A Duality Theory for Quantitative Semantics

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Abstract. A continuous predicate on a domain, or more generally a topological space, can be concretely described as an open or closed set, or less obviously, as the set of all predicates consistent with it. Generalizing this scenario to quantitative predicates, we obtain under certain well-understood hypotheses an isomorphism between continuous functions on points and supremum preserving functions on open sets, both with values in a fixed lattice. The functions on open sets provide a topological foundation for possibility theories in Artificial Intelligence, revealing formal analogies of quantitative predicates with continuous valuations. Three applications of this duality demonstrate its usefulness: we prove a universal property for the space of quantitative predicates, we characterize its inf-irreducible elements, and we show that bicontinuous lattices and Scott-continuous maps form a cartesian closed category.

1 Introduction

It is well-known that a predicate p on a set X, i.e. a function $p : X \to \{0, 1\}$, can be considered as a subset A of X. It is less well-known yet quite obvious that predicates p on X correspond to disjunctive predicates P on the powerset $\mathcal{P}(X)$, where disjunctive means $P(\bigcup_{i \in I} S_i) = \bigvee_{i \in I} P(S_i)$. The predicate P can be obtained from p by existential quantification: $P(S) = \bigvee_{x \in S} p(x)$, while $p(x)$ can be simply obtained as $P(\{x\})$, or in a more elaborated way as $\bigwedge_{S \ni x} P(S)$.

This simple scenario can be generalized from sets (discrete topological spaces) to arbitrary topological spaces [13, 15, 1]. According to the spirit of topology, only continuous predicates should be considered. Of course, continuity of a function from a topological space X to $\{0, 1\}$ depends on the topology defined for $\{0, 1\}$. Continuous functions from X to the Sierpinski space or two-point lattice $\mathbf{2}$ correspond to open sets of X, i.e. the lattice $[X \to \mathbf{2}]$ of continuous functions from X to $\mathbf{2}$ is isomorphic to the lattice $\mathcal{O}(X)$ of open sets of X. In contrast, continuous functions to the opposite $\mathbf{2}^{\mathrm{op}}$ of the Sierpinski space correspond to closed sets of X. To obtain a lattice isomorphism with the closed sets ordered by inclusion, the functions have to be turned around again: $[X \to \mathbf{2}^{\mathrm{op}}]^{\mathrm{op}}$ is therefore another representation for the lattice of closed sets with the inclusion order. The existentially quantified version of these 'closed' predicates are disjunctive predicates on open sets, i.e. $[X \to \mathbf{2}^{\mathrm{op}}]^{\mathrm{op}} \cong \mathcal{O}(X) \multimap \mathbf{2}$, where the symbol $\multimap$ indicates the restriction to disjunctive or supremum preserving functions. Like above, $P(O) = \bigvee_{x \in O} p(x)$ holds for open sets O, while $p(x) = \bigwedge_{O \ni x} P(O)$. The simpler equation $p(x) = P(\{x\})$ is no longer adequate, as singleton sets are not open in most cases.

In this paper, we generalize the above scenario from conventional qualitative predicates with values in $\{0, 1\}$ to *quantitative predicates* with values in more general complete lattices L. In the center of interest is the situation where L is the unit interval $[0, 1]$ or the full domain $[0, \infty]$ of non-negative reals, and X is some 'domain' modeling incremental gain of information. This setting is quite general and encompasses a variety of actual semantic meaning functions which occur e.g. in probabilistic semantics [5, 8–10].

After introducing some notions from domain theory and topology in Section 2, we start out with the most general situation, where X is an arbitrary space and L an arbitrary complete lattice, and compare the two posets $\mathsf{F}_L(X) = [X \to L^{\mathrm{op}}]^{\mathrm{op}}$ and $\mathsf{P}_L(X) = \mathcal{O}(X) \multimap L$. In full generality, these posets are not necessarily isomorphic. To obtain an isomorphism, the space X or the lattice L have to satisfy certain constraints. In Section 3.3, we prove the isomorphism for co-continuous lattices L and arbitrary spaces X. This covers the important cases $L = \mathbf{2}$, $[0, 1]$, and $[0, \infty]$. In Section 3.4, the isomorphism is shown for continuous domains X and arbitrary complete lattices L. In Section 3.5, we shall see that these restrictions are necessary. In the short section 4, the functions in $\mathsf{P}_L(X)$ are extended from open sets of X to arbitrary subsets of X. This result is used in Section 7. The object constructors F and P are extended to functors in both arguments in Section 5.

In Section 6, step functions are introduced that are particularly simple elements of $\mathsf{P}_L(X)$. These step functions are used in Section 7 to characterize the sup-irreducible and sup-prime elements of $\mathsf{P}_L(X)$. Via the isomorphism of $[X \to L]$ with the order-dual of $\mathsf{P}_{L^{\mathrm{op}}}(X)$, this provides information about the inf-irreducible elements of the function space $[X \to L]$.

In the final sections 8 and 9, we concentrate on the situation where X and L are continuous. In this case, finite disjunctions of step functions form a basis of $\mathsf{P}_L(X)$, making $\mathsf{P}_L(X)$ continuous again. As a nice consequence, it follows that bicontinuous lattices and Scott-continuous maps form a cartesian closed category. The knowledge of the basis of $\mathsf{P}_L(X)$ allows the proof of a universal property for this construction in Section 9. Finally, we mention some related work in Section 10.

Note that the duality presented in this paper is based on the equivalence of two representations of complete lattices. This is in contrast to, say, the usual Stone duality, where objects can be seen as either spatial frames (certain complete lattices) or sober spaces (certain topological spaces). Thus, even for $L = \mathbf{2}$ the objects $\mathsf{F}_L(X)$ and $\mathsf{P}_L(X)$ are both frames.

Disjunctive functions from opens sets of X to complete lattices L are known as *possibility measures* in Artificial Intelligence, where they are studied under the following assumptions: L is the unit interval, and X is a finite set in its discrete topology. Thus, our work generalizes this framework considerably. However, we would like to stress that we are *not* suggesting to replace established probabilistic reasoning based on probability measures; yet no matter what one might think of possibility measures, they simply show up if one studies continuous functions into continuous lattices such as $[0, 1]$ or $[0, \infty]$.

2 Dcpos, Lattices, and Topological Spaces

In this section, we introduce basic definitions and facts from topology, domain theory, and lattice theory, as far as they are used in the remainder of the paper. This information is added to make the paper more self-contained. For a more detailed reference see [1]. There is nothing new in here except for the notion of *continuous space* introduced near the end of the section.

2.1 Dcpos and Complete Lattices

A *poset* is a set P with a partial order '$\leq$', i.e. a reflexive, transitive, and anti-symmetric binary relation on P. For an element a of P, $\uparrow a$ denotes the set $\{b \in P \mid a \leq b\}$. A subset A of P is an *upper set* if from $a \in A$ and $a \leq b$, it follows $b \in A$. A function $f : P \to Q$ between two posets is *monotonic* if $a \leq b$ in P implies $fa \leq fb$ in Q.

A subset D of a poset P is *directed* if it is not empty, and for all a, b in D, there is c in D such that $a \leq c$ and $b \leq c$. Ascending sequences $a_1 \leq a_2 \leq \cdots$ are important examples of directed sets.

For every poset P, there is an opposite poset P^{op}, where $a \leq b$ in P^{op} iff $b \leq a$ in P. If $f : P \to Q$ is monotonic, then so is $f : P^{\mathrm{op}} \to Q^{\mathrm{op}}$. A *lower set* of P is defined as an upper set of P^{op}, and a *filtered set* of P as a directed set of P^{op}.

A poset is a *dcpo* (*directed complete partial order*) if every directed set has a least upper bound, and a *complete lattice* if any subset whatsoever has a least upper bound. Least upper bounds are also called *suprema*. We write $\bigvee A$ for the supremum of the set A and $\bigvee_{i \in I} a_i$ for the supremum of an indexed family. In a complete lattice, every subset A has a greatest lower bound or infimum $\bigwedge A$ as well. Thus, the notion of a complete lattice is symmetric: if L is a complete lattice, then so is L^{op}. In contrast, the notion of a dcpo is not symmetric.

Examples for complete lattices are abundant. All powersets are complete lattices, and for every topological space X (see below), the collection $\mathcal{O}(X)$ of open sets forms a complete lattice with the inclusion ordering. Special examples of interest are the two-point lattice $\mathbf{2} = \{0, 1\}$ with $0 < 1$, and the intervals $[0, 1]$ and $[0, \infty]$ with the numeric ordering.

A function $f : X \to Y$ between two dcpos is *Scott-continuous* if for every directed subset D of X, $f(\bigvee D) = \bigvee f(D)$ holds. Every Scott-continuous function is monotonic. The set of all Scott-continuous functions from X to Y is denoted by $[X \to Y]$. When ordered pointwise, i.e. $f \leq g$ iff $fx \leq gx$ for all x in X, this set becomes a dcpo with the pointwise supremum. The category of dcpos and Scott-continuous functions is called DCPO.

A function $f : L \to M$ between two complete lattices is *disjunctive* or a *sup map* if $f(\bigvee A) = \bigvee f(A)$ holds for all $A \subseteq L$. Clearly, sup maps are Scott-continuous. The set of all sup maps from L to M is denoted by $L \multimap M$. When ordered pointwise, it is a complete lattice with pointwise suprema. The category of complete lattices and sup maps is called SUP.

