defining property of multi-continuous domains can be generalized from points to compact sets:

Proposition 7.6 Let $\mathbf{X}$ be multi-continuous. For all compact sets K and open sets O with $K \subseteq O$, there are a finitary set F and an open set O' such that $K \subseteq O' \subseteq F \subseteq O$. Thus, every compact set in $\mathbf{X}$ is strongly compact.

Proof: Assume $K \subseteq O$. For every x in K, there is a finitary set F_x and an open set O'_x such that $x \in O'_x \subseteq F_x \subseteq O$. The sets O'_x form an open cover of K. By compactness, there is a finite subset E of K such that $K \subseteq \bigcup_{x \in E} O'_x \subseteq \bigcup_{x \in E} F_x \subseteq O$. Finite unions of finitary sets are finitary. □

Sobriety is a topological notion that occurs for instance in [Smy83, Law88, Hec90a]. Every multi-continuous domain is sober, and sober domains have the following property [Hec90a]:

Proposition 7.7 Let $\mathbf{X}$ be a sober domain, $\mathcal{K}$ a $\supseteq$-directed set of compact upper sets of $\mathbf{X}$, and O an open set of $\mathbf{X}$. If $\bigcap \mathcal{K} \subseteq O$, then $K \subseteq O$ holds for some K in $\mathcal{K}$.

7.3 The Final Upper Power Construction

The final upper power construction can be described concretely in terms of sets for sober ground domains $\mathbf{X}$. For such $\mathbf{X}$, an order isomorphism is established in [Hec90a, Hec93] between the linear second order predicates to $\mathbf{U}$ and the compact upper subsets of $\mathbf{X}$ ordered by inverse inclusion. This order isomorphism is given by

$$\begin{array}{lll} \text{Compact upper set } K & \mapsto & \lambda p.\, \text{if } (\exists a \in K : pa = 1) \text{ then } 1 \text{ else } 0 \\ \text{Predicate } P & \mapsto & \bigcap \{p^{-1}[\{0\}] \mid Pp = 0\} \end{array}$$

On the one hand, the situation is completely analogous to the case of the lower power construction, since the formulae for the isomorphisms are equal. On the other hand, there is a big difference: the final $\mathbf{L}$-construction can always be described in terms of subsets of $\mathbf{X}$, whereas this is possible for the final $\mathbf{U}$-construction iff $\mathbf{X}$ is sober.

In [Hec90a, Hec93], we derived formulae for the power operations in terms of compact upper sets.

Theorem 7.8 For sober ground domains $\mathbf{X}$, the final upper power domain $\mathcal{U}_f\mathbf{X} = [[\mathbf{X} \to \mathbf{U}] \stackrel{rlin}{\to} \mathbf{U}]$ is isomorphic to the domain $\mathcal{U}_K\mathbf{X}$ defined as follows:

(1) $\mathcal{U}_K\mathbf{X}$ is the set of all compact upper sets of $\mathbf{X}$,

(2) $K \le K'$ iff $K \supseteq K'$,

(3) $\sqcup \mathcal{D} = \bigcap \mathcal{D}$ for directed sets $\mathcal{D}$ in $\mathcal{U}_K\mathbf{X}$,

(4) $0 = \emptyset$,

(5) $A + B = A \cup B$,

(6) $\{\!|x|\!\} = \uparrow x$ for all $x \in \mathbf{X}$,

(7) if both $\mathbf{X}$ and $\mathbf{Y}$ are sober, and $f : [\mathbf{X} \to \mathcal{U}_K\mathbf{Y}]$ is continuous and A is in $\mathcal{U}_K\mathbf{X}$, then $\mathsf{E}fA = \bigcup_{a \in A} fa = \bigcup f[A]$.

All these operations are well defined and continuous. The derived operation M is given by $\mathsf{M}fA = \uparrow f[A]$.

