On the Relationship between Filter Spaces and Equilogical Spaces

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Abstract

It was already known that the category of $\mathcal{T}_0$ topological spaces is not itself cartesian closed, but can be embedded into the cartesian closed categories FIL of *filter spaces* and EQU of *equilogical spaces* where the latter embeds into the cartesian closed category ASSM of assemblies over algebraic lattices. Here, we first clarify the notion of filter space—there are at least three versions $\mathrm{FIL}^a \supseteq \mathrm{FIL}^b \supseteq \mathrm{FIL}^c$ in the literature. We establish adjunctions between FIL^a and ASSM and between FIL^c and ASSM, and show that FIL^b and FIL^c are equivalent to reflective full subcategories of ASSM. The corresponding categories FIL^b_0 and FIL^c_0 of $\mathcal{T}_0$ spaces are equivalent to full subcategories of EQU.

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1 Introduction

It is well-known that the categories TOP of *topological spaces* and TOP_0 of $\mathcal{T}_0$ topological spaces are not cartesian closed, i.e., do not admit a function space construction such that typed λ-calculus could be interpreted in the category. In December 1996, Dana Scott proposed to remedy the situation by extending TOP_0 to the cartesian closed category EQU of equilogical spaces [7, 2, 3]. There are several equivalent descriptions of EQU; one of them is by *modest sets over algebraic lattices* [2] (see also Section 3.2). *Assemblies* are an immediate generalisation of modest sets; the cartesian closed category ASSM of assemblies over algebraic lattices contains the whole of TOP, not just TOP_0.

Yet another such extension was already known in the seventies: *filter spaces* [6, 5, 1] also form a cartesian closed extension of TOP. In fact, there are three such extensions as each of the three cited texts comes with its own notion of filter space; the resulting three categories are contained in each other: $\mathrm{FIL}^a \supseteq \mathrm{FIL}^b \supseteq$

$\mathrm{FIL}^c \supseteq \mathrm{TOP}$. All three are cartesian closed, but the function spaces in FIL^b are different from those in FIL^a.

Filter spaces are mentioned in [2] as an alternative approach to equilogical spaces, but the authors of this paper seemed not to be aware of any relationship between filter spaces and equilogical spaces. Indeed, there is no obvious relationship between these kinds of spaces: equilogical spaces (or better, modest sets) have strong links with domain theory and are easily cut down to countably based spaces, while this is not immediate for filter spaces. On the other hand, filter spaces are closer to topological reasoning; they are built around a notion of convergence which easily allows the definition of a notion of compactness that smoothly extends the familiar topological notion. In contrast, it is hard to see any sensible kind of convergence in equilogical spaces or modest sets, and nobody has ever defined compact subsets in these spaces.

In view of the situation described above, it is quite surprising that Hyland already found an equivalence between FIL^b and a category of so-called *convergence structures*, which in modern terminology, is a full subcategory of ASSM (well, not quite; Hyland's category consists of certain assemblies over *continuous* lattices, but this does not make an essential difference). The equivalence is mentioned in [5, Section 2] without proof. The reader is referred to another paper for details which, as it seems, was never formally published.

The purpose of the paper at hand is to clarify the situation w.r.t. the various different notions of filter spaces, and to establish a strong link between filter spaces and assemblies / equilogical spaces, thereby elaborating, extending, and generalising Hyland's nearly forgotten results.

In Section 2, we introduce the various notions of filter spaces and study their relationship. Section 3 handles assemblies and modest sets (over algebraic lattices). In Section 4, we establish functors $\mathrm{A_F} : \mathrm{FIL}^a \to \mathrm{ASSM}$ and $\mathrm{F_A} : \mathrm{ASSM} \to \mathrm{FIL}^a$ such that $\mathrm{A_F}$ is left adjoint to $\mathrm{F_A}$. The image of $\mathrm{F_A}$ is FIL^b, and the image of $\mathrm{A_F}$ is (the algebraic version of) Hyland's convergence structures. There is a closely related adjunction between FIL^c and ASSM. Both adjunctions embed FIL^c as a full subcategory into ASSM which can be characterised independently.

The functor $\mathrm{A_F} : \mathrm{FIL}^a \to \mathrm{ASSM}$ yields a modest set if and only if it is applied to a $\mathcal{T}_0$ filter space. Thus, using the equivalence between modest sets and equilogical spaces, it induces a functor from the category FIL^a_0 of $\mathcal{T}_0$ filter spaces to the category EQU of equilogical spaces. The restriction of this functor to FIL^b_0 and FIL^c_0 gives equivalences between these categories of filter spaces and full subcategories of EQU.