2.2 Adjoints

Let P and Q be two posets. A pair of monotonic functions $f : P \to Q$ and $g : Q \to P$ is an *adjoint pair* or *Galois connection* if $fa \leq b \Leftrightarrow a \leq gb$ for a in P and b in Q. An equivalent characterization is $g \circ f \geq \mathrm{id}_P$ and $f \circ g \leq \mathrm{id}_Q$. Here, $g \circ f$ is the function $x \mapsto g\,(f\,x)$, etc. In an adjoint pair (f, g), function f is called *left* or *lower adjoint* of g, and g *right* or *upper adjoint* of f.

If (f, g) and (f', g') are adjoint pairs between the same two posets, then $f \leq f'$ holds iff $g \geq g'$. Consequently, $f = f'$ holds iff $g = g'$, i.e. adjoints determine each other uniquely.

All lower adjoints are sup maps (preserve suprema), and all upper adjoints are inf maps (preserve infima). If P and Q are complete lattices, the converse holds: every sup map $f : P \to Q$ has an upper adjoint $g : Q \to P$, and every inf map $g : Q \to P$ has a lower adjoint $f : P \to Q$. Since infima are suprema in the opposite ordering, this implies that the complete lattices $P \multimap Q$ and $Q^{op} \multimap P^{op}$ are isomorphic.

2.3 Topological Spaces and Sobriety

A *topological space* is a set X together with a set $\mathcal{O}(X)$ of subsets of X that is closed under arbitrary unions and finite intersections. This set is called the *topology* of X, and its elements are called *open sets*. Since unions of open sets are open, every subset A of a topological space has a largest open subset, called its *interior* A°. The complements of the open sets are known as *closed sets*. A function $f : X \to Y$ between two topological spaces is *continuous* if $V \in \mathcal{O}(Y)$ implies $f^{-1}(V) \in \mathcal{O}(X)$, i.e. the inverse image of open sets is open. Topological spaces and continuous functions form a category TOP.

A space is *discrete* if its topology is the full powerset. Any function from a discrete space to any space is continuous. Hence, the category SET of sets and functions can be considered as a full subcategory of TOP if sets are considered as discrete spaces.

For points x of a topological space X, let $\mathcal{F}_x$ be the set of all opens $O \in \mathcal{O}(X)$ containing x (*open neighborhoods* of x). This notion can be used to define a preorder (a reflexive, transitive relation) on X: for x, y in X, $x \leq y$ holds iff $\mathcal{F}_x \subseteq \mathcal{F}_y$. This preorder is called the *specialization preorder* of X. Every continuous function from X to Y is monotonic in the specialization preorders of X and Y. A space is T_0 if the specialization preorder is a partial order, i.e. $\mathcal{F}_x = \mathcal{F}_y$ implies $x = y$. A space is T_1 if the preorder is degenerated to the equality relation. A space is *Hausdorff* if for $x \neq y$, there are $U \in \mathcal{F}_x$ and $V \in \mathcal{F}_y$ such that $U \cap V = \emptyset$, i.e. distinct points have disjoint neighborhoods. All Hausdorff spaces are T_1 spaces, while dcpos and lattices are T_0 but generally not T_1 when considered as spaces (see below).

For any point x, $\mathcal{F}_x$ is a *completely prime filter*: it is an upper set, closed under finite intersections, and inaccessible by unions ($\bigcup_{i \in I} O_i \in \mathcal{F}_x \Rightarrow \exists i \in I\colon O_i \in \mathcal{F}_x$). A space is *sober* if there are no other completely prime filters, or more exactly, if for every completely prime filter $\mathcal{F}$ of opens, there is a unique point x such that $\mathcal{F} = \mathcal{F}_x$. Uniqueness is just the T_0 property, while existence is the important fact. Therefore, sobriety should be understood as a completeness property; there are enough points to generate all completely prime filters of opens. Every Hausdorff space is sober, and every sober space is T_0. Furthermore, every sober space is a dcpo in its specialization preorder, every open set in a sober space is Scott-open (see below), and any continuous function between sober spaces is Scott-continuous w.r.t. the induced dcpos.

2.4 Scott-topology

A subset O of a dcpo X is *Scott-open* if it is an upper set inaccessible by directed suprema (if D is directed with $\bigvee D \in O$, then D meets O: $D \cap O \neq \emptyset$). Arbitrary unions and finite intersections of Scott-open sets are Scott-open again. Hence, the Scott-open sets form a topology, the *Scott-topology* on X. The two-point lattice with the Scott-topology is known as the *Sierpinski space*. The Scott-topology of the unit interval $[0,1]$ is *not* the usual Hausdorff or interval topology; for, the only Scott-open sets are $\emptyset$, $[0,1]$ itself, and the sets $\{s \in [0,1] \mid s > r\}$ for fixed r in $[0,1]$.

The specialization preorder of a dcpo with its Scott-topology is identical with the original order of the dcpo. A function between dcpos is Scott-continuous, i.e. preserves directed suprema, if and only if it is topologically continuous in the two Scott-topologies. In this paper, we shall always consider dcpos, and thus complete lattices as well, as topological spaces with their Scott-topology. In this sense, DCPO is a full subcategory of TOP.

2.5 Continuous Dcpos and Spaces

Next, we introduce *continuity* of dcpos. This notion has nothing to do with continuity of functions. For two elements x and y of a dcpo X, we say that x is *way below* y, in formulae $x \ll y$, if whenever $y \leq \bigvee D$ holds for some directed set D, then $x \leq d$ holds for some d in D. In $[0,1]$ for instance, $x \ll y$ holds iff $x = 0$ or $x < y$. For x in X, let $\Downarrow x = \{y \in X \mid y \ll x\}$ and $\Uparrow x = \{y \in X \mid x \ll y\}$. A dcpo X is *continuous* if for every x in X, $\Downarrow x$ is directed with supremum x. As complete lattices are special dcpos, the same definition applies to them, but can be simplified since $\Downarrow x$ is directed in all complete lattices. All powersets are continuous lattices, and so are the special lattices $\mathbf{2}$, $[0,1]$, and $[0,\infty]$. Topologies

$\mathcal{O}(X)$ are continuous if the underlying space X is locally compact. Continuous dcpos are often called *continuous domains*.

A subset B of a dcpo X is a *basis* if for all x in X, $B \cap \Downarrow x$ is directed with supremum x. Every continuous dcpo X has at least one basis, namely X itself, and the other way round, every dcpo that has a basis is continuous. The defining condition for continuity, $x = \bigvee \Downarrow x$, or for a basis, $x = \bigvee(B \cap \Downarrow x)$, can be used to derive a criterion for the order relation: in a continuous dcpo, $x \leq y$ holds if and only if for all (basis) elements $z \ll x$ implies $z \leq y$. This criterion will be used many times in our paper.

The way-below relation of a continuous dcpo satisfies the *interpolation property*: whenever $x \ll z$, there is a point y such that $x \ll y \ll z$. From this property, it follows that $\Uparrow x$ is Scott-open for all x, and moreover, $\Uparrow x$ is the interior $(\uparrow x)^\circ$ of $\uparrow x$. The special open sets $\Uparrow x$ form a topological basis, i.e. all open sets are unions of sets $\Uparrow x$: for every Scott-open set O, $O = \bigcup_{x \in O} \Uparrow x$ holds.

The remarks in the previous paragraph show how the notion of continuity can be transferred from dcpos to topological spaces. For a topological space X, define $\Uparrow x$ to be $(\uparrow x)^\circ$, and call X *continuous* if $O = \bigcup_{x \in O} \Uparrow x$ for all open sets O. In every continuous space, the sets $\Downarrow x = \{y \mid x \in \Uparrow y\}$ are directed with least upper bound x. For a dcpo with Scott-topology, the original order-theoretic definition of continuity is equivalent to the new topological definition. Moreover, the (order-theoretically) continuous dcpos with their Scott-topologies are exactly the (topologically) continuous sober spaces. All posets are continuous spaces in their Alexandroff topology (open sets are upper sets). This provides examples for continuous spaces that are not continuous domains.

3 The Basic Isomorphism

In this section, we define the lattices $\mathsf{F}_L(X)$ of fuzzy predicates and $\mathsf{P}_L(X)$ of possibility measures. We show that they are isomorphic if L is *co-continuous*, i.e. L^{op} is continuous, or if X is a continuous space. Finally, we show that these conditions are necessary.

3.1 Fuzzy Predicates and Possibility Measures

For a topological space X and a complete lattice L, let $[X \to L]$ be the set of continuous functions from X to L (equipped with the Scott-topology). When ordered pointwise, $[X \to L]$ is always a dcpo, and if X is a dcpo or L a continuous lattice, it is even a complete lattice.

In the special case $L = \mathbf{2} = \{0, 1\}$, $[X \to L]$ is a complete lattice isomorphic to the topology $\mathcal{O}(X)$. For every open O, the characteristic function χ_O that maps elements of O to 1 and non-elements to 0 is in $[X \to \mathbf{2}]$. Conversely, for every continuous $f: X \to \mathbf{2}$, $f^{-1}(\{1\})$ is an open set. The subset ordering of the opens corresponds to the pointwise ordering of $[X \to \mathbf{2}]$.

If we want to capture the closed sets of X in a similar fashion, we have to turn $\mathbf{2}$ around, for the characteristic function of a closed set is a continuous function from X to $\mathbf{2}^{op}$. Of course, $\mathbf{2}$ and $\mathbf{2}^{op}$ are isomorphic, but we prefer to work with $\mathbf{2}^{op}$ explicitly so that 1 means membership and 0 means non-membership as before. Since subset inclusion of closed sets corresponds to the pointwise ordering of functions *with respect to* $\mathbf{2}$, we must turn $[X \to \mathbf{2}^{op}]$ around again: the complete lattice of closed sets of X ordered by inclusion (also known as Hoare powerdomain of X) is isomorphic to the complete lattice $[X \to \mathbf{2}^{op}]^{op}$.