7.4 The Core of the Final U-Construction

In this subsection, we show that the final $\mathbf{U}$-construction is reduced for multi-continuous ground domains. It is *not* reduced for arbitrary ground domains. The set $\mathcal{U}_S\mathbf{X}$ of strongly compact subsets of $\mathbf{X}$ forms an $\mathbf{U}$-$\mathbf{X}$-submodule of $\mathcal{U}_K\mathbf{X}$. In [Hec92], a domain was presented that is not multi-continuous but sober. This domain has a compact subset that is not strongly compact. Hence, $\mathcal{U}_K\mathbf{X}$ is not reduced for this domain $\mathbf{X}$. It is an open problem whether $\mathcal{U}_S\mathbf{X}$ is the core of $\mathcal{U}_K\mathbf{X}$ for every sober domain $\mathbf{X}$.

For multi-continuous domains however, we show that every compact set is a directed join of finite unions of formal singletons $\{\!|x|\!\} = \uparrow x$. Such finite unions are the finitary sets.

Proposition 7.9 Let $\mathbf{X}$ be a multi-continuous domain. For all compact upper sets K of $\mathbf{X}$, let $\mathcal{F}(K)$ be the set of all finitary environments of K, i.e., the set of all finitary sets F such that $K \subseteq O \subseteq F$ holds for some open set O. Then $\mathcal{F}(K)$ is directed in $\mathcal{U}_K\mathbf{X}$ and $\bigcap \mathcal{F}(K) = K$ holds.

Proof: By Prop. 7.6 applied to $K \subseteq \mathbf{X}$, the set $\mathcal{F}(K)$ is not empty.

If F_1 and F_2 are two members of $\mathcal{F}(K)$, then $K \subseteq O_i \subseteq F_i$ holds for some open sets O_1 and O_2. By Prop. 7.6 applied to $K \subseteq O_1 \cap O_2$, there are an open set O' and a finitary set F such that $K \subseteq O' \subseteq F \subseteq O_1 \cap O_2 \subseteq F_1 \cap F_2$. Thus, $\mathcal{F}(K)$ is directed in $\mathcal{U}_K\mathbf{X}$.

The inclusion $\bigcap \mathcal{F}(K) \supseteq K$ is obvious. For the opposite inclusion, we apply Lemma 7.2. $K \subseteq O$ implies $K \subseteq O' \subseteq F \subseteq O$ for some finitary F and open O' by Prop. 7.6. Then $F \in \mathcal{F}(K)$, whence $\bigcap \mathcal{F}(K) \subseteq F \subseteq O$. $\square$

Thus, we obtain:

Theorem 7.10 $\mathcal{U}_S\mathbf{X}$, $\mathcal{U}_K\mathbf{X}$, and their cores coincide for multi-continuous ground domains $\mathbf{X}$.

7.5 The Initial Upper Power Construction

In this subsection, we prove that initial and final $\mathbf{U}$-constructions coincide for continuous ground domains. In the next subsection, we present a multi-continuous but not continuous ground domain where the two constructions differ.

In order to point out clearly where continuity is required in our proof, we start with a sober domain and restrict it to a more special class when needed.

Let $\mathbf{X}$ be a sober ground domain. Thus, we can take $\mathcal{U}_K\mathbf{X}$ as a representation of the final upper power domain over $\mathbf{X}$. We try to prove that $\mathcal{U}_K\mathbf{X}$ is initial, i.e., that it is a free

$\mathbf{U}$-$\mathbf{X}$-module. By similar arguments as in the case of $\mathbf{L}$, the addition of $\mathbf{U}$-modules is a binary meet operation and 0 is a top element. Thus, $\mathbf{U}$-module domains are domains (dcpo's) with a top element and a continuous binary meet operation, and $\mathbf{U}$-linear morphisms are functions preserving directed joins and finite meets.

Let $(M, \sqcap, \top)$ be an arbitrary $\mathbf{U}$-module, and let $f : [\mathbf{X} \to M]$ be a morphism. We have to provide a linear map $\bar{f} : [\mathcal{U}_K\mathbf{X} \to M]$ with $\bar{f} \circ \mathsf{s} = f$ and then show its uniqueness.

