Domain Environments

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Abstract

We show some theorems about domain environments of topological and metric spaces.

1 Domain Environments for Topological Spaces

1.1 The Basic Situation

The basic situation we consider is the following: D is a continuous dcpo (not necessarily bounded complete) with a basis B (not necessarily countable), M is its subset of maximal elements, and X is a subset of M. We say that (D, B, M, X) is a (generalized) *domain environment*. We consider X and M as topological spaces whose topologies result as subspace topologies from the Scott topology of D.

Proposition 1.1 Space X is a $\mathcal{T}_1$-space.

Proof: The specialization preorder of X is the restriction of the order of D onto X. Since X is a subset of M, $x \sqsubseteq x'$ in X implies $x = x'$. $\Box$

1.2 Characterization of the Subspace Topology on X

Let $\mathcal{T}$ be any topology on X. We want to compare it with the subspace topology $\mathcal{S}$ on X induced by a domain environment (D, B, M, X).

Proposition 1.2 The following are equivalent:

(1) $\mathcal{S} \subseteq \mathcal{T}$;

(2) If $x \gg d$ for x in X and d in D, then there is an open set U in $\mathcal{T}$ such that $x \in U \subseteq \Uparrow d$;

(3) If $x \gg b$ for x in X and b in B, then there is an open set U in $\mathcal{T}$ such that $x \in U \subseteq \Uparrow b$.

Proof:

(1) $\Rightarrow$ (2) : $\twoheaduparrow d$ is open in D, whence $U = \twoheaduparrow d \cap X$ is in $\mathcal{S} \subseteq \mathcal{T}$. The claimed property $x \in U \subseteq \twoheaduparrow d$ obviously holds.

(2) $\Rightarrow$ (3) : Obvious.

(3) $\Rightarrow$ (1) : Let O be in $\mathcal{S}$. Since $\mathcal{S}$ is the subspace topology, there is V Scott open in D such that $O = V \cap X$. For every x in O, there is b_x in B such that $x \gg b_x \in V$. By (3), there is U_x in $\mathcal{T}$ such that $x \in U_x \subseteq \twoheaduparrow b_x \subseteq V$, whence $x \in U_x \subseteq O$. Thus, $O = \bigcup_{x \in O} U_x$ is in $\mathcal{T}$. □

Proposition 1.3 The following are equivalent:

(1) $\mathcal{T} \subseteq \mathcal{S}$;

(2) If $x \in O \in \mathcal{T}$, then there is d in D such that $x \gg d$ and $\twoheaduparrow d \cap X \subseteq O$;

(3) If $x \in O \in \mathcal{T}$, then there is b in B such that $x \gg b$ and $\twoheaduparrow b \cap X \subseteq O$.

Proof:

(1) $\Rightarrow$ (3) : If $x \in O \in \mathcal{T}$, then $x \in O \in \mathcal{S}$. Since $\mathcal{S}$ is the subspace topology, there is V Scott open in D such that $O = V \cap X$. Since x in V, there is b in B such that $x \gg b \in V$. Hence $\twoheaduparrow b \subseteq V$, whence $\twoheaduparrow b \cap X \subseteq V \cap X = O$.

(3) $\Rightarrow$ (2) : Obvious.

(2) $\Rightarrow$ (1) : Let O be in $\mathcal{T}$. For every x in O, there is d_x in D such that $x \gg d_x$ and $\twoheaduparrow d_x \cap X \subseteq O$ by (2). Let $U_x = \twoheaduparrow d_x \cap X$. Since $\twoheaduparrow d_x$ is Scott open, U_x is in $\mathcal{S}$. For every x in X, $x \in U_x \subseteq O$ holds, whence $O = \bigcup_{x \in X} U_x$ is in $\mathcal{S}$. □

1.3 Closed Approximation

A domain environment (D, B, M, X) has the *closed approximation* property if for every d in D, $\uparrow d \cap X$ is closed in X. The relevance of closed approximation can be seen by the following result:

Proposition 1.4 If (D, B, M, X) has closed approximation, then X is regular.

Proof: Let $x \in U$ where U is an open set of X. By Prop. 1.3 (2), there is d in D such that $x \gg d$ and $\twoheaduparrow d \cap X \subseteq U$. By interpolation, there is d' in D such that $x \gg d' \gg d$. Then $x \in \twoheaduparrow d' \cap X \subseteq \uparrow d' \cap X \subseteq \twoheaduparrow d \cap X \subseteq U$. Set $\twoheaduparrow d' \cap X$ is open in X, and $\uparrow d' \cap X$ is closed in X by the closed approximation property. □

The opposite of Prop. 1.4 does not hold; see Example 3 of Subsection 1.10.

Closed approximation can be checked by considering the basis only:

Proposition 1.5 (D, B, M, X) has closed approximation (i.e., $\uparrow d \cap X$ is closed in X for every d in D) if and only if $\uparrow b \cap X$ is closed in X for every b in B.

Proof: The 'only if' is trivial. For the if part, let d be an arbitrary member of D. Since B is a basis, $d = \bigsqcup B(d)$ holds, where $B(d) = \{b \in B \mid b \ll d\}$. Then $\uparrow d = \bigcap_{b \in B(d)} \uparrow b$, whence $\uparrow d \cap X = \bigcap_{b \in B(d)} (\uparrow b \cap X)$ is closed in X. □

Next, we consider closed approximation from the point of view of a fixed element d of D.

Proposition 1.6 For a fixed d in D, the following are equivalent:

(1) $\uparrow d \cap X$ is closed in X;

(2) there is a closed set C of D such that $\uparrow d \cap X = C \cap X$;

(3) $\uparrow d \cap X = \mathsf{cl}_D\,(\uparrow d \cap X) \cap X$;

(4) $\mathsf{cl}_D\,(\uparrow d \cap X) \cap X \subseteq \uparrow d$.