If we want to generalize closed sets to fuzzy closed sets, we must replace $\mathbf{2}$ by the unit interval $[0,1]$ in the formula above, thus allowing for degrees of membership in between 0 (no membership at all) and 1 (full membership). As announced in the introduction, we generalize further and define $\mathsf{F}_L(X) = [X \to L^{op}]^{op}$ for arbitrary spaces X and arbitrary complete lattices L. Note that because of the double reversal of the ordering, $f \le g$ holds in $\mathsf{F}_L(X)$ iff for all x in X, $fx \le gx$ holds in L.

It is folklore knowledge that the lattice of closed sets, or $\mathsf{F}_{\mathbf{2}}(X)$, is isomorphic to $\mathcal{O}(X) \multimap \mathbf{2}$ (for a proof see [4]). This suggests to define $\mathsf{P}_L(X) = \mathcal{O}(X) \multimap L$ for arbitrary spaces X and complete lattices L, and to compare $\mathsf{F}_L(X)$ and $\mathsf{P}_L(X)$. We call the elements of $\mathsf{P}_L(X)$, i.e. sup-maps from the lattice of opens of X to L, *possibility measures* in analogy with a notion from Artificial Intelligence.

Note that the complete lattice $\mathsf{P}_L(X)$ sits inside the complete lattice $[\mathcal{O}(X) \to L]$ such that the inclusion is Scott-continuous. The same is true for the dcpo $\mathcal{V}_X$ of continuous valuations [6] which consists of all strict maps $\mu \in [\mathcal{O}(X) \to L]$ such that μ satisfies the classical modular law $\mu(U \cup V) + \mu(U \cap V) = \mu(U) + \mu(V)$ for all opens U and V. If we interpret addition in this equation in a fuzzy way as supremum, then, noting that μ is monotone, this modified modular law $\mu(U \cup V) \vee \mu(U \cap V) = \mu(U) \vee \mu(V)$ reduces to the equation $\mu(U \cup V) = \mu(U) \vee \mu(V)$. Thus, we can see that $\mu \in \mathcal{V}_X$ and $\nu \in \mathsf{P}_L(X)$ are both strict Scott-continuous maps, but that μ satisfies the classical modular law, whereas ν fulfills the modified modular law above. The presence of suprema in this modified modular law is the reason why $\mathsf{P}_L(X)$, unlike $\mathcal{V}_X$, is amenable to lattice-theoretic techniques.

Convention. For the remainder of this paper X stands for a topological space with lattice of opens $\mathcal{O}(X)$, and L denotes a complete lattice. When dcpos and in particular complete lattices are considered as topological spaces, they are always equipped with the Scott-topology. Further, we talk about order, suprema, infima, lower and upper sets *with respect to* L whenever we consider the range of functions in $\mathsf{P}_L(X)$ *and* $\mathsf{F}_L(X)$, unless stated otherwise.

3.2 An Adjoint Pair

We start out by establishing an adjoint pair between $\mathsf{F}_L(X)$ and $\mathsf{P}_L(X)$ for any X and L. In the next two subsections, the adjoints introduced here will be proved to be isomorphisms if X or L^{op} is continuous.

Theorem 1. *For every topological space X and complete lattice L, the maps $\Psi\colon \mathsf{F}_L(X) \to \mathsf{P}_L(X)$ defined by $\Psi\, f\, O = \bigvee_{x \in O} f(x)$, and $\Phi\colon \mathsf{P}_L(X) \to \mathsf{F}_L(X)$ with*

$\Phi\,\mu\,x = \bigwedge_{O\in\mathcal{F}_x} \mu(O)$ *are well-defined, monotone, and form an adjoint pair, i.e.* $\Psi f \le \mu \Leftrightarrow f \le \Phi\mu$ *for all* f *in* $\mathsf{F}_L(X)$ *and* μ *in* $\mathsf{P}_L(X)$.

Proof. Given $f \in [X \to L^{op}]$, the function $\Psi\, f$ preserves all suprema by the general associativity of suprema. Therefore Ψ is well-defined as a set-theoretic map. To see that Ψ is actually monotone, recall that $f \le g$ in $\mathsf{F}_L(X)$ means $f(x) \le g(x)$ in L for all $x \in X$, which readily entails $\bigvee_{x\in O} f(x) \le \bigvee_{x\in O} g(x)$ for all $O \in \mathcal{O}(X)$, whence $\Psi\, f \le \Psi\, g$ in $\mathsf{P}_L(X)$.

Next, we show that Φ is well-defined. Given $\mu \in \mathsf{P}_L(X)$ we need to show that f, defined as $\Phi\,\mu$, is a continuous map between X and L^{op} in its Scott-topology. To this end let V be a lower set of L inaccessible by filtered infima. We are done if $f^{-1}(V)$ is open in X. But $x \in f^{-1}(V)$ means that $\bigwedge_{O\in\mathcal{F}_x} \mu(O) \in V$. Since $\mathcal{F}_x$ is a filter, i.e. an upper set in $\mathcal{O}(X)$ closed under finite intersections, we see that the infimum is filtered and since V is inaccessible by filtered infima we conclude that there must be some $O \in \mathcal{F}_x$ such that $\mu(O) \in V$. Since $x \in O$ it suffices to show $O \subseteq f^{-1}(V)$. Given any $x' \in O$ we have $O \in \mathcal{F}_{x'}$ so $f(x') = \bigwedge_{O'\in\mathcal{F}_{x'}} \mu(O') \le \mu(O) \in V$ follows. But then $f(x') \in V$ since V is a lower set.

To prove monotonicity of Φ, let $\mu \le \nu$ in $\mathsf{P}_L(X)$. Then $\mu(O) \le \nu(O)$ for all $O \in \mathcal{O}(X)$ implies that the infimum $\bigwedge_{O\in\mathcal{F}_x} \mu(O)$ is less than or equal to $\bigwedge_{O\in\mathcal{F}_x} \nu(O)$, so $\Phi\,\mu \le \Phi\,\nu$ holds in $\mathsf{F}_L(X)$.

For the adjoint property, we show $\Psi\circ\Phi \le \mathrm{id}$ and $\Phi\circ\Psi \ge \mathrm{id}$. First, for any $\mu \in \mathsf{P}_L(X)$ and $O \in \mathcal{O}(X)$, we have $\Psi(\Phi\,\mu)\,(O) = \bigvee_{x\in O} \Phi\,\mu\,x = \bigvee_{x\in O} \bigwedge_{U\in\mathcal{F}_x} \mu\,U \le \mu\,O$ because for all $x \in O$, we obtain $\bigwedge_{U\in\mathcal{F}_x} \mu\,U \le \mu\,O$. Second, for any $f \in \mathsf{F}_L(X)$ and $x \in X$, we have $\Phi(\Psi\, f)\,(x) = \bigwedge_{O\in\mathcal{F}_x} \Psi\, f\, O = \bigwedge_{O\in\mathcal{F}_x} \bigvee_{y\in O} f\,y \ge f\,x$ since for all $O \in \mathcal{F}_x$ we have $\bigvee_{y\in O} f\,y \ge f\,x$. □

3.3 The Isomorphism for Co-Continuous L

Theorem 2. *Let* L^{op} *be continuous, which includes the cases* $L = \mathbf{2}$, $[0,1]$, *and* $[0,\infty]$. *Then for all spaces* X, *the maps of Theorem 1 are order-inverses of each other. In particular, the partial order* $\mathsf{F}_L(X)$ *is a complete lattice.*

Proof. We only have to show the two inequalities not yet covered by Theorem 1, namely $\Psi\circ\Phi \ge \mathrm{id}$ and $\Phi\circ\Psi \le \mathrm{id}$.

First, let μ be any element of $\mathsf{P}_L(X)$ and let ν be the map $\Psi(\Phi\,\mu)$. For any $O \in \mathcal{O}(X)$ we have $\nu\,O = \bigvee_{x\in O} \Phi\,\mu\,x = \bigvee_{x\in O} \bigwedge_{U\in\mathcal{F}_x} \mu\,U$. To establish $\nu\,O \ge \mu\,O$, we use the fact that the infimum $\bigwedge_{U\in\mathcal{F}_x} \mu\,U$ is filtered *and* that L^{op} is continuous. We write $l' \prec l$ for l and l' in L if l is way-below l' in the lattice L^{op}. Since L^{op} is continuous it suffices to prove that $\nu\,O \prec l$ implies $\mu\,O \le l$. Given $\nu\,O \prec l$ we have in particular that $\bigwedge_{U\in\mathcal{F}_x} \mu\,U \prec l$ for all $x \in O$. Since the latter infimum is filtered we know that there is some $U_x \in \mathcal{F}_x$ such that $\mu\,U_x \le l$. But $x \in U_x$ entails $O \subseteq \bigcup_{x\in O} U_x$. Using that μ is monotone and preserves suprema we conclude $\mu\,O \le \mu(\bigcup_{x\in O} U_x) \le \bigvee_{x\in O} \mu(U_x) \le l$.

