Lower Bag Domains

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Abstract. Two lower bag domain constructions are introduced: the initial construction which gives free lower monoids, and the final construction which is defined explicitly in terms of second order functions. The latter is analyzed closely. For sober dcpo's, the elements of the final lower bag domains can be described concretely as bags. For continuous domains, initial and final lower bag domains coincide. They are continuous again and can be described via a basis which is constructed from a basis of the argument domain. The lower bag domain construction preserves algebraicity and the properties I and M, but does not preserve bounded completeness, property L, or bifiniteness.

1 Introduction

Power domain constructions [13, 15, 16] were introduced to describe the denotational semantics of non-deterministic programming languages. A power domain construction is a domain constructor $\mathcal{P}$, which maps domains to domains, together with some families of continuous operations. If X is the semantic domain of values of type τ, then $\mathcal{P}X$ is the domain of `computations' of type τ, which because of non-determinism roughly correspond to sets of values. One of the families of continuous operations belonging to a power domain construction is (the domain-theoretic) analogue of union $+_X : \mathcal{P}X \times \mathcal{P}X \to \mathcal{P}X$. This operation is used to model the semantics of binary non-deterministic choice. In the classical power domain constructions, `$+$' is idempotent, i.e., $A + A = A$ holds for all A in $\mathcal{P}X$. This means that multiplicities of results are not taken into account.

In contrast, we propose to count results, thus considering bags instead of sets. Mathematically, this means that we drop the idempotence of `$+$'. As a consequence, the semantic values carry more information. They not only indicate the possibility that some result is produced, but also how often it is produced. There is another unexpected source of more information: in all the classical power domains, different finite sets are sometimes identified. In contrast, we shall see that this does not happen to finite bags in our bag domains.

The classical power domain constructions come in three flavors: the union operation of the lower or Hoare power domain construction satisfies $A \sqsubseteq A + B$, where `$\sqsubseteq$' is the domain order; in upper or Smyth power domains, $A + B \sqsubseteq A$ holds; and in convex or Plotkin power domains, A and $A + B$ are incomparable in general.

A lower power domain semantics describes the set of possible results of a non-deterministic program; it does not provide information about the termination properties of the program. An upper power domain semantics tells us whether a non-deterministic program is terminating (no matter how the choices are done), and describes the set of possible results (only) in this case. A convex power domain semantics combines both kinds of information. Thus, each power domain construction describes a different kind of observable behavior of a program. Of course, all three descriptions are computationally adequate. However, only the lower power domain semantics is fully abstract for its kind of observable behavior in case of a simple call-by-value non-deterministic functional language [14]. The other two power domain semantics are not fully abstract for the behavior they ought to describe, neither in case of the simple language nor in any reasonable extension thereof.

Mathematically, the lower power domain construction is the simplest of the three classical constructions, and it has the nicest properties. As can be seen from the full abstraction results cited above, this mathematical beauty also has practical impacts. For all these reasons, we decided to investigate the non-idempotent analogue of the lower power domain construction, the *lower bag domain construction*, instead of possible upper or convex bag constructions.

According to the general theory of power constructions (which is briefly recalled in Subsection 2.4), the lower bag construction can be introduced in two ways: as `initial' construction $\mathcal{L}_{\mathrm{i}}^*$, where $\mathcal{L}_{\mathrm{i}}^*\mathbf{X}$ is the free *lower monoid* over $\mathbf{X}$ in $\mathcal{DCPO}$ (Section 3), or as `final' construction $\mathcal{L}_{\mathrm{f}}^*$, where $\mathcal{L}_{\mathrm{f}}^*\mathbf{X} = [[\mathbf{X} \to \mathrm{N}_0^\infty] \stackrel{add}{\to} \mathrm{N}_0^\infty]$ is the dcpo of continuous additive functions from $[\mathbf{X} \to \mathrm{N}_0^\infty]$ to N_0^∞ (Section 3). Here, N_0^∞ denotes the domain of natural numbers plus infinity, ordered by $\{0 \sqsubset 1 \sqsubset \cdots \sqsubset \infty\}$. The description of the bag domain $\mathcal{L}_{\mathrm{f}}^*\mathbf{X}$ can be simplified to $\mathcal{L}_{\mathrm{m}}^*\mathbf{X} = [\Omega\mathbf{X} \stackrel{mod}{\to} \mathrm{N}_0^\infty]$, the dcpo of continuous modular functions (*evaluations*) from the lattice $\Omega\mathbf{X}$ of Scott open sets of $\mathbf{X}$ to N_0^∞ (Section 4). A function $\mu : \Omega\mathbf{X} \to \mathrm{N}_0^\infty$ is *modular* iff it satisfies $\mu\,\emptyset = 0$ and $\mu(U \cup V) + \mu(U \cap V) = \mu U + \mu V$. These evaluations look like measures, not like bags. Their meaning is as follows: If the semantics of a non-deterministic expression E is μ, then $\mu O = n$ means that n results of E are in the open set O.

In Section 5, we show that every concrete bag A of elements of $\mathbf{X}$ induces an evaluation A^*, where $A^*(O)$ is the number of elements of A in O. We investigate which bags induce the same evaluation, and find as a corollary, that each finite bag induces a different evaluation; only certain infinite bags are identified. We also present criteria for $A^* \sqsubseteq B^*$ in terms of A and B. In Section 6, this is done for finite bags A and B with more interesting results than in the general case. In Section 7, we show that for sober dcpo's $\mathbf{X}$, each element μ of $\mathcal{L}_{\mathrm{m}}^*\mathbf{X}$ can be represented by some bag A as $\mu = A^*$, or equivalently can be written as a (possibly infinite) sum of `singletons' (point measures). This is not true for the non-sober dcpo of [6].

For continuous $\mathbf{X}$, $\mathcal{L}_{\mathrm{m}}^*\mathbf{X}$ is again continuous, and is isomorphic to $\mathcal{L}_{\mathrm{i}}^*\mathbf{X}$ (Section 8). In this special case, we introduce the common name $\mathcal{L}^*$ for $\mathcal{L}_{\mathrm{i}}^*$ and $\mathcal{L}_{\mathrm{m}}^*$. Given a basis B of $\mathbf{X}$, a basis of $\mathcal{L}^*\mathbf{X}$ consists of finite sums of singletons from B. Equivalently, it consists of all finite bags of elements of B ordered by an analogue of the lower order on sets. For algebraic $\mathbf{X}$, $\mathcal{L}^*\mathbf{X}$ is again algebraic (Subsection 9.1). The basis construction developed for the continuous case maps the canonical basis of $\mathbf{X}$ (the one consisting of the compact elements of $\mathbf{X}$) into the canonical basis of $\mathcal{L}^*\mathbf{X}$.

The construction $\mathcal{L}^*$ preserves properties I and M (Subsections 9.2 and 9.3), but it does not preserve the property to be a Scott domain, nor property L, nor bifiniteness (Subsection 9.4). These negative results can be seen at a small finite example. Moreover, we can show that there is no other power domain construction with characteristic semiring N_0^∞ that preserves these domain classes (Subsection 9.5).

In Section 10, we compare our approach with various other proposals of bag domains [11, 2, 17, 7].

2 Mathematical Background

After fixing some set-theoretical notation (Subsection 2.1), we briefly recall the definition of the category $\mathcal{DCPO}$ of dcpo's (domains) and continuous functions (Subsection 2.2), introduce some algebraic structures in this category (Subsection 2.3), and recall the definition and basic properties of power domain constructions from [4, 5] (Subsection 2.4).

2.1 Some Set-Theoretic Notation

We only mention some slightly non-standard notation. If $f : X \to Y$ is a function, then we denote the image of a set $A \subseteq X$ by $f^+A = \{fx \mid x \in A\}$, and the inverse image of a set $B \subseteq Y$ by $f^-B = \{x \in X \mid fx \in B\}$. The set f^+X, which is the image of the function, is also denoted by $\Im f$. If f is injective and y is in $\Im f$, then we denote the (unique) element of $f^-\{y\}$ by $f^{-1}y$.

2.2 Dcpo's and Topological Spaces

We use the standard definitions: a poset $(X, \sqsubseteq)$ is a set X together with a reflexive, anti-symmetric, and transitive relation '$\sqsubseteq$'. For a subset A of a poset X, we define $\downarrow A = \{x \in X \mid \exists a \in A : x \sqsubseteq a\}$ and $\uparrow A = \{x \in X \mid \exists a \in A : a \sqsubseteq x\}$. We shall often abbreviate $\downarrow\{a\}$ by $\downarrow a$ and $\uparrow\{a\}$ by $\uparrow a$. The set A is *lower* if $A = \downarrow A$, and *upper* if $A = \uparrow A$. The *least upper bound* or *join* of A is denoted by $\bigsqcup A$ (if it exists).

A subset D of a poset is *directed* if it is non-empty, and for all x, y in D, there is z in D with $x, y \sqsubseteq z$. A *dcpo* is a poset where every directed set has a least upper bound. We sometimes call dcpo's *domains*, especially if they are continuous (Section 8). A function $f : \mathbf{X} \to \mathbf{Y}$ between dcpo's $\mathbf{X}$ and $\mathbf{Y}$ is *continuous* if for all directed subsets D of $\mathbf{X}$, $f(\bigsqcup D) = \bigsqcup f^+D$ holds. The category of dcpo's and continuous functions is called $\mathcal{DCPO}$. This category is small complete and cartesian closed. The function space $[\mathbf{X} \to \mathbf{Y}]$ for instance consists of the continuous functions from $\mathbf{X}$ to $\mathbf{Y}$ ordered by $f \sqsubseteq g$ iff $fx \sqsubseteq gx$ for all x in $\mathbf{X}$.

A *topological space* is a set X together with a set ΩX of subsets of X which is closed under finite intersections and arbitrary unions. The sets in ΩX are called *open*, and their complements are called *closed*. A function $f : X \to Y$ between two topological spaces X and Y is *continuous* iff f^-V is open in X for every open set V of Y.

A subset O of a dcpo $\mathbf{X}$ is *Scott open* (or briefly *open*) if it is upper, and for all directed sets D, $\bigsqcup D \in O$ implies $D \cap O \neq \emptyset$. With this definition, every dcpo becomes a topological space. A function $f : \mathbf{X} \to \mathbf{Y}$ between dcpo's is dcpo continuous iff it is topologically continuous. Thus, $\mathcal{DCPO}$ can be considered as a full subcategory of the category of topological spaces.

Every topological space can be preordered by defining $x \sqsubseteq x'$ iff every open set which contains x also contains x'. This is called the *specialization preorder* of the space. Every continuous function is in particular monotonic. A space is a $\mathcal{T}_0$*-space* iff this preorder is a partial order, i.e., anti-symmetric. The specialization preorder of a dcpo with its Scott topology is the original order of the dcpo. Hence, every dcpo is a $\mathcal{T}_0$-space.

2.3 Some Algebraic Structures

A *monoid* (in the category $\mathcal{DCPO}$) is an algebraic structure $(M, *, e)$ where M is a dcpo, $* : M \times M \to M$ is a continuous associative function, and $e \in M$ is its neutral element. A monoid is *commutative* iff '$*$' is commutative. A *semilattice* is a commutative monoid where '$*$' is idempotent, i.e., $m * m = m$ holds for all m in M. A function $f : M \to M'$ between two monoids is *additive* iff $fe = e$ and $f(a * b) = fa * fb$ for all a, b in M.

A *semiring* (in $\mathcal{DCPO}$) is a structure $(R, +, 0, \cdot, 1)$ where $(R, +, 0)$ is a commutative monoid, $(R, \cdot, 1)$ is a monoid, and the following distributive and zero laws hold:

$$a \cdot (b + b') = a \cdot b + a \cdot b' \qquad (a + a') \cdot b = a \cdot b + a' \cdot b \qquad a \cdot 0 = 0 \cdot b = 0.$$

In other words, the functions $\lambda b.\ a \cdot b$ and $\lambda a.\ a \cdot b$ are additive w.r.t. $(R, +, 0)$.

Let R be a semiring. A (left) R-module (in $\mathcal{DCPO}$) is a structure $(M, +, 0, \cdot)$ where $(M, +, 0)$ is a commutative monoid and $\cdot : R \times M \to M$ is continuous with $1 \cdot m = m$ and $(r \cdot s) \cdot m = r \cdot (s \cdot m)$ for all r, s in R and m in M, and distributive and zero laws which formally look like those in a semiring. A function $f : M \to M'$ between left R-modules is *left R-linear* or shortly *linear* iff it is additive and satisfies $f(r \cdot m) = r \cdot fm$ for all r in R and m in M.

Right R-modules and *right linear* functions are defined analogously; the only difference is that the external multiplication has type $\cdot : M \times R \to M$.