Proof:

(1) $\Leftrightarrow$ (2) since X is a subspace of D.

(2) $\Rightarrow$ (4) : $\mathsf{cl}\,(\uparrow d \cap X) \cap X = \mathsf{cl}\,(C \cap X) \cap X \subseteq \mathsf{cl}\,C \cap X = C \cap X = \uparrow d \cap X \subseteq \uparrow d$.

(4) $\Rightarrow$ (3) : The inclusion '$\supseteq$' is an immediate consequence of (4). The opposite one follows from $\uparrow d \cap X = \uparrow d \cap X \cap X \subseteq \mathsf{cl}\,(\uparrow d \cap X) \cap X$.

(3) $\Rightarrow$ (2) : Let $C = \mathsf{cl}\,(\uparrow d \cap X)$. □

1.4 Strong Closed Approximation

By Prop. 1.6 (4), (D, B, M, X) has closed approximation (CA) iff for all d in D, $\mathsf{cl}_D\,(\uparrow d \cap X) \cap X \subseteq \uparrow d$. By Prop. 1.5, it suffices to check this for all b in B.

We define: (D, B, M, X) has *strong closed approximation* (SCA) if even $\mathsf{cl}_D \uparrow d \cap X \subseteq \uparrow d$ holds for all d in D. Obviously, SCA implies CA. Again, it suffices to check a basis:

Proposition 1.7 (D, B, M, X) has strong closed approximation if and only if $\mathsf{cl} \uparrow b \cap X \subseteq \uparrow b$ for every b in B.

Proof: The 'only if' is trivial. For the if part, let d be an arbitrary member of D. Since B is a basis, $d = \bigsqcup B(d)$ holds, where $B(d) = \{b \in B \mid b \ll d\}$. For every b in $B(d)$, $\mathsf{cl} \uparrow d \cap X \subseteq \mathsf{cl} \uparrow b \cap X \subseteq \uparrow b$ holds. Hence $\mathsf{cl} \uparrow d \cap X \subseteq \bigcap_{b \in B(d)} \uparrow b = \uparrow d$. □

Next, we consider CA and SCA for varying subsets X of M.

Proposition 1.8 If (D, B, M, X) has (strong) closed approximation and $X' \subseteq X$, then (D, B, M, X') has (strong) closed approximation.

Proof:
For CA: for all d in D, $\mathsf{cl}\,(\uparrow d \cap X') \cap X' \subseteq \mathsf{cl}\,(\uparrow d \cap X) \cap X \subseteq \uparrow d$.
For SCA: for all d in D, $\mathsf{cl} \uparrow d \cap X' \subseteq \mathsf{cl} \uparrow d \cap X \subseteq \uparrow d$. □

The significance of the SCA property lies in the following:

Proposition 1.9 Let $(X_i)_{i \in I}$ be a family of subsets of M with the SCA property. Then $\bigcup_{i \in I} X_i$ has SCA. Hence, there is a largest $X \subseteq M$ which enjoys the SCA property.

Proof: If $\mathsf{cl} \uparrow d \cap X_i \subseteq \uparrow d$ for all i in I, then $\mathsf{cl} \uparrow d \cap \bigcup_{i \in I} X_i = \bigcup_{i \in I} (\mathsf{cl} \uparrow d \cap X_i) \subseteq \uparrow d$. □

In the next subsection, we characterize the largest X with SCA further. In general, there is no largest X with CA. For, there are examples (see Example 3 in Subsection 1.10) where M itself does not have CA. On the other hand, $\{m\}$ has CA for every m in M, since every subset of $\{m\}$ is closed.

1.5 Strongly Maximal Elements

The notion of a strongly maximal element of a continuous dcpo is defined in the following proposition.

Proposition 1.10 For an element x of a continuous dcpo D with basis B, the following are equivalent:

(1) For every a in D: x in $\mathsf{cl}\uparrow a$ implies x in $\uparrow a$.

(2) For every a in B: x in $\mathsf{cl}\uparrow a$ implies x in $\uparrow a$.

(3) For every $a \gg b$ in D: x in $\mathsf{cl}\uparrow a$ implies x in $\twoheaduparrow b$.

(4) For every $a \gg b$ in B: x in $\mathsf{cl}\uparrow a$ implies x in $\twoheaduparrow b$.

(5) For every $a \gg b$ in D: x in $\mathsf{cl}\uparrow a$ implies x in $\uparrow b$.

(6) For every $a \gg b$ in B: x in $\mathsf{cl}\uparrow a$ implies x in $\uparrow b$.

(7) For every $a \gg b$ in D: x in $\mathsf{cl}\twoheaduparrow a$ implies x in $\twoheaduparrow b$.

(8) For every $a \gg b$ in B: x in $\mathsf{cl}\twoheaduparrow a$ implies x in $\twoheaduparrow b$.

An element x with these properties is called *strongly maximal*. The set of strongly maximal elements of a continuous dcpo D is called M^*D.

Proof:

(1) $\Rightarrow$ (3) : If $a \gg b$, then $\uparrow a \subseteq \twoheaduparrow b$.

(3) $\Rightarrow$ (4) , (5) $\Rightarrow$ (6), and (7) $\Rightarrow$ (8) are obvious.

(3) $\Rightarrow$ (5) and (4) $\Rightarrow$ (6) hold since $\twoheaduparrow b \subseteq \uparrow b$.

(3) $\Rightarrow$ (7) and (4) $\Rightarrow$ (8) hold since $\twoheaduparrow a \subseteq \uparrow a$.