Second, for $f \in \mathsf{F}_L(X)$ we define g to be $\Phi(\Psi\, f)$. Then for any $x \in X$ we have $g\,x = \bigwedge_{O\in\mathcal{F}_x} \Psi\, f\, O = \bigwedge_{O\in\mathcal{F}_x} \bigvee_{y\in O} f\,y$. Again, we use the continuity of L^{op} to show $g\,x \le f\,x$. So let $f\,x \prec l$ be given. We are done if $g\,x \le l$. The set

$V = \{a \in L \mid a \prec l\}$ is Scott-open in L^{op} as L^{op} is continuous, so $f^{-1}(V)$ is open in X since $f\colon X \to L^{op}$ is continuous. For all $y \in f^{-1}(V)$ we have $f\,y \in V$ and in particular $f\,y \le l$. Thus, $\bigvee_{y \in f^{-1}(V)} f\,y \le l$. But since $f\,x \prec l$ we have $x \in f^{-1}(V)$, i.e. $f^{-1}(V) \in \mathcal{F}_x$ and therefore $g\,x \le \bigvee_{y \in f^{-1}(V)} f\,y$, where the latter is below l. □

3.4 The Isomorphism for Continuous X

In this section we assume that X is a continuous space and show the isomorphism above for *all* complete lattices L.

Theorem 3. *Let X be a continuous space. Then for all complete lattices L, the maps of Theorem 1 are order-inverses of each other. In particular, the partial order $\mathsf{F}_L(X)$ is a complete lattice.*

Proof. First, let μ and ν be as in the previous proof. To show $\mu \le \nu$, it suffices to show $\mu(\Uparrow x) \le \nu(\Uparrow x)$ for all $x \in X$ since μ and ν preserve suprema and $O = \bigcup_{x \in O} \Uparrow x$ for all opens O by continuity. In particular, we have $\Uparrow x = \bigcup_{y \in \Uparrow x} \Uparrow y$, so it suffices to show $\mu(\Uparrow y) \le \nu(\Uparrow x)$ for all $y \in \Uparrow x$ since μ preserves suprema. Since $\nu(\Uparrow x) = \bigcup_{y \in \Uparrow x} \Phi\,\mu\,y$, it is enough to prove $\mu(\Uparrow y) \le \Phi\,\mu\,y$ for all y. Now, $\Phi\,\mu\,y$ is $\bigcap_{U \in \mathcal{F}_y} \mu\,U$, and because $U \in \mathcal{F}_y$ means $y \in U$ and thus implies $\Uparrow y \subseteq U$, the desired inequality $\mu(\Uparrow y) \le \Phi\,\mu\,y$ follows from monotonicity of μ.

Second, let f and g be as in the previous proof. To show $g\,x \le f\,x$ in L or $f\,x \le g\,x$ in L^{op}, we use the fact that the ordering of L^{op} is the specialization preorder of L^{op} in its Scott-topology. Thus, let $fx \in V$ for some Scott-open set V of L^{op}. Then $x \in f^{-1}(V)$ and this set is open in X by continuity of f. By continuity of X, there is $y \in f^{-1}(V)$ such that x is in $\Uparrow y$. For all $z \in \Uparrow y$, we have $z \ge y$, and since $f\colon X \to L$ is *antitone*, $f\,z \le f\,y$ follows. Hence, $g\,x = \bigcap_{O \in \mathcal{F}_x} \bigcup_{z \in O} f\,z \le \bigcup_{z \in \Uparrow y} f\,z \le f\,y$ holds. Since $f\,y$ is in V, and V is a lower set of L, it follows $g\,x \in V$ as required. □

3.5 The Necessity of Continuity

It is not accidental that the isomorphism between fuzzy predicates $\mathsf{F}_L(X)$ and possibility measures $\mathsf{P}_L(X)$ reflects a certain tradeoff: assuming that the complete lattice L^{op} is continuous, we had no restrictions on the topological space X; conversely, assuming that X is a continuous space, we had no restrictions on the complete lattice L in securing the isomorphism. In this section we demonstrate that the condition of L^{op} or X being continuous is as general as it could be.

Theorem 4. *For a complete lattice L, the following are equivalent:*

1. *The lattice L^{op} is continuous.*
2. *The lattices $\mathsf{F}_L(X)$ and $\mathsf{P}_L(X)$ are isomorphic via Ψ and Φ for all X.*
3. *We have $\Phi(\Psi\,\mathrm{id}_{L^{op}}) = \mathrm{id}_{L^{op}}$ for the space $X = L^{op}$.*

Proof. We already showed that (1) implies (2), and (2) $\Rightarrow$ (3) is merely a restriction. To show (3) $\Rightarrow$ (1) note that $\Phi(\Psi\, \mathrm{id}_{L^{\mathrm{op}}}) = \mathrm{id}_{L^{\mathrm{op}}}$ says $x = \bigwedge_{O \in \mathcal{F}_x} \bigvee_{y \in O} y$ for all $x \in L^{\mathrm{op}}$. Given any $O \in \mathcal{F}_x$ and a filtered set $(y_i)_{i \in I}$ in L with $\bigwedge_{i \in I} y_i \leq x$ we have $\bigwedge_{i \in I} y_i \in O$ as O is a lower set in L. Since O is Scott-open in L^{op} we obtain some $y_{i_0} \in O$ which implies $y_{i_0} \leq \bigvee_{y \in O} y$. Therefore, all elements $\bigvee_{y \in O} y$ with $O \in \mathcal{F}_x$ are way-below x in L^{op}, and their supremum in L^{op} ($\bigwedge$ in L) is x. Since L^{op} is a complete lattice the set of elements way-below x in L^{op} is directed. $\square$

Thus, we cannot expect to extend this isomorphism to a larger class of complete lattices, unless we make restrictions on the class of topological spaces X.

Theorem 5. *For a topological space X, the following are equivalent:*

1. *The space X is continuous.*
2. *The lattices $\mathsf{F}_L(X)$ and $\mathsf{P}_L(X)$ are isomorphic via Ψ and Φ for all L.*
3. *We have $\Psi(\Phi\, \mathrm{id}_{\mathcal{O}(X)}) = \mathrm{id}_{\mathcal{O}(X)}$ for the complete lattice $L = \mathcal{O}(X)$.*

Proof. As before, it remains to show (3) $\Rightarrow$ (1). Now (3) means $\bigcup_{x \in U} \bigwedge_{V \in \mathcal{F}_x} V = U$ for all $U \in \mathcal{O}(X)$. Since $\bigwedge$ is the infimum in $\mathcal{O}(X)$ we readily see $\bigwedge_{V \in \mathcal{F}_x} V = (\uparrow x)^\circ = \Uparrow x$. Thus, we have $U = \bigcup_{x \in U} \Uparrow x$ and X is a continuous space. $\square$

Example. Here is an example where the isomorphism above *does not* hold. Take X to be the unit interval $[0,1]$ in its Hausdorff/Interval-topology. Let L be the complete lattice of opens of X. Note that the latter is a continuous lattice since X is locally compact. However, the isomorphism cannot hold since X would then be a continuous space, i.e. a continuous domain as X is sober. Therefore it is neither necessary nor sufficient to assume that L be continuous. Our results also imply that the lattice L^{op} is not continuous.

4 Extension of Possibility Measures

The possibility measures μ in $\mathsf{P}_L(X)$ map open sets of X to values in L. There is a simple way to extend them to the full powerset X: mimicking Ψ, we define $\bar{\mu}\, S = \bigvee_{x \in S} (\Phi\, \mu)\, x$ for arbitrary subsets S of X.

Proposition 6. *Let $\mu \in \mathsf{P}_L(X)$. For the map $\bar{\mu}$ defined above we have:*

1. *$\bar{\mu}(\uparrow x) = (\Phi\, \mu)\, x$ for all $x \in X$,*
2. *$\bar{\mu}\colon \mathcal{P}(X) \to L$ preserves suprema,*
3. *and $\bar{\mu}$ extends μ if $\mathsf{P}_L(X) \cong \mathsf{F}_L(X)$ via Φ and Ψ.*

Proof. Part 1 holds since $\Phi\, \mu\colon X \to L$ is antitone, part 2 follows from the general associativity of suprema, and part 3 holds since for open sets O, we have $\bar{\mu}\, O = \Psi(\Phi\, \mu)(O)$ by definition. $\square$

Fixing $\mu \in \mathsf{P}_L(X)$ we can define a family $\mu_S \in \mathsf{P}_L(X)$ indexed over $S \subseteq X$ such that $S \mapsto \mu_S$ from $\mathcal{P}(X)$ to $\mathsf{P}_L(X)$ preserves suprema. The map μ_S sends $O \in \mathcal{O}(X)$ to $\bar{\mu}(O \cap S)$. This insight will be used in Section 7 to characterize the inf-irreducible elements in $[X \to L]$.

5 Functoriality and Naturality

We defined a dcpo of fuzzy predicates $\mathsf{F}_L(X)$ and a complete lattice of possibility measures $\mathsf{P}_L(X)$ for every topological space X and complete lattice L. These definitions can be extended to functors in both arguments.

5.1 Functoriality in X

First, we extend $\mathsf{P}_L(\cdot)$ to a functor. If $f: X \to Y$ is a continuous function between topological spaces X and Y, $\mathsf{P}_L(f): \mathsf{P}_L(X) \to \mathsf{P}_L(Y)$ is defined as $\mathsf{P}_L(f)\,\mu\,O = \mu(f^{-1}(O))$ for all $\mu \in \mathsf{P}_L(X)$ and all $O \in \mathcal{O}(Y)$. This definition corresponds to those used for valuations [7] and in conventional measure theory [3].