2.4 Power Constructions

In this subsection, we briefly recall the algebraic theory of power domain constructions from [4, 5]. A power domain construction can be seen as a triple $(\mathcal{P}, \mathsf{E}, \mathsf{s})$, where $\mathcal{P}$ maps every dcpo $\mathbf{X}$ into a commutative monoid $(\mathcal{P}\mathbf{X}, +, 0)$ in $\mathcal{DCPO}$, and $\mathsf{E} : [\mathbf{X} \to \mathcal{P}\mathbf{Y}] \to [\mathcal{P}\mathbf{X} \to \mathcal{P}\mathbf{Y}]$ and $\mathsf{s} : \mathbf{X} \to \mathcal{P}\mathbf{X}$ are (families of) continuous functions, which satisfy the axioms of a *Kleisli triple* $\mathsf{E}f \circ \mathsf{s} = f$, $\mathsf{E}\,\mathsf{s} = \mathsf{id}$, and $\mathsf{E}g \circ \mathsf{E}f = \mathsf{E}(\mathsf{E}g \circ f)$. Additionally, the functions $\mathsf{E}f : \mathcal{P}\mathbf{X} \to \mathcal{P}\mathbf{Y}$ as well as the functions E themselves have to be additive. For the latter, the monoid structure on $\mathcal{P}\mathbf{Y}$ is extended to $[\mathbf{X} \to \mathcal{P}\mathbf{Y}]$ and $[\mathcal{P}\mathbf{X} \to \mathcal{P}\mathbf{Y}]$ by pointwise operations.

Power constructions may be classified according to their effect on the one-point domain $\mathbf{1} = \{\diamond\}$. Defining $\cdot : \mathcal{P}\mathbf{1} \times \mathcal{P}\mathbf{1} \to \mathcal{P}\mathbf{1}$ by $a \cdot b = \mathsf{E}(\lambda\diamond.\ b)a$ and $1 = \mathsf{s}(\diamond) \in \mathcal{P}\mathbf{1}$, we obtain a semiring structure $(\mathcal{P}\mathbf{1}, +, 0, \cdot, 1)$. This is called the *characteristic semiring* of $\mathcal{P}$. Each power domain $\mathcal{P}\mathbf{X}$ becomes a $\mathcal{P}\mathbf{1}$*-module* by defining $\cdot : \mathcal{P}\mathbf{1} \times \mathcal{P}\mathbf{X} \to \mathcal{P}\mathbf{X}$ by $a \cdot B = \mathsf{E}(\lambda\diamond.\ B)a$, and all functions $\mathsf{E}f$ are linear.

Conversely, for every given semiring R in $\mathcal{DCPO}$, there are two distinguished power domain constructions with $\mathcal{P}\mathbf{1} \cong R$, namely the *initial* R-construction $\mathcal{P}_{\mathrm{i}}^R$ and the *final* R-construction $\mathcal{P}_{\mathrm{f}}^R$.

The initial R-construction maps every argument domain $\mathbf{X}$ into the free R-module $\mathcal{P}_{\mathrm{i}}^R\mathbf{X}$ over $\mathbf{X}$. It is characterized by the following universal property: $\mathcal{P}_{\mathrm{i}}^R\mathbf{X}$ is an R-module, and there is a continuous function $\mathsf{s} : \mathbf{X} \to \mathcal{P}_{\mathrm{i}}^R\mathbf{X}$ (the singleton function); and for every R-module $\mathbf{M}$ and continuous $f : \mathbf{X} \to \mathbf{M}$, there is a unique continuous R-linear $F : \mathcal{P}_{\mathrm{i}}^R\mathbf{X} \to \mathbf{M}$ with $F \circ \mathsf{s} = f$. In particular, we can extend $\mathsf{s} : \mathbf{X} \to \mathcal{P}\mathbf{X}$ where $\mathcal{P}$ is any R-construction, to a homomorphism $H : \mathcal{P}_{\mathrm{i}}^R\mathbf{X} \to \mathcal{P}\mathbf{X}$. This indicates why $\mathcal{P}_{\mathrm{i}}^R$ is the initial R-construction.

Apart from existence and universal property, the general theory does not provide an explicit description of $\mathcal{P}_{\mathrm{i}}^R\mathbf{X}$ except for $\mathcal{P}_{\mathrm{i}}^R\mathbf{1} = R$.

The final R-construction maps every argument domain $\mathbf{X}$ into the domain $\mathcal{P}_{\mathrm{f}}^R\mathbf{X} = [[\mathbf{X} \to R] \overset{rlin}{\to} R]$ of continuous right linear functions from the right R-module $[\mathbf{X} \to R]$

(with pointwise operations) to the right R-module R. The operations of $\mathcal{P}_{\mathrm{f}}^R$ are given by the following λ-expressions:

$$\begin{array}{rcll} 0 &=& \lambda f.\, 0 & \\ A+B &=& \lambda f.\, Af+Bf & (A, B \in \mathcal{P}_{\mathrm{f}}^R\mathbf{X}) \\ \mathsf{s}\, x &=& \lambda f.\, fx & (x \in \mathbf{X}) \\ \mathsf{E}\, gA &=& \lambda f.\, A\,(\lambda a.\, gaf) & (A \in \mathcal{P}_{\mathrm{f}}^R\mathbf{X},\ g : \mathbf{X} \to \mathcal{P}_{\mathrm{f}}^R\mathbf{Y}) \end{array}$$

If $\mathcal{P}$ is any R-construction, then $\mathsf{E} : [\mathbf{X} \to \mathcal{P}\mathbf{1}] \to [\mathcal{P}\mathbf{X} \to \mathcal{P}\mathbf{1}]$ can be rearranged into $H : \mathcal{P}\mathbf{X} \to [[\mathbf{X} \to R] \to R]$, which indicates why $\mathcal{P}_{\mathrm{f}}^R$ is the final R-construction. The details can be found in [3, 4, 5].

3 Lower Monoids

The lower power domain construction produces free lower semilattices, i.e., semilattices in $\mathcal{DCPO}$ where 0 is the least element (or equivalently, the inherent semilattice order coincides with the order of the underlying dcpo).

Dropping idempotence, we reach at the following definition: A *lower monoid* is a commutative monoid $(M,\ +,\ 0)$ (in the category $\mathcal{DCPO}$) with the additional property $0 \sqsubseteq m$ for all m in M, or equivalently $a \sqsubseteq a+b$ for all a, b in M.

By the definition, every lower monoid has a least element, namely 0. A lower monoid M has a largest element, too, since M is directed by $a, b \sqsubseteq a+b$, and directed sets have joins.

Every lower monoid M admits infinite summation. For an indexed family $(m_i)_{i\in I}$ of members of M, we define

$$\sum_{i\in I} m_i = \bigsqcup\{\sum_{i\in F} m_i \mid F \subseteq_{fin} I\}.$$

The join exists since the set is directed because of the property $a \sqsubseteq a+b$. Additive continuous functions preserve such infinite sums: $f(\sum_{i\in I} m_i) = \sum_{i\in I} fm_i$.

Let us consider some examples of lower monoids. Every complete lattice L implies a lower monoid $(L,\ \sqcup,\ \bot)$. Another example is $(\mathbf{N}_0^\infty,\ +,\ 0)$ where $\mathbf{N}_0^\infty = \{0 \sqsubset 1 \sqsubset 2 \sqsubset \cdots \sqsubset \infty\}$ and `+' is the usual addition extended to ∞ by $n+\infty = \infty+n = n$ for all n in $\mathbf{N}_0^\infty$. Moreover, this example forms a semiring in $\mathcal{DCPO}$ if the usual multiplication is extended by $0\cdot\infty = \infty\cdot 0 = 0$ and $n\cdot\infty = \infty\cdot n = \infty$ for $n \neq 0$.

Every lower monoid becomes an $\mathbf{N}_0^\infty$-module by the inductive definition $0\cdot m = 0$, $(n+1)\cdot m = n\cdot m + m$, and $\infty\cdot m = \bigsqcup_{n\in\mathbf{N}} n\cdot m$. This implies that $\mathbf{N}_0^\infty$ is the free lower monoid in $\mathcal{DCPO}$ with one generator, namely 1. For, if M is a lower monoid with a particular element x, then $\lambda n.\, n\cdot x$ is the unique additive function from $\mathbf{N}_0^\infty$ to M which maps 1 to x.

Conversely, every $\mathbf{N}_0^\infty$-module M is a lower monoid since $0 = 0\cdot m \sqsubseteq 1\cdot m = m$ holds for all m in M. Every additive continuous function between lower monoids is linear between the corresponding $\mathbf{N}_0^\infty$-modules; conversely, linear functions are obviously additive. Thus, the category of lower monoids with continuous additive functions is isomorphic to the category of $\mathbf{N}_0^\infty$-modules with continuous linear functions.

This isomorphism shows that we may concentrate on power constructions with characteristic semiring $\mathbf{N}_0^\infty$, when we want to study constructions producing lower monoids. We call such power constructions *lower bag constructions*. The general theory of [4, 5] implies the existence of two distinguished lower bag constructions (possibly among many others): the initial $\mathbf{N}_0^\infty$-construction which we call $\mathcal{L}_{\mathrm{i}}^*$, and the final $\mathbf{N}_0^\infty$-construction which we call $\mathcal{L}_{\mathrm{f}}^*$.

As mentioned in the previous section, the initial lower bag domains $\mathcal{L}_{\mathrm{i}}^*\mathbf{X}$ are characterized by their universal property; $\mathcal{L}_{\mathrm{i}}^*\mathbf{X}$ is the free lower monoid over $\mathbf{X}$ in $\mathcal{DCPO}$. The final lower bag domains $\mathcal{L}_{\mathrm{f}}^*\mathbf{X}$ are given by $\mathcal{L}_{\mathrm{f}}^*\mathbf{X} = [[\mathbf{X} \to \mathbf{N}_0^\infty] \stackrel{rlin}{\to} \mathbf{N}_0^\infty]$. Since the semiring $\mathbf{N}_0^\infty$ is commutative, and continuous $\mathbf{N}_0^\infty$-linear functions coincide with continuous additive functions, we may simplify to $\mathcal{L}_{\mathrm{f}}^*\mathbf{X} = [[\mathbf{X} \to \mathbf{N}_0^\infty] \stackrel{add}{\to} \mathbf{N}_0^\infty]$, i.e., the final lower bag domain over $\mathbf{X}$ consists of the continuous additive functions from the lower monoid $[\mathbf{X} \to \mathbf{N}_0^\infty]$ (with pointwise operations) to the lower monoid $\mathbf{N}_0^\infty$.

Apart from existence and universal property, the general theory does not provide an explicit description of $\mathcal{L}_{\mathrm{i}}^*\mathbf{X}$ except for $\mathcal{L}_{\mathrm{i}}^*\mathbf{1} = \mathbf{N}_0^\infty$. In this paper, we make $\mathcal{L}_{\mathrm{i}}^*\mathbf{X}$ explicit by comparing it with $\mathcal{L}_{\mathrm{f}}^*\mathbf{X}$; we show $\mathcal{L}_{\mathrm{i}}^*\mathbf{X} \cong \mathcal{L}_{\mathrm{f}}^*\mathbf{X}$ for continuous domains $\mathbf{X}$. To this end, we study $\mathcal{L}_{\mathrm{f}}^*\mathbf{X}$ in greater detail in the following sections.

Many properties of $\mathcal{L}_{\mathrm{f}}^*\mathbf{X}$ do not depend on the fact that $\mathbf{X}$ is a dcpo. Instead, they hold for more general classes of topological spaces. Thus, we generalize $\mathcal{L}_{\mathrm{f}}^*$ with $\mathcal{L}_{\mathrm{f}}^*X = [[X \to \mathbf{N}_0^\infty] \stackrel{add}{\to} \mathbf{N}_0^\infty]$ to a construction mapping a topological space X into a dcpo $\mathcal{L}_{\mathrm{f}}^*X$, where $[X \to \mathbf{N}_0^\infty]$ is understood as the dcpo of continuous functions from X to $\mathbf{N}_0^\infty$ ordered pointwise. (One easily verifies that for a topological space X and a dcpo $\mathbf{Y}$, the poset of continuous functions from X to $\mathbf{Y}$ ordered pointwise is indeed a dcpo.)

We introduce this generalization only because it does not cause additional difficulties in the following sections. We do not intend to develop a theory of lower bag *spaces*; for such a theory, the function spaces should be equipped with a more suitable topology than the Scott topology, and lower monoids should be defined in the category of topological spaces. Such lower monoids differ from those in $\mathcal{DCPO}$ in the continuity of $+ : M \times M \to M$, since the topological product differs from the $\mathcal{DCPO}$ product in general.

4 The Final Construction in terms of Modular Functions

In this section, we present a simpler description of the final lower bag domains. This description is inspired by [8, 9], where a *probabilistic power construction* $\mathcal{P}$ is defined by $\mathcal{P}\mathbf{X} = [\Omega\mathbf{X} \stackrel{mod}{\to} \mathbf{I}]$, the dcpo of continuous modular functions from $\Omega\mathbf{X}$, the lattice of open sets of $\mathbf{X}$, to $\mathbf{I}$, the unit interval of the reals with the Scott topology. We define an analogous construction $\mathcal{L}_{\mathrm{m}}^*X = [\Omega X \stackrel{mod}{\to} \mathbf{N}_0^\infty]$, and show $\mathcal{L}_{\mathrm{m}}^*X \cong \mathcal{L}_{\mathrm{f}}^*X$ for all topological spaces X.

For a topological space X, let ΩX be its lattice of open sets. A function μ from ΩX to $\mathbf{N}_0^\infty$ is *modular* if it is strict ($\mu\,\emptyset = 0$) and satisfies the *modular law*

$$\mu(U \cap V) + \mu(U \cup V) = \mu U + \mu V.$$

This property makes μ similar to an (integer-valued) measure, but it is defined on the opens of X and not on the Borel sets. The dcpo $\mathcal{L}_{\mathrm{m}}^*X$ consists of the Scott continuous modular functions from the dcpo ΩX to the dcpo $\mathbf{N}_0^\infty$. In analogy to [8, 9], we call the members of $\mathcal{L}_{\mathrm{m}}^*X$ *evaluations*. (The exact name would be integer-valued continuous evaluations.)