At first, we define a function f' on finite subsets of $\mathbf{X}$:

- For finite sets E, let $f'\{e_1, \ldots, e_n\} = fe_1 \sqcap \cdots \sqcap fe_n$.

This function is well defined because '$\sqcap$' is commutative and associative. The neutral element $\top$ in M can be taken as the result of applying f' to $\emptyset$. f' has the following properties:

(1) $f'(\emptyset) = \top$

(2) $f'(A \cup B) = f'A \sqcap f'B$ for all finite sets A and B.

(3) $f'\{x\} = fx$ for all x in $\mathbf{X}$.

(4) If $\uparrow A \supseteq \uparrow B$, then $f'A \leq f'B$ for all finite sets A and B.

Properties (1) and (3) are obvious. For (2), idempotence of '$\sqcap$' is needed since some operands occurring once in $f'(A \cup B)$ may occur twice in $f'A \sqcap f'B$. The precondition of (4) means $B \subseteq \uparrow A$, whence for all b in B, there is $\alpha(b)$ in A with $b \geq \alpha(b)$. By monotonicity of f, $f'B = \sqcap_{b \in B} fb \geq \sqcap_{b \in B} f(\alpha(b))$ holds. By idempotence of '$\sqcap$', the last term equals $\sqcap_{a \in \alpha[B]} fa$. By adding the remaining elements of A, the last term can only become smaller. Hence, it is above $\sqcap_{a \in A} fa = f'A$.

Because of property (4), $\uparrow A = \uparrow B$ implies $f'A = f'B$. Hence, f' implies a function $\widehat{f}$ mapping all finitary sets of $\mathbf{X}$ into M. $\widehat{f}$ is defined by $\widehat{f}(\uparrow E) = f'E$ for finite sets E. Properties (1) through (4) of f' directly translate into corresponding properties of $\widehat{f}$:

(1) $\widehat{f}(\emptyset) = 0$

(2) $\widehat{f}(A \cup B) = \widehat{f}A \sqcap \widehat{f}B$ for all finitary sets A and B.

(3) $\widehat{f}(\uparrow x) = fx$

(4) If $A \supseteq B$, then $\widehat{f}A \leq \widehat{f}B$ for all finitary sets A and B.

The next step should be the extension of $\widehat{f}$ to all members K of $\mathcal{U}_K\mathbf{X}$. In view of the results of the previous section, we consider the set of all values $\widehat{f}A$ where A is a finitary environment of K (cf. Prop. 7.9). We first investigate the set $\mathcal{F}(K)$ of finitary environments of K more closely and then turn to $\widehat{f}[\mathcal{F}(K)] \subseteq M$.

Proposition 7.11 Let $\mathbf{X}$ be a sober domain. Then

(1) $\emptyset$ is in $\mathcal{F}(\emptyset)$.

(2) $\mathcal{F}(K_1 \cup K_2) = \{A_1 \cup A_2 \mid A_i \in \mathcal{F}(K_i)\}$.

(3) $K \supseteq K'$ implies $\mathcal{F}(K) \subseteq \mathcal{F}(K')$.

(4) Let $\mathcal{K}$ be a directed set in $\mathcal{U}_K\mathbf{X}$. Then $\mathcal{F}(\bigcap \mathcal{K}) = \bigcup_{K \in \mathcal{K}} \mathcal{F}(K)$.

Proof:

(1) $\emptyset$ is a finitary open superset of $\emptyset$.

(2) Let A be a member of the left hand side. Then there is some open set O with $K_1 \cup K_2 \subseteq O \subseteq A$. This implies $K_i \subseteq O \subseteq A$, whence A is in both $\mathcal{F}(K_1)$ and $\mathcal{F}(K_2)$. Because of $A = A \cup A$, it is a member of the right hand side.

Conversely, if A is in the set on the right, then there are finitary sets A_i and open sets O_i such that $K_i \subseteq O_i \subseteq A_i$. This implies $K_1 \cup K_2 \subseteq O_1 \cup O_2 \subseteq A_1 \cup A_2 = A$.

