Lower and Upper Power Domain Constructions Commute on all Cpos

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Abstract

The initial *lower* and *upper power domain* constructions $\mathcal{P}_\vee$ and $\mathcal{P}_\wedge$ commute under composition for all cpos. The common result $\mathcal{P}_\vee(\mathcal{P}_\wedge \mathbf{X})$ and $\mathcal{P}_\wedge(\mathcal{P}_\vee \mathbf{X})$ is the *free frame* over the cpo $\mathbf{X}$.

1 Introduction

In [FM90], the lower and upper power domain constructions were shown to commute on bounded complete algebraic cpos. For the proof, these cpos were represented by information systems. In this paper, we show that the two constructions $\mathcal{P}_\vee$ and $\mathcal{P}_\wedge$ commute on *all* cpos $\mathbf{X}$, and moreover, that the common result $\mathcal{P}_\vee(\mathcal{P}_\wedge \mathbf{X}) \cong \mathcal{P}_\wedge(\mathcal{P}_\vee \mathbf{X})$ is the free frame over $\mathbf{X}$. Our proof differs completely from the proof of [FM90]. It uses algebraic methods and does not rely on any explicit representations of the power domains.

2 The used algebraic structures

In this section, we define the categories of algebraic structures that will be used in our proof. We suppose the usual definitions of *poset* $\mathbf{P} = (P, \leq)$, directed set, greatest lower bound (meet), and least upper bound (join).

Definition 2.1

(1) A *cpo* is a poset where all directed sets D have a join $\bigvee D$. A function f between two cpos $\mathbf{X}$ and $\mathbf{Y}$ is *continuous* iff it preserves directed joins. Continuous functions are also called *morphisms*.

(2) A $\vee$-*cpo* is a cpo $\mathbf{X}$ in which finite joins exist. A $\vee$-*morphism* is a morphism f between two $\vee$-cpos that preserves finite joins.

(3) A $\wedge$-*cpo* is a cpo $\mathbf{X}$ in which finite meets exist and distribute over directed joins. A $\wedge$-*morphism* is a morphism f between two $\wedge$-cpos that preserves finite meets.

(4) A $\vee$-$\wedge$-*cpo* is a cpo $\mathbf{X}$ that is both a $\vee$-cpo and a $\wedge$-cpo such that the distributive law $a \wedge (b \vee b') = (a \wedge b) \vee (a \wedge b')$ holds. A $\vee$-$\wedge$-*morphism* is a morphism between two $\vee$-$\wedge$-cpos that is both a $\vee$- and a $\wedge$-morphism.

$\vee$-cpos are just posets where all joins exist. $\vee$-morphisms preserve all joins. The empty join is the least element $\perp$. The binary join operation $\vee : \mathbf{X} \times \mathbf{X} \to \mathbf{X}$ is always continuous such that its continuity need not be required in contrast to $\wedge$-cpos.

$\wedge$-cpos are posets where finite meets and directed joins exist. The $\wedge$-morphisms preserve these operations. The empty meet is the greatest element $\top$. Distributivity of finite meets over directed joins means $a \wedge \bigvee D = \bigvee_{d \in D} a \wedge d$ for directed sets D, i.e. continuity of the binary meet operation $\wedge : \mathbf{X} \times \mathbf{X} \to \mathbf{X}$.

$\vee$-$\wedge$-cpos are posets with finite meets and arbitrary joins. The finite distributive law and continuity of '$\wedge$' combine to the infinite distributive law $a \wedge \bigvee S = \bigvee_{b \in S} a \wedge b$ for all sets S. According to [Vic89], such posets are called *frames*. $\vee$-$\wedge$-morphisms preserve finite meets and arbitrary joins, i.e. they are *frame homomorphisms* in the sense of [Vic89]. If only objects are considered, frames are nothing else than *complete Heyting algebras*, but the corresponding arrows are different.

In the sequel, we shall introduce the lower and upper power constructions $\mathcal{P}_\vee$ and $\mathcal{P}_\wedge$ as constructions producing free $\vee$-cpos and $\wedge$-cpos from given cpos. For all cpos $\mathbf{X}$, we then show that both $\mathcal{P}_\vee(\mathcal{P}_\wedge \mathbf{X})$ and $\mathcal{P}_\wedge(\mathcal{P}_\vee \mathbf{X})$ are free frames over $\mathbf{X}$, whence they are isomorphic.