(8) $\Rightarrow$ (4) : Let $a \gg b$ in B and assume x in $\mathsf{cl}\uparrow a$. By interpolation, there is a' in B such that $a \gg a' \gg b$. Then $\uparrow a \subseteq \twoheaduparrow a'$, whence x in $\mathsf{cl}\twoheaduparrow a'$. From (8), x in $\twoheaduparrow b$ follows.

(6) $\Rightarrow$ (2) : By continuity, $a = \bigsqcup B(a)$ where $B(a) = \{b \in B \mid b \ll a\}$. If x is in $\mathsf{cl}\uparrow a$, then x is in $\uparrow b$ for all b in $B(a)$ by (6). Hence, x is in $\bigcap_{b \in B(a)} \uparrow b = \uparrow a$.

(2) $\Rightarrow$ (1) : By continuity, $a = \bigsqcup B(a)$ where $B(a) = \{b \in B \mid b \ll a\}$. If x is in $\mathsf{cl}\uparrow a$, then x is in $\mathsf{cl}\uparrow b$ for all b in $B(a)$, whence x in $\uparrow b$ for all b in $B(a)$ by (2). Hence, x is in $\bigcap_{b \in B(a)} \uparrow b = \uparrow a$. $\square$

The name 'strongly maximal' is appropriate:

Proposition 1.11 Every strongly maximal element is maximal.

Proof: Let x be strongly maximal and assume $x \sqsubseteq y$. Then x is in $\downarrow y \subseteq \downarrow\uparrow y \subseteq \mathsf{cl}\uparrow y$. By strong maximality, x is in $\uparrow y$, i.e., $y \sqsubseteq x$ holds. Thus $x = y$ as required. $\square$

The relevance for domain environments comes from the following facts:

Lemma 1.12 An element m of M is strongly maximal iff $(D, B, M, \{m\})$ has SCA.

Proof: The subspace $\{m\}$ has SCA iff for all d in D, $\mathsf{cl}\uparrow d \cap \{m\} \subseteq \uparrow d$. This is equivalent to $m \notin \mathsf{cl}\uparrow d$ or $m \in \uparrow d$ which means m is strongly maximal. $\square$

Theorem 1.13 For every continuous dcpo D, the set M^*D is the largest subset of MD which has the SCA property.

Proof: We have to show two things:

(1) If $X \subseteq M$ has SCA, then $X \subseteq M^*$;

(2) M^* itself has SCA.

For (1), assume $X \subseteq M$ has SCA. Then for every x in X, $\{x\}$ has SCA by Prop. 1.8. By Lemma 1.12, every such x is in M^*.

By Lemma 1.12, $\{x\}$ has SCA for every x in M^*. By Prop. 1.9, $M^* = \bigcup_{x \in M^*} \{x\}$ has SCA. □

Summarizing, we see that for continuous dcpo's D, the set M^*D has nicer properties than MD. Whereas MD does not have the CA property for every D (see Example 3 in Subsection 1.10), the set M^*D always has the stronger SCA property, and it is the largest such set. For some continuous D, the subspace MD is not Hausdorff (see Example 4 in Subsection 1.10), whereas M^*D is always regular. Moreover, strong maximality in the description of Prop. 1.10 (4) may be better suited for an effective treatment than maximality.

1.6 Density

In the previous subsection, we sometimes considered subspaces $\{m\}$ of M. Normally, we want to consider 'big' subspaces of M. A possible criterion for bigness is that X is dense in M.

Proposition 1.14 In a domain environment (D, B, M, X), X is dense in M iff X is dense in D, i.e., $\mathsf{cl}_D\, X = D$.

Proof: First assume X is dense in M. Let O be a non-empty open subset of D. Since there is a maximal element above every element of D (Zorn!), $O \cap M$ is a non-empty open set of M. Since X is dense in M, $O \cap M$ meets X.

For the opposite direction, assume $X \subseteq M$ is dense in D. Let U be a non-empty open subset of M. Since U is open in M, there is V open in D such that $V \cap M = U$. The open set V is non-empty as a superset of U. Since X is dense in D, there is x in $V \cap X = V \cap M \cap X = U \cap X$. □

Because of this result it is appropriate to study the dense subsets of a continuous dcpo a bit closer. Notice that dense subsets of a continuous dcpo may be quite small; e.g., $\{\top\}$ is a dense subset of every continuous lattice.

Proposition 1.15 For a subset S of a continuous dcpo D with basis B, the following are equivalent:

(1) S is dense in D, i.e., $\mathsf{cl}_D\, S = D$.

(2) Every non-empty open subset of D meets S.

(3) For every $d \gg d'$ in D, $\Uparrow d'$ meets S.

(4) For every $b \gg b'$ in B, $\Uparrow b'$ meets S.

Proof: Properties (1) and (2) are equivalent by definition of cl.

(2) $\Rightarrow$ (3) : The set $\Uparrow d'$ contains d. Thus it is a non-empty open subset of D.

(3) $\Rightarrow$ (4) is obvious.

(4) $\Rightarrow$ (2) : Let V be a non-empty open subset of D. For d in V, continuity gives b and b' in B such that $d \gg b \gg b'$ and b' in V. By (4), $\Uparrow b'$ meets S. Since $\Uparrow b' \subseteq V$, V meets S, too. □

Lemma 1.16 Let O be a dense open subset of a continuous dcpo D. For every $a \gg b$ in D, there are $a' \gg b'$ in $O \cap \Uparrow b$.

Proof: By Prop. 1.15 (3), $\Uparrow b$ meets O. Let a' be in $\Uparrow b \cap O$. Since $\Uparrow b \cap O$ is open, it contains some b' with $b' \ll a'$. □

The following theorem is Baire's Theorem for continuous dcpo's.

Theorem 1.17 Let $(O_n)_{n \in \mathbf{N}}$ be a countable family of dense open subsets of a continuous dcpo D. Then $\bigcap_{n \in \mathbf{N}} O_n$ is dense in D.