Notation. For $f : X \to Y$ and $A \subseteq X$, let $f^+A = \{fa \mid a \in A\}$, and for $B \subseteq Y$, let $f^-B = \{a \in X \mid fa \in B\}$.

2 Filter Spaces

2.1 Filters

A *filter* $\mathcal{F}$ on a set X is a subset of the powerset $\mathcal{P}X$ of X which is closed under finite intersection (in particular contains X) and extension to supersets. Arbitrary intersections and directed unions of filters are filters, so the collection ΦX of all filters on X forms a complete lattice. The bottom element of ΦX is $\{X\}$, while the top element is the *improper filter* $\mathcal{P}X$, the (unique) filter containing $\emptyset$.

For a subset $\mathcal{S}$ of $\mathcal{P}X$, let $[\mathcal{S}]$ be the least filter containing $\mathcal{S}$, i.e., the intersection of all filters $\mathcal{F} \supseteq \mathcal{S}$. For $A \subseteq X$, we abbreviate $[\{A\}] = \{B \subseteq X \mid A \subseteq B\}$ by $[A]$. Note that $[A] \subseteq [B]$ iff $A \supseteq B$ and $[A] \cap [B] = [A \cup B]$. For x, y in X, we further abbreviate $[\{x, y\}]$ by $[x, y]$, and $[\{x\}] = \{B \subseteq X \mid x \in B\}$ by $[x]$. Note that $[x, y]$ equals $[x] \cap [y]$.

The filters $[A]$ for $A \subseteq X$ are the compact elements of ΦX. Since $\mathcal{F} = \bigcup_{A \in \mathcal{F}} [A]$ with directed union, ΦX is an algebraic lattice. As usual, a basis for its Scott topology is given by the sets $B_A = \{\mathcal{F} \in \Phi X \mid [A] \subseteq \mathcal{F}\} = \{\mathcal{F} \in \Phi X \mid A \in \mathcal{F}\}$ where A varies over the subsets of X.

For every function $f : X \to Y$, there is a function $f^* : \Phi X \to \Phi Y$ given by $f^*\mathcal{A} = [\{f^+A \mid A \in \mathcal{A}\}] = \{B \subseteq Y \mid f^-B \in \mathcal{A}\}$. The description via inverse image shows that f^* is $(f^-)^-$, and thus distributes over all set operations. Hence, f^* is Scott continuous. We also have $f^*[A] = [f^+A]$ for $A \subseteq X$, hence $f^*[\emptyset] = [\emptyset]$ and $f^*[x] = [fx]$ for x in X. The assignment $f \mapsto f^*$ is functorial.

2.2 Filter Spaces in the Broadest Sense

There are several notions of filter spaces in the literature. The most liberal one is the following [6]: A *filter space* is a set X together with a relation '$\downarrow$' between ΦX and X such that $[x] \downarrow x$ holds for all x in X, and $\mathcal{A} \downarrow x$ and $\mathcal{A} \subseteq \mathcal{B}$ implies $\mathcal{B} \downarrow x$. A function $f : X \to Y$ between two filter spaces is *continuous* if $\mathcal{A} \downarrow x$ implies $f^*\mathcal{A} \downarrow fx$. The category of filter spaces and continuous functions is called FIL^a. (The letter a distinguishes this particular variant of filter spaces from others introduced below.)

$\mathcal{A} \downarrow x$ is usually read as '$\mathcal{A}$ converges to x', or 'x is a limit of $\mathcal{A}$'. A filter has many different limits in general; in particular, the conditions for filter spaces imply that the improper filter $[\emptyset]$ converges to every x in X. Usually, it is omitted, but it does not cause any harm in the definition of the category because $f^*[\emptyset] = [\emptyset]$, and so $f^*[\emptyset] \downarrow fx$ is guaranteed for any f.

The *product* of a family $(X_i, \downarrow)_{i \in I}$ of filter spaces is $\prod_{i \in I} X_i$ with $\mathcal{A} \downarrow (x_i)_{i \in I}$ iff $\pi_i^*\mathcal{A} \downarrow x_i$ for all i in I, where $\pi_i : \prod_{i \in I} X_i \to X_i$ are the projection functions.

Any subset Y of a filter space X becomes a *subspace* of X by defining $\mathcal{B} \downarrow y$ in Y iff $e^*\mathcal{B} \downarrow ey$ in X, where $e : Y \to X$ is the inclusion map. A special case is the construction of the *equaliser* of continuous $f, g : X \to U$ as the subspace $\{x \in X \mid fx = gx\}$ of X.