Theorem 7. *The map $\mathsf{P}_L(\cdot)$ is well-defined and results in a monotonic functor from TOP to SUP, the category of complete sup-lattices and sup-maps. When restricted to sober spaces, it is even locally continuous, i.e. Scott-continuous in its action on morphisms.*

Proof. Clearly, $\mathsf{P}_L(X)$ is a complete lattice, where the supremum operation is the pointwise one. Given a continuous function $f: X \to Y$ and $\mu \in \mathsf{P}_L(X)$ we see that $\mathsf{P}_L(f)\,\mu$ is a well-defined map from $\mathcal{O}(Y)$ to L. It preserves suprema since μ and f^{-1} preserve suprema. Thus, $\mathsf{P}_L(\cdot)$ is well-defined and clearly functorial.

For monotonicity of $\mathsf{P}_L(\cdot)$, assume the inequality $f \leq g$ in the pointwise ordering of the set $[X \to Y]$ of continuous functions from X to Y. Since the order of Y is the specialization preorder, $f\,x \leq g\,x$ for all x in X implies $f^{-1}(O) \subseteq g^{-1}(O)$ for all open sets O of Y, whence $\mathsf{P}_L(f) \leq \mathsf{P}_L(g)$ follows.

For local continuity, let $(f_i)_{i \in I}$ be a directed family in $[X \to Y]$. Since all open sets O in the sober space Y are Scott-open in the specialization preorder, a routine calculation shows $(\mathsf{P}_L(\bigvee_{i \in I} f_i)\,\mu)\,O = ((\bigvee_{i \in I} \mathsf{P}_L(f_i))\,\mu)\,O$. □

We also want to extend $\mathsf{F}_L(\cdot)$ to a functor from TOP to SUP. A slight difficulty lies in the fact that it is not clear that $\mathsf{F}_L(X)$ is a complete lattice for any X and L. Thus, we introduce, for fixed L, the category TOP-L as the full subcategory of TOP of all topological spaces X such that $\mathsf{F}_L(X)$ is a complete lattice. Given a morphism $f: X \to Y$ in TOP-L, we define $\mathsf{F}_L(f)$ as the lower adjoint of the map $\tilde{f}: \mathsf{F}_L(Y) \to \mathsf{F}_L(X)$ defined by $\tilde{f}(g) = g \circ f$.

Lemma 8. *For any L, the category TOP-L subsumes all dcpos and continuous spaces. If L^{op} is continuous then TOP-L is just TOP. On TOP-L, the map $\mathsf{F}_L(\cdot)$ is well-defined and results in a functor from TOP-L to SUP.*

Proof. If D is a dcpo and L a complete lattice, then $[D \to L]$ is a complete lattice. If X or L^{op} are continuous, $\mathsf{F}_L(X)$ is a complete lattice because it is order-isomorphic to the complete lattice $\mathsf{P}_L(X)$ by Theorems 3 and 2.

Given spaces X and Y in TOP-L and continuous $f: X \to Y$, the function $\hat{f}: [Y \to L^{\mathrm{op}}] \to [X \to L^{\mathrm{op}}]$ maps $g \in [Y \to L^{\mathrm{op}}]$ to $g \circ f$. Since suprema in $[X \to L^{\mathrm{op}}]$ and $[Y \to L^{\mathrm{op}}]$ are pointwise over L^{op} we see that $\hat{f}$ preserves suprema. In other words, $\tilde{f}: \mathsf{F}_L(Y) \to \mathsf{F}_L(X)$ with $\tilde{f}(g) = g \circ f$ is a map preserving infima. But $\mathsf{F}_L(Y)$ is a complete lattice, so $\tilde{f}$ has a lower adjoint $\mathsf{F}_L(f): \mathsf{F}_L(X) \to \mathsf{F}_L(Y)$ which preserves suprema. Since the process of taking lower adjoints is a contravariant functor and since we composed that with the contravariant functor $f \mapsto \hat{f}$ in TOP, we see that $\mathsf{F}_L(\cdot)$ is a well-defined functor from TOP-L into SUP. □

5.2 Naturality of the Isomorphisms in X

We showed that the objects $\mathsf{F}_L(X)$ and $\mathsf{P}_L(X)$ are isomorphic whenever X is a continuous space or L^{op} a continuous lattice. These results extend to morphisms as well. Fixing a complete lattice L, we parametrize the maps Φ and Ψ with X. Given a continuous function $f: X \to Y$ we need to show that $\mathsf{F}_L(f)$ and $\mathsf{P}_L(f)$ correspond via the isomorphisms for X and Y. Thus, we have to show the commuting diagrams $\mathsf{F}_L(f) \circ \Phi_X = \Phi_Y \circ \mathsf{P}_L(f)$ and $\mathsf{P}_L(f) \circ \Psi_X = \Psi_Y \circ \mathsf{F}_L(f)$. Since Φ and Ψ are inverses of each other for X and Y, it suffices to show $\mathsf{F}_L(f) = \Phi_Y \circ \mathsf{P}_L(f) \circ \Psi_X$. Since lower adjoints are unique, we are done if $\Phi_Y \circ \mathsf{P}_L(f) \circ \Psi_X$ is a lower adjoint of $\tilde{f}$, i.e. we need to show $\tilde{f} \circ (\Phi_Y \circ \mathsf{P}_L(f) \circ \Psi_X) \geq \mathrm{id}_{\mathsf{F}_L(X)}$ and $(\Phi_Y \circ \mathsf{P}_L(f) \circ \Psi_X) \circ \tilde{f} \leq \mathrm{id}_{\mathsf{F}_L(Y)}$. Using the fact that Φ_X and Ψ_X are inverses of each other, we may rewrite the first inequality as $\tilde{f} \circ \Phi_Y \circ \mathsf{P}_L(f) \geq \Phi_X$ or equivalently $\tilde{f}(\Phi_Y(\mathsf{P}_L(f)\,\mu)) \geq \Phi_X\,\mu$ for all $\mu \in \mathsf{P}_L(X)$. Recalling the definitions of $\tilde{f}$ and $\mathsf{P}_L(f)$ the latter is equivalent to $\Phi_Y(\mu \circ f^{-1}) \circ f \geq \Phi_X\,\mu$. Since this is an inequality in $\mathsf{F}_L(X)$ this is equivalent to $\Phi_Y(\mu \circ f^{-1})(f\,x) \geq \Phi_X\,\mu\,x$ for all $x \in L$, where '$\geq$' is taken in L. But this always holds since $\Phi_Y(\mu \circ f^{-1})(f\,x) = \bigwedge_{V \in \mathcal{F}_{fx}} \mu(f^{-1}(V)) = \bigwedge\{\mu(f^{-1}(V)) \mid f^{-1}(V) \in \mathcal{F}_x\} \geq \bigwedge_{U \in \mathcal{F}_x} \mu\,U = \Phi_X\,\mu\,x$.

Unwinding the meaning of the second inequality $(\Phi_Y \circ \mathsf{P}_L(f) \circ \Psi_X) \circ \tilde{f} \leq \mathrm{id}_{\mathsf{F}_L(Y)}$ we may similarly rewrite this as $\mathsf{P}_L(f) \circ \Psi_X \circ \tilde{f} \leq \Psi_Y$, which means $\mathsf{P}_L(f)(\Psi_X(\tilde{f}(g))) \leq \Psi_Y\,g$ for all $g \in \mathsf{F}_L(Y)$. Applying the definitions of $\tilde{f}$ and $\mathsf{P}_L(f)$ this rewrites to $\Psi_X(g \circ f) \circ f^{-1} \leq \Psi_Y\,g$, which is an inequality in $\mathsf{P}_L(Y)$. Thus, it is equivalent to $\Psi_X(g \circ f)(f^{-1}(V)) \leq \Psi_X\,g\,V$ for all $V \in \mathcal{O}(Y)$, where '$\leq$' refers to the order in L. The latter inequality is satisfied since $\Psi_X(g \circ f)(f^{-1}(V)) = \bigvee_{x \in f^{-1}(V)} g(fx) = \bigvee\{g(fx) \mid fx \in V\} \leq \bigvee_{y \in V} g\,y = \Psi_Y\,g\,V$.

Theorem 9. *Let L^{op} be continuous. Then the functors $\mathsf{F}_L(\cdot)$ and $\mathsf{P}_L(\cdot)$ are naturally isomorphic via Φ and Ψ. If L' is any complete lattice, then we get a similar equivalence of functors if we restrict $\mathsf{F}_{L'}(\cdot)$ and $\mathsf{P}_{L'}(\cdot)$ to the class of continuous spaces.*

5.3 Functoriality in L

So far we studied $\mathsf{F}_L(\cdot)$ and $\mathsf{P}_L(\cdot)$ for arbitrary, but fixed, complete lattices L. This was motivated by a desire for mathematical generality and by the fact that we want to compare such functors for different choices of L. The most prominent candidates for L are $[0,1]$ and $[0,\infty]$. Exploiting the uniform definitions of $\mathsf{F}_L(\cdot)$ and $\mathsf{P}_L(\cdot)$ in L, such comparisons are now relatively straightforward.

First, consider a map $h: L \to L'$ preserving *suprema* between two complete lattices L and L'. Then we get a functorial action $\mathcal{P}_h: \mathsf{P}_L(X) \to \mathsf{P}_{L'}(X)$ for any topological space X defined by $\mathcal{P}_h\, \mu = h \circ \mu$.

Second, if a map $h: L \to L'$ preserves *filtered infima*, it is just a Scott-continuous map $h: L^{\mathrm{op}} \to L'^{\mathrm{op}}$. So given any continuous map $f: X \to L^{\mathrm{op}}$ we see that the map $\mathcal{F}_h\, f = h \circ f: X \to L'^{\mathrm{op}}$ is continuous as well. Thus, $\mathcal{F}_h$ is a well-defined functorial action.