Proof: Let $S = \bigcap_{n \in \mathbf{N}} O_n$. We use Prop. 1.15 (3) to prove that S is dense. Let $a_0 \gg b_0$ in D. By Lemma 1.16, there are $a_1 \gg b_1$ in $O_1 \cap \Uparrow b_0$. Hence, there are $a_2 \gg b_2$ in $O_2 \cap \Uparrow b_1$. Continuing, we obtain two sequences $(a_n)_{n \in \mathbf{N}_0}$ and $(b_n)_{n \in \mathbf{N}_0}$ with $b_n \in O_n \cap \Uparrow b_{n-1}$. Hence, $b_0 \ll b_1 \ll \cdots$ is an ascending sequence. Let b be its least upper bound. For every n, $b \gg b_n \in O_n$, whence b is in S. Of course, $b \gg b_0$ holds. Thus b witnesses that $\Uparrow b_0$ meets S. □

Finally, we present some examples for dense open sets.

Proposition 1.18 For $a \gg b$ in a continuous dcpo D, the set $O_{ab} = \neg \mathsf{cl} \uparrow a \cup \Uparrow b$, where '$\neg$' means complement, is a dense open subset of D.

Proof: The set is open as the union of two open sets. Let U be a non-empty open set. If U meets $\neg \mathsf{cl} \uparrow a$, then it meets O_{ab}. If not, it is a subset of $\mathsf{cl} \uparrow a$. Since O is non-empty, it is an open set which meets $\mathsf{cl} \uparrow a$. Thus it meets $\uparrow a$. The relation $a \gg b$ implies $\uparrow a \subseteq \Uparrow b \subseteq O_{ab}$, whence O meets O_{ab}. □

1.7 Density and Closed Approximation

For dense X, CA and SCA are equivalent. To prove this, we need a lemma.

Lemma 1.19 Let S be a dense subset of a continuous dcpo D. If $a \gg b$ in D, then $\mathsf{cl} \uparrow a \subseteq \mathsf{cl}\,(\uparrow b \cap S)$.

Proof: For arbitrary subsets A and B of a topological space, $\mathsf{cl}\, A \subseteq \mathsf{cl}\, B$ holds iff for all open O, $O \cap A \neq \emptyset$ implies $O \cap B \neq \emptyset$. Hence, assume O meets $\uparrow a$. The relation $a \gg b$ implies $\uparrow a \subseteq \Uparrow b$. Thus $O \cap \Uparrow b$ is a non-empty open set. Since S is dense, $O \cap \Uparrow b \cap S$ is non-empty. Since $\Uparrow b \subseteq \uparrow b$, O meets $\uparrow b \cap S$. □

Proposition 1.20 If X is dense in M and has CA, then it has SCA.

Proof: By Theorem 1.13, we have to show $X \subseteq M^*$. To do this, we use Prop. 1.10 (5). Let x be in X and $a \gg b$ in D. We have to show that $x \in \uparrow b$ if $x \in \mathsf{cl} \uparrow a$. Thus assume $x \in \mathsf{cl} \uparrow a$. By Lemma 1.19, $\mathsf{cl} \uparrow a \subseteq \mathsf{cl}\,(\uparrow b \cap X)$ holds. Hence, $x \in \mathsf{cl}\,(\uparrow b \cap X) \cap X$ which is a subset of $\uparrow b$ by Prop. 1.6 (4). □

Corollary 1.21 The set M itself has CA iff it has SCA.

Proof: M is dense since there is a maximal element above every element (Zorn). $\square$

1.8 The Countable Case

In the case of an ω-continuous dcpo, something more can be said.

Theorem 1.22 For every ω-continuous dcpo D, M^* is a dense G_δ-set of D. It is the largest subset of M with SCA, and it is maximal among the subsets of M with CA.

Proof: Let B be a countable basis of D. For $a \gg b$ in B let $O_{ab} = \neg \mathsf{cl} \uparrow a \cup \Uparrow b$. By Prop. 1.18, these are dense open sets. By Prop. 1.10 (4), $M^* = \bigcap_{a,b \in B,\, a \gg b} O_{ab}$ holds. Since B is countable, this is a countable intersection of open sets, i.e., a G_δ-set. By Theorem 1.17, it is dense.

From Theorem 1.13, we already know that M^* is the largest SCA subset of M. To prove that it is a maximal CA subset, let $M^* \subseteq X$ where X has CA. Since M^* is dense, X is dense as well. By Prop. 1.20, it has SCA. From Theorem 1.13, $X \subseteq M^*$ follows. $\square$

Among other things, the theorem tells that for every ω-continuous D, M^*D is a G_δ-set of D. It is not known whether MD is a G_δ-set of D for every ω-continuous D. At least we can say the following:

Proposition 1.23 If D is ω-continuous such that MD has CA, then MD is a dense G_δ-set of D.

Proof: Since $M^* \subseteq M$ and M^* is maximal with CA, $M = M^*$ follows. $\square$

This result was already found by Lawson with a completely different proof.

1.9 A Hierarchy of Continuous Domains

Here, we consider possible additional properties of the continuous dcpo's D in a domain environment (D, B, M, X). In particular, we look at:

(1) Distributive bounded completeness: Every bounded subset of D has a least upper bound in D, and $\downarrow d$ is a distributive lattice for every d in D.

(2) Bounded completeness: Every bounded subset of D has a least upper bound in D.

(3) FS: D is an FS-domain.

(4) Compact intersection: For every a and b in D, $\uparrow a \cap \uparrow b$ is compact in the Scott topology.

(5) Closed consistence: For every d in D, $\downarrow\uparrow d$ is Scott closed.