(3) If $K \subseteq O \subseteq A$ for some open O, then $K' \subseteq K \subseteq O \subseteq A$.

(4) The union is a subset of $\mathcal{F}(\bigcap \mathcal{K})$ because of (3).

Assume $\bigcap \mathcal{K} \subseteq O \subseteq A$ for some open set O. By Prop. 7.7, there is some K in $\mathcal{K}$ with $K \subseteq O$. Hence, A is in $\mathcal{F}(K)$. □

If the ground domain is multi-continuous, $\mathcal{F}(K)$ is directed by Prop. 7.9, whence $\hat{f}[\mathcal{F}(K)]$ is directed, too, by monotonicity of $\hat{f}$. Hence, the set $\hat{f}[\mathcal{F}(K)]$ has a least upper bound in M. Thus, we can define $\bar{f}K = \bigsqcup \hat{f}[\mathcal{F}(K)]$. We have to show that this function is continuous, additive, and $\bar{f}(\uparrow x) = fx$ holds.

By property (1) of $\mathcal{F}(K)$, the value $\top = \hat{f}\emptyset$ is in $\hat{f}[\mathcal{F}(\emptyset)]$, whence $\bar{f}(\emptyset) = \top$. Using property (2) and continuity of '$\sqcap$' in M, one obtains

$$\begin{aligned}
\bar{f}(K_1 \cup K_2) &= \bigsqcup \hat{f}[\mathcal{F}(K_1 \cup K_2)] \\
&= \bigsqcup \{\hat{f}(A_1 \cup A_2) \mid A_i \in \mathcal{F}(K_i)\} \\
&= \bigsqcup \{\hat{f}A_1 \sqcap \hat{f}A_2 \mid A_i \in \mathcal{F}(K_i)\} \\
&= \bigsqcup \{\hat{f}A_1 \mid A_1 \in \mathcal{F}(K_1)\} \sqcap \bigsqcup \{\hat{f}A_2 \mid A_2 \in \mathcal{F}(K_2)\} \\
&= \bar{f}K_1 \sqcap \bar{f}K_2
\end{aligned}$$

By property (3) of Prop. 7.11, $K \le K'$ implies $\mathcal{F}(K) \subseteq \mathcal{F}(K')$, whence $\hat{f}[\mathcal{F}(K)] \subseteq \hat{f}[\mathcal{F}(K')]$. From this, $\bar{f}K \le \bar{f}K'$ follows. Hence, $\bar{f}$ is monotonic.

Continuity of $\bar{f}$ is shown using part (4) of Prop. 7.11. Let $\mathcal{K}$ be a directed set in $\mathcal{U}_K\mathbf{X}$.

$$\begin{aligned}
\bar{f}(\bigcap \mathcal{K}) &= \bigsqcup \hat{f}[\mathcal{F}(\bigcap \mathcal{K})] = \bigsqcup \hat{f}[\bigcup_{K \in \mathcal{K}} \mathcal{F}(K)] \\
&= \bigsqcup \bigcup_{K \in \mathcal{K}} \hat{f}[\mathcal{F}(K)] = \bigsqcup_{K \in \mathcal{K}} \bigsqcup \hat{f}[\mathcal{F}(K)] = \bigsqcup_{K \in \mathcal{K}} \bar{f}K
\end{aligned}$$

Finally, we have to show $\bar{f}(\uparrow x) = fx$ for all x in $\mathbf{X}$. For all A in $\mathcal{F}(\uparrow x)$, $A \supseteq \uparrow x$ holds, whence $\hat{f}A \subseteq \hat{f}(\uparrow x) = fx$. Thus, $\bar{f}(\uparrow x) = \bigsqcup \hat{f}[\mathcal{F}(\uparrow x)] \le fx$ follows.