If a map $h: L \to L'$ preserves suprema *and* filtered infima, we have two well-defined functorial actions $\mathcal{P}_h$ and $\mathcal{F}_h$. One readily sees that they correspond via the isomorphisms Φ and Ψ: we have $\Psi(\mathcal{F}_h\, f)\, O = \bigvee_{x \in O} h(f\, x) = h(\bigvee_{x \in O} f\, x) = \mathcal{P}_h(\Psi\, f\, O)$ since h preserves suprema and similarly $\Phi(\mathcal{P}_h\, \mu) = \mathcal{F}_h(\Phi\, \mu\, x)$ since h preserves filtered infima. Note that this combined condition on h means that $h: L^{\mathrm{op}} \to L'^{\mathrm{op}}$ is Lawson continuous [2, Theorem 1.8].

Embedding $[0,1]$ into $[0,\infty]$. As an example let us take the embedding of $[0,1]$ into $[0,\infty]$. The map $e: [0,1] \to [0,\infty]$ with $e(r) = r$ preserves all suprema and non-empty infima. Thus, it has an upper adjoint $p: [0,\infty] \to [0,1]$ mapping s to $s \wedge 1$. The latter preserves suprema and infima, so both p and e have functorial actions for $\mathcal{F}$ and $\mathcal{P}$ corresponding via Φ and Ψ. Since these actions are monotone and functorial we get $\mathcal{P}_p \circ \mathcal{P}_e = \mathrm{id}$, $\mathcal{P}_e \circ \mathcal{P}_p \le \mathrm{id}$ and likewise for $\mathcal{F}$.

Scalar Multiplication. For a fixed $r \in [0,1]$ we define $l_r: [0,1] \to [0,1]$ as left multiplication: $l_r(s) = r \cdot s$. This function preserves suprema and non-empty infima, so $\mathcal{P}_{l_r}$ is a well-defined function on $\mathsf{P}_{[0,1]}(X)$ for all topological spaces X. Using $\mathcal{P}_{l_r}$ we may define a multiplicative action of $[0,1]$ on $\mathsf{P}_{[0,1]}(X)$ by $r * \mu = \mathcal{P}_{l_r}\, \mu$ which is the possibility measure over $[0,1]$ sending $O \in \mathcal{O}(X)$ to $r \cdot (\mu\, O)$. Since $\mathcal{F}_{l_r}$ and $\mathcal{P}_{l_r}$ correspond via our isomorphisms, we know that the multiplicative action of $[0,1]$ on $\mathsf{F}_{[0,1]}(X)$ defined by $\mathcal{F}_{l_r}$ is also pointwise: $(r * f)\, x = r \cdot (f\, x)$.

However, things become less trivial if we try to figure out what to do with $r = \infty$ in $[0,\infty]$. How should we extend the multiplication in $[0,\infty)$ to $[0,\infty]$? If l_∞ is to preserve suprema, we need $\infty \cdot 0 = 0$. If l_∞ is to preserve filtered infima, then $\infty \cdot 0$ must be ∞. In the theory of cones for *valuations* [6] one chooses $\infty \cdot 0 = 0$, so $\mathcal{P}_{l_\infty}$ is well-defined, whereas $\mathcal{F}_{l_\infty}$ is not, i.e. interpreting $\mathcal{P}_{l_\infty}$ via the isomorphism does not result in pointwise scalar multiplications. For example, let X be the ordinal $\omega + 1$ in its Scott-topology. The function $f: \omega + 1 \to [0,\infty]^{\mathrm{op}}$ with $f(n) = \frac{1}{n}$ and $f(\omega) = 0$ is continuous. Pointwise multiplication with ∞ yields a function $g: \omega + 1 \to [0,\infty]^{\mathrm{op}}$ with $g(n) = \infty$ and $g(\omega) = 0$, so g is *not* continuous. This mismatch does not occur for any $r \in [0,\infty)$, for $s \mapsto r \cdot s: [0,\infty] \to [0,\infty]$ preserves suprema and filtered infima.

6 Point Valuations as Step Functions

In the special cases $L = [0,1]$ or $L = [0,\infty]$, a *point valuation* $\eta_x: \mathcal{O}(X) \to L$ may be defined for every x in X. It maps all $O \in \mathcal{F}_x$ to 1 and all $U \in \mathcal{O}(X) \setminus \mathcal{F}_x$ to 0. For any r in L, one may further define $r * \eta_x: \mathcal{O}(X) \to L$ using pointwise multiplication as considered above. These definitions suggest that a distinguished element 1 and some multiplication on L are instrumental in defining the map $r * \eta_x$. Yet, note that $r * \eta_x$ maps opens in $\mathcal{F}_x$ to r and all other opens to 0, the least element of $[0,1]$ and $[0,\infty]$. Thus, a function like $r * \eta_x$ may be defined for every L without any need for a multiplicative structure on L or a distinguished element 1.

Definition 10. Let X be any topological space and L any complete lattice with least element 0_L. For x in X and r in L, we define a map $\mathcal{F}_x \searrow r\colon \mathcal{O}(X) \to L$ by $(\mathcal{F}_x \searrow r)\, O = r$ if $x \in O$, and $= 0_L$ otherwise.

One readily sees that the maps $\mathcal{F}_x \searrow r$ are possibility measures of type $\mathsf{P}_L(X)$. We will later show that they are the building blocks of a basis for $\mathsf{P}_L(X)$ if X and L are continuous. For that we need to characterize the order between such possibility measures:

Lemma 11. *Let $x, y \in X$ and $r, s \in L$. Then $\mathcal{F}_x \searrow r \leq \mathcal{F}_y \searrow s$ if, and only if, $r \leq s$ and ($r = 0_L$ or $x \leq y$).*

Proof. We concentrate on the 'only if' direction. If $\mathcal{F}_x \searrow r \leq \mathcal{F}_y \searrow s$, then $r = (\mathcal{F}_x \searrow r)\, X \leq (\mathcal{F}_y \searrow s)\, X = s$. Suppose that $r \neq 0_L$. We show $x \leq y$ via $\mathcal{F}_x \subseteq \mathcal{F}_y$. If $U \in \mathcal{F}_x$, then $(\mathcal{F}_x \searrow r)\, U = r \neq 0_L$ implies that $(\mathcal{F}_y \searrow s)\, U \neq 0_L$, as $(\mathcal{F}_y \searrow s)\, U \geq r$. Thus, $U \in \mathcal{F}_y$ follows. □

We conclude this section with a characterization of the possibility measures $\mathcal{F}_x \searrow r$, where $r > 0_L$.

Proposition 12. *Let X be a sober space. For any $\mu \in \mathsf{P}_L(X)$ the following are equivalent:*

1. *The map $\mu\colon \mathcal{O}(X) \to L$ preserves finite infima and its image contains exactly two elements.*
2. *There is some $r > 0_L$ and a (unique) $x \in X$ such that $\mu = \mathcal{F}_x \searrow r$.*

Proof. Assuming item 1 above we know that 0_L is in the image of μ since the sup-map μ has to map $\emptyset$ to 0_L. Let r be the other element in its image (clearly $r > 0_L$ must hold). The set $\mathcal{F} = \{O \in \mathcal{O}(X) \mid \mu\, O = r\}$ is a completely prime filter in $\mathcal{O}(X)$. Since X is sober there exists a unique $x \in X$ such that $\mathcal{F} = \mathcal{F}_x$. Hence $\mu = \mathcal{F}_x \searrow r$. The validity of the converse should be clear. □

It is worth pointing out that neither the preservation of finite infima nor the fact that μ's image has exactly two elements is sufficient for concluding that $\mu = \mathcal{F}_x \searrow r$ for some x and some $r > 0_L$. For example, if we take $X = \mathbf{2} \times \mathbf{2}$ in its Scott-topology and $L = [0,1]$ then $\mathcal{F}_{(1,0)} \searrow 1 \vee \mathcal{F}_{(0,1)} \searrow 1$ has two elements in its image but is not of the form $\mathcal{F}_x \searrow r$; neither is the possibility measure $\mathcal{F}_{(1,0)} \searrow 1 \vee \mathcal{F}_{(1,1)} \searrow 0.5$ which preserves finite infima.

7 Prime and Irreducible Elements

An element p of a lattice L is a *sup-prime* if for all finite subsets F of L, $p \leq \bigvee F$ implies $p \leq x$ for some x in F. In this definition, F may be restricted to the empty case $F = \emptyset$ and the binary case $F = \{x, y\}$. Thus, p is sup-prime if $p \neq 0_L$ and $p \leq x \vee y$ implies $p \leq x$ or $p \leq y$.

The element p is *sup-irreducible* if $p = \bigvee F$ with finite F implies $p \in F$. Again, the empty case and the binary case are sufficient. Every sup-prime element is sup-irreducible. The converse holds in distributive lattices such as powersets $\mathcal{P}(X)$, topologies $\mathcal{O}(X)$, and the intervals $[0, 1]$ and $[0, \infty]$.

The inf-primes of L are defined as the sup-primes of L^{op}, i.e. $p \leq \bigvee F$ is replaced by $p \geq \bigwedge F$ etc. Likewise, the inf-irreducible elements of L are the sup-irreducible elements of L^{op}.