(6) $M^*D = MD$, or equivalently MD has SCA, or equivalently, MD has CA. (The first equivalence is due to Theorem 1.13, and the second due to Cor. 1.21.)

(7) MD is a regular space.

Proposition 1.24 The properties listed above form a hierarchy: (1) $\Rightarrow$ (2) $\Rightarrow \cdots \Rightarrow$ (7).

Proof:

(1) $\Rightarrow$ (2) is obvious.

(2) $\Rightarrow$ (3) and (3) $\Rightarrow$ (4) were shown by Achim Jung.

(2) $\Rightarrow$ (4) can also be shown directly:

Let D be bounded complete. If a and b are bounded above, then $\uparrow a \cap \uparrow b = \uparrow(a \sqcup b)$ is compact. Otherwise, $\uparrow a \cap \uparrow b = \emptyset$ which is also compact.

(4) $\Rightarrow$ (5) : We have to show $\mathsf{cl}\uparrow d \subseteq \downarrow\uparrow d$. Let $a \in \mathsf{cl}\uparrow d$. For every $b \ll a$, a is in the open set $\Uparrow b$, whence $\Uparrow b$ meets $\uparrow d$. Thus, $\uparrow b \cap \uparrow d$ is non-empty for every $b \ll a$. By (4) and continuity, $\{\uparrow b \cap \uparrow d \mid b \ll a\}$ is a filtered set of non-empty compact sets. Continuous dcpo's are sober, and in sober spaces, filtered intersections of non-empty compact sets are non-empty. Let c be in $\bigcap_{b \ll a} \uparrow b \cap \uparrow d$. Then c in $\bigcap_{b \ll a} \uparrow b$ which equals $\uparrow a$ by continuity. From $a \sqsubseteq c \sqsupseteq d$, a in $\downarrow\uparrow d$ follows.

(5) $\Rightarrow$ (6) : We show that every maximal element is strongly maximal. Let m be maximal, and let m be in $\mathsf{cl}\uparrow d$ for some d in D. By (5), m is in $\downarrow\uparrow d$. Hence, there is some a in D such that $m \sqsubseteq a$ and $a \sqsupseteq d$. By maximality of m, $m = a \sqsupseteq d$ follows, whence m in $\uparrow d$.

(6) $\Rightarrow$ (7) by Prop. 1.4. □

1.10 Examples

Example 1

Let $D = \{a_1, a_2, \ldots, a_\infty, b_1, b_2, \ldots\}$ ordered by $a_i \sqsubseteq a_j$ iff $i \le j$, $b_i \sqsubseteq b_j$ iff $i = j$, and $a_i \sqsubseteq b_j$ iff $i \le j$. This domain is ω-algebraic and distributive bounded complete, i.e., in class (1). The subspace $MD = M^*D = \{b_1, b_2, \ldots, a_\infty\}$ is a compact Hausdorff space. The subspace $\{b_1, b_2, \ldots\}$ is a dense G_δ-set with SCA which is isomorphic to the discrete space of natural numbers. The example shows that there may be several dense G_δ-sets with SCA.

Example 2

Let $D = \{\bot, a_1, a_2, \ldots, b_1, b_2, \ldots\}$ where $\bot$ is below everything, $a_i \sqsubseteq a_j$ iff $i = j$, $b_i \sqsubseteq b_j$ iff $i = j$, and $a_i \sqsubseteq b_j$ iff $i \le j$. This is an ω-algebraic dcpo where all elements are compact. Since D has finite height, all lower subsets of D are closed. In particular, all $\downarrow\uparrow d$ for d in D are closed, i.e., D is in class (5) of the domain hierarchy. The set $\uparrow a_1 \cap \uparrow a_2 = \{b_2, b_3, \ldots\}$ is not compact. Thus, D is not in class (4) of the hierarchy. The subspace $MD = M^*D = \{b_1, b_2, \ldots\}$ is isomorphic to the discrete space of natural numbers.

Example 3

Let $D = \{\bot, c, a_1, a_2, \ldots, a_\infty, b_1, b_2, \ldots\}$ where $\bot$ is below everything, c is below itself and all b_i, and the order among the a_i and b_j is as in Example 1. This domain is ω-algebraic. The subspace $MD = \{b_1, b_2, \ldots, a_\infty\}$ is a compact Hausdorff space. Thus, D is in class (7) of the hierarchy.

The maximal element a_∞ is not strongly maximal. For, it is in $\mathsf{cl}\uparrow c$, but not in $\uparrow c$. Thus, D is not in class (6). This also means that MD does not have CA; the counterexample is $\uparrow c \cap MD$ which is not closed in MD. The subspace M^*D is $\{b_1, b_2, \ldots\}$ which is isomorphic to the discrete space of natural numbers.

Example 4

Let $D = \{\bot, a_1, a_2, \ldots, a_\infty, b_1, b_2, \ldots, b_\infty, c_1, c_2, \ldots\}$ where $\bot$ is below everything, $a_i \sqsubseteq a_j$ iff $i \le j$, $b_i \sqsubseteq b_j$ iff $i \le j$, $c_i \sqsubseteq c_j$ iff $i = j$, $a_i \sqsubseteq c_j$ iff $i \le j$, and $b_i \sqsubseteq c_j$ iff $i \le j$. This domain is ω-algebraic. The subspace $MD = \{c_1, c_2, \ldots, a_\infty, b_\infty\}$ is not a Hausdorff space. For, every neighborhood of a_∞ contains almost all c_i, and likewise for b_∞. Thus, D is not in class (7) of the hierarchy.

The maximal elements a_∞ and b_∞ are not strongly maximal. For, a_∞ is in $\mathsf{cl}\,\uparrow b_1$, but not in $\uparrow b_1$, and likewise with b_∞ and a_1. The subspace M^*D is $\{c_1, c_2, \ldots\}$ which is isomorphic to the discrete space of natural numbers.