For the opposite direction, we assume $fx \in O$ for some open set O, and have to show $\bar{f}(\uparrow x) \in O$ by Lemma 7.3. $fx \in O$ implies $x \in O' = f^{-1}[O]$. By continuity of f, O' is an open environment of x. By multi-continuity, there are a finite set E and an open set O'' with $x \in O'' \subseteq \uparrow E \subseteq O'$. Thus, $\uparrow E$ is in $\mathcal{F}(\uparrow x)$, and $fe \in O$ holds for all e in E. Unfortunately, this does not generally imply $\hat{f}(\uparrow E) = \bigsqcap_{e \in E} fe \in O$. Thus, the proof cannot be completed.

If $\mathbf{X}$ is even continuous, then the set E may be assumed to be a singleton by Th. 7.4. Thus, $\hat{f}(\uparrow E) = fe$ is in O, whence $\bar{f}(\uparrow x) = \bigsqcup \hat{f}[\mathcal{F}(\uparrow x)]$ is in O, and the proof is completed. Summarizing, we obtain

Theorem 7.12

For continuous ground domain $\mathbf{X}$, initial and final upper power domain over $\mathbf{X}$ coincide.

7.6 Initial and Final U-Construction Differ

In the previous subsection, we have seen that initial and final **U**-construction coincide for continuous ground domains. We now present an example of a domain that is not continuous but still multi-continuous, where initial and final **U**-construction differ. This example was not yet published in written form before.

The ground domain is $\mathbf{X} = \{0, 1\}^* \cup \{\top\}$. It consists of all finite sequences of 'bits' and an additional element $\top$. The sequences are ordered by the prefix order, and $\top$ is greater than all sequences. When drawn, the domain looks like an infinite binary tree where all branches converge to the single point $\top$.

The domain is not continuous since its way-below relation is empty except for $\epsilon \ll \epsilon$ where ϵ denotes the empty sequence. We claim that all open sets of $\mathbf{X}$ are finitary. For, closed sets that do not contain $\top$ do not contain any infinite branches, whence they are finite by König's Lemma. The complement of a finite lower subset E of $\mathbf{X}$ is finitary; its minimal elements result from the maximal elements of E by appending 0 or 1.

To show that $\mathcal{U}_K\mathbf{X}$ is not initial, we present a **U**-module M and a morphism $f : [\mathbf{X} \to M]$ such that there is no **U**-linear $\bar{f} : [\mathcal{U}_K\mathbf{X} \to M]$ with $\bar{f} \circ \mathsf{s} = f$.

The **U**-module M for our example is the set of all *regular closed* subsets of the unit interval $[0, 1]$ of real numbers, ordered by inverse inclusion. A set C is regular closed iff $\mathsf{cl}\ \mathsf{int}\, C = C$ holds. In [GHK+80], M is introduced and shown to be a complete Boolean algebra, whence in particular a **U**-module.[2] In M, $A \sqcap B$ is $A \cup B$, and $\bigsqcup_{i \in I} C_i$ is given by $\mathsf{cl}\ \mathsf{int} \bigcap_{i \in I} C_i$.

The morphism $f : [\mathbf{X} \to M]$ is defined by mapping the sequence $a_1 \ldots a_n$ to the interval $[r, r + 2^{-n}]$ where $r = \sum_{i=1}^{n} a_i \cdot 2^{-i}$. The additional element $\top$ is mapped to $\emptyset$. Thus, we obtain for instance $f(\epsilon) = [0, 1]$, $f(010) = [2/8, 3/8]$, and $f(111) = [7/8, 8/8]$. The only non-trivial limits in $\mathbf{X}$ are induced by infinite ascending sequences converging to $\top$. If $\alpha_1 = a_1$, $\alpha_2 = a_1 a_2$, $\alpha_3 = a_1 a_2 a_3$ etc. is such a sequence, then

$$\bigsqcup_{i \in \mathbf{N}} f(\alpha_i) = \mathsf{cl}\ \mathsf{int} \bigcap_{i \in \mathbf{N}} f(\alpha_i) = \mathsf{cl}\ \mathsf{int}\, \{r\} = \emptyset = f(\top)$$

where $r = \sum_{i=1}^{\infty} a_i \cdot 2^{-i}$. Thus, f is continuous.