Obviously, the function $\lambda U.\, 0$ is continuous and modular, and the pointwise sum of two continuous and modular functions is again so. With these operations, $\mathcal{L}_{\mathrm{m}}^*X$ becomes a lower monoid. Additionally, for every x in X, we get an evaluation $\mathsf{s}_{\mathrm{m}}x = \lambda U.\, `x \in U'$, where $`x \in U'$ is 1 if x is in U, and 0 otherwise. The `singleton' evaluations $\mathsf{s}_{\mathrm{m}}x$ will be called *point evaluations*. Note that continuity of the function s_{m} itself is only guaranteed if X is a dcpo, but this does not matter in the following. We can make $\mathcal{L}_{\mathrm{m}}^*$ into a functor by defining $\mathcal{L}_{\mathrm{m}}^*\, g\, \mu = \lambda U.\, \mu\,(g^- U)$. Also, we note that for every μ in $\mathcal{L}_{\mathrm{m}}^*X$ and open W in ΩX, the *restriction* μ_W defined by $\mu_W U = \mu(U \cap W)$ is again in $\mathcal{L}_{\mathrm{m}}^*X$.

Our goal is to prove:

Theorem 4.1
For every topological space X, the two lower monoids $\mathcal{L}^*_\mathrm{f} X = [[X \to \mathbf{N}_0^\infty] \stackrel{add}{\to} \mathbf{N}_0^\infty]$ and $\mathcal{L}^*_\mathrm{m} X = [\Omega X \stackrel{mod}{\to} \mathbf{N}_0^\infty]$ are isomorphic. The isomorphism φ also respects the `singleton functions' (i.e., $\varphi \circ \mathsf{s}_\mathrm{f} = \mathsf{s}_\mathrm{m}$) and the functor structure (i.e., $\varphi \circ \mathcal{L}^*_\mathrm{f}\, g = \mathcal{L}^*_\mathrm{m}\, g \circ \varphi$).

In the idempotent case, where $\mathbf{N}_0^\infty$ is replaced by $\mathbf{2} = \{0 \sqsubset 1\}$, an analogous theorem trivially holds: the dcpo's $[X \to \mathbf{2}]$ and ΩX are isomorphic, and if `$+$' is join, then modularity $\mu(U \cap V) + \mu(U \cup V) = \mu U + \mu V$ and additivity $\mu(U \cup V) = \mu U + \mu V$ are obviously equivalent.

In the non-idempotent case, more work is required. The isomorphism $\varphi : \mathcal{L}^*_\mathrm{f} X \to \mathcal{L}^*_\mathrm{m} X$ is easily defined; its inverse will be more complex. For A in $\mathcal{L}^*_\mathrm{f} X$, let $\varphi A = \lambda U.\, A\langle U \rangle$, where $\langle U \rangle : X \to \mathbf{N}_0^\infty$ is defined by $\langle U \rangle x =$ `$x \in U$'. Obviously, φ is Scott continuous as well as every φA. The functions φA are strict since $\varphi A\, \emptyset = A(\lambda x.\, 0) = 0$ by additivity of A. For the modular law, we have to show

$$A\langle U \cap V \rangle + A\langle U \cup V \rangle = A\langle U \rangle + A\langle V \rangle.$$

By additivity of A, it suffices to prove

$$\langle U \cap V \rangle + \langle U \cup V \rangle = \langle U \rangle + \langle V \rangle \quad \text{or}$$
$$\text{`}x \in U \cap V\text{'} + \text{`}x \in U \cup V\text{'} = \text{`}x \in U\text{'} + \text{`}x \in V\text{'}$$

which can be shown by case analysis. Thus, φ is well-defined and continuous. Additivity of φ is obvious. Furthermore, we have

$$\varphi(\mathsf{s}_\mathrm{f} x)U \;=\; \mathsf{s}_\mathrm{f} x \langle U \rangle \;=\; \langle U \rangle x \;=\; \text{`}x \in U\text{'} \;=\; \mathsf{s}_\mathrm{m} x U$$
$$\varphi(\mathcal{L}^*_\mathrm{f} g A)U \;=\; \mathcal{L}^*_\mathrm{f} g A \langle U \rangle \;=\; A(\langle U \rangle \circ g) \;=\; A\langle g^- U \rangle \;=\; \varphi A(g^- U) \;=\; \mathcal{L}^*_\mathrm{m} g(\varphi A)U$$

which shows that φ preserves s and the functor structure.

To define $\psi : \mathcal{L}^*_\mathrm{m} X \to \mathcal{L}^*_\mathrm{f} X$, we have to compute a number $\psi(\mu)(f)$ for every continuous $\mu : \Omega X \stackrel{mod}{\to} \mathbf{N}_0^\infty$ and $f : X \to \mathbf{N}_0^\infty$. This is done by *integration*: $\psi(\mu)(f) = \int f \mathsf{d}\mu$. Informally, the integral $\int f\, \mathsf{d}\mu$ is defined by

$$\sum_{n \in \mathbf{N}_0^\infty} n \cdot \mu\{x \mid fx = n\} \qquad \text{(informal).}$$

Unfortunately, the set $\{x \mid fx = n\}$ is not open, and μ can only be applied on open sets. We thus write this set as the difference $\{x \mid fx \geq n\} \setminus \{x \mid fx > n\}$. This is a difference of open sets for finite n. A little more work is needed to have a formal definition of integration. We avoid the case $n = \infty$ by approximating f by the functions $f_k = \lambda x.\, k \sqcap fx$ for k in $\mathbf{N}_0$, where `$\sqcap$' denotes minimum in $\mathbf{N}_0^\infty$. Using this approximation, we define

$$\textstyle\int f\, \mathsf{d}\mu \;=\; \bigsqcup_{k \in \mathbf{N}} \int_k f_k\, \mathsf{d}\mu \qquad \text{where}$$
$$\textstyle\int_k g\, \mathsf{d}\mu \;=\; \sum_{n=0}^{k} n \cdot (\mu\{x \mid gx \geq n\} - \mu\{x \mid gx > n\}).$$

We have to show that $\psi(\mu)$ is continuous and additive, i.e., integration is continuous and additive in its functional argument.

For $n \leq k$, $f_k\, x \geq n$ and $f x \geq n$ are equivalent, whereas $f_k\, x \geq k + 1$ never holds. Thus we can simplify

$$\begin{aligned} \textstyle\int_k f_k\, \mathsf{d}\mu \;&=\; \textstyle\sum_{n=0}^{k} n \cdot (\mu\{x \mid f_k\, x \geq n\} - \mu\{x \mid f_k\, x \geq n+1\}) \\ &=\; \textstyle\sum_{n=1}^{k} \mu\{x \mid fx \geq n\} \end{aligned}$$

whence

$$\textstyle\int f\, \mathsf{d}\mu \;=\; \sum_{n \in \mathbf{N}} \mu\{x \mid fx \geq n\}.$$

Note that this sum does not range over the cases $n = 0$ or $n = \infty$.

We use this last formula to show continuity of $\int f\, \mathsf{d}\mu$ in f. Directed joins of continuous functions are continuous, and by the continuity of `$+$', pointwise sums of continuous functions are continuous. Thus we only need to show the continuity of $\lambda f.\ \mu\{x \mid fx \geq n\}$ for fixed n. Let $(f_i)_{i\in I}$ be a directed family of functions with join f. Then $fx \geq n$, iff $\bigsqcup_{i\in I} f_i\, x \geq n$, iff $\exists i \in I : f_i\, x \geq n$. Thus $\{x \mid fx \geq n\} = \bigcup_{i\in I}\{x \mid f_i\, x \geq n\}$. Since this union is directed, continuity of μ yields $\mu\{x \mid fx \geq n\} = \bigsqcup_{i\in I} \mu\{x \mid f_i\, x \geq n\}$.

The equality $\int(\lambda x.\ 0)\,\mathsf{d}\mu = 0$ is trivial. The equality $\int(f+g)\,\mathsf{d}\mu = \int f\,\mathsf{d}\mu + \int g\,\mathsf{d}\mu$ need only be shown for bounded functions f and g, since $f = \bigsqcup_{k\in\mathbf{N}} f_k$, and integration is continuous as shown above. Bounded functions to $\mathbf{N}_0^\infty$ are `simple', i.e., have a finite image not containing ∞. For simple functions to $\mathbf{R}_0^\infty$, additivity of integration was shown in [8]. Simple functions to $\mathbf{N}_0^\infty$ form just a special instance of this.

We now know that $\psi : \mathcal{L}^*_{\mathrm{m}}X \to \mathcal{L}^*_{\mathrm{f}}X$ is a well-defined function. Continuity of ψ is continuity of integration in the μ-argument. It can be shown by a reduction argument similar to the one above; the functions $\lambda\mu.\ \mu\{x \mid fx \geq n\}$ are obviously continuous.

Next, we show that φ and ψ are inverse of each other. For μ in $\mathcal{L}^*_{\mathrm{m}}X$ and U in ΩX, we may compute

$$\begin{array}{rcl} \varphi\,(\psi\mu)U &=& \psi\,(\mu)\,\langle U\rangle \\ &=& \int(\lambda x.\ `x \in U'\,)\,\mathsf{d}\mu \\ &=& \sum_{n\in\mathbf{N}} \mu\{x \mid `x \in U' \geq n\} \;=\; \mu U \end{array}$$

where the last equality holds since $\{x \mid `x \in U' \geq n\}$ is U if $n = 1$, and empty if $n > 1$.

Conversely, let A be in $\mathcal{L}^*_{\mathrm{f}}X$ and f in $[X \to \mathbf{N}_0^\infty]$. Let $`fx \geq n'$ be the number 1 if $fx \geq n$, and 0 otherwise. Then

$$\begin{array}{rcl} \psi\,(\varphi A)\,f &=& \int f\,\mathsf{d}(\lambda U.\ A\langle U\rangle) \\ &=& \sum_{n\in\mathbf{N}} A\,\langle\{x \mid fx \geq n\}\rangle \\ &=& \sum_{n\in\mathbf{N}} A\,(\lambda x.\ `fx \geq n'\,) \\ &=& A\,(\lambda x.\ \sum_{n\in\mathbf{N}} `fx \geq n'\,) \\ &=& A\,(\lambda x.\ fx) \;=\; Af \end{array}$$

where we used continuity and additivity of A for the fourth equality.

Additivity of ψ and the properties $\psi \circ \mathsf{s}_{\mathrm{m}} = \mathsf{s}_{\mathrm{f}}$ and $\psi \circ \mathcal{L}^*_{\mathrm{m}}\, g = \mathcal{L}^*_{\mathrm{f}}\, g \circ \psi$ follow from the corresponding properties of its inverse φ in the usual way. This concludes the proof of Theorem 4.1.

5 From Bags to Evaluations

In this section, we show that every bag A of elements of a space X induces an evaluation A^* in $\mathcal{L}^*_{\mathrm{m}}X$. For $\mathcal{T}_0$-spaces X, we present conditions which exactly tell when two bags induce the same evaluation. As a corollary, we obtain the fact that for two *finite* bags A and B on a $\mathcal{T}_0$-space, the equality $A^* = B^*$ implies $A = B$.

First, we have to define what bags on X are. A *bag* on a set X is a function $A : X \to \mathbf{N}_0^\infty$. For every x in X, Ax tells the number of occurrences of x in A. Note that this number is at most countably infinite. A bag is finite if Ax is finite for all x in X and the carrier set set $A = \{x \in X \mid Ax > 0\}$ is finite. A bag A is a *subbag* of a bag B — $A \subseteq B$ — iff for all x in X, $Ax \leq Bx$ holds. The *empty bag* $\emptyset$ is the function $\lambda x.\ 0$. The *union* $\bigcup_{i\in I} A_i$ of a family of bags maps every point x to $\bigsqcup_{i\in I} A_i x$. Thus, $(A \cup B)x$ is the maximum of Ax and Bx. The *sum* $A + B$ of two bags A and B is defined pointwise: $(A+B)x = Ax + Bx$. A bag A on a topological space X is just a bag $A : X \to \mathbf{N}_0^\infty$ on the underlying set X; the function A need not be continuous or even monotonic.

Let X be an arbitrary topological space. Given a bag A on X and a subset S of X, we define $A[S] = \sum_{x\in S} Ax \in \mathbf{N}_0^\infty$. The elementary properties of this notion are as follows:

Proposition 5.1 For all bags A and B on X and subsets S and T of X:

(1) $\emptyset[S] = 0$.

(2) $(A+B)[S] = A[S] + B[S]$.

(3) If $A \subseteq B$, then $A[S] \leq B[S]$.

(4) $A[\emptyset] = 0$.

(5) If $S \cap T = \emptyset$, then $A[S \cup T] = A[S] + A[T]$.

(6) $A[S \cup T] + A[S \cap T] = A[S] + A[T]$.

(7) If $(S_i)_{i \in I}$ is a directed family of subsets of X, then $A[\bigcup_{i \in I} S_i] = \bigsqcup_{i \in I} A[S_i]$.

(8) If $(A_i)_{i \in I}$ is a directed family of bags w.r.t. the subbag ordering `$\subseteq$', then $(\bigcup_{i \in I} A_i)[S] = \bigsqcup_{i \in I}(A_i[S])$.