For two filter spaces X and Y, the function space $[X \to Y]$ is the set of continuous functions from X to Y with $\mathcal{F} \downarrow f$ iff for all $\mathcal{A} \downarrow x$ in X, $\mathcal{F}(\mathcal{A}) \downarrow fx$ holds in Y. Here, $\mathcal{F}(\mathcal{A})$ is $[\{F(A) \mid F \in \mathcal{F},\, A \in \mathcal{A}\}]$, where $F(A) = \{fa \mid f \in F,\, a \in A\}$. With these constructions, FIL^a becomes a cartesian closed category with limits.

2.3 A Subclass of Filter Spaces

In [5], Hyland defines filter spaces in a more restrictive way than in [6], requiring the additional condition $\mathcal{A} \downarrow x \Rightarrow \mathcal{A} \cap [x] \downarrow x$. We call the resulting category FIL^b. It is closed under subspace and product of FIL^a, but unfortunately, it is not closed under the function space construction of FIL^a (there is a finite counterexample). Thus it is not immediate that FIL^b is cartesian closed at all.

Given an arbitrary $X = (X, \downarrow)$ in FIL^a, there are two ways to obtain an object of FIL^b from X. The convergence relation of X may be extended to $X^b = (X, \downarrow^b)$ where $\mathcal{A} \downarrow^b x$ iff there is $\mathcal{B} \downarrow x$ such that $\mathcal{A} \supseteq \mathcal{B} \cap [x]$. Alternatively, it may be restricted to $X_b = (X, \downarrow_b)$ where $\mathcal{A} \downarrow_b x$ iff $\mathcal{A} \cap [x] \downarrow x$. The following properties are easily verified:

(1) X_b and X^b are well-defined objects of FIL^b.

(2) $\mathcal{A} \downarrow_b x \;\Rightarrow\; \mathcal{A} \downarrow x \;\Rightarrow\; \mathcal{A} \downarrow^b x$.

(3) $X_b = X$ iff $X^b = X$ iff $X \in \mathrm{FIL}^b$.

(4) If $f : X \to Y$ is continuous, then so are $f : X_b \to Y_b$ and $f : X^b \to Y^b$.

(5) The functors $(.)^b$ and $(.)_b$ are left and right adjoint to the inclusion functor $\mathrm{FIL}^b \hookrightarrow \mathrm{FIL}^a$.

If X, Y, and Z are in FIL^b, and $[Y \to Z]$ is the function space of Y and Z in FIL^a, then $\mathrm{Hom}(X \times Y, Z) \cong \mathrm{Hom}(X, [Y \to Z])$ holds. By (5), the latter equals $\mathrm{Hom}(X, [Y \to Z]_b)$. This shows that FIL^b is cartesian closed, but it is not a sub-ccc of FIL^a because the function space is different. (Hyland did not comment on the different filter space notions in [6] and [5], nor on the difference in the function space constructions.)

2.4 Other Subclasses of Filter Spaces

In the textbook of Asperti and Longo [1], yet another filter space notion is used. The authors require that the filters converging to x form a filter again, i.e., they extend the definition of FIL^a objects by the additional requirement that $\mathcal{A} \downarrow x$ and $\mathcal{B} \downarrow x$ implies $\mathcal{A} \cap \mathcal{B} \downarrow x$. Obviously, this condition implies the FIL^b condition. Thus, we get a full subcategory FIL^c of FIL^b. It is closed under subspace and product of FIL^a (and FIL^b), and moreover, it is closed under exponentiation in FIL^a: If Y is in FIL^a and Z in FIL^c, then the FIL^a function space $[Y \to Z]$ is in FIL^c, and thus coincides with the FIL^b function space $[Y \to Z]_b$. Therefore, FIL^c is a sub-ccc of FIL^a and of FIL^b, though FIL^b is not a sub-ccc of FIL^a.

Given a FIL^a object $X = (X, \downarrow)$, its convergence relation may be extended to $X^c = (X, \downarrow^c)$ where $\mathcal{A} \downarrow^c x$ iff there are $\mathcal{B}_1 \downarrow x, \ldots, \mathcal{B}_n \downarrow x$ such that $\mathcal{A} \supseteq$

$\mathcal{B}_1 \cap \cdots \cap \mathcal{B}_n$. The result is a well-defined object of FIL^c. The properties of the construction $X \mapsto X^c$ are analogous to those of $X \mapsto X^b$ listed above. In addition, $(X^b)^c = X^c$ holds, but in general, $(X_b)^c$ is different from X^c.

There are several slightly different $\mathcal{T}_0$ notions for filter spaces. One of the weakest is the following: $(X, \downarrow)$ is $\mathcal{T}_0$ if $[x, y] \downarrow x$ and $[x, y] \downarrow y$ implies $x = y$. It is preserved by subspace, product, and exponentiation. Let FIL_0^a, FIL_0^b, and FIL_0^c be the full subcategories of $\mathcal{T}_0$ filter spaces of the three categories FIL^a, FIL^b, and FIL^c.