As an application of our duality results, we first characterize the sup-prime and sup-irreducible elements of $\mathsf{P}_L(X)$. Employing our basic isomorphism, we obtain a characterization of the inf-prime and inf-irreducible elements of $[X \to L]$. The proofs here are given for primes but readily adapt to proofs about irreducible elements.

Theorem 13. *Let X be a sober space such that $\mathsf{F}_L(X)$ and $\mathsf{P}_L(X)$ are isomorphic via Φ and Ψ. Then $\mu \in \mathsf{P}_L(X)$ is sup-irreducible (sup-prime) in $\mathsf{P}_L(X)$ if and only if $\mu = \mathcal{F}_x \searrow r$ for some sup-irreducible (sup-prime) $r \in L$ and some $x \in X$.*

Proof. First, let $x \in X$ and consider a sup-prime r in L. Then $r \neq 0_L$ implies $\mathcal{F}_x \searrow r \neq 0_{\mathsf{P}_L(X)}$. So suppose that $\mathcal{F}_x \searrow r \leq \mu \vee \nu$ in $\mathsf{P}_L(X)$. Assume that $\mathcal{F}_x \searrow r \not\leq \mu$ and $\mathcal{F}_x \searrow r \not\leq \nu$. Thus, there must be some $O \in \mathcal{O}(X)$ such that $(\mathcal{F}_x \searrow r)\, O \not\leq \mu\, O$. This implies $(\mathcal{F}_x \searrow r)\, O = r$, and so $x \in O$. Likewise, there must exist some $O' \in \mathcal{O}(X)$ such that $(\mathcal{F}_x \searrow r)\, O' \not\leq \nu\, O'$ and $(\mathcal{F}_x \searrow r)\, O' = r$ follows. Thus, x is contained in the open set $O \cap O'$ and we compute $r = (\mathcal{F}_x \searrow r)\,(O \cap O') \leq (\mu \vee \nu)\,(O \cap O') = \mu(O \cap O') \vee \nu(O \cap O') \leq \mu(O) \vee \nu(O')$ which contradicts $r \not\leq \mu(O)$ and $r \not\leq \nu(O')$, for r is a sup-prime in L.

Second, let π be a sup-prime in $\mathsf{P}_L(X)$. Consider $\mathcal{F}$, defined as the set of all $O \in \mathcal{O}(X)$ with $\pi(O) > 0_L$. We claim that $\mathcal{F}$ is a completely prime filter in $\mathcal{O}(X)$. Since π is monotone we see that $\mathcal{F}$ is an upper set in $\mathcal{O}(X)$.

In Section 4 we saw that π defines a family of possibility measures $\pi_S \in \mathsf{P}_L(X)$, $S \subseteq X$, defined by $\pi_S(O) = \bar{\pi}(O \cap S)$. For U and V in $\mathcal{F}$ we obtain $\pi = \pi_{X \setminus (U \cap V)} \vee \pi_{U \cap V}$. Now, if $U \cap V$ is not in $\mathcal{F}$ we obtain $\pi(U \cap V) = 0_L$ and so $\pi_{U \cap V}(O) = \bar{\pi}(O \cap (U \cap V)) \leq \pi(U \cap V) = 0_L$ for all $O \in \mathcal{O}(X)$ since $\bar{\pi}$ extends π. Therefore π equals $\pi_{X \setminus (U \cap V)}$ and the latter is just $\pi_{X \setminus U} \vee \pi_{X \setminus V}$ since $\bar{\pi}$ preserves suprema. Since π is a sup-prime in $\mathsf{P}_L(X)$ we may assume $\pi = \pi_{X \setminus U}$ without loss of generality. But then $\pi\, U = 0_L$ is immediate and a contradiction. Hence $U \cap V \in \mathcal{F}$.

If $\bigcup \mathcal{O}$ is in $\mathcal{F}$ for some $\mathcal{O} \subseteq \mathcal{O}(X)$ then $\pi(O) = 0_L$ for all $O \in \mathcal{O}$ implies $\pi(\bigcup \mathcal{O}) = \bigvee_{O \in \mathcal{O}} \pi(O) = 0_L$ contradicting $\bigcup \mathcal{O} \in \mathcal{F}$. Thus, the filter $\mathcal{F}$ is completely prime. Since X is sober we know that there exists a unique $x \in X$ such that $\mathcal{F} = \mathcal{F}_x$. Thus, $\pi(U) = 0_L$ for all U not containing x. Yet, for any open

$O \in \mathcal{F}_x$ we have $\pi(O) > 0_L$. As before we may reason that $\pi = \pi_{X\setminus O} \vee \pi_O$. Since π is a sup-prime in $\mathsf{P}_L(X)$ we must have $\pi = \pi_{X\setminus O}$ or $\pi = \pi_O$. The former is impossible since $\pi(O) > 0_L$. Thus, $\pi = \pi_O$ renders $\pi(O) = \pi_O(O) = \pi_O(X) = \pi(X)$. Thus, $\pi = \mathcal{F}_x \searrow \pi(X)$.

To see that $\pi(X)$ is a sup-prime in L, let $\pi(X) \leq l \vee k$ in L. Then $\pi = \mathcal{F}_x \searrow \pi(X) \leq \mathcal{F}_x \searrow l \vee \mathcal{F}_x \searrow k$ follows. Since π is a sup-prime in $\mathsf{P}_L(X)$ we obtain $\pi \leq \mathcal{F}_x \searrow l$ without loss of generality. Thus, $\pi(X) \leq (\mathcal{F}_x \searrow l)\, X = l$. $\square$

Assuming the isomorphism $\mathsf{F}_{L^{\mathrm{op}}}(X) \cong \mathsf{P}_{L^{\mathrm{op}}}(X)$ we therefore get a characterization of inf-irreducible elements and inf-primes in the lattice $[X \to L]$.

Definition 14. Let $x \in X$ and $r \in L$. We define a map $x \nearrow r \colon X \to L$ such that $(x \nearrow r)\, y = r$ if $y \leq x$ in the specialization order, and $(x \nearrow r)\, y = 1_L$ otherwise, where 1_L is the greatest element of L. We call $x \nearrow r$ a *co-step function.*

Corollary 15. *Let X be a sober space such that $\mathsf{F}_{L^{\mathrm{op}}}(X) \cong \mathsf{P}_{L^{\mathrm{op}}}(X)$ via Φ and Ψ. Then an element $f \in [X \to L]$ is inf-irreducible (inf-prime) in the lattice $[X \to L]$ if, and only if, $f = x \nearrow r$ for some $x \in X$ and some r which is inf-irreducible (inf-prime) in L.*

Proof. Since $[X \to L]$ is dually isomorphic to $\mathsf{P}_{L^{\mathrm{op}}}(X)$ we may apply the version of Theorem 13 which characterizes sup-primes in $\mathsf{P}_{L^{\mathrm{op}}}(X)$ as $\mathcal{F}_x \searrow r$ where r is sup-prime in L^{op}. One readily computes that the isomorphisms Φ and Ψ between $\mathsf{F}_{L^{\mathrm{op}}}(X)$ and $\mathsf{P}_{L^{\mathrm{op}}}(X)$ restrict to bijections between co-step functions $x \nearrow r$ and step functions $\mathcal{F}_x \searrow r$. The rest is now clear. $\square$

8 Continuity of $\mathsf{P}_L(X)$

In this section and the next one, we assume that X is a continuous domain and L a continuous lattice. By Theorem 3, this setting ensures the isomorphism between $\mathsf{F}_L(X)$ and $\mathsf{P}_L(X)$. Our aim is to show that the sup-semilattice generated by all step functions $\mathcal{F}_x \searrow r$ is a basis for $\mathsf{P}_L(X)$. Of course, such a result presupposes a detailed understanding of how the way-below relations on X and L establish way-below relations in $\mathsf{P}_L(X)$ via $\mathcal{F}_\cdot \searrow \cdot \colon X \times L \to \mathsf{P}_L(X)$.

Lemma 16. *Let $x, y \in X$, $r, s \in L$ and $\mu \in \mathsf{P}_L(X)$. If $r \ll \mu(\Uparrow x)$ then $\mathcal{F}_x \searrow r \ll \mu$. In particular, if $x \ll y$ and $r \ll s$ then $\mathcal{F}_x \searrow r \ll \mathcal{F}_y \searrow s$.*

Proof. First, let $(\mu_i)_{i \in I}$ be directed in $\mathsf{P}_L(X)$ with $\mu \leq \bigvee_{i \in I} \mu_i$. Since the supremum in $\mathsf{P}_L(X)$ is pointwise, and using $r \ll \mu(\Uparrow x)$ we get some $i_0 \in I$ such that $r \leq \mu_{i_0}(\Uparrow x)$. Now, if $U \in \mathcal{F}_x$ then $\Uparrow x \subseteq U$ and so $r \leq \mu_{i_0} U$. Thus, $\mathcal{F}_x \searrow r \leq \mu_{i_0}$. The second claim follows readily from the first one since $r \ll s = (\mathcal{F}_y \searrow s)\,(\Uparrow x)$. $\square$

Theorem 17. *If X is a continuous domain and L a continuous lattice, the lattice $\mathsf{P}_L(X)$ is continuous with basis elements of the form $\bigvee_{i=1}^{n} \mathcal{F}_{x_i} \searrow r_i$.*

Proof. Since $\mathsf{P}_L(X)$ is a complete lattice, and since $\eta \ll \mu$ and $\eta' \ll \mu$ imply $\eta \vee \eta' \ll \mu$, it suffices to show that every element $\mu \in \mathsf{P}_L(X)$ is the supremum of such elements $\bigvee_{i=1}^{n} \mathcal{F}_{x_i} \searrow r_i$ which are way-below μ. By Lemma 16, it therefore suffices to show $\mu = \mu'$ where $\mu' = \bigvee\{\mathcal{F}_x \searrow r \mid r \ll \mu(\Uparrow x)\}$.