Proof: Recall that the infinite sum $A[S]$ is defined as $\bigsqcup_{F \subseteq_{fin} S} \sum_{x \in F} Ax$. Then it is not difficult to prove (1) through (5) and (7).

Property (6) follows from (5) because of the following disjoint partitions: $S = (S \setminus T) \cup (S \cap T)$, $T = (T \setminus S) \cup (S \cap T)$, $S \cup T = (S \setminus T) \cup (T \setminus S) \cup (S \cap T)$.

Property (8) obviously holds for singletons $S = \{x\}$. By (4) and (5) together with continuity of `$+$' in $\mathbf{N}_0^\infty$, it holds for all finite sets S. By (7), it extends to all sets S using $S = \bigcup_{F \subseteq_{fin} S} F$.□

Because of parts (4), (6), and (7) of the proposition, every bag A on X induces an evaluation A^* in $\mathcal{L}^*_{\mathrm{m}}X$ by defining $A^*(O) = A[O]$ for O in ΩX.

By Prop. 5.1, $A \subseteq B$ implies $A^* \sqsubseteq B^*$ in $\mathcal{L}^*_{\mathrm{m}}X$. The inverse implication does not hold in general: consider the dcpo $X = \{\perp \sqsubset \top\}$ and two sample bags A and B:

	$\perp$	$\top$
A	1	0
B	0	1

	$\emptyset$	$\{\top\}$	$\{\perp, \top\}$
A^*	0	0	1
B^*	0	1	1

In this example, $A \subseteq B$ does not hold, but $A^* \sqsubseteq B^*$. In the sequel, we are interested in criteria for $A^* = B^*$. Later, we shall look for criteria for $A^* \sqsubseteq B^*$.

Let us consider two examples. In the first, let X be the flat dcpo consisting of an infinite number of incomparable points $x_1, x_2, \ldots$, together with a least element $\perp$. Let A be the bag with $Ax_i = 1$ for all i and $A\perp = 0$, and B the bag with $Bx_i = 1$ for all i and $B\perp = \infty$. For opens O with $\perp \notin O$, $A^*O = B^*O$ obviously holds. The only open set which contains $\perp$ is the whole dcpo X. Since X contains infinitely many x_i, $A^*X = \infty$ holds. The additional occurrences of $\perp$ in B cannot increase this number any further; B^*X is also ∞. Thus, $A^* = B^*$ holds. In contrast, any bag C which differs from A by $Cx_i \neq Ax_i$ for some i yields an evaluation C^* different from A^* since $\{x_i\}$ is open, and $C^*\{x_i\} = Cx_i$.

In the second example, let X be the dcpo $\{x_1 \sqsubset x_2 \sqsubset \cdots \sqsubset x_\infty\}$. Let A be the bag with $Ax_i = 0$ for finite i and $Ax_\infty = \infty$. Every non-empty Scott open set O of X contains x_∞, whence $A^*O = \infty$. Let B be the bag with $Bx_i = 1$ for finite i and $Bx_\infty = 0$. Every non-empty Scott open set O of X contains infinitely many points x_i, whence $B^*O = \infty$. Thus, $A^* = B^*$ holds. Any superbag C of A or B will yield the same evaluation. The biggest such C has $Cx = \infty$ for all x in X.

Summarizing, we see that A^* does not change if the number Ax is increased at a point x where $A^*(O)$ is already ∞ for every neighborhood O of x.

Definition 5.2 Let A be a bag in a topological space X.

- The *infinite part* $\mathcal{I}(A)$ of A is the set of all points x such that $A^*(O) = \infty$ for every open O containing x. The *finite part* $\mathcal{F}(A)$ is the complement of $\mathcal{I}(A)$.

- From A, we define a bag $\overline{A}$ with $\overline{A}x = \infty$ for all x in $\mathcal{I}(A)$, and $\overline{A}x = Ax$ for x in $\mathcal{F}(A)$.
- The bag A is called ∞-*closed* iff $Ax = \infty$ for every x in $\mathcal{I}(A)$.

Our goal is to show that the map $A \mapsto \overline{A}$ is a closure operator which belongs to the notion of being ∞-closed, and that $A^* = B^*$ iff $\overline{A} = \overline{B}$. We present a list of simple properties of our notions. Obvious proofs are omitted.

(1) For every bag A, $\mathcal{F}(A)$ is open, and hence $\mathcal{I}(A)$ is closed.

Proof: For every x in $\mathcal{F}(A)$, there is an open O_x such that $x \in O_x$ and $A^*(O_x) < \infty$. Then $O_x \subseteq \mathcal{F}(A)$, whence $\mathcal{F}(A) = \bigcup_{x\in\mathcal{F}(A)} O_x$. □

Note that though $A^*(O_x) < \infty$ for all x in $\mathcal{F}(A)$, $A^*(\mathcal{F}(A))$ may be ∞. In the example with $X = \{\bot, x_1, x_2, \ldots\}$ and $Ax_i = 1$, $A\bot = 0$, the finite part $\mathcal{F}(A)$ is $X \setminus \{\bot\}$, and $A^*(\mathcal{F}(A))$ is ∞.

(2) If $A^* \sqsubseteq B^*$, then $\mathcal{I}(A) \subseteq \mathcal{I}(B)$.

(3) $A \subseteq \overline{A}$.

(4) $\overline{A}^* = A^*$.

Proof: By (3), `$\sqsupseteq$' holds. For `$\sqsubseteq$', we have to show $\overline{A}[O] \leq A[O]$ for all opens O. If $A[O] < \infty$, then $O \subseteq \mathcal{F}(A)$, whence $\overline{A}x = Ax$ for all x in O, whence $\overline{A}[O] = A[O]$. □

(5) $\mathcal{I}(\overline{A}) = \mathcal{I}(A)$ and $\mathcal{F}(\overline{A}) = \mathcal{F}(A)$.

Proof: By (4) and (2). □

(6) $\overline{A}$ is ∞-closed.

Proof: If x in $\mathcal{I}(\overline{A})$, then x in $\mathcal{I}(A)$ by (5), whence $\overline{A}x = \infty$ by definition of $\overline{A}$. □

(7) If $A \subseteq B$ and B is ∞-closed, then $\overline{A} \subseteq B$ holds.

Proof: By (2), we know $\mathcal{I}(A) \subseteq \mathcal{I}(B)$. If x in $\mathcal{I}(A) \subseteq \mathcal{I}(B)$, then $Bx = \infty$. If x in $\mathcal{F}(A)$, then $\overline{A}x = Ax \leq Bx$. □

(8) $A \mapsto \overline{A}$ is a closure operator, and $\overline{A} = A$ iff A is ∞-closed.

Proof: By (3), (6), and (7). □

From property (4), we know that $\overline{A} = \overline{B}$ implies $A^* = B^*$. For the opposite implication, we need two auxiliary lemmas. The order-theoretic notions ($\uparrow x$ etc.) in these lemmas refer to the specialization preorder of the underlying topological space.

Lemma 5.3 For every x in $\mathcal{F}(A)$, there is an open set V with $x \in V$, $A[V] = A[\uparrow x] < \infty$, and $A[V \setminus \uparrow x] = 0$.

Proof: Let V be an open set with $x \in V$ such that $A[V]$ is minimal in $\mathbf{N}_0^\infty$. Then $A[V] < \infty$ since x is in $\mathcal{F}(A)$. For y in $V \setminus \uparrow x$, let $W = V \setminus \downarrow y$. The set W is open since $\downarrow y$ is closed. Since $x \in W \subseteq V$ holds, the minimality property of V implies $A[W] = A[V]$. Since both are finite, $A[V \setminus W] = 0$ follows. From $y \in V \setminus W$, we conclude $Ay = 0$. This holds for all y in $V \setminus \uparrow x$, whence $A[V \setminus \uparrow x] = 0$ and thus $A[V] = A[\uparrow x]$. □

Lemma 5.4 Let X be a $\mathcal{T}_0$-space, A a bag on X, x a point and V an open of X with $x \in V$ and $A[V] = A[\uparrow x] < \infty$. Then $Ax = A^*(V) - A^*(V \setminus \downarrow x)$ holds.

Proof: The notation with A^* is appropriate since $V \setminus \downarrow x$ is open. We have to show $Ax + A[V \setminus \downarrow x] = A[V]$, or since x is not in $V \setminus \downarrow x$, $A[\{x\} \cup (V \setminus \downarrow x)] = A[V]$. This equality follows from $A[\uparrow x] = A[V]$ and the inclusions $\uparrow x \subseteq \{x\} \cup (V \setminus \downarrow x) \subseteq V$. The second inclusion follows from $x \in V$. For the first, let $y \sqsupseteq x$. Then y is in V. If $y \not\sqsubseteq x$, we are done. Otherwise, $y \sqsupseteq x$ and $y \sqsubseteq x$ holds. In a $\mathcal{T}_0$-space, this implies $y = x$. □

Now we can prove the criterion for $A^* = B^*$:

Theorem 5.5
For two bags A and B in a $\mathcal{T}_0$-space, the following statements are equivalent:
(1) $\overline{A} = \overline{B}$.
(2) $\mathcal{F}(A) = \mathcal{F}(B)$, and for all x in $\mathcal{F}(A)$, $Ax = Bx$ holds.
(3) $A^* = B^*$.

Proof:
$(1) \Rightarrow (3)$ holds since $\overline{A}^* = A^*$.
$(3) \Rightarrow (2)$: $\mathcal{F}(A) = \mathcal{F}(B)$ holds since $\mathcal{F}(A)$ is defined via A^*. Let x be in $\mathcal{F}(A) = \mathcal{F}(B)$. By Lemma 5.3, there is an open U with $x \in U$ and $A[U] = A[\uparrow x] < \infty$. Similarly, there is an open V with $x \in V$ and $B[V] = B[\uparrow x] < \infty$. Let $W = U \cap V$. Then $x \in W$, and $A[W] = A[\uparrow x]$ and $B[W] = B[\uparrow x]$. From Lemma 5.4, $Ax = A^*(W) - A^*(W \setminus \downarrow x)$ and $Bx = B^*(W) - B^*(W \setminus \downarrow x)$ follows. Thus, Ax and Bx are equal.
$(2) \Rightarrow (1)$: For x in $\mathcal{F}(A) = \mathcal{F}(B)$, $\overline{A}x = Ax = Bx = \overline{B}x$ holds. For other points x, we have $\overline{A}x = \infty = \overline{B}x$. □

Corollary 5.6 For ∞-closed bags A and B in a $\mathcal{T}_0$-space, $A^* = B^*$ implies $A = B$.

A bag A is finite iff $A[X] < \infty$. Thus, $\mathcal{I}(A)$ is empty if A is finite, whence A is ∞-closed.

Corollary 5.7 For finite bags A and B in a $\mathcal{T}_0$-space, $A^* = B^*$ implies $A = B$.

The $\mathcal{T}_0$ property is really needed. Consider the space $X = \{a, b\}$ where $\emptyset$ and X are the only open sets. The bags A with $Aa = 1$ and $Ab = 0$, and B with $Ba = 0$ and $Bb = 1$ are finite and different, but induce the same evaluation μ with $\mu(\emptyset) = 0$ and $\mu(X) = 1$.

Next, we consider addition of bags.

Proposition 5.8 Let A and B be two bags in a topological space X.
(1) $(A + B)^* = A^* + B^*$.
(2) $\mathcal{I}(A + B) = \mathcal{I}(A) \cup \mathcal{I}(B)$.
(3) $\overline{A + B} = \overline{A} + \overline{B}$.
(4) If A and B are ∞-closed, then so is $A + B$.

Proof:
(1) By Prop. 5.1 (2).
(2) From $A + B \supseteq A, B$, we may conclude $\mathcal{I}(A + B) \supseteq \mathcal{I}(A), \mathcal{I}(B)$. For $\mathcal{I}(A + B) \subseteq \mathcal{I}(A) \cup \mathcal{I}(B)$, we prove $\mathcal{F}(A + B) \supseteq \mathcal{F}(A) \cap \mathcal{F}(B)$. If x is a member of the intersection, then there are open sets U and V such that x in U and $A^*(U) < \infty$, and x in V and $B^*(V) < \infty$. With $W = U \cap V$, x in W and $(A + B)^*(W) < \infty$ holds.
(3) If x is in $\mathcal{I}(A) \subseteq \mathcal{I}(A + B)$, then $(\overline{A + B})x = \infty = \overline{A}x = \overline{A}x + \overline{B}x$. Similarly for x in $\mathcal{I}(B)$. If x is in $\mathcal{F}(A) \cap \mathcal{F}(B) = \mathcal{F}(A + B)$, then $(\overline{A + B})x = (A + B)x = Ax + Bx = \overline{A}x + \overline{B}x$.
(4) From (3): $\overline{A + B} = \overline{A} + \overline{B} = A + B$. □

Our final goal in this section is to find a criterion for $A^* \sqsubseteq B^*$. The first step is to generalize Lemma 5.3 from upper cones $\uparrow x$ to arbitrary upper sets.

Lemma 5.9 Let A be an arbitrary bag in an arbitrary topological space X, and let S be an upper set of X.
(1) If $S \subseteq \mathcal{F}(A)$, then there is an open V such that $S \subseteq V$ and $A[S] = A[V]$.

(2) If A is ∞-closed, then there is an open V such that $S \subseteq V$ and $A[S] = A[V]$.