2.5 Topological Spaces as Filter Spaces

In a topological space $(X, \mathcal{O})$, a filter $\mathcal{A} \in \Phi X$ converges to x in X if $\mathcal{A}$ contains all opens that contain x. This can be expressed differently: A set $N \subseteq X$ is a *neighbourhood* of a point x of X if there is some open O in $\mathcal{O}$ such that $x \in O \subseteq N$. The collection $\mathcal{N}(x)$ of all neighbourhoods of x is a filter in ΦX, and the above definition of convergence amounts to saying $\mathcal{A} \downarrow_{\mathcal{O}} x$ iff $\mathcal{N}(x) \subseteq \mathcal{A}$. Clearly, this definition makes $(X, \downarrow_{\mathcal{O}})$ to a well-defined object of FIL^c.

A function $f : (X, \mathcal{O}) \to (Y, \mathcal{O}')$ is continuous in the topological sense if and only if $f : (X, \downarrow_{\mathcal{O}}) \to (Y, \downarrow_{\mathcal{O}'})$ is continuous in the filter space sense. Thus, we get a full and faithful functor $\mathrm{F_T} : \mathrm{TOP} \to \mathrm{FIL}^c$, and TOP can be considered as a full subcategory of FIL^c.

By definition, $[x, y] \downarrow_{\mathcal{O}} x$ is equivalent to $\mathcal{N}(x) \subseteq [x, y] = [x] \cap [y]$. Since $\mathcal{N}(x) \subseteq [x]$ holds anyway, this is equivalent to $\mathcal{N}(x) \subseteq [y]$, i.e., x is below y in the specialisation preorder of $(X, \mathcal{O})$. Therefore, $(X, \mathcal{O})$ is a $\mathcal{T}_0$ topological space if and only if $(X, \downarrow_{\mathcal{O}})$ is a $\mathcal{T}_0$ filter space.

2.6 Algebraic Lattices as Filter Spaces

Any algebraic lattice L can be considered as a topological space with its Scott topology, and thus induces a filter space in FIL_0^c. Convergence in this filter space has a nice characterisation:

$$
\begin{aligned}
\mathcal{A} \downarrow x \quad &\Leftrightarrow \quad \mathcal{N}(x) \subseteq \mathcal{A} \quad \Leftrightarrow \quad \text{for all compact } a \leq x\text{: } \uparrow a \in \mathcal{A} \\
&\Leftrightarrow \quad \text{for all compact } a \leq x\text{, there is an } A \text{ in } \mathcal{A} \text{ such that } \uparrow a \supseteq A \\
&\Leftrightarrow \quad \text{for all compact } a \leq x\text{, there is an } A \text{ in } \mathcal{A} \text{ such that } a \leq \textstyle\bigwedge A \\
&\Leftrightarrow \quad \text{for all compact } a \leq x\text{: } a \leq \textstyle\bigvee_{A \in \mathcal{A}} \bigwedge A \quad \Leftrightarrow \quad x \leq \textstyle\bigvee_{A \in \mathcal{A}} \bigwedge A
\end{aligned}
$$

Thus, every filter $\mathcal{A}$ in ΦL has a greatest limit $\lim \mathcal{A} = \bigvee_{A \in \mathcal{A}} \bigwedge A$. We have the special cases $\lim[A] = \bigwedge A$ and $\lim[x] = x$. Since directed joins in ΦL are given by union, it is easy to see that $\lim : \Phi L \to L$ is Scott continuous.

3 Equilogical Spaces

were introduced by Scott as equivalence relations on topological $\mathcal{T}_0$ spaces. We prefer to work with the equivalent category of *modest sets* (on algebraic lattices), and also consider the larger category of *assemblies* (on algebraic lattices).

3.1 Assemblies

An *assembly* (X, L, E) consists of a *carrier set* X, an algebraic lattice L, and a function E from X to the powerset of L such that $E(x)$ is non-empty for all x in X. The elements of $E(x)$ are *realisers* for x. We write $E(X)$ for $\bigcup_{x\in X} E(x)$.

Given two assemblies (X, L, E) and (Y, M, F), a function $f : X \to Y$ is *realisable* if there is a (Scott) continuous function $\varphi : L \to M$ such that $a \in E(x)$ implies $\varphi a \in F(fx)$. Let ASSM be the category of assemblies and realisable functions.

Two assemblies $\mathcal{X}$ and $\mathcal{Y}$ are *equivalent* ($\mathcal{X} \cong \mathcal{Y}$) if they have the same carrier set, and the identity on this set is realisable in both directions. Two equivalent assemblies are isomorphic in the usual category-theoretic sense, but not vice-versa.