3 Lower and upper power constructions

As indicated in [HP79], lower and upper power domains may be characterized as free structures w.r.t. the algebraic theories of $\vee$- and $\wedge$-cpos.

Theorem 3.1

$\vee$: For every cpo $\mathbf{X}$, there is a *free* $\vee$-*cpo* $\mathcal{P}_\vee \mathbf{X}$ and a morphism $\eta_\vee : \mathbf{X} \to \mathcal{P}_\vee \mathbf{X}$ such that for every morphism $f : \mathbf{X} \to \mathbf{Y}$ to a $\vee$-cpo $\mathbf{Y}$ there is exactly one $\vee$-morphism $\mathsf{E}_\vee f : \mathcal{P}_\vee \mathbf{X} \to \mathbf{Y}$ with $\mathsf{E}_\vee f \circ \eta_\vee = f$.

$\wedge$: For every cpo $\mathbf{X}$, there is a *free* $\wedge$-*cpo* $\mathcal{P}_\wedge \mathbf{X}$ with analogous properties.

This existence theorem is proved in [Hoo87] and in [Hec90, Hec91]. $\mathcal{P}_\vee \mathbf{X}$ is usually called *lower power domain* over the ground domain $\mathbf{X}$, whereas $\mathcal{P}_\wedge \mathbf{X}$ is the *initial upper power domain* over $\mathbf{X}$. η is then a formal singleton map, and E denotes extension of set-valued functions from points to sets.

$\mathcal{P}_\vee \mathbf{X}$ may be explicitly described as the poset of *Scott closed subsets* of $\mathbf{X}$ ordered by inclusion '$\subseteq$' [Hec90]. A subset of $\mathbf{X}$ is Scott closed iff it is a lower set that is closed w.r.t. directed joins. In this concrete representation, $\eta_\vee x$ is given by $\downarrow x = \{u \in X \mid u \leq x\}$, and $\mathsf{E}_\vee f A$ is $\bigvee_{a \in A} f a$ for subsets A of $\mathbf{X}$.

If $\mathbf{X}$ is a continuous cpo, then $\mathcal{P}_\wedge \mathbf{X}$ is explicitly given by the set of Scott compact upper sets of $\mathbf{X}$ ordered by opposite inclusion '$\supseteq$' [Hec90]. For non-continuous $\mathbf{X}$, the free $\wedge$-cpo in general differs from the cpo of compact upper subsets of $\mathbf{X}$. In this case, there is no known explicit representation of $\mathcal{P}_\wedge \mathbf{X}$. Thus, we also ignore the explicit description of $\mathcal{P}_\vee \mathbf{X}$ to obtain a symmetric theory.

4 Properties of the free power domains and extension

In this section, we present some properties of the construction $\mathcal{P}_\vee$ and the operations $\eta_\vee$ and $\mathsf{E}_\vee$. For simplification, we drop the index '$\vee$'. $\mathcal{P}_\wedge$ has completely analogous properties, which are not explicitly stated.

Theorem 4.1 Let $\mathbf{X}$ be a cpo and $\mathbf{Y}$ a $\vee$-cpo. If two $\vee$-morphisms $F, G : \mathcal{P}\mathbf{X} \to \mathbf{Y}$ satisfy $F \circ \eta = G \circ \eta$, then they are equal.

Proof: By the uniqueness of extended maps, $F = \mathsf{E}(F \circ \eta) = G$ holds. □

Theorem 4.2 Let S be a subset of $\mathcal{P}\mathbf{X}$ with the following properties:

(1) ηx is in S for all x in $\mathbf{X}$;

(2) $\perp$ is in S;

(3) if a and b are in S, then $a \vee b$ is in S;

(4) if D is a directed subset of S, then $\bigvee D$ is in S.

Then S equals $\mathcal{P}\mathbf{X}$.

Proof: By (2) through (4), S induces a $\vee$-cpo $\mathbf{S}$, and the natural inclusion map $i : \mathbf{S} \to \mathcal{P}\mathbf{X}$ is a $\vee$-morphism. By (1), η restricts to a morphism $\eta' : \mathbf{X} \to \mathbf{S}$ with $\eta = i \circ \eta'$. By Th. 3.1, we get the $\vee$-morphism $\mathsf{E}\eta' : \mathcal{P}\mathbf{X} \to \mathbf{S}$. Composition yields a $\vee$-morphism $i \circ \mathsf{E}\eta' : \mathcal{P}\mathbf{X} \to \mathcal{P}\mathbf{X}$ with $i \circ \mathsf{E}\eta' \circ \eta = i \circ \eta' = \eta = id \circ \eta$, whence $i \circ \mathsf{E}\eta' = id$ by Prop. 4.1. Thus, every y in $\mathcal{P}\mathbf{X}$ may be written as $y = i\, y'$, whence $S = \mathcal{P}\mathbf{X}$. □

Lemma 4.3 Let $\mathbf{X}$ be a cpo and $\mathbf{Y}$ a $\vee$-cpo. If two $\vee$-morphisms $F, G : \mathcal{P}\mathbf{X} \to \mathbf{Y}$ satisfy $F \circ \eta \leq G \circ \eta$, then $F \leq G$ holds.