1.11 The Notion of Domain Environments

From what we have learned so far, we revise the notion of domain environment. A *(weak) domain environment* is (D, X) where D is a continuous dcpo and X is embedded into the maximal elements of D. In a weak domain environment, closed approximation may fail, and X may not be Hausdorff.

A *strong domain environment* is (D, X) where D is a continuous dcpo and X is embedded into the *strongly* maximal elements of D. A strong domain environment always has the SCA property, whence X is regular. Additional good properties which may be looked for are density of X in D, equality of X and M^*D, and the G_δ-property of X in D.

2 Forcing Bounded Completeness

Here, we consider the question: given a (strong) domain environment (D, X), can we find a (strong) domain environment (D', X') such that X and X' are isomorphic, and D' is distributive bounded complete? For strong domain environments, the answer is yes; we shall construct D' and X' in the sequel. The construction preserves density, but unfortunately, $X = M^*D$ does not imply $X' = M^*D'$. The behaviour w.r.t. the G_δ-property is not known. The construction does not work for weak domain environments. *Remark: In the sequel, some proofs are omitted due to lack of time.*

2.1 Consistent Sets

Let D be a continuous dcpo. A subset A of D is called *consistent* if for all open sets $O_1, \ldots, O_n$ which meet A, $O_1 \cap \cdots \cap O_n$ is non-empty. Equivalently, if $a_1 \gg b_1, \ldots, a_n \gg b_n$ with a_i in A, then there is some c in D such that $c \gg b_1, \ldots, b_n$. *Warning: This notion differs from other definitions of 'consistent'. These other notions do not enjoy property (4) below.*

Proposition 2.1

(1) If $A \subseteq B$ and B is consistent, then A is consistent.

(2) $\emptyset$ is consistent if $D \neq \emptyset$.

(3) If $(A_i)_{i \in I}$ is a $\subseteq$-directed family of consistent sets, then $\bigcup_{i \in I} A_i$ is consistent.

(4) If A is consistent, then so is $\mathsf{cl}\, A$.

(5) For every d in D, $\downarrow d$ is consistent.

2.2 The Domain of Consistent Closed Sets

For every continuous dcpo D, let LD be the set of closed subsets of D ordered by set inclusion. (We include $\emptyset$ in LD.) The poset LD is a complete distributive continuous lattice, the *lower power domain* of D. The lub of a directed family $(C_i)_{i\in I}$ in LD is $\mathsf{cl} \bigcup_{i\in I} C_i$ (not $\bigcup_{i\in I} C_i$). If B is a basis of D, then $\{\downarrow F \mid F \subseteq_{\mathit{fin}} B\}$ is a basis of LD. The relation $\downarrow F \ll \downarrow G$ holds iff for every a in F, there is b in G such that $a \ll b$. The function $\mathsf{s} : D \to LD$ with $\mathsf{s}d = \downarrow d$ is a topological embedding and thus an order embedding.

Let $L'D$ be the poset of consistent closed sets of D. By Prop. 2.1, $L'D$ is a Scott closed subset of LD. Hence, it is a continuous dcpo again. It is easily verified to be distributive bounded complete. The function $\mathsf{s} : D \to LD$ factors through $L'D$. Thus, $\mathsf{s} : D \to L'D$ is a topological embedding.

2.3 Maximality and Strong Maximality in $L'D$

If m is maximal in D, then $\downarrow m$ is not necessarily maximal in $L'D$. Consider Example 3 in Subsection 1.10. Point a_∞ is maximal in D (but not strongly maximal). The corresponding set $\downarrow a_\infty$ is not maximal in $L'D$, however, since the larger closed set $\{c\} \cup \downarrow a_\infty$ is consistent. On the other hand, we have the following:

Theorem 2.2 If m is strongly maximal in D, then $\downarrow m$ is strongly maximal in $L'D$.

Proof: Let C be a member of $L'D$. Assuming $\downarrow m \in \mathsf{cl} \uparrow\{C\}$ (where 'cl' and '$\uparrow$' are formed in $L'D$), we have to show $\downarrow m \in \uparrow\{C\}$, or equivalently $C \subseteq \downarrow m$.

We show that $a \gg b$ in C implies $m \in \mathsf{cl} \uparrow b$. To do this, assume $m \in O$ for a Scott open set O of D. The set $\diamond O = \{A \in L'D \mid A \cap O \neq \emptyset\}$ is Scott open in $L'D$ and contains $\downarrow m$. Since $\downarrow m$ is in $\mathsf{cl} \uparrow\{C\}$, $\diamond O$ meets $\uparrow\{C\}$. Let C' be a member of $\diamond O \cap \uparrow\{C\}$. Then C' meets O, and since $a \in C \subseteq C'$, C' meets $\Uparrow b$. Since C' is consistent, O meets $\Uparrow b$ and hence $\uparrow b$.

We have just shown $a \in C, a \gg b \Rightarrow m \in \mathsf{cl} \uparrow b$. Since m is strongly maximal, $a \in C, a \gg b \Rightarrow m \in \uparrow b$ follows. Thus, for every a in C, m is in $\bigcap_{b \ll a} \uparrow b = \uparrow a$. Hence for every a in C, $a \leq m$ holds, i.e., $C \subseteq \downarrow m$. □

2.4 Construction of a Distributive Bounded Complete Environment

Given a strong domain environment (D, X) (a space X embedded into the strongly maximal elements of a continuous dcpo D), we define $D' = L'D$ and $X' = \{\downarrow x \mid x \in X\}$. By Theorem 2.2, X' embeds into the strongly maximal elements of D'. Hence, we get a new strong domain environment (D', X'). (This does not work for weak domain environments since X' would not always embed into the maximal elements of D'; see the counterexample at the beginning of the previous subsection.)