Proof: First assume $S \subseteq \mathcal{F}(A)$. By Lemma 5.3, for every x in S, there is an open set V_x such that x in V_x and $Ay = 0$ for all y in $V_x \setminus \uparrow x$. Let $V = \bigcup_{x \in S} V_x$. Then $S \subseteq V$ follows.

If y is in $V \setminus S$, then there is x in S such that $y \in V_x \setminus S \subseteq V_x \setminus \uparrow x$. Thus, Ay is 0. Hence, $A[V] = A[V \setminus S] + A[S] = A[S]$ follows.

For (2), let A be ∞-closed and $S \not\subseteq \mathcal{F}(A)$. Since there is a point x in $S \cap \mathcal{I}(A)$, $A[S] \geq Ax = \infty$ holds. With $V = X \supseteq S$, $A[V] = A[S]$ follows. □

Now we are ready for the theorem about $A^* \sqsubseteq B^*$:

Theorem 5.10 For two bags A and B in a topological space X, where B is ∞-closed, the following statements are equivalent:

(1) $A^* \sqsubseteq B^*$.

(2) $A[O] \leq B[O]$ holds for all open sets O.

(3) $A[S] \leq B[S]$ holds for all upper sets S.

(4) $A[F] \leq B[\uparrow F]$ holds for all finite sets F.

(5) $A[F] \leq B[\uparrow F]$ holds for all finite sets $F \subseteq \mathsf{set}\, A$.

Proof:

$(1) \Leftrightarrow (2)$ by the definition of A^* and B^*.

$(2) \Rightarrow (3)$: Let S be an upper set. By Lemma 5.9, there is an open set V such that $S \subseteq V$ and $B[V] = B[S]$. Then $A[S] \leq A[V] \leq B[V] = B[S]$.

$(3) \Rightarrow (5)$: $A[F] \leq A[\uparrow F] \leq B[\uparrow F]$.

$(5) \Rightarrow (4)$: Recall $\mathsf{set}\, A = \{x \in X \mid Ax > 0\}$. Thus, for arbitrary finite sets F, we have $A[F] = A[F \cap \mathsf{set}\, A] \leq B[\uparrow(F \cap \mathsf{set}\, A)] \leq B[\uparrow F]$.

$(4) \Rightarrow (2)$: For every $F \subseteq_{\mathit{fin}} O$, we have $A[F] \leq B[\uparrow F] \leq B[O]$. By Prop. 5.1 (7), we obtain $A[O] \leq B[O]$ using $O = \bigcup_{F \subseteq_{\mathit{fin}} O} F$. □

The criterion for $A^* \sqsubseteq B^*$ in terms of finite sets can be used to find a criterion for $A^* \sqsubseteq B^*$ in terms of finite subbags of A and B.

Proposition 5.11 Let A be a finite bag, and C an ∞-closed bag in a space X. If $A^* \sqsubseteq C^*$, then there is a finite subbag B of C such that $A^* \sqsubseteq B^*$.

Proof: We use part (5) of Theorem 5.10: For every $F \subseteq \mathsf{set}\, A$, $A[F] \leq C[\uparrow F]$ holds. By Prop. 5.1 (8), $C[\uparrow F]$ equals $\bigsqcup_{B \subseteq_{\mathit{fin}} C} B[\uparrow F]$. For every $F \subseteq \mathsf{set}\, A$, $A[F]$ is a finite number. Thus, there is a finite subbag B_F of C such that $A[F] \leq B_F[\uparrow F]$. Since $\mathsf{set}\, A$ is finite, there are finitely many sets $F \subseteq \mathsf{set}\, A$. Let $B = \bigcup_{F \subseteq \mathsf{set}\, A} B_F$. Then B is a finite subbag of C with $A[F] \leq B[\uparrow F]$ for all $F \subseteq \mathsf{set}\, A$. By Theorem 5.10 (5), $A^* \sqsubseteq B^*$ holds. □

Using this proposition, we get:

Theorem 5.12 For two bags A and B in a topological space X, where B is ∞-closed, $A^* \sqsubseteq B^*$ holds iff for every finite subbag A' of A, there is a finite subbag B' of B such that $A'^* \sqsubseteq B'^*$.

Proof: First assume $A^* \sqsubseteq B^*$ holds. If A' is a finite subbag of A, then $A'^* \sqsubseteq A^* \sqsubseteq B^*$, whence by Prop. 5.11, there is a finite subbag B' of B such that $A'^* \sqsubseteq B'^*$.

For the opposite direction, let O be an open set. For every finite subbag A' of A, there is a subbag B' of B such that $A'[O] \leq B'[O] \leq B[O]$. By Prop. 5.1 (8), $A[O] \leq B[O]$ follows using the fact $A = \bigcup_{A' \subseteq_{\mathit{fin}} A} A'$. □

6 Finite Bags and Evaluations

At the end of the previous section, we have developed a criterion for $A^* \sqsubseteq B^*$ in terms of finite subbags of A and B. In this section, we present a criterion for $A^* \sqsubseteq B^*$ where A and B are finite bags.

In the previous section, arbitrary bags A were represented as functions $A : X \to \mathbf{N}_0^\infty$. For finite bags, we also use another kind of representation. A (finite) *bag enumeration* is a function $\alpha : I \to X$ where $I = \mathsf{dom}\,\alpha$ is a finite index set. We often write α_i for αi. A bag enumeration α *enumerates* the finite bag $\hat{\alpha}$ with $\hat{\alpha} x = \#\{i \in I \mid \alpha_i = x\}$, where $\#S$ is the cardinality of S. Two enumerations $\alpha : I \to X$ and $\beta : J \to X$ enumerate the same bag iff there is a bijection $\varphi : I \to J$ with $\beta \circ \varphi = \alpha$; we write $\alpha \cong \beta$ in this situation. Every finite bag A can be enumerated; let e.g., $I = \{(n,\, x) \in \mathbf{N} \times X \mid n \le Ax\}$ and $\alpha(n,\, x) = x$.

The action of an enumerated bag $\hat{\alpha}$ on a set S can be described in terms of α:

Proposition 6.1 Let $\alpha : I \to X$ be a finite bag enumeration. Then $\hat{\alpha}[S] = \#\{i \in I \mid \alpha_i \in S\}$ holds for all subsets S of X.

The evaluation induced by an enumerated bag $\hat{\alpha}$ should be written as $\hat{\alpha}^*$. We simplify this notation to α^*.

Theorem 6.2 Let X be an arbitrary topological space. For two bag enumerations $\alpha : I \to X$ and $\beta : J \to X$, the following statements are equivalent:

(1) $\alpha^* \sqsubseteq \beta^*$;

(2) for all U in ΩX: $\#\{i \in I \mid \alpha_i \in U\} \le \#\{j \in J \mid \beta_j \in U\}$, or shortly $\#\alpha^- U \le \#\beta^- U$;

(3) as (2), but even for all upper sets U of X;

(4) for all $T \subseteq I$: $\#T \le \#\{j \in J \mid \beta_j \in \uparrow\{\alpha_i \mid i \in T\}\}$, or shortly $\#T \le \#\beta^- \uparrow \alpha^+ T$;

(5) there is an injective function $\varphi : I \to J$ with $\alpha_i \sqsubseteq \beta(\varphi i)$, or shortly $\alpha \sqsubseteq \beta \circ \varphi$;

(6) there is a finite bag enumeration $\gamma : K \to X$ with $\alpha \cong \gamma$, $K \subseteq J$, and $\gamma_k \sqsubseteq \beta_k$ for all k in K.

Proof: The equivalence of (1) through (3) follows from Theorem 5.10 with Prop. 6.1.

(3) $\Rightarrow$ (4) The set $U = \uparrow\{\alpha_i \mid i \in T\}$ is an upper set. If i is in T, then α_i is in U. Thus

$$\#T \;=\; \#\{i \in I \mid i \in T\} \;\le\; \#\{i \in I \mid \alpha_i \in U\} \;\le\; \#\{j \in J \mid \beta_j \in U\}.$$

(4) $\Rightarrow$ (5) We apply Hall's theorem for bipartite graphs [12, Theorem 1.1.3]. From α and β, we construct a graph with node set $I \cup J$ (where I and J are assumed to be disjoint). There is an edge from i in I to j in J iff $\alpha_i \sqsubseteq \beta_j$. By (4), the number of nodes adjacent to some node in $T \subseteq I$ is at least $\#T$. By Hall's theorem, there is a `perfect matching' from I into J, i.e., an injective function $\varphi : I \to J$ such that there is an edge from i to φi for every i in I.

(5) $\Rightarrow$ (6) Let $K = \varphi^+ I \subseteq J$, and $\gamma_k = \alpha(\varphi^{-1} k)$. Then $\alpha \cong \gamma$ holds by witness of φ, and $\gamma_k = \alpha(\varphi^{-1} k) \sqsubseteq \beta(\varphi(\varphi^{-1} k)) = \beta_k$ holds for all k in K.

(6) $\Rightarrow$ (1) First, $\alpha \cong \gamma$ implies $\alpha^* = \gamma^*$. Second, $\gamma^* \sqsubseteq \beta^*$ holds, since for all open sets U,

$$\begin{array}{rcll} \gamma^* U & = & \#\{k \in K \mid \gamma_k \in U\} & \\ & \le & \#\{k \in K \mid \beta_k \in U\} & \text{since } \gamma_k \sqsubseteq \beta_k \\ & \le & \#\{j \in J \mid \beta_j \in U\} & \text{since } K \subseteq J. \end{array}$$ □

The pre-order on finite bags which is described by Theorem 6.2 is the same as the order introduced in [11]. Our Cor. 5.7 proves that it is anti-symmetric if the underlying space is a $\mathcal{T}_0$-space. This anti-symmetry is already proved in [11] by other means. Proposition 5.2 of

[11] states that $\alpha^* \sqsubseteq \beta^*$ can be checked in time $O(n^{5/2} + n^2 \cdot k)$, where n is the sum of the sizes of the domains of α and β, and k is the time for comparisons in the argument space X.

7 From Evaluations to Bags

In the previous sections, we studied the mapping $A \mapsto A^*$. In particular, we proved that it is injective when restricted to ∞-closed bags. In this section, we investigate its surjectivity: given an evaluation μ in $\mathcal{L}^*_{\mathrm{m}}X$, can we find a bag A in X such that $\mu = A^*$? Once we know that $\mu = A^*$ for some bag A, we also know that μ can be obtained as a (possibly infinite) sum of singletons $\sum_{i \in I} \mathsf{s}\alpha_i$, where $\alpha : I \to X$ is an enumeration of the (possibly infinite) bag A.

In the idempotent case, where sum is join, the members of the lower power domains $\mathcal{L}\mathbf{X}$ can be represented as closed sets for all dcpo's $\mathbf{X}$, and every closed set of $\mathbf{X}$ is a join of `singletons' $\downarrow x$.

In contrast to this, we will see that sobriety of X is needed in order to represent all evaluations in $\mathcal{L}^*_{\mathrm{m}}X$ by bags. Since there are non-sober dcpo's [6], the situation does not become better by restricting to the case that X is a dcpo.

We start our investigations by classifying the evaluations in $\mathcal{L}^*_{\mathrm{m}}X$. Recall that $\Im\mu = \{\mu O \mid O \in \Omega X\}$ denotes the image of μ.

Definition 7.1 An evaluation $\mu : \Omega X \overset{mod}{\to} \mathbf{N}_0^\infty$ is

- *idempotent* if $\mu + \mu = \mu$ (or equivalently $\Im\mu \subseteq \{0, \infty\}$),
- *simple* if $\Im\mu$ is finite and does not contain ∞ (or equivalently $\mu X < \infty$),
- *primitive* if $\Im\mu = \{0, 1\}$.

The equivalence in the definition of idempotent is due to the fact that 0 and ∞ are the only idempotent members of $\mathbf{N}_0^\infty$. The other equivalence holds because our evaluations are integer-valued.

In the sequel we show:

(1) Every evaluation can be written as the sum of an idempotent evaluation and a directed join of simple evaluations.

(2) Every idempotent evaluation is an infinite sum of point evaluations.

(3) Every simple evaluation is a finite sum of primitive evaluations.

(4) Every point evaluation is primitive. The converse holds iff X is sober.

Step 1: Every evaluation is the sum of an idempotent evaluation and a directed join of simple evaluations.

Let $\mu : \Omega X \overset{mod}{\to} \mathbf{N}_0^\infty$ be an evaluation. Let $\mathcal{V} = \{V \in \Omega X \mid \mu V < \infty\}$, and $W = \bigcup \mathcal{V}$. Then we define $\mu' U = 0$ if $U \subseteq W$, and $= \infty$ otherwise. The function μ' is easily verified to be an idempotent evaluation. Remember the definition $\mu_V U = \mu(U \cap V)$. The evaluations μ_V with V in $\mathcal{V}$ are simple by $\mu_V X = \mu V < \infty$. By modularity, the set $\mathcal{V}$ is directed, whence the family $(\mu_V)_{V \in \mathcal{V}}$ is directed. By continuity of μ, its join is μ_W.

We claim $\mu U = \mu' U + \mu_W U$ for all opens U. For $U \subseteq W$, the equality holds since $\mu' U = 0$ and $\mu_W U = \mu U$. For $U \not\subseteq W$, $\mu' U$ and hence the sum are ∞. We also have $\mu U = \infty$, since otherwise, $U \subseteq W$ would hold.

Step 2: Every idempotent evaluation is an infinite sum of point evaluations.