Proof: Let $S = \{a \in \mathcal{P}\mathbf{X} \mid Fa \leq Ga\}$. We show that S satisfies the properties of Th. 4.2, whence $S = \mathcal{P}\mathbf{X}$ follows.

(1) For $x \in \mathbf{X}$, ηx is in S since $F(\eta x) \leq G(\eta x)$.

(2) $F(\perp) = \perp_{\mathbf{Y}} = G(\perp)$ whence $\perp \in S$.

(3) Let $a, b \in S$. Then $F(a \vee b) = Fa \vee Fb \leq Ga \vee Gb = G(a \vee b)$.

(4) Let D be a directed subset of S with limit a w.r.t. $\mathcal{P}\mathbf{X}$. Then for all $d \in D$, $Fd \leq Gd \leq Ga$ holds, whence $Fa \leq Ga$ by continuity of F. □

If $\mathbf{Y}$ is a $\vee$-cpo, then for every morphism $f : \mathbf{X} \to \mathbf{Y}$, there is a unique $\vee$-morphism $\mathsf{E}f : \mathcal{P}\mathbf{X} \to \mathbf{Y}$ with $\mathsf{E}f \circ \eta = f$. Thus, E itself is a function between the function spaces $[\mathbf{X} \to \mathbf{Y}]$ and $[\mathcal{P}\mathbf{X} \to \mathbf{Y}]$. We claim that this function is a morphism itself.

Theorem 4.4
For every cpo $\mathbf{X}$ and $\vee$-cpo $\mathbf{Y}$, the mapping $\mathsf{E} : [\mathbf{X} \to \mathbf{Y}] \to [\mathcal{P}\mathbf{X} \to \mathbf{Y}]$ is continuous.

Proof: E is monotonic by Lemma 4.3.

Let $\mathcal{D}$ be a directed set of morphisms from $\mathbf{X}$ to $\mathbf{Y}$, and let f be its join. We have to show $\mathsf{E}f = \bigvee_{d \in \mathcal{D}} \mathsf{E}\, d$. This join exists by monotonicity of E and directedness of $\mathcal{D}$. The function on the right hand side is a $\vee$-morphism by continuity of '$\vee$'. It maps ηx to $f x$ by continuity of application and $\mathsf{E}\, d\, (\eta x) = d\, x$. By Th. 4.1, it thus equals $\mathsf{E}f$. $\square$

Finally, we show some properties of extension.

Proposition 4.5 $\mathsf{E}\eta = id$

Proof: Both $\mathsf{E}\eta$ and the identity are $\vee$-morphisms from $\mathcal{P}\mathbf{X}$ to itself. $\mathsf{E}\eta \circ \eta = \eta = id \circ \eta$ holds, whence the claim follows from Th. 4.1. $\square$

Proposition 4.6 Let f and g be two morphisms from $\mathbf{X}$ to a $\vee$-cpo $\mathbf{Y}$. Then
$\mathsf{E}\,(\lambda x.\, f x \vee g\, x) \;=\; \lambda X.\, \mathsf{E}f X \;\vee\; \mathsf{E}g X$ holds.

Proof: The left hand side is a $\vee$-morphism as all extended maps are. The right hand side is easily shown to be a $\vee$-morphism, too. Composed with η, both sides yield $\lambda x.\, f x \vee g\, x$. Thus, they are equal by Th. 4.1. $\square$

5 Power domains of $\wedge$- and $\vee$-cpos

The construction $\mathcal{P}_\vee$ maps cpos into $\vee$-cpos. In this section, we show that it does not destroy an eventual $\wedge$-operation in its argument.

Theorem 5.1

(1) If $\mathbf{X}$ is a $\wedge$-cpo, then $\mathcal{P}_\vee\, \mathbf{X}$ is a $\vee$-$\wedge$-cpo, and $\eta_\vee : \mathbf{X} \to \mathcal{P}_\vee\, \mathbf{X}$ is a $\wedge$-morphism.