Since X' is the image of X under the topological embedding s, it is topologically isomorphic to X. Thus we can identify X and X', and obtain:

Corollary 2.3 If X has a strong domain environment, then it also has a distributive bounded complete strong domain environment.

2.5 Additional Properties

Proposition 2.4 If X is dense in D, then X' is dense in D'.

Proof: We apply Prop. 1.15 (4), taking as basis of D' the set $\{\downarrow F \mid F \subseteq_{fin} D \text{ consistent}\}$. Let $\downarrow F \gg \downarrow G$ in D'. We have to find some x in X such that $\downarrow x \gg \downarrow G$.

For every b in G, there is a in F such that $a \gg b$. Hence, $\Uparrow b$ meets F for every b in G. Since F is consistent, $\bigcap_{b \in G} \Uparrow b$ is a non-empty open set of D. Since X is dense in D, there is some x in X such that $x \in \bigcap_{b \in G} \Uparrow b$, which implies $\downarrow x \gg \downarrow G$. □

Problem 1 If X is a G_δ-set of D, is then X' a G_δ-set of D'?

Unfortunately, $X = M^*D$ does not imply $X' = M^*D'$. The reason is that D' may contain strongly maximal elements which are different from the elements $\downarrow m$ with strongly maximal m. Consider the continuous dcpo D of Example 2 in Subsection 1.10. It even has the property $M^*D = MD$. We also have $M^*D' = MD'$ by Prop. 1.24 since D' is bounded complete. The set $C = \{\perp, a_1, a_2, \ldots\}$ is maximal in D', i.e., a maximal consistent closed subset of D. Obviously, there is no d in D such that $C = \downarrow d$.

In the sequel, we show that this creation of new strongly maximal elements does not happen if D has compact intersection.

2.6 The Case of Compact Intersection

If D has compact intersection, then $MD = M^*D$ holds by Prop. 1.24. Because of bounded completeness, $MD' = M^*D'$ holds as well. We claim that MD' and MD are topologically isomorphic.

Lemma 2.5 Let D be a continuous dcpo with compact intersection, and let $a_1, \ldots, a_n$ in D with $n > 0$. Then $\uparrow a_1 \cap \cdots \cap \uparrow a_n$ is compact.

Theorem 2.6 Let D be a non-empty continuous dcpo with compact intersection, and let $D' = L'D$. Let further $M = MD = M^*D$ and $M' = MD' = M^*D'$. Then $M' = \{\downarrow m \mid m \in M\}$, and thus M and M' are topologically isomorphic.

Proof: By Theorem 2.2, $\{\downarrow m \mid m \in M\} \subseteq M'$ holds. For the opposite inclusion, let C be in M'. Since $D \neq \emptyset$, C cannot be empty. Let $\mathcal{F}$ be the set of all finite non-empty subsets F of C with the property that for all a in F, there is b in C such that $a \ll b$. Set $\mathcal{F}$ is $\subseteq$-directed.

For every F in $\mathcal{F}$, let $U_F = \bigcap_{a \in F} \Uparrow a$ and $K_F = \bigcap_{a \in F} \uparrow a$. Since C is consistent, all sets U_F are non-empty. Since $U_F \subseteq K_F$, all sets K_F are non-empty as well. Since D has compact intersection, all sets K_F are compact by Lemma 2.5. Since $F \subseteq G$ in $\mathcal{F}$ implies $K_F \supseteq K_G$, the family $(K_F)_{F \in \mathcal{F}}$ is filtered. Every continuous dcpo is sober, and in a sober space, filtered intersections of non-empty compact sets are non-empty. Hence, $\bigcap_{F \in \mathcal{F}} K_F$ contains an element d.

Let a be an arbitrary element of C. For every $b \ll a$, $\{b\}$ is in $\mathcal{F}$, whence d is in $K_{\{b\}} = \uparrow b$. Hence $d \in \bigcap_{b \ll a} \uparrow b = \uparrow a$. Thus $C \subseteq \downarrow d$ holds, whence by maximality of C, $C = \downarrow d$ follows. We still have to show that d is maximal in D. If $d \sqsubseteq d'$, then $C = \downarrow d \subseteq \downarrow d'$, whence by maximality of C, $C = \downarrow d = \downarrow d'$ follows, whence $d = d'$. □

2.7 Classes of Max-Spaces

From Theorem 2.6, we can conclude:

Corollary 2.7 If $X \cong MD$ (in $\mathcal{TOP}$) for some continuous (ω-continuous / algebraic / ω-algebraic) D with compact intersection, then $X \cong MD'$ (in $\mathcal{TOP}$) for some continuous (ω-continuous / algebraic / ω-algebraic) distributive bounded complete D'. Here, M may be replaced by M^* without changing the meaning.

This strengthens a result of Kamimura and Tang who showed that if $X \cong MD$ (in $\mathcal{TOP}$) for some SFP-domain D (a special kind of ω-algebraic dcpo with compact intersection), then $X \cong MD'$ (in $\mathcal{TOP}$) for some ω-algebraic bounded complete D'.

3 Domain Environments for Metric Spaces

In this section, we show some special properties of domain environments for metric spaces, in particular, a G_δ-theorem.

3.1 Notation for Metric Spaces

Let (X, d) be a metric space. For x, x' in X, $d(x, x') \in \mathbf{R}_+$ gives the distance of x and x'. For x in X and r in $\mathbf{R}_+$, let $O(x, r) = \{x' \in X \mid d(x, x') < r\}$ and $C(x, r) = \{x' \in X \mid d(x, x') \leq r\}$. The sets $O(x, r)$ form a basis of a topology on X. In this topology, the sets $O(x, r)$ are open and the sets $C(x, r)$ are closed.