Assume there is a continuous **U**-linear mapping $\bar{f} : [\mathcal{U}_K\mathbf{X} \to M]$ with $\bar{f} \circ \mathsf{s} = f$. Consider the sets $E_n = \{\alpha \in \{0, 1\}^* \mid |\alpha| = n\}$. Then the sets $K_n = \uparrow E_n$ form an ascending sequence in $\mathcal{U}_K\mathbf{X}$ with limit $\bigcap_{n \in \mathbf{N}} K_n = \{\top\}$. For every n in $\mathbf{N}$, **U**-linearity of $\bar{f}$ implies $\bar{f}(K_n) = \bigcup_{\alpha \in E_n} \bar{f}(\uparrow \alpha) = \bigcup_{\alpha \in E_n} f(\alpha) = \bigcup_{0 \le i < 2^n} [i/2^n, (i+1)/2^n] = [0, 1]$, whence $\bigsqcup_{n \in \mathbf{N}} \bar{f}(K_n) = [0, 1]$. On the other hand, $\bar{f}(\bigsqcup_{n \in \mathbf{N}} K_n) = \bar{f}\{\top\} = \bar{f}(\mathsf{s}\top) = f(\top) = \emptyset$ holds. This is the greatest possible contradiction to the continuity of $\bar{f}$.

The example shows that the initial upper power domain $\mathcal{U}_i\mathbf{X}$ and the final upper power domain $\mathcal{U}_K\mathbf{X}$ differ. Since $\mathcal{U}_K\mathbf{X}$ is reduced for multi-continuous $\mathbf{X}$, $\mathcal{U}_i\mathbf{X}$ cannot be a subdomain of $\mathcal{U}_K\mathbf{X}$. On the contrary, the mapping $F : [\mathcal{U}_i\mathbf{X} \to \mathcal{U}_K\mathbf{X}]$ with $F\{\!|x|\!\} = \uparrow\{x\}$, which results from initiality, can be shown to be epi in the category of dcpo's. The proof is a simple consequence of $\mathcal{U}_K\mathbf{X}$ being reduced.

7.7 Observable and Non-Observable U-Modules

The result of the previous subsection indicates that there are **U**-modules that are not observable. Indeed, the **U**-module M of regular closed sets is not observable since otherwise, the extended morphism $\bar{f}$ would exist by Th. 5.8. A second example is provided by the initial upper power domain $\mathcal{U}_i\mathbf{X}$ of the domain $\mathbf{X}$ of the previous subsection. If it were observable,

[2] Actually, regular open sets ordered by inclusion are considered in this reference.

then the singleton map $\mathsf{s}_i : [\mathbf{X} \to \mathcal{U}_i\mathbf{X}]$ would have an extension $h : [\mathcal{U}_K\mathbf{X} \to \mathcal{U}_i\mathbf{X}]$, whence we could construct $\bar{f}$ by composing h and the $\mathcal{U}_i$-extension of f.

There is an interesting criterion for observability of $\mathbf{U}$-modules:

Theorem 7.13

An $\mathbf{U}$-module $\mathbf{M} = (M, +, 0, \cdot)$ is observable iff it can be embedded into a power set, i.e., if there is a set S and a map $\varphi : M \to 2^S$ such that

(1) $a \le b$ iff $\varphi a \supseteq \varphi b$,

(2) $\varphi(0) = \emptyset$ and $\varphi(a + b) = \varphi a \cup \varphi b$,

(3) $\varphi(\bigsqcup_{i \in I} a_i) = \bigcap_{i \in I} \varphi a_i$ for directed families.