Let μ be an idempotent evaluation. Let $\mathcal{V} = \{V \in \Omega X \mid \mu V = 0\}$ and $W = \bigcup \mathcal{V}$. By modularity, $\mathcal{V}$ is directed, and by continuity, $\mu W = 0$ holds. Thus $\mu V = 0$ iff $V \subseteq W$.

Let C be the complement of W, a closed set. Let $\mu' = \sum_{x \in C} \infty \cdot \mathsf{s}_{\mathrm{m}} x$. Then $\mu' V = 0$ iff V does not meet C, i.e., $V \subseteq W$. Otherwise, $\mu' V$ is ∞. Thus μ equals μ', an infinite sum of point evaluations.

Note that this description of idempotent evaluations by a closed set C is similar to the representation of the lower power domain elements by closed sets.

Step 3: Every simple evaluation is a finite sum of primitive evaluations.

Let μ be a simple evaluation. We use induction on the finite natural number μX. If μX is 0, then μ is the empty sum of primitive evaluations. Otherwise, let k be the least non-zero element of $\Im\mu$, and let W be some open with $\mu W = k$. The evaluation μ_W with $\mu_W U = \mu(U \cap W)$ assumes the two values 0 and k only because of the minimality of k. Thus $\mu_1 = 1/k \cdot \mu_W$ is a well-defined primitive evaluation. Let $\mu' = \mu - \mu_W$. Once we have shown that μ' is an evaluation, we are done since $\mu = k \cdot \mu_1 + \mu'$. The induction argument works since $\mu' X = \mu X - k$.

Modularity of μ' easily follows from modularity of μ and μ_W. Similarly, $\mu'\emptyset = \mu\emptyset - \mu_W\emptyset = 0$ holds. Next, we show monotonicity of μ'. Let $U \subseteq V$. Then $\mu(U \cup W) \leq \mu(V \cup W)$ follows from monotonicity of μ. By modularity, we obtain $\mu U + \mu W - \mu(U \cap W) \leq \mu V + \mu W - \mu(V \cap W)$, whence $\mu' U \leq \mu' V$. Positivity follows from monotonicity as $\mu' U \geq \mu'\emptyset = 0$. For continuity, let $(U_i)_{i \in I}$ be a directed family of open sets with join U. If $\mu_W U_i = 0$ for all i in I, then $\mu_W U = 0$ by continuity of μ_W. If $\mu_W U_i = k$ for some i in I, then $\mu_W U = k$. In both cases, the μ_W-part of μ' stays constant. Thus continuity of μ implies continuity of μ'.

Step 4: Every point evaluation is primitive. The converse holds iff X is sober.

The primitivity of point evaluations is obvious. Arbitrary functions $\mu : \Omega X \to \{0, 1\} \subseteq \mathbf{N}_0^\infty$ are in one-to-one correspondence with subsets $\mathcal{O}$ of ΩX by $U \in \mathcal{O}$ iff $\mu U = 1$. The function μ is continuous iff $\mathcal{O}$ is Scott open in $(\Omega X, \subseteq)$, strict iff $\mathcal{O}$ does not contain $\emptyset$, and modular iff $\mathcal{O}$ is closed under binary intersection and inaccessible by binary union. Altogether, μ is an evaluation iff $\mathcal{O}$ is a completely prime filter. Trivially, $\mu = \mathsf{s}_{\mathrm{m}} x$ holds iff $\mathcal{O} = \mathcal{O}(x) = \{U \in \Omega X \mid x \in U\}$. Thus, every primitive evaluation is a point evaluation iff every completely prime filter is given by $\mathcal{O}(x)$ for some point x. For $\mathcal{T}_0$-spaces, the latter statement is equivalent to the sobriety of X.

So far, our results can be summarized as follows:

Theorem 7.2 Every evaluation in $\mathcal{L}_{\mathrm{m}}^* X$ can be written as a sum of primitive evaluations. If X is sober, then every primitive evaluation is a point evaluation.

If X is sober, then every evaluation μ can be written as a (possibly infinite) sum $\sum_{i \in I} \mathsf{s} x_i$ of point evaluations. By defining $Ax = \#\{i \in I \mid x_i = x\}$, we obtain a bag A with $\mu = \sum_{x \in X} Ax \cdot \mathsf{s} x$, whence $\mu = A^*$.

If X is not sober, then we already know that there is a primitive evaluation μ which is not a point evaluation $\mathsf{s}x$ for any point x. Moreover, there is no way to obtain μ as a *sum* of point evaluations, since any sum of more than one summand would not be primitive. Assume there were a bag A such that $\mu = A^*$. From $\mu X = 1$, $A[X] = 1$ would follow, whence A would be a singleton bag, i.e., there would be some point x with $Ax' = 1$ if $x' = x$, and $= 0$ otherways. Then $\mu = \mathsf{s}x$ would hold. Thus, we can conclude:

Theorem 7.3 For a $\mathcal{T}_0$-space X, the following statements are equivalent:

(1) X is sober.

(2) Every evaluation in $\mathcal{L}_{\mathrm{m}}^* X$ can be written as a sum of point evaluations.

(3) For every evaluation μ in $\mathcal{L}_{\mathrm{m}}^* X$, there is a bag A such that $\mu = A^*$.

8 The Continuous Case

For two elements x and y of a dcpo $\mathbf{X}$, x is *way-below* y ($x \ll y$) iff for all directed $D \subseteq \mathbf{X}$, $y \sqsubseteq \bigsqcup D$ implies $x \in {\downarrow} D$. The dcpo $\mathbf{X}$ is *continuous* iff for all x in $\mathbf{X}$, the set $\{a \in \mathbf{X} \mid a \ll x\}$ is directed with join x. A subset B of $\mathbf{X}$ is a *basis* of $\mathbf{X}$ iff for all x in $\mathbf{X}$, the set $\{b \in B \mid b \ll x\}$ is directed with join x. By definition, every continuous domain $\mathbf{X}$ has a basis, namely $\mathbf{X}$ itself. In general, there are many other bases. $\mathbf{X}$ is ω-*continuous* iff it has a countable basis. In the sequel, we sometimes need the property that in a continuous domain $\mathbf{X}$, the set ${\Uparrow} A = \{x \in \mathbf{X} \mid \exists a \in A : a \ll x\}$ is Scott open for every $A \subseteq \mathbf{X}$.

Throughout this section, we assume that $\mathbf{X}$ is a continuous domain with a basis B. We shall prove that $\mathcal{L}^*_{\mathrm{m}}\mathbf{X}$ is again continuous with a basis consisting of all finite sums of singletons from B. This description is then used to prove that $\mathcal{L}^*_{\mathrm{m}}\mathbf{X}$ is the free lower monoid over $\mathbf{X}$, i.e., $\mathcal{L}^*_{\mathrm{i}}\mathbf{X}$ and $\mathcal{L}^*_{\mathrm{m}}\mathbf{X}$ are isomorphic for continuous $\mathbf{X}$.

We start with a characterization of the way-below relation between simple evaluations in $\mathcal{L}^*_{\mathrm{m}}\mathbf{X}$. Note the analogy to Theorem 6.2.

Theorem 8.1 Let $\mathbf{X}$ be a continuous domain, and $\alpha : I \to \mathbf{X}$ and $\beta : J \to \mathbf{X}$ two finite bag enumerations. The following statements are equivalent:

(1) In $\mathcal{L}^*_{\mathrm{m}}\mathbf{X}$, $\alpha^* \ll \beta^*$ holds;

(2) For all $T \subseteq I$, the relation $\#T \leq \beta^*({\Uparrow}\alpha^+ T)$ holds;

(3) There is an injection $\varphi : I \to J$ with $\alpha_i \ll \beta(\varphi i)$ for all i in I.

Proof:

$(1) \Rightarrow (3)$ By continuity of $\mathbf{X}$, s, and '+', $\sum_{j \in J} \mathsf{s}\beta_j = \bigsqcup\{\sum_{j\in J} \mathsf{s}\beta'_j \mid \beta'_j \ll \beta_j\}$ holds. By $\alpha^* \ll \beta^*$, α^* is below some member of this set. By Theorem 6.2, we obtain an injection $\varphi : I \to J$ with $\alpha_i \sqsubseteq \beta'(\varphi i) \ll \beta(\varphi i)$.

$(3) \Rightarrow (2)$ Let $T \subseteq I$. For all i in T, $\beta(\varphi i)$ is in ${\Uparrow}\alpha^+ T$. Thus

$$\begin{aligned} \#T &= \#\{i \in I \mid i \in T\} \\ &\leq \#\{i \in I \mid \beta(\varphi i) \in {\Uparrow}\alpha^+ T\} \\ &\leq \#\{j \in J \mid \beta_j \in {\Uparrow}\alpha^+ T\} \\ &= \beta^*({\Uparrow}\alpha^+ T). \end{aligned}$$

$(2) \Rightarrow (1)$ Let D be a directed subset of $\mathcal{L}^*_{\mathrm{m}}\mathbf{X}$ with $\beta^* \sqsubseteq \bigsqcup D$. Then for all $T \subseteq I$, $\#T \leq \bigsqcup_{\mu \in D} \mu({\Uparrow}\alpha^+ T)$ holds. Since $\#T$ is finite, we find μ_T in D with $\#T \leq \mu_T({\Uparrow}\alpha^+ T)$. Let $\mu \in D$ be a common upper bound of all μ_T. We claim $\alpha^* \sqsubseteq \mu$. For every open set U, let $T = \{i \in I \mid \alpha_i \in U\}$. Then ${\Uparrow}\alpha^+ T \subseteq U$, whence $\alpha^* U = \#T \leq \mu({\Uparrow}\alpha^+ T) \leq \mu U$ follows. □

In the sequel, we need a simple consequence of the theorem.

Proposition 8.2
If $\alpha, \beta : I \to \mathbf{X}$ are bag enumerations with $\alpha_i \ll \beta_i$ for all i in I, then $\alpha^* \ll \beta^*$ holds.

Proof: Identity $\mathsf{id} : I \to I$ is an injection with $\alpha_i \ll \beta(\mathsf{id}\, i)$ for all i in I. □

Now we are ready to prove continuity of $\mathcal{L}^*_{\mathrm{m}}\mathbf{X}$:

Theorem 8.3 Let $\mathbf{X}$ be a (ω-)continuous domain with a basis B. Then $\mathcal{L}^*_{\mathrm{m}}\mathbf{X}$ is (ω-)continuous with basis $B^* = \{\mathsf{s}x_1 + \cdots + \mathsf{s}x_n \in \mathcal{L}^*_{\mathrm{m}}\mathbf{X} \mid n \in \mathbf{N}_0,\ x_i \in B\}$.

Proof: Let μ be a member of $\mathcal{L}^*_{\mathrm{m}}\mathbf{X}$. By Theorem 7.2, μ is a sum of singletons, i.e., $\mu = \sum_{k \in K} \mathsf{s}x_k$, where the index set K may be infinite. Let A be the set of all $\sum_{i \in I} \mathsf{s}\alpha_i$, where I is a finite subset of K, $\alpha_i \in B$, and $\alpha_i \ll x_i$ for all i in I. By definition, A is a subset of B^*. By Prop. 8.2, the condition $\alpha_i \ll x_i$ implies $\sum_{i \in I} \mathsf{s}\alpha_i \ll \sum_{i \in I} \mathsf{s}x_i \sqsubseteq \sum_{k \in K} \mathsf{s}x_k$. Thus all members of A are way-below μ.

The set A is non-empty since it contains the empty sum 0. It is directed: for $\sum_{i \in I} \mathsf{s}\alpha_i$ and $\sum_{j \in J} \mathsf{s}\beta_j$ in A, define $\gamma : I \cup J \to \mathbf{X}$ by $\gamma_i = \alpha_i$ if i in $I \setminus J$, $\gamma_j = \beta_j$ for j in $J \setminus I$, and for k in $I \cap J$, let γ_k be some point with $\alpha_k, \beta_k \sqsubseteq \gamma_k \ll x_k$. Such a point exists by continuity of $\mathbf{X}$. By witness of the inclusion maps $I \to I \cup J$ and $J \to I \cup J$, $\alpha^*, \beta^* \sqsubseteq \gamma^*$ holds.

The join of A is μ: for a fixed index set I, the join of all α^* in A indexed by I is $\sum_{i \in I} \mathsf{s}x_i$ because of continuity of $\mathbf{X}$, of s, and of '$+$'. The join of these evaluations $\sum_{i \in I} \mathsf{s}x_i$ is $\sum_{k \in K} \mathsf{s}x_k$ by the definition of infinite sums. □

Next, we show that for continuous $\mathbf{X}$, the bag domain $\mathcal{L}^*_{\mathrm{m}}\mathbf{X}$ has the universal property of a free lower monoid over $\mathbf{X}$, whence $\mathcal{L}^*_{\mathrm{m}}\mathbf{X} \cong \mathcal{L}^*_{\mathrm{i}}\mathbf{X}$.

Theorem 8.4 For continuous domains $\mathbf{X}$, $\mathcal{L}^*_{\mathrm{m}}\mathbf{X}$ is a free lower monoid over $\mathbf{X}$ in $\mathcal{DCPO}$.

Proof: Let $\mathbf{X}$ be a continuous domain, $\mathbf{Y}$ a lower monoid (whose carrier dcpo need not be continuous), and $f : \mathbf{X} \to \mathbf{Y}$ a continuous function. We have to show that there is a unique continuous additive function $F : \mathcal{L}^*_{\mathrm{m}}\mathbf{X} \to \mathbf{Y}$ with $F \circ \mathsf{s} = f$.