The *diameter* δA of a subset A of a metric space (X, d) is $\bigsqcup_{a, a' \in A} d(a, a')$.

3.2 Complete Metric Spaces

We need the following lemma about complete metric spaces:

Lemma 3.1 Let $\mathcal{C}$ be a filtered set of non-empty closed subsets of a complete metric space X with $\bigsqcap_{C \in \mathcal{C}} \delta C = 0$. Then there is a point x of X such that $\bigcap \mathcal{C} = \{x\}$.

Proof: The lemma follows from Theorem 5, Page 188 in [Engelking].

A direct proof can be given as follows: Because of $\bigsqcap_{C \in \mathcal{C}} \delta C = 0$, for every n in $\mathbf{N}$, there is A_n in $\mathcal{C}$ such that $\delta A_n < 1/n$. Since $\mathcal{C}$ is filtered, we may assume $A_1 \supseteq A_2 \supseteq \cdots$. All sets in $\mathcal{C}$ are non-empty. Choose some element x_n of A_n.

The sequence $(x_n)_{n \in \mathbf{N}}$ is Cauchy. For, given $\epsilon > 0$, let $N > 1/\epsilon$. For all $n, m \geq N$, x_n and x_m are in A_N, whence $d(x_n, x_m) < 1/N < \epsilon$.

Let x be the limit of the sequence $(x_n)_{n\in\mathbf{N}}$. We show $x \in \bigcap\mathcal{C}$. Fix some C in $\mathcal{C}$. It meets all A_n since $\mathcal{C}$ is a filtered set of non-empty sets. Let y_n be a member of $C \cap A_n$. From $d(y_n,x) \le d(x,x_n) + d(x_n,y_n) \le d(x,x_n) + 1/n$, we conclude $y_n \to x$. Since C is closed, x is in C.

We also have to show $\bigcap\mathcal{C} \subseteq \{x\}$. If y is in $\bigcap\mathcal{C}$, then x and y are in A_n for all n, whence $d(x,y) \le 1/n$ for all n, whence $y = x$. $\square$

3.3 The G_δ-Theorem for Complete Metric Spaces

Now, we prove that complete metric spaces X which are dense in a strong domain environment with an ω-continuous dcpo D are G_δ-sets in D. We start with a lemma.

Lemma 3.2 If (D,X) is a strong domain environment where X is dense in D and homeomorphic to a complete metric space, then there is a G_δ-set H of D such that $X \subseteq H$ and $MD \cap H \subseteq X$.

Proof: For every n in $\mathbf{N}$, let $U_n = \bigcup\{\twoheaduparrow d \mid \delta(\twoheaduparrow d \cap X) < 1/n\}$. As a union of open sets $\twoheaduparrow d$, U_n is open. (The set $\twoheaduparrow d \cap X$ is a subset of the metric space X; hence it has a well-defined diameter $\delta(\twoheaduparrow d \cap X)$ in X.) Let $H = \bigcap_{n\in\mathbf{N}} U_n$, a G_δ-set.

For $X \subseteq H$, let x be a point of X. For every n in $\mathbf{N}$, the open ball $O = O(x, 1/2n)$ is a neighborhood of x in X with diameter $< 1/n$. Since X is topologically embedded into the continuous dcpo D, there is d in D with $x \in \twoheaduparrow d \cap X \subseteq O$. Since $\delta(\twoheaduparrow d \cap X) \le \delta O < 1/n$, point x is in U_n.

For $M \cap H \subseteq X$, let h be an element of $M \cap H$. Let $\mathcal{C} = \{\uparrow a \cap X \mid a \ll h\}$. A strong domain environment has the CA property. Thus, every element of $\mathcal{C}$ is closed in X. Since the set of all a with $a \ll h$ is directed and $a \sqsubseteq a'$ implies $\uparrow a \supseteq \uparrow a'$, the set $\mathcal{C}$ is filtered. If $a \ll h$, then $\twoheaduparrow a$ meets X by density of X in D, and thus its superset $\uparrow a$ meets X. Consequently, all sets in $\mathcal{C}$ are non-empty. In $\mathcal{C}$, there are sets of arbitrarily small diameter. For, if $\epsilon > 0$, take n with $1/n < \epsilon$. Since h is in U_n, there is $a \ll h$ with $\delta(\twoheaduparrow a \cap X) < 1/n$. By interpolation, there is b such that $a \ll b \ll h$. Then $\uparrow b \cap X$ is in $\mathcal{C}$, and $\delta(\uparrow b \cap X) \le \delta(\twoheaduparrow a \cap X) < 1/n$.

Summarizing, $\mathcal{C}$ is a filtered set of non-empty closed subsets of X with $\sqcap_{C\in\mathcal{C}} \delta C = 0$. By completeness of X, there is x in X such that $\bigcap\mathcal{C} = \{x\}$. Whenever $a \ll h$, then x is in $\uparrow a \cap X$. Thus $x \in \bigcap_{a \ll h} \uparrow a = \uparrow h$ by continuity. Maximality of h implies $h = x \in X$. $\square$

In the lemma, we did not need ω-continuity of D. This is different for the G_δ-Theorem itself.

Theorem 3.3 If (D,X) is a strong domain environment where D is ω-continuous and X is dense in D and homeomorphic to a complete metric space, then X is a G_δ-set of D.

Proof: By Theorem 1.22, M^*D is a G_δ-set of D. Since we are in a strong environment, $X \subseteq M^*$ holds. The G_δ-set H of Lemma 3.2 satisfies $X \subseteq H$ and $M \cap H \subseteq X$. Hence, $X \subseteq M^* \cap H \subseteq M \cap H \subseteq X$ follows. Thus, $X = M^* \cap H$ is a G_δ-set. $\square$