Proof: By Th. 5.6, an R-module is observable iff it can be embedded into $\prod_{i \in I} R$. The product $\prod_{i \in I} \mathbf{U}$ and the powerset $(2^I, \supseteq)$ are isomorphic by $(u_i)_{i \in I} \leftrightarrow \{i \in I \mid u_i = 1\}$. The powerset 2^I has to be ordered by '$\supseteq$' since $1 < 0$ in $\mathbf{U}$. □

This theorem again shows that $\mathcal{U}_f\mathbf{X}$ is observable for sober $\mathbf{X}$ because of its concrete representation as $\mathcal{U}_K\mathbf{X}$ in terms of subsets of $\mathbf{X}$. For non-sober $\mathbf{X}$, $\mathcal{U}_f\mathbf{X}$ is still observable by Prop. 5.5 although it cannot be represented in terms of subsets of $\mathbf{X}$. There is a representation by subsets of $[\mathbf{X} \to \mathbf{U}]$ however, i.e., sets of open sets of $\mathbf{X}$.

If $\mathcal{U}_i\mathbf{X}$ is not observable — e.g., for the example domain $\mathbf{X}$ of the previous subsection — then it cannot be represented in terms of subsets of any set S without violating property (2) or (3) of the representation function φ in Th. 7.13. This does not exclude completely an explicit representation of the implicitly defined free $\mathbf{U}$-$\mathbf{X}$-module $\mathcal{U}_i\mathbf{X}$, but it indicates that such a representation will be difficult to find. Indeed, nobody has ever found one to the best of our knowledge.

8 Further Semirings

Finally, we want to consider two further semirings. We do not skip into details here; all proofs will be omitted. They can be found in [Hec90a, Hec91, Hec93].

8.1 The Boolean Semiring

Recently, Buneman and Gunter defined two power domain constructions, the *Sandwich* construction $\mathcal{S}$ [BDW88] and the *mixed* construction $\mathcal{M}$ [Gun89, Gun90]. These constructions were defined for algebraic ground domains only via base and ideal completion. Although they differ even for small finite ground domains, they coincide for $\mathbf{1}$, i.e., they share the same characteristic semiring. This semiring is $\mathbf{B} = \{\perp, 0, 1\}$, where $\perp$ is the least element, and 0 and 1 are two incomparable maximal elements. Addition is defined like 'parallel or', and multiplication like 'parallel and'.

In [Gun89, Gun90], Gunter showed that mixed powerdomains $\mathcal{M}\mathbf{X}$ are free *mix algebras* over $\mathbf{X}$. In [Hec91], it is shown that mix algebras are nothing else than $\mathbf{B}$-modules. Thus, $\mathcal{M}$ is the initial $\mathbf{B}$-construction. On the other hand, $\mathcal{S}$ is shown to be the final $\mathbf{B}$-construction in [Hec93]. Since (for algebraic ground domains) $\mathcal{M}$ is a subconstruction of $\mathcal{S}$, the core of $\mathcal{S}$ coincides with $\mathcal{M}$.

Our theory allows the extension of the two constructions $\mathcal{M}$ and $\mathcal{S}$ to arbitrary ground domains. Whereas $\mathcal{S}$ should be generalized as the final $\mathbf{B}$-construction, there are two possible generalizations of $\mathcal{M}$: the initial $\mathbf{B}$-construction $\mathcal{M}_i$ producing free $\mathbf{B}$-modules and the core of the final $\mathbf{B}$-construction $\mathcal{M}_c$ producing free observable $\mathbf{B}$-modules. For continuous ground domains, they can be shown to coincide, whereas they generally differ. The example for this difference is the same as in Subsection 7.6. The reason for this is that powerdomains $\mathcal{S}\mathbf{X}$ and $\mathcal{M}_c\mathbf{X}$ can be explicitly represented as subdomains of $\mathcal{L}\mathbf{X} \times \mathcal{U}_f\mathbf{X}$. For powerdomains $\mathcal{M}_i\mathbf{X}$, no explicit description is known in general.