(2) If $\mathbf{X}$ is a $\vee$-cpo, then $\mathcal{P}_\wedge\, \mathbf{X}$ is a $\vee$-$\wedge$-cpo, and $\eta_\wedge : \mathbf{X} \to \mathcal{P}_\wedge\, \mathbf{X}$ is a $\vee$-morphism.

Proof: As usual, we prove only part (1). Part (2) may be shown analogously. As in the previous section, $\mathcal{P}_\vee$ is abbreviated to $\mathcal{P}$, and $\eta_\vee$ to η.

As a $\wedge$-cpo, $\mathbf{X}$ has a greatest element $\top$. We have to show that $\eta\top$ is the greatest element of $\mathcal{P}\mathbf{X}$. Let $S = \{A \in \mathcal{P}\mathbf{X} \mid A \le \eta\top\}$. We apply Th. 4.2.

(1) For all x in $\mathbf{X}$, $\eta x \le \eta\top$ holds by monotonicity of η.

(2) $\bot \le \eta\top$ is obvious.

(3) $A, B \le \eta\top$ implies $A \vee B \le \eta\top$.

(4) S is obviously closed w.r.t. directed joins.

Next, we have to construct the operation '$\wedge$' in $\mathcal{P}\mathbf{X}$. To this end, we construct a continuous operation '&' by double extension of '$\wedge_{\mathbf{X}}$' and show that it represents the meet operation of $\mathcal{P}\mathbf{X}$. For A, B in $\mathcal{P}\mathbf{X}$, we define

$A \,\&\, B \;=\; \mathsf{E}\,(\lambda a.\, \mathsf{E}\,(\lambda b.\, \eta\,(a \wedge b))\; B)\; A.$

In formulas like this one, we let application associate to the left, i.e. $\mathsf{E}f A$ means $(\mathsf{E}f)\, A$. Note that the definition above is only possible since we know that E is continuous (Th. 4.4); otherwise, we would not get the continuity of $\lambda a. \ldots$.

Using the rule $\mathsf{E}f \circ \eta = f$ twice, one gets

(1) $\eta u \,\&\, \eta v = \eta\,(u \wedge v)$.

Since E creates $\vee$-morphisms, one obtains

(2) $(A_1 \vee A_2) \,\&\, B = (A_1 \,\&\, B) \vee (A_2 \,\&\, B)$.

Using Prop. 4.6 at the outer extension, one also gets

(3) $A \,\&\, (B_1 \vee B_2) = (A \,\&\, B_1) \vee (A \,\&\, B_2)$.

Once $\& = \wedge$ is shown, (1) implies that η is a $\wedge$-morphism, and (2) proves the necessary distributive law in $\mathcal{P}\mathbf{X}$.

In the sequel, we show that $A \,\&\, B$ is the meet of A and B in four steps.

Step 1: $A \,\&\, B \leq B$

$$
\begin{array}{rll}
A \,\&\, B & \leq & \quad \text{(since } a \wedge b \leq b\text{)} \\
 & & \mathsf{E}\,(\lambda a.\, \mathsf{E}\,(\lambda b.\, \eta b)\, B)\, A \\
 & = & \quad \text{(by Prop. 4.5)} \\
 & & \mathsf{E}\,(\lambda a.\, B)\, A \\
 & \leq & \quad \text{(since } A \leq \eta\top \text{ as shown above)} \\
 & & \mathsf{E}\,(\lambda a.\, B)\,(\eta\top) \\
 & = & (\lambda a.\, B)\top \;=\; B
\end{array}
$$

Step 2: $A \,\&\, B \leq A$

$$
\begin{array}{rll}
A \,\&\, B & \leq & \quad \text{(since } a \wedge b \leq a\text{)} \\
 & & \mathsf{E}\,(\lambda a.\, \mathsf{E}\,(\lambda b.\, \eta a)\, B)\, A \\
 & \leq & \quad \text{(since } B \leq \eta\top \text{ as shown above)} \\
 & & \mathsf{E}\,(\lambda a.\, \mathsf{E}\,(\lambda b.\, \eta a)\,(\eta\top))\, A \\
 & = & \mathsf{E}\,(\lambda a.\, (\lambda b.\, \eta a)\top)\, A \\
 & = & \mathsf{E}\,(\lambda a.\, \eta a)\, A \\
 & = & \quad \text{(by Prop. 4.5)} \\
 & & A
\end{array}
$$

Step 3: $A \,\&\, A = A$

We apply Th. 4.2 to the set $S = \{A \in \mathcal{P}\mathbf{X} \mid A \,\&\, A \;=\; A\}$.