In the definition of equivalence, it is not required that the two realisers compose to the identities on the two lattices. Hence, the lattices of two equivalent assemblies may be of radically different size and shape. This is quite different from many familiar categories such as topological spaces where equivalent objects are equal.

The *product* of a family $(X_i, L_i, E_i)_{i\in I}$ of assemblies is $(\prod_{i\in I} X_i, \prod_{i\in I} L_i, E)$ where $E((x_i)_{i\in I}) = \prod_{i\in I} E_i(x_i)$. (Note that choice is needed to establish the non-emptiness of this set.)

Given an assembly $\mathcal{X} = (X, L, E)$, any subset S of X induces a *subspace* $\mathcal{X}|_S = (S, L, E|_S)$ of $\mathcal{X}$. Again, *equalisers* are special subspaces.

The function space of $\mathcal{X} = (X, L, E)$ and $\mathcal{Y} = (Y, M, F)$ is $(Z, [L \to M], G)$ where Z is the set of realisable functions from $\mathcal{X}$ to $\mathcal{Y}$, $[L \to M]$ is the algebraic lattice of Scott continuous functions from L to M, and $G(f)$ is the set of realisers of f. With these constructions, ASSM becomes a cartesian closed category with limits.

3.2 Subclasses of Assemblies

- An assembly (X, L, E) is *modest* (a modest set) if the sets $E(x)$ are pairwise disjoint. The full subcategory of modest sets (on algebraic lattices) is equivalent to Scott's category EQU of equilogical spaces [2].
- (X, L, E) is *join-closed* if all sets $E(x)$ are closed under non-empty joins of L. In this case, each $E(x)$ has a greatest element $e(x)$ (and the product construction can be performed without appealing to the axiom of choice).
- (X, L, E) is *meet-closed* if all sets $E(x)$ are closed under binary meets.
- (X, L, E) is *order-convex* if for $a \le b \le c$, a and c in $E(x)$ implies b in $E(x)$.

All these properties (and their combinations) are preserved by subspace, product, and exponentiation. Hyland [5, Section 2] already defined the category of join-closed order-convex assemblies (on *continuous* lattices) under the name *convergence structures*, without introducing the larger category of all assemblies.

3.3 Topological Spaces as Assemblies

The following generalises the embedding of $\mathcal{T}_0$ topological spaces into modest sets from [2]. For a topological space $(X, \mathcal{O})$, the powerset $\mathcal{PO}$ is an algebraic lattice. Let $\mathcal{S}$ be its Scott topology. By defining $ex = \{O \in \mathcal{O} \mid x \in O\}$, we get a continuous function $e : (X, \mathcal{O}) \to (\mathcal{PO}, \mathcal{S})$ such that $e^- : \mathcal{S} \to \mathcal{O}$ is surjective. We call such functions *pre-embeddings*; with this notion, a (topological) embedding is an injective pre-embedding.

It is well-known that for any embedding $e : X \to Y$, any continuous function from X to a continuous lattice L can be extended to a continuous function from Y to L [4]. It is easy to see that this extension property holds for pre-embeddings as well.

The assembly $\mathrm{A_T}(X, \mathcal{O})$ belonging to $(X, \mathcal{O})$ is defined as $(X, \mathcal{PO}, E)$ where $E(x) = \{ex\}$. The above-mentioned extension property ensures that any continuous $f : X \to Y$ is realisable from $\mathrm{A_T}X$ to $\mathrm{A_T}Y$. This gives a full and faithful functor $\mathrm{A_T} : \mathrm{TOP} \to \mathrm{ASSM}$. The assemblies $\mathrm{A_T}X$ obviously are join-closed, meet-closed, and order-convex. They are modest iff e is injective, i.e., iff X is $\mathcal{T}_0$.

4 An Adjunction between FIL^a and ASSM

4.1 From Filter Spaces to Assemblies

Given a filter space $(X, \downarrow)$, it is an obvious idea to use the algebraic lattice ΦX to define an assembly $(X, \Phi X, E)$. The first idea for E is $E(x) = \{\mathcal{A} \in \Phi X \mid \mathcal{A} \downarrow x\}$. This idea has the drawback that the improper filter $[\emptyset]$ would be in all sets $E(x)$, and so, any function from X to X could be realised by the constant $[\emptyset]$-valued function. (Even if the improper filter is banned, the function that exchanges bottom and top in the Sierpinski space would be realisable.)