8.2 The Convex Semiring

If Plotkin's power construction is extended by the empty set, then the empty set will become incomparable to any other set. Thus, the characteristic semiring of this construction is $\mathbf{C} = \{0, 1\}$, where 0 and 1 are incomparable. If we define Plotkin's construction as free construction of semilattices following [HP79] and add a neutral element, then we arrive at the initial $\mathbf{C}$-construction $\mathcal{C}_i$.

In contrast to all the final constructions we met so far, the final $\mathbf{C}$-construction is degenerated. The reason is that a domain $\mathbf{X}$ with least element $\perp$ has just two $\mathbf{C}$-predicates, namely the two constant functions $\lambda x.\, 0$ and $\lambda x.\, 1$. Thus, there are just two (right) linear second order predicates, i.e., the final $\mathbf{C}$-powerdomain of a domain $\mathbf{X}$ with $\perp$ has two elements, no matter how large $\mathbf{X}$ is. Consequently, the core of the final $\mathbf{C}$-construction is also degenerated.

Nevertheless, there is a non-degenerated $\mathbf{C}$-construction $\mathcal{C}_p$ defined in terms of second-order predicates. The trick is to use $\mathbf{B}$-predicates instead of $\mathbf{C}$-predicates [Hec93]. The construction $\mathcal{C}_p$ is a modified core of the final $\mathbf{B}$-construction $\mathcal{S}$, where the modification consists of dropping condition (4) of Def. 4.2. Equivalently put, we do not form the $\mathbf{B}$-core of the $\mathbf{B}$-module $\mathcal{S}\mathbf{X}$, but the $\mathbf{C}$-core of $\mathcal{S}\mathbf{X}$ considered as a $\mathbf{C}$-module. (Notice that $\mathbf{C}$ is a subsemiring of $\mathbf{B}$).

Similarly to the cases of $\mathbf{U}$ and $\mathbf{B}$, one can show that $\mathcal{C}_i$ and $\mathcal{C}_p$ coincide for continuous ground domains, but differ in general. Again, the example of Subsection 7.6 can be employed to show the difference. The question remains how to characterize the $\mathcal{C}_p$-powerdomains as free algebraic structures.

9 Conclusion

The general theory of power domain constructions shows that every construction is equipped with a so-called characteristic semiring. Conversely, every semiring R induces three power domain constructions: the initial R-construction, the final R-construction, and the core of the latter. From previous publications [Hec90a, Hec91], the initial R-construction was known to produce free R-modules. In this paper, we were able to show that the core of the final R-construction produces free *observable* R-modules.

We then investigated two semirings that seem to be simple: the 'lower' semiring $\mathbf{L} = \{0 < 1\}$ and the 'upper' semiring $\mathbf{U} = \{1 < 0\}$. Whereas the three $\mathbf{L}$-constructions induced by the general theory coincide, we found that the initial **U**-construction $\mathcal{U}_i$, the final one $\mathcal{U}_f$, and its core $\mathcal{U}_c$ are all different when applied to arbitrary ground domains. For continuous ground domains however, they still coincide.

From these three constructions, $\mathcal{U}_c$ produces the smallest power domains. The power domains of $\mathcal{U}_f$ may contain junk, i.e., elements that cannot be reached from singletons by finite unions and limits. The power domains of $\mathcal{U}_i$ on the other hand may distinguish between elements that cannot be distinguished by any linear **U**-valued predicate.

Furthermore, construction $\mathcal{U}_c$ incorporates both the advantages of $\mathcal{U}_i$ and $\mathcal{U}_f$. Like $\mathcal{U}_f$, it can be concretely represented in terms of second order predicates or in terms of certain sets, and like $\mathcal{U}_i$, it can be implicitly characterized as a certain free construction. Compared to this, $\mathcal{U}_i$ lacks an explicit representation, and $\mathcal{U}_f$ lacks a characterization as a free construction other than the self-referential description in terms of $\mathcal{U}_f$-algebras.

The investigation of the two further semirings **B** and **C** reveals a situation similar to the case of **U**. For continuous ground domains, the initial power domains can be characterized in terms of second order predicates, whereas for more general ground domains, this is not possible.

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