Since the whole of $\mathbf{X}$ is a basis of $\mathbf{X}$, the set $\mathbf{X}^*$ of finite sums of point evaluations is a basis of $\mathcal{L}^*_{\mathrm{m}}\mathbf{X}$ by Theorem 8.3. This immediately implies the uniqueness statement.

For existence, we first extend f to a function $f^* : \mathbf{X}^* \to \mathbf{Y}$ by defining $f^*(\sum_{i \in I} \mathsf{s}x_i) = \sum_{i \in I} f x_i$. We show monotonicity of f^* which also proves that it is well-defined. If $\sum_{i \in I} \mathsf{s}x_i \sqsubseteq \sum_{j \in J} \mathsf{s}y_j$, then there is an injection $\varphi : I \to J$ with $x_i \sqsubseteq y_{\varphi i}$ by Theorem 6.2, whence

$$\sum_{i \in I} f x_i \sqsubseteq \sum_{i \in I} f y_{\varphi i} = \sum_{j \in \varphi^+ I} f y_j \sqsubseteq \sum_{j \in J} f y_j$$

Next, we define $F : \mathcal{L}^*_{\mathrm{m}}\mathbf{X} \to \mathbf{Y}$ by $F\mu = \bigsqcup\{f^* a \mid a \in \mathbf{X}^*,\ a \ll \mu\}$. By monotonicity of f^*, the set is directed and the join exists. For directed sets D in $\mathcal{L}^*_{\mathrm{m}}\mathbf{X}$, $a \ll \bigsqcup D$ holds iff $a \ll d$ for some d in D. Thus $F(\bigsqcup D) = \bigsqcup \bigcup_{d \in D}\{f^* a \mid a \ll d\} = \bigsqcup_{d \in D} Fd$, i.e., F is continuous.

Our next goal is to show that the restriction of F to $\mathbf{X}^*$ equals f^*. Let $a = \sum_{i \in I} \mathsf{s}a_i$ be a member of $\mathbf{X}^*$. By continuity of $\mathbf{X}$, f, and '$+$' in $\mathbf{Y}$, $f^* a = \sum_{i \in I} f a_i = \bigsqcup\{\sum_{i \in I} f b_i \mid b_i \ll a_i\}$ holds. By Prop. 8.2, $\sum_{i \in I} \mathsf{s}b_i \ll a$ holds, whence the join above is below $\bigsqcup\{f^* b \mid b \ll a\} = Fa$. By monotonicity of f^*, this last term is below $f^* a$ again.

The fact that F extends f^* immediately yields $F \circ \mathsf{s} = f^* \circ \mathsf{s} = f$, and additivity of F on elements of $\mathbf{X}^*$. By continuity of F and '$+$' in $\mathcal{L}^*_{\mathrm{m}}\mathbf{X}$ and $\mathbf{Y}$, additivity of F on the whole of $\mathcal{L}^*_{\mathrm{m}}\mathbf{X}$ follows. (More exactly: the two continuous functions $\lambda(a, b).\ F(a + b)$ and $\lambda(a, b).\ Fa + Fb$ agree on the basis $\mathbf{X}^* \times \mathbf{X}^*$ of $\mathcal{L}^*_{\mathrm{m}}\mathbf{X} \times \mathcal{L}^*_{\mathrm{m}}\mathbf{X}$, whence they agree on the whole of $\mathcal{L}^*_{\mathrm{m}}\mathbf{X} \times \mathcal{L}^*_{\mathrm{m}}\mathbf{X}$.) □

For non-sober dcpo's $\mathbf{X}$, $\mathcal{L}^*_{\mathrm{i}}\mathbf{X}$ differs from $\mathcal{L}^*_{\mathrm{m}}\mathbf{X}$, since all elements of the free lower monoid $\mathcal{L}^*_{\mathrm{i}}\mathbf{X}$ can be reached by directed joins of finite sums of singletons, which is not the case for $\mathcal{L}^*_{\mathrm{m}}\mathbf{X}$. We have no results for dcpo's which are sober but not continuous.

Problem 1 Does $\mathcal{L}^*_{\mathrm{i}}\mathbf{X} \cong \mathcal{L}^*_{\mathrm{m}}\mathbf{X}$ hold for all sober dcpo's $\mathbf{X}$?

Now that we know $\mathcal{L}^*_{\mathrm{i}}\mathbf{X} \cong \mathcal{L}^*_{\mathrm{f}}\mathbf{X} \cong \mathcal{L}^*_{\mathrm{m}}\mathbf{X}$ for continuous domains $\mathbf{X}$, we may use the simpler notation $\mathcal{L}^*\mathbf{X}$ without index.

9 Preservation of Domain Classes

In the previous section, we have seen that the lower bag domain construction $\mathcal{L}^*$ preserves continuity of domains. Now we investigate some more restricted classes of domains: algebraic domains, I-domains, M-domains, bifinite domains, Scott domains, and L-domains.

9.1 Algebraic Domains

An element a of a dcpo is *compact* iff $a \ll a$. The set of compact elements of $\mathbf{X}$ is denoted by $\mathsf{K}\mathbf{X}$. A continuous domain $\mathbf{X}$ is *algebraic*, iff there is a basis $B \subseteq \mathsf{K}\mathbf{X}$ of $\mathbf{X}$, iff $\mathsf{K}\mathbf{X}$ itself is a basis of $\mathbf{X}$.

We can apply our results about continuous domains, and immediately obtain:

Theorem 9.1 If $\mathbf{X}$ is an (ω-)algebraic domain, then so is $\mathcal{L}^*\mathbf{X}$. Its set of compact elements is $(\mathsf{K}\mathbf{X})^* = \{\mathsf{s}x_1 + \cdots + \mathsf{s}x_n \in \mathcal{L}^*\mathbf{X} \mid n \in \mathbf{N}_0,\ x_i \in \mathsf{K}\mathbf{X}\}$.

Proof: Because of Theorem 8.3, we only need to show that the elements of $(\mathsf{K}\mathbf{X})^*$ really are compact. For x_i in $\mathsf{K}\mathbf{X}$, $x_i \ll x_i$ holds, whence by Prop. 8.2, $\sum_{i\in I} \mathsf{s}x_i \ll \sum_{i\in I} \mathsf{s}x_i$ follows.□

Alternatively, we may say that the basis of $\mathcal{L}^*\mathbf{X}$ consists of all finite bags on $\mathsf{K}\mathbf{X}$ ordered according to Theorem 6.2 (4) or (5). Note that in contrast to the classical power domain constructions with idempotence, no different finite bags are identified by the order (Cor. 5.7).

9.2 I-Domains

An algebraic domain $\mathbf{X}$ is an *I-domain* iff for all compact elements a of $\mathbf{X}$, the lower cone $\downarrow a$ is finite. For this to hold, it suffices that $\downarrow a \cap \mathsf{K}\mathbf{X}$ is finite.

Proposition 9.2 If $\mathbf{X}$ is an I-domain, then so is $\mathcal{L}^*\mathbf{X}$.

Proof: Let $\alpha : I \to \mathsf{K}\mathbf{X}$ be a finite bag enumeration. By Theorem 6.2 (6), every compact bag $B \sqsubseteq \alpha^*$ can be enumerated by some $\beta : J \to \mathsf{K}\mathbf{X}$, where $J \subseteq I$ and $\beta_j \sqsubseteq \alpha_j$ for all j in J. By finiteness of I and all sets $\downarrow \alpha_j$, the number of such enumerations is finite. □

9.3 M-Domains

An algebraic domain $\mathbf{X}$ is an *M-domain* (or more elaborated: a domain, whose basis has property M) if for all finite subsets F of $\mathsf{K}\mathbf{X}$, the set $\mathsf{mub}\, F$ of minimal upper bounds of F is finite and complete in the sense that every upper bound of F is above some m in $\mathsf{mub}\, F$. Equivalently, $\mathbf{X} = \uparrow E$ with finite E, and for all a, b in $\mathsf{K}\mathbf{X}$, $\uparrow a \cap \uparrow b$ equals $\uparrow E$ for some finite E. (These are the cases $F = \emptyset$ and $F = \{a,\ b\}$, respectively.)

Proposition 9.3 If $\mathbf{X}$ is an M-domain, then so is $\mathcal{L}^*\mathbf{X}$.

Proof: Since 0 is the least element of $\mathcal{L}^*\mathbf{X}$, we have $\mathcal{L}^*\mathbf{X} = \uparrow\{0\}$. Thus, we only have to consider the binary case. Let A and B be two finite bags of compact elements of $\mathbf{X}$, let $n = \#A$ and $m = \#B$, and let $\mathcal{U} = \uparrow A \cap \uparrow B \cap \mathsf{K}(\mathcal{L}^*\mathbf{X})$ be the set of common upper bounds of A and B relative to the basis of $\mathcal{L}^*\mathbf{X}$. Then we consider the set $\mathcal{C}$ of bags in $\mathcal{U}$, whose size is at most $n + m$, and whose elements are taken from A, from B, or from $\mathsf{mub}\,\{a,\ b\}$ for some a in A and b in B. This set is finite since it consists of bags of bounded size whose elements are drawn from a finite set. By definition, $\mathcal{C} \subseteq \mathcal{U}$ holds, and we are done once $\mathcal{U} \subseteq \uparrow\mathcal{C}$ is proven. Note that we do not claim that $\mathcal{C}$ *is* $\mathrm{mub}\,\{A,\,B\}$; in general, it is only a superset thereof, which is sufficient for our proof.

Let C be in $\mathcal{U}$, i.e., $C \sqsupseteq A, B$, and let $\gamma : K \to \mathsf{K}\mathbf{X}$ be an enumeration of C. By Theorem 6.2 (6), there are enumerations $\alpha : I \to \mathsf{K}\mathbf{X}$ of A and $\beta : J \to \mathsf{K}\mathbf{X}$ of B with $I, J \subseteq K$ and $\alpha_i \sqsubseteq \gamma_i$ for all i in I and $\beta_j \sqsubseteq \gamma_j$ for all j in J. We construct an enumeration $\delta : I \cup J \to \mathsf{K}\mathbf{X}$ by setting $\delta_i = \alpha_i$ for i in $I \setminus J$ and $\delta_j = \beta_j$ for j in $J \setminus I$. For k in $I \cap J$, $\alpha_k, \beta_k \sqsubseteq \gamma_k$ holds, whence there is some δ_k in $\mathsf{mub}\,\{\alpha_k,\,\beta_k\}$ below γ_k. Summarizing, $\alpha_i \sqsubseteq \delta_i \sqsubseteq \gamma_i$ holds for all i in I, and $\beta_j \sqsubseteq \delta_j \sqsubseteq \gamma_j$ for all j in J, whence $A, B \sqsubseteq \delta^* \sqsubseteq C$. By construction, δ^* is a member of $\mathcal{C}$. □

9.4 Scott Domains, Bifinite Domains, and L-Domains

A *Scott domain* is a bounded complete algebraic domain. It can be characterized as an M-domain $\mathbf{X}$ where $\mathsf{mub}\,F$ contains at most one element for every finite $F \subseteq \mathsf{K}\mathbf{X}$. A *bifinite domain* is an M-domain where iterated computation of mub terminates after finitely many steps. An *L-domain* is an algebraic domain where every subset $\downarrow x$ forms a complete lattice. Every Scott domain is bifinite as well as an L-domain. Every bifinite domain is an M-domain, whereas L-domains may not be M-domains.

In contrast to the positive results of the previous subsections, the three classes considered here are not preserved under $\mathcal{L}^*$. This is particularly bad since these three classes are cartesian closed categories of algebraic domains, whereas the classes of the previous subsections are not. Non-preservation of these classes can be shown by means of examples which contain a Λ like pattern of compact elements.

Proposition 9.4 Let $\mathbf{X}$ be an M-domain which contains three compact elements a, b, and c, where a and b are incomparable, and $c = a \sqcup b$ holds. Then $\mathcal{L}^*\mathbf{X}$ is neither bifinite nor an L-domain.

The simplest Scott domain which satisfies the hypothesis of Prop. 9.4 is $\mathbf{X} = \{\perp,\,a,\,b,\,c\}$, where c is the largest element, and a and b are incomparable. It is even a finite complete distributive lattice. The least element $\perp$ is not required for our purposes; the dcpo $\{a,\,b,\,c\}$ would do the job as well.

Let $\mathbf{X}$ be a domain satisfying the hypothesis of Prop. 9.4. By Prop. 9.3, $\mathcal{L}^*\mathbf{X}$ is an M-domain. We shall exhibit the structure of a certain part of $\mathcal{L}^*\mathbf{X}$ by presenting the minimal upper bounds of some finite sets. Since we do not know how to compute minimal upper bounds, we have guessed them and verified our guesses using the following lemma, which can be easily proved.

Lemma 9.5 Let M and U be two subsets of a poset such that the elements of M are pairwise incomparable and $M \subseteq U \subseteq \uparrow M$ holds. Then M is the set of minimal elements of U. □

To simplify our notation in the following computation, we shall suppress the singleton function s and use natural coefficients, thus abbreviating e.g., $\mathsf{s}a + \mathsf{s}c + \mathsf{s}c$ by $a + 2c$. We prove two statements about minimal upper bounds.