The solution is to consider only those filters with limit x that are below $[x]$, i.e., to define $E_\downarrow(x) = \{\mathcal{A} \in \Phi X \mid \mathcal{A} \downarrow x,\ \mathcal{A} \subseteq [x]\}$ [6, 5]. The non-emptiness condition is satisfied since $[x]$ is in any $E_\downarrow(x)$. We denote the resulting assembly by $\mathrm{A_F}X$.

Every continuous $f : (X, \downarrow) \to (Y, \downarrow)$ is a function $f : \mathrm{A_F}X \to \mathrm{A_F}Y$ realised by the Scott continuous $f^* : \Phi X \to \Phi Y$. Hence, $\mathrm{A_F}$ is a faithful functor from FIL^a to ASSM. Later, we shall see that the restriction of $\mathrm{A_F}$ to FIL^b is full, embedding FIL^b as a subcategory into ASSM. Note that the assemblies $\mathrm{A_F}X$ are join-closed and order-convex. Later, we shall prove that all such assemblies are in the image of $\mathrm{A_F}$ (up to equivalence).

If $\mathcal{A}$ is in $E_\downarrow(x) \cap E_\downarrow(y)$, then $\mathcal{A} \downarrow x$, $\mathcal{A} \downarrow y$, and $\mathcal{A} \subseteq [x] \cap [y] = [x, y]$. This implies $[x, y] \downarrow x$ and $[x, y] \downarrow y$. Conversely, if $[x, y]$ converges to x and to y, then it is in $E_\downarrow(x) \cap E_\downarrow(y)$. These arguments show that $\mathrm{A_F}X$ is modest if and only if X is $\mathcal{T}_0$.

Do $\mathrm{F_T} : \mathrm{TOP} \to \mathrm{FIL}^a$ and $\mathrm{A_F} : \mathrm{FIL}^a \to \mathrm{ASSM}$ compose to $\mathrm{A_T} : \mathrm{TOP} \to \mathrm{ASSM}$? The assembly $\mathrm{A_F}(\mathrm{F_T}X)$ uses the lattice ΦX, while $\mathrm{A_T}X$ uses $\mathcal{PO}$, thus equality is not possible in general. Yet $\mathrm{A_F}(\mathrm{F_T}X)$ and $\mathrm{A_T}X$ are equivalent for any X; the equivalence is realised by the two Scott continuous functions $\varphi : \Phi X \to \mathcal{PO}$ with $\varphi(\mathcal{A}) = \mathcal{A} \cap \mathcal{O}$ and $\psi : \mathcal{PO} \to \Phi X$ with $\psi(\mathcal{B}) = [\mathcal{B}]$.

4.2 From Assemblies to Filter Spaces

Given a modest set (X, L, E), a filter space structure on X can be obtained by restricting the filter space structure of L to the subspace $E(X)$, and then forming the quotient filter space modulo the equivalence relation given by the partition $(E(x))_{x \in X}$ of $E(X)$. This quite informal idea can be made precise in a way that also applies to assemblies.

Consider an assembly (X, L, E). For $B \subseteq L$, let $E^-(B) = \{x \in X \mid E(x) \cap B \neq \emptyset\}$ be the set of points of X that are realised by elements of B. Generalising this to filters, let $E^-(\mathcal{B}) = [\{E^-(B) \mid B \in \mathcal{B}\}]$ for filters $\mathcal{B}$ in ΦL. Then define $\mathcal{A} \downarrow x$ in X if there is $\mathcal{B}$ in ΦL converging to some b in $E(x)$ such that $\mathcal{A} \supseteq E^-(\mathcal{B})$.

Since the filter space structure on L is topological, we may replace the filter $\mathcal{B}$ converging to b by the neighbourhood filter $\mathcal{N}(b)$. Thus, $\mathcal{A} \downarrow x$ if there is b in $E(x)$ such that $E^-(\mathcal{N}(b)) \subseteq \mathcal{A}$. The latter inclusion may be replaced by the condition that for all Scott open sets O containing b, $E^-(O)$ is in $\mathcal{A}$. In this case, we call b a *witness* for the convergence $\mathcal{A} \downarrow x$.

Since the set $E(x)$ is non-empty, and b in $E(x)$ implies $E^-(O) \in [x]$ for all O containing b, we have $[x] \downarrow x$. Moreover, since $E^-(O)$ is in $[x]$ anyway, it is in $\mathcal{A}$ if and only if it is in $\mathcal{A} \cap [x]$. Thus, $\mathcal{A} \downarrow x$ implies $\mathcal{A} \cap [x] \downarrow x$, and so, $(X, \downarrow)$ is a well-defined object of FIL^b. We call it $\mathrm{F_A}(X, L, E)$. If $f : \mathcal{X} \to \mathcal{Y}$ is realisable by φ, and $\mathcal{A} \downarrow x$ in $\mathrm{F_A}\mathcal{X}$ is witnessed by a in $E_\mathcal{X}(x)$, then φa in $E_\mathcal{Y}(fx)$ is a witness for $f^*\mathcal{A} \downarrow fx$ in $\mathrm{F_A}\mathcal{Y}$. Thus, $\mathrm{F_A}$ is a functor from ASSM to FIL^b.