(1) All singletons are in S because of property (1) above: $\eta x \,\&\, \eta x = \eta(x \wedge x) = \eta x$.

(2) $\perp$ is in S, since $\perp \,\&\, \perp\, = \mathsf{E}\,(\ldots) \perp\, = \,\perp$ holds because E produces $\vee$-morphisms.

(3) Let A and B be two members of S. We have to show $A \vee B$ in S. By distributivity of '&' (properties (2) and (3)), we obtain

$$
\begin{array}{rcl}
(A \vee B) \,\&\, (A \vee B) & = & (A \,\&\, A) \vee (A \,\&\, B) \vee (B \,\&\, A) \vee (B \,\&\, B) \\
 & = & A \vee (A \,\&\, B) \vee (B \,\&\, A) \vee B
\end{array}
$$

Since '$\vee$' is least upper bound and we already showed $A \geq A \,\&\, B$ and $B \geq B \,\&\, A$ in Step 1 and Step 2, the result simplifies to $A \vee B$. Thus, $A \vee B$ is back in S again.

(4) Limits of directed subsets of S are in S by continuity of '&'.

Step 4: $A \,\&\, B = A \wedge B$.

By $A \,\&\, B \leq A, B$, the element $A \,\&\, B$ is a lower bound of A and B. If C is another lower bound of A and B, then $A \,\&\, B \geq C \,\&\, C = C$ follows from monotonicity of '&' and Step 3.□

6 Extension of $\wedge$- and $\vee$-morphisms

In the previous section, we showed that the $\vee$-power construction preserves the property to be a $\wedge$-cpo. In this section, we show the corresponding statement for morphisms.

Theorem 6.1

(1) Let $\mathbf{X}$ be a $\wedge$-cpo, $\mathbf{Y}$ a $\vee$-$\wedge$-cpo, and $f : \mathbf{X} \to \mathbf{Y}$ a $\wedge$-morphism. Then its (unique) extension $\mathsf{E}_\vee f : \mathcal{P}_\vee \mathbf{X} \to \mathbf{Y}$ is a $\vee$-$\wedge$-morphism.

(2) Let $\mathbf{X}$ be a $\vee$-cpo, $\mathbf{Y}$ a $\vee$-$\wedge$-cpo, and $f : \mathbf{X} \to \mathbf{Y}$ a $\vee$-morphism. Then its (unique) extension $\mathsf{E}_\wedge f : \mathcal{P}_\wedge \mathbf{X} \to \mathbf{Y}$ is a $\vee$-$\wedge$-morphism.

Proof: Again, we only show statement (1) and drop the '$\vee$' indexes.

As an extension generated by $\mathsf{E} = \mathsf{E}_\vee$, $\mathsf{E}f$ is a $\vee$-morphism.

Since η and f are $\wedge$-morphisms, we get $\mathsf{E}f\top = \mathsf{E}f(\eta\top) = f(\top) = \top$. The last equation to show is $\mathsf{E}f(A \wedge B) = \mathsf{E}fA \wedge \mathsf{E}fB$. Let F be defined by $F(A, B) = \mathsf{E}f(A \wedge B)$ and G by $G(A, B) = \mathsf{E}fA \wedge \mathsf{E}fB$. By Th. 5.1, $\mathcal{P}\mathbf{X}$ is a $\vee$-$\wedge$-cpo as $\mathbf{Y}$ is. By the distributive law in part (3) of the definition of $\vee$-$\wedge$-cpos, '$\wedge$' is a $\vee$-morphism in both arguments, whence both F and G are $\vee$-morphisms in both arguments. They coincide for singletons:

$$\begin{aligned}
F(\eta a, \eta b) &= \mathsf{E}f(\eta a \wedge \eta b) & \\
&= \mathsf{E}f(\eta(a \wedge b)) & \text{since } \eta \text{ is a } \wedge\text{-morphism} \\
&= f(a \wedge b) & \text{by Th. 3.1} \\
&= fa \wedge fb & \text{since } f \text{ is a } \wedge\text{-morphism} \\
&= \mathsf{E}f(\eta a) \wedge \mathsf{E}f(\eta b) & \\
&= G(\eta a, \eta b). &
\end{aligned}$$