(1) $\quad \mathsf{mub}\,\{a + kc,\; b + kc\} \;=\; \{a + b + kc,\; (k+1)\,c\}$

Proof: Always remember $a \sqsubseteq c \sqsupseteq b$. Let U be the set of common upper bounds of $a + kc$ and $b + kc$, and let M be $\{a + b + kc, (k+1)\,c\}$. We apply Lemma 9.5. First, we show that the two elements of M are incomparable by considering them as functions from $\mathbf{\Omega X}$ to $\mathbf{N}_0^\infty$ and applying them to certain open sets.

$$(a+b+kc)(\uparrow\{a,\,b\}) = k+2 \qquad (a+b+kc)(\uparrow c) = k$$
$$((k+1)\,c)(\uparrow\{a,\,b\}) = k+1 \qquad ((k+1)\,c)(\uparrow c) = k+1$$

Next, we show $M \subseteq U$. The relations $a+kc \sqsubseteq a+b+kc$ and $b+kc \sqsubseteq a+b+kc$ hold by the lower axiom, and the relations $a+kc \sqsubseteq c+kc$ and $b+kc \sqsubseteq c+kc$ hold since $a, b \sqsubseteq c$.

Finally, we have to check $U \subseteq \uparrow M$. Let μ be a member of U, considered as a modular function $\mu : \mathbf{\Omega X} \overset{mod}{\to} \mathbf{N}_0^\infty$. From $\mu \sqsupseteq a+kc$ and $\mu \sqsupseteq b+kc$, we may conclude

$$\mu(\uparrow c) \geq k \qquad \mu(\uparrow a) \geq k+1$$
$$\mu(\uparrow\{a,\,b\}) \geq k+1 \qquad \mu(\uparrow b) \geq k+1$$

If even $\mu(\uparrow c) \geq k+1$ holds, then $\mu \sqsupseteq (k+1)\,c$ follows. Otherwise, we have $\mu(\uparrow c) = k$. Since $c = a \sqcup b$, $\uparrow a \cap \uparrow b = \uparrow c$ holds. By modularity,

$$\mu(\uparrow\{a,\,b\}) = \mu(\uparrow a) + \mu(\uparrow b) - \mu(\uparrow c) \geq (k+1)+(k+1)-k = k+2,$$

whence $\mu \sqsupseteq a+b+kc$. □

(2) $\quad \mathsf{mub}\,\{a+b+kc,\ (k+1)\,c\} \ = \ \{a+(k+1)\,c,\ b+(k+1)\,c\}$

Proof: The proof follows the same lines as the proof of (1). □

By combining (1) and (2), we see $\mathsf{mub}\,(\mathsf{mub}\,\{a+kc,\ b+kc\}) = \{a+(k+1)\,c,\ b+(k+1)\,c\}$, so that there is an infinite ladder of mub -formation, starting from $\{a,\,b\}$. This ladder shows that $\mathcal{L}^*\mathbf{X}$ is neither bifinite nor an L-domain.

Prop. 9.4 has the unfortunate consequence that there is no cartesian closed category of algebraic domains which would be closed under $\mathcal{L}^*$ as well: Starting with the one-point domain $\mathbf{1}$, we may form $\mathbf{X} = \mathcal{L}^*\mathbf{1} \times \mathcal{L}^*\mathbf{1} \cong \mathbf{N}_0^\infty \times \mathbf{N}_0^\infty$. The domain $\mathbf{X}$ satisfies the hypothesis of Prop. 9.4 with $a = (1,\,0)$, $b = (0,\,1)$, and $c = (1,\,1)$. Thus, $\mathbf{Y} = \mathcal{L}^*\mathbf{X}$ is neither bifinite nor an L-domain. By Cor. 3.7 in [10], the function space $[\mathbf{Y} \to \mathbf{Y}]$ is not algebraic.

9.5 Any Alternatives?

Maybe there is another lower bag domain construction which preserves bifiniteness or property L. To investigate this, let us assume another power domain construction $\widetilde{\mathcal{L}}$ with characteristic semiring $\mathbf{N}_0^\infty$. Let $\mathbf{X}$ be a domain satisfying the hypothesis of Prop. 9.4. Since $\mathcal{L}^*\mathbf{X} \cong \mathcal{L}_\mathrm{i}^*\mathbf{X}$ where $\mathcal{L}_\mathrm{i}^*$ is the initial $\mathbf{N}_0^\infty$-construction, there is an additive continuous function $F : \mathcal{L}^*\mathbf{X} \to \widetilde{\mathcal{L}}\mathbf{X}$ with $F \circ \mathsf{s}_{\mathcal{L}^*} = \mathsf{s}_{\widetilde{\mathcal{L}}}$. Since $\mathcal{L}^*\mathbf{X} \cong \mathcal{L}_\mathrm{f}^*\mathbf{X}$ where $\mathcal{L}_\mathrm{f}^*$ is the final $\mathbf{N}_0^\infty$-construction, there is an additive continuous function $G : \widetilde{\mathcal{L}}\mathbf{X} \to \mathcal{L}^*\mathbf{X}$ with $G \circ \mathsf{s}_{\widetilde{\mathcal{L}}} = \mathsf{s}_{\mathcal{L}^*}$. By the universal property of $\mathcal{L}^*\mathbf{X}$ as the free lower monoid over $\mathbf{X}$, $G \circ F = \mathsf{id}_{\mathcal{L}^*\mathbf{X}}$ follows. Thus, $\mathcal{L}^*\mathbf{X}$ is a retract of $\widetilde{\mathcal{L}}\mathbf{X}$. Remember that $\mathcal{L}^*\mathbf{X}$ is algebraic by Theorem 9.1. If $\widetilde{\mathcal{L}}\mathbf{X}$ were an L-domain, then so would be its retract $\mathcal{L}^*\mathbf{X}$ by [1, Prop. 4.3]. If $\widetilde{\mathcal{L}}\mathbf{X}$ were bifinite, then so would be its retract $\mathcal{L}^*\mathbf{X}$ by [1, Prop. 4.14], which says that retracts of bifinite domains are FS-domains, and algebraic FS-domains are bifinite.

10 Related Work

We already mentioned Libkin and Wong [11] who investigate the expressibility of various query languages for data bases. To compare informations with respect to their partiality, they introduce an order on bags that is identical to our order (the one characterized in Theorem 6.2).

Apart from defining the order, they do not introduce a bag domain construction as a monad producing monoids, nor do they study the preservation of domain classes.

A different order is used by Broy [2]. Remember that we have $\alpha^* \sqsubseteq \beta^*$ iff there is an injective function $\varphi : \mathsf{dom}\,\alpha \to \mathsf{dom}\,\beta$ with $\alpha \sqsubseteq \beta \circ \varphi$, which means that bags may become bigger by increasing their elements or by acquiring new elements. Broy defines $\alpha^* \sqsubseteq \beta^*$ iff there is a surjective function $\psi : \mathsf{dom}\,\beta \to \mathsf{dom}\,\alpha$ with $\alpha \circ \psi \sqsubseteq \beta$, which means that bags may become bigger by increasing their elements or by duplicating existing elements. Broy's order always is a subset of our order. In general, it is a proper subset, e.g., if $\perp \sqsubseteq \mathsf{T}, \mathsf{F}$, then $\mathsf{sT} \sqsubseteq \mathsf{sT} + \mathsf{sF}$ holds in our order, but not in Broy's. For bags containing at least one occurrence of $\perp$, both orders are equivalent. Whereas we derived our order from general considerations, Broy introduces his order in an ad-hoc manner. Nevertheless, he also obtains a continuous singleton function s, extension functional E, and bag addition `$+$'. Broy excludes the empty bag, probably since it is incomparable to every other bag in his order. If the empty bag were added, then the characteristic semiring of Broy's construction would be $\{0\} \cup \{1 \sqsubset 2 \sqsubset \cdots \sqsubset \infty\}$. Broy uses his construction to specify the denotational semantics of a programming language. He does not give a universal property of his bag domains, nor a description in terms of second order functions. Concerning the preservation of domain classes, Broy's construction suffers from the same problem as ours. This can be shown at the argument domain $\{\perp, a, b, c\}$; all bags occurring in the demonstration have to be augmented by one occurrence of $\perp$.

Quite a different kind of `lower bag domain' is introduced in [17], which is discussed in a more general context in [7]. Vickers's notion of domain is far more general than ours; his bag domains are small categories, which are not just posets. Given an algebraic domain (= dcpo) $\mathbf{X}$, he constructs a category whose objects are the enumerations of finite bags of compact elements of $\mathbf{X}$. The morphisms from a bag enumeration α to a bag enumeration β are all functions $\varphi : \mathsf{dom}\,\alpha \to \mathsf{dom}\,\beta$ with $\alpha \sqsubseteq \beta \circ \varphi$. The bag `domain' is the categorical `ideal completion' of this category. In contrast to our approach, Vickers considers *all* functions between the index sets of the bag enumerations, not just the *injective* functions as we do. Thus, a bag increases in the sense of Vickers if its elements become bigger, new elements appear, *or different occurrences of the same element are identified.* This order does not take into account the multiplicities of elements of a bag. In fact, the poset reflection of Vickers's bag `domain' is not our bag domain, but the lower power domain.

11 Conclusion

We define two lower bag domain constructions: the initial construction $\mathcal{L}_\mathrm{i}^*$ whose bag domains are characterized implicitly as free lower monoids, and the final construction $\mathcal{L}_\mathrm{f}^*$ whose bag domains can be explicitly described as $\mathcal{L}_\mathrm{f}^*\mathbf{X} = [[\mathbf{X} \to \mathbf{N}_0^\infty] \overset{add}{\to} \mathbf{N}_0^\infty]$, or by another isomorphic description as $\mathcal{L}_\mathrm{m}^*\mathbf{X} = [\Omega\mathbf{X} \overset{mod}{\to} \mathbf{N}_0^\infty]$.

We have investigated the connection between concrete bags of elements of $\mathbf{X}$ and the members of the bag domain $\mathcal{L}_\mathrm{m}^*\mathbf{X}$. If $\mathbf{X}$ is sober, then every member of $\mathcal{L}_\mathrm{m}^*\mathbf{X}$ can be represented by some bag. Different finite bags represent different elements of $\mathcal{L}_\mathrm{m}^*\mathbf{X}$; only certain infinite bags are identified. This is in contrast to the classical power domains, where certain mathematically different finite sets are identified to guarantee anti-symmetry of the order. In the lower power domain for instance, two finite sets F and G are identified iff $\downarrow F = \downarrow G$ holds.

For continuous domains $\mathbf{X}$, the two lower bag domains $\mathcal{L}_{\mathrm{i}}^{*}\mathbf{X}$ and $\mathcal{L}_{\mathrm{m}}^{*}\mathbf{X}$ are isomorphic. For non-sober dcpo's $\mathbf{X}$, $\mathcal{L}_{\mathrm{i}}^{*}\mathbf{X}$ and $\mathcal{L}_{\mathrm{m}}^{*}\mathbf{X}$ differ from each other. We have no knowledge about the relationship between $\mathcal{L}_{\mathrm{i}}^{*}\mathbf{X}$ and $\mathcal{L}_{\mathrm{m}}^{*}\mathbf{X}$ in case of sober non-continuous dcpo's $\mathbf{X}$.

For continuous domains $\mathbf{X}$, the lower bag domain $\mathcal{L}^{*}\mathbf{X}$ $(= \mathcal{L}_{\mathrm{i}}^{*}\mathbf{X} = \mathcal{L}_{\mathrm{m}}^{*}\mathbf{X})$ can be characterized likewise as the free lower monoid over $\mathbf{X}$ in $\mathcal{DCPO}$, as dcpo of functions $[[\mathbf{X} \to \mathrm{N}_0^{\infty}] \overset{add}{\to} \mathrm{N}_0^{\infty}]$ or $[\Omega\mathbf{X} \overset{mod}{\to} \mathrm{N}_0^{\infty}]$, or in terms of bags of elements of $\mathbf{X}$. We also showed how to construct a basis of $\mathcal{L}^{*}\mathbf{X}$ from a basis of $\mathbf{X}$.

The construction $\mathcal{L}^{*}$ preserves algebraicity and the properties I and M. Unfortunately, neither L-domains, nor Scott domains, nor bifinite domains are closed under the construction $\mathcal{L}^{*}$. Worse, this cannot be repaired by considering another power domain construction with the same characteristic semiring N_0^{∞}. This does not mean that the construction $\mathcal{L}^{*}$ is useless. As indicated in the previous section, Broy's construction suffers from the same problems. Nevertheless, it is used in [2] to describe the denotational semantics of a programming language.

As a meta-result of our analysis, we see that the strong mathematical properties of the lower power domain construction do not carry over to its non-idempotent analogue. All elements of the lower power domain $\mathcal{L}\mathbf{X}$ of any dcpo $\mathbf{X}$ can be represented by sets; in case of $\mathcal{L}_{\mathrm{m}}^{*}\mathbf{X}$, this is only possible for sober $\mathbf{X}$. Initial and final lower power domains coincide for all argument dcpo's $\mathbf{X}$; for the bag domains, such a coincidence is only known for continuous $\mathbf{X}$, doubtful for sober non-continuous $\mathbf{X}$, and wrong for non-sober $\mathbf{X}$. The lower power domain construction maps every algebraic domain to a Scott domain, and thus preserves the classes of Scott domains, L-domains, and bifinite domains; in contrast, the lower bag domain construction does not preserve any of these classes.

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