4.3 The Effect of the Composite Functor on Filter Spaces

For X in FIL^a, $\mathrm{F_A}(\mathrm{A_F}X)$ is in FIL^b. Thus, we cannot expect $\mathrm{F_A}(\mathrm{A_F}X) = X$ in general. In the following, we show that $\mathrm{F_A}(\mathrm{A_F}X)$ is X_b instead.

The assembly $\mathrm{A_F}(X, \downarrow)$ is $(X, \Phi X, E)$ where $E(x) = \{\mathcal{A} \in \Phi X \mid \mathcal{A} \downarrow x, \mathcal{A} \subseteq [x]\}$. Recall that the sets $B_A = \{\mathcal{B} \in \Phi X \mid A \in \mathcal{B}\}$ form a basis of the Scott topology of ΦX. We need to calculate $E^-(B_A)$.

$$x \in E^-(B_A) \;\Leftrightarrow\; \exists \mathcal{A} \in E(x) : \mathcal{A} \in B_A \;\Leftrightarrow\; [x] \in B_A \;\Leftrightarrow\; A \in [x] \;\Leftrightarrow\; x \in A$$

Using $E^-(B_A) = A$, we obtain for $(X, \downarrow^*) = \mathrm{F_A}(\mathrm{A_F}(X, \downarrow))$:

$$\begin{aligned} \mathcal{A} \downarrow^* x \;&\Leftrightarrow\; \exists \mathcal{B} \in E(x) : \forall A \subseteq X : (\mathcal{B} \in B_A \Rightarrow E^-(B_A) \in \mathcal{A}) \\ &\Leftrightarrow\; \exists \mathcal{B} \in E(x) : \forall A \subseteq X : (A \in \mathcal{B} \Rightarrow A \in \mathcal{A}) \\ &\Leftrightarrow\; \exists \mathcal{B} \in \Phi X : \mathcal{B} \downarrow x, \mathcal{B} \subseteq [x], \mathcal{B} \subseteq \mathcal{A} \;\Leftrightarrow\; \mathcal{A} \cap [x] \downarrow x \end{aligned}$$

This proves $\mathrm{F_A}(\mathrm{A_F}X) = X_b$. Hence, the identity on the set X is continuous from $\mathrm{F_A}(\mathrm{A_F}X)$ to X for any X in FIL^a, and $\mathrm{F_A}(\mathrm{A_F}X) = X$ holds if and only if X is in FIL^b. This implies that $\mathrm{A_F}$ is full when restricted to FIL^b. For, if $f : \mathrm{A_F}X \to \mathrm{A_F}Y$ is realisable, then $f : X = \mathrm{F_A}(\mathrm{A_F}X) \to \mathrm{F_A}(\mathrm{A_F}Y) = Y$ is continuous.

4.4 The Effect of the Composite Functor on Assemblies

Now, we show that for any assembly $\mathcal{X} = (X, L, E)$, the identity on the set X is a realisable function from $\mathcal{X}$ to $\mathrm{A_F}(\mathrm{F_A}\mathcal{X})$.

Let $\mathrm{F_A}\mathcal{X} = (X, \downarrow)$, and $\mathrm{A_F}(\mathrm{F_A}X) = (X, \Phi X, F)$. Define $\varphi : L \to \Phi X$ by $\varphi b = [\{E^{-}(O) \mid O \text{ open} \ni b\}] = \{B \subseteq X \mid B \supseteq E^{-}(O) \text{ for some open } O \ni b\}$. Function φ is Scott continuous. If b is in $E(x)$, then $\varphi b \downarrow x$ is witnessed by b, and as b in O implies x in $E^{-}(O)$, we have $\varphi b \subseteq [x]$. Therefore, b in $E(x)$ implies φb in $F(x)$, i.e., φ realises the identity on X.