Thus, F and G coincide by Prop. 4.1. □

7 The main theorem

Theorem 7.1

(1) For every cpo $\mathbf{X}$, every $\vee$-$\wedge$-cpo $\mathbf{Y}$, and every morphism $f : \mathbf{X} \to \mathbf{Y}$, there is exactly one $\vee$-$\wedge$-morphism F from $\mathcal{P}_\wedge(\mathcal{P}_\vee \mathbf{X})$ to $\mathbf{Y}$ such that $F \circ (\eta_\wedge \circ \eta_\vee) = f$. F is given by $\mathsf{E}_\wedge(\mathsf{E}_\vee f)$.
Thus, $\mathcal{P}_\wedge(\mathcal{P}_\vee \mathbf{X})$ is the free $\vee$-$\wedge$-cpo over the cpo $\mathbf{X}$.

(2) For every cpo $\mathbf{X}$, every $\vee$-$\wedge$-cpo $\mathbf{Y}$, and every morphism $f : \mathbf{X} \to \mathbf{Y}$, there is exactly one $\vee$-$\wedge$-morphism F from $\mathcal{P}_\vee(\mathcal{P}_\wedge \mathbf{X})$ to $\mathbf{Y}$ such that $F \circ (\eta_\vee \circ \eta_\wedge) = f$. F is given by $\mathsf{E}_\vee(\mathsf{E}_\wedge f)$.
Thus, $\mathcal{P}_\vee(\mathcal{P}_\wedge \mathbf{X})$ is the free $\vee$-$\wedge$-cpo over the cpo $\mathbf{X}$, too.

(3) For every ground domain $\mathbf{X}$, the iterated power domains $\mathcal{P}_\wedge(\mathcal{P}_\vee \mathbf{X})$ and $\mathcal{P}_\vee(\mathcal{P}_\wedge \mathbf{X})$ are isomorphic by a $\vee$-$\wedge$-morphism, which maps $\eta_\wedge(\eta_\vee x)$ to $\eta_\vee(\eta_\wedge x)$.

Proof: We first show (1). $\mathsf{E}_\wedge f$ is a $\wedge$-morphism, whence $\mathsf{E}_\vee(\mathsf{E}_\wedge f)$ is $\vee$-$\wedge$-morphism by Th. 6.1. Applying Th. 3.1 twice, we get

$$\mathsf{E}_\vee(\mathsf{E}_\wedge f) \circ \eta_\vee \circ \eta_\wedge = \mathsf{E}_\wedge f \circ \eta_\wedge = f.$$

Let F_1 and F_2 be two such $\vee$-$\wedge$-morphisms, i.e. $F_i \circ \eta_\vee \circ \eta_\wedge = f$ holds. We have to show $F_1 = F_2$. Let $F_i' = F_i \circ \eta_\vee$. The mapping $\eta_\vee$ is a $\wedge$-morphism by Th. 5.1. Thus, F_i' are two $\wedge$-morphisms from $\mathcal{P}_\wedge \mathbf{X}$ to $\mathbf{Y}$ with $F_i' \circ \eta_\wedge = f$. By Th. 4.1, $F_1' = F_2'$ follows. Thus, F_1 and F_2 are two $\vee$-morphisms from $\mathcal{P}_\vee(\mathcal{P}_\wedge \mathbf{X})$ to $\mathbf{Y}$ with $F_1 \circ \eta_\vee = F_2 \circ \eta_\vee$. By Th. 4.1, $F_1 = F_2$ follows.

Part (2) is proved analogously. Parts (1) and (2) together imply part (3). The isomorphism is given by $\mathsf{E}_\wedge(\mathsf{E}_\vee(\eta_\vee \circ \eta_\wedge))$. $\square$

Since $\vee$-$\wedge$-cpos are nothing else than frames, the theorem shows that the iterated power domains form two representations of the free frame $\mathcal{F}\mathbf{X}$ over the cpo $\mathbf{X}$. The representation $\mathcal{P}_\wedge(\mathcal{P}_\vee \mathbf{X})$ intuitively describes $\mathcal{F}\mathbf{X}$ in terms of conjunctive normal forms, whereas the opposite representation $\mathcal{P}_\vee(\mathcal{P}_\wedge \mathbf{X})$ corresponds to a disjunctive normal form.

[Hec] contains a similar commutation theorem that also generalizes the result of [FM90]. It however uses an upper power domain different from the initial one, and is proved by topological methods using explicit representations of the power domains. The topological proof does not work out for all cpos, and there is no connection to frames.

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