Next, we show that the identity from $\mathrm{A_F}(\mathrm{F_A}\mathcal{X})$ to $\mathcal{X}$ is realisable provided that $\mathcal{X}$ is join-closed and order-convex, or equivalently, each $E(x)$ is order-convex and has a greatest element $e(x)$. In this case $E^{-}(O) = e^{-}(O)$ holds for all opens O. Define $\psi : \Phi X \to L$ as the composition of the two Scott continuous functions $e^{*} : \Phi X \to \Phi L$ and $\lim : \Phi L \to L$ (the latter was introduced in Section 2.6). We have to show that $\mathcal{A}$ in $F(x)$ implies $\psi\mathcal{A}$ in $E(x)$. First, $\mathcal{A}$ in $F(x)$ implies $\mathcal{A} \subseteq [x]$, whence $\psi(\mathcal{A}) \leq \psi[x] = \lim[ex] = ex \in E(x)$. Second, $\mathcal{A}$ in $F(x)$ implies $\mathcal{A} \downarrow x$. Let b in $E(x)$ be a witness for this convergence. Then for all opens O containing b, $e^{-}(O)$ is in $\mathcal{A}$, i.e., O is in $e^{*}(\mathcal{A})$. Hence, $e^{*}(\mathcal{A}) \downarrow b$ holds in the filter space structure coming from the Scott topology on L. The characteristic property of lim (see Section 2.6) implies $\psi(\mathcal{A}) = \lim(e^{*}(\mathcal{A})) \geq b \in E(x)$. Order-convexity of $E(x)$ implies $\psi(\mathcal{A}) \in E(x)$.

Summarising, we saw that the set-theoretic identity from $\mathcal{X}$ to $\mathrm{A_F}(\mathrm{F_A}\mathcal{X})$ is realisable for any assembly $\mathcal{X}$, and $\mathcal{X} \cong \mathrm{A_F}(\mathrm{F_A}\mathcal{X})$ if $\mathcal{X}$ is join-closed and order-convex.

4.5 Main Results

Theorem 4.1 The functor $\mathrm{F_A} : \mathrm{ASSM} \to \mathrm{FIL}^a$ is left adjoint to the functor $\mathrm{A_F} : \mathrm{FIL}^a \to \mathrm{ASSM}$. The adjunction cuts down to equivalences between

- FIL^b and the full subcategory of join-closed order-convex assemblies,
- FIL^b_0 and the full subcategory of join-closed order-convex modest sets,
- FIL^c and the full subcategory of join-closed meet-closed order-convex assemblies,
- FIL^c_0 and the full subcategory of join-closed meet-closed order-convex modest sets.

The adjunction and the first two items are consequences of the results of the two previous sections. If X is in FIL^c, then $\mathrm{A_F}X$ clearly is meet-closed. Conversely, if $\mathcal{X}$ is a meet-closed assembly, then $\mathcal{A} \downarrow x$ and $\mathcal{B} \downarrow x$ witnessed by $a, b \in E(x)$, implies $\mathcal{A} \cap \mathcal{B} \downarrow x$ witnessed by $a \wedge b \in E(x)$, and thus, $\mathrm{F_A}\mathcal{X}$ is in FIL^c. This proves the last two items of Theorem 4.1.

If one prefers to work with FIL^c altogether, one may compose $\mathrm{F_A}$ with $X \mapsto X_c$ to a functor $\mathrm{F^c_A} : \mathrm{ASSM} \to \mathrm{FIL}^c$ that is left adjoint to $\mathrm{A_F} : \mathrm{FIL}^c \to \mathrm{ASSM}$. Clearly, $\mathrm{F^c_A}$ induces the same equivalences for FIL^c and FIL^c_0 as $\mathrm{F_A}$ does.

One has to take care of a few points when working with Theorem 4.1. First, $\mathrm{A_F}X$ is modest if and only if X is $\mathcal{T}_0$, but even if $\mathcal{Y}$ is modest, $\mathrm{F_A}\mathcal{Y}$ is not $\mathcal{T}_0$ in general. Second, none of the listed subcategories of assemblies is closed under equivalence. Third, as a right adjoint, $\mathrm{A_F}$ preserves products—but only up to equivalence. For instance, $\mathrm{A_F}(X \times Y)$ uses the lattice $\Phi(X \times Y)$, while $\mathrm{A_F}X \times \mathrm{A_F}Y$ uses the (smaller) lattice $\Phi X \times \Phi Y$ of "rectangular" filters.

Finally, recall that the category of modest sets (over algebraic lattices) is equivalent to Scott's category EQU of equilogical spaces [2]. Thus, we obtain:

Corollary 4.2 The small-complete and cartesian closed categories FIL_0^b and FIL_0^c of $\mathcal{T}_0$ filter spaces are equivalent to certain full subcategories of EQU which are closed under limits and exponentiation.

Historical Comment and Acknowledgement. I first found the adjunction $\mathrm{F_A^c}/\mathrm{A_F}$ between FIL^c and ASSM, without being aware of Hyland's work. When I made this result known to a few interested people, Alex Simpson and Giuseppe Rosolini pointed me to Hyland's paper [5], indicating that it may contain related work. After reading that paper and [6], I found the remaining results.

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