$$\begin{aligned}
\mu'\,U &= \bigvee_{x \in U} \bigvee\{r \mid r \ll \mu(\Uparrow x)\} \\
&= \bigvee_{x \in U} \mu(\Uparrow x) \quad \text{as } L \text{ is continuous} \\
&= \mu(\bigcup_{x \in U} \Uparrow x) \quad \text{as } \mu \in \mathsf{P}_L(X) \\
&= \mu\,U \quad \text{as } X \text{ is continuous.}
\end{aligned}$$

□

Corollary 18. *Let X be a continuous domain and L^{op} a continuous lattice. Then the order-dual of $[X \to L]$ is a continuous lattice.*

Proof. By Theorem 17, the lattice $\mathsf{P}_{L^{\mathrm{op}}}(X)$ is continuous. By Theorem 3, the latter is isomorphic to the order-dual of $[X \to L^{\mathrm{op\,op}}] \cong [X \to L]$. □

A lattice L is *bicontinuous* [2] if both L and L^{op} are continuous. Let BCONT be the category of bicontinuous lattices with Scott-continuous maps as morphisms. Since continuous lattices and Scott-continuous maps form a cartesian closed category [2] and since BCONT has finite products we may conclude that BCONT is cartesian closed as well, provided that $[L \to M]^{\mathrm{op}}$ is a continuous lattice for bicontinuous lattices L and M. Yet this is true by Corollary 18.

Corollary 19. *The category BCONT is cartesian closed.*

Using Theorem 17, one can show that the way-below relation on $\mathsf{P}_L(X)$ is induced by the one on $[\mathcal{O}(X) \to L]$.

Proposition 20. *Let X and L be continuous. Then $\mathsf{P}_L(X)$ is a Scott-continuous projection of $[\mathcal{O}(X) \to L]$. In particular, the way-below relation on $\mathsf{P}_L(X)$ is induced by the way-below relation on $[\mathcal{O}(X) \to L]$.*

Proof. We define $P\colon [\mathcal{O}(X) \to L] \to \mathsf{P}_L(X)$ by $P\,\mu = \bigvee\{\mathcal{F}_x \searrow r \mid r \ll \mu(\Uparrow x)\}$. Clearly, P is a well-defined and monotone map. If $E\colon \mathsf{P}_L(X) \to [\mathcal{O}(X) \to L]$ denotes the canonical embedding, then $E \circ P \leq \mathrm{id}_{[\mathcal{O}(X)\to L]}$ is obvious. By Theorem 17, we also have $P \circ E = \mathrm{id}_{\mathsf{P}_L(X)}$ and, since E is monotone as well, we realize E as the lower adjoint of P. Thus, it is a sup-map. Since the supremum in $[\mathcal{O}(X) \to L]$ is pointwise and since the way-below relation on L satisfies the interpolation property, one can readily calculate that P is Scott-continuous. □

The *way-way-below relation* on a complete lattice is defined analogously to the way-below relation. The difference is that one refers to arbitrary suprema instead of directed suprema. A complete lattice is *completely distributive* if every element is the supremum of all elements way-way-below it [11, 12].

If L is completely distributive then we obtain similarly that $\mathsf{P}_L(X)$ is a sup-projection of $[\mathcal{O}(X) \to L]$ and the way-way-below relation on $\mathsf{P}_L(X)$ is induced by the way-way-below relation on $[\mathcal{O}(X) \to L]$. Let us also point out that for continuous X and completely distributive lattices L our isomorphic functors restrict to completely distributive lattices in their range.

Theorem 21. *Let L be a continuous lattice. The functors $\mathsf{P}_L(\cdot)$ and $\mathsf{F}_L(\cdot)$ restrict to naturally equivalent locally continuous endofunctors of CONT, the category of continuous domains and Scott-continuous maps. If L is completely distributive, then $\mathsf{P}_L(\cdot)$ and $\mathsf{F}_L(\cdot)$ map from CONT into CD, the category of completely distributive lattices and sup-maps.*

Proof. The first part follows from Theorem 17. Local continuity was already stated in Theorem 7. For the statement about CD, we know that $\mathcal{O}(X)$ is a completely distributive lattice since X is continuous [1]; but it is folklore knowledge that $L \multimap M$ is completely distributive if L and M are completely distributive. $\square$

Note that the local continuity of these functors allows us to solve domain equations with the standard machinery of locally continuous functors [1].

9 A Universal Property for $\mathsf{P}_L(X)$

Before we state the universal property, we consider how well $\mathcal{F}_\cdot \searrow \cdot$ embeds the topological structure X and the lattice-theoretical structure L into $\mathsf{P}_L(X)$.

Lemma 22. *Let X be a dcpo and L a complete lattice. Then $\mathcal{F}_\cdot \searrow \cdot : X \times L \to \mathsf{P}_L(X)$ is Scott-continuous in its first coordinate and preserves all suprema in its second coordinate.*

Proof.
1. Let $r \in L$. By Lemma 11, the map $x \mapsto \mathcal{F}_x \searrow r : X \to \mathsf{P}_L(X)$ is monotone. Thus, it suffices to show that $\mathcal{F}_{\bigvee_{i\in I} x_i} \searrow r \leq \bigvee_{i \in I} \mathcal{F}_{x_i} \searrow r$ for any directed family $(x_i)_{i\in I}$ in X. This is now immediate from $\mathcal{F}_{\bigvee_{i\in I} x_i} = \bigcup_{i\in I} \mathcal{F}_{x_i}$.
2. If $U \in \mathcal{F}_x$ then $(\mathcal{F}_x \searrow \bigvee_{i\in I} r_i) U = \bigvee_{i\in I} r_i = (\bigvee_{i\in I} \mathcal{F}_x \searrow r_i) U$. Otherwise, both sides equal 0_L. $\square$

Theorem 23. *If X is a continuous domain and L a continuous lattice, then $\mathsf{P}_L(X)$ is a free construction in the following sense: for every complete lattice M and every function $f: X \times L \to M$ which is Scott-continuous in the first and a sup-map in the second coordinate, there is a unique sup-map $F: \mathsf{P}_L(X) \to M$ such that $F \circ \mathcal{F}_\cdot \searrow \cdot = f$.*

Proof. First, assuming that F is a sup-map with $F \circ \mathcal{F}_\cdot \searrow \cdot = f$, we compute $F(\mu) = F(\bigvee\{\mathcal{F}_x \searrow r \mid r \ll \mu(\Uparrow x)\}) = \bigvee\{(F \circ \mathcal{F}_\cdot \searrow \cdot)(x, r) \mid r \ll \mu(\Uparrow x)\} = \bigvee_{r \ll \mu(\Uparrow x)} f(x, r)$. The first equation above is a consequence of the proof of Theorem 17; the second one follows since F is a sup-map. This implies the uniqueness of such an F and suggests to define $F(\mu) = \bigvee_{r \ll \mu(\Uparrow x)} f(x, r)$.

Second, $F(0_{\mathsf{P}_L(X)}) = \bigvee_{x\in X} f(x, 0_L) = 0_M$ as f is a sup-map in its second coordinate.

Third, to show that F is monotone, let $\mu \leq \mu'$ and $r \ll \mu(\Uparrow x)$. This implies $r \ll \mu'(\Uparrow x)$ and so $F(\mu) \leq F(\mu')$.

Fourth, let $(\mu_i)_{i\in I}$ be a directed family in $\mathsf{P}_L(X)$. Since L is continuous we know that for every $r \ll \bigvee_{i\in I} \mu_i(\Uparrow x)$ there is some $s \in L$ with $r \ll s \ll \bigvee_{i\in I} \mu_i(\Uparrow x)$. Thus, a routine calculation ensures that F is Scott-continuous.

Fifth, we show $F \circ \mathcal{F}_{\cdot} \searrow \cdot = f$. For $y \in X$ and $s \in L$ we have $F(\mathcal{F}_y \searrow s) = \bigvee\{f(x,r) \mid r \ll (\mathcal{F}_y \searrow s)(\Uparrow x)\} \leq \bigvee\{f(x,r) \mid r \ll s,\ x \ll y\} \leq f(y,s)$ as f is monotone. The justification for the first inequality above is that $r \ll (\mathcal{F}_y \searrow s)(\Uparrow x)$ implies $r \ll s$ and $x \ll y$, or $x \not\ll y$ and $r = 0_L$. Since $r = 0_L$ implies $f(x,r) = 0_M$ this case is immaterial. Conversely, since X and L are continuous and since f is continuous in its first coordinate, we have $f(y,s) = \bigvee_{r\ll s} \bigvee_{x \ll y} f(x,r)$. From $r \ll s$ and $x \ll y$, we get $r \ll s = (\mathcal{F}_y \searrow s)(\Uparrow x)$, which gives us $f(x,r) \leq F(\mathcal{F}_y \searrow s)$ by the definition of F.

Sixth, it remains to demonstrate that F is a sup-map. Given $\mu_1, \mu_2 \in \mathsf{P}_L(X)$, it suffices to show that $F(\mu_1 \vee \mu_2) \leq F(\mu_1) \vee F(\mu_2)$. Since X is a continuous dcpo and the function $x \mapsto f(x,r) : X \to L$ is Scott-continuous for all $r \in L$, we have $f(x,r) = \bigvee_{y \ll x} f(y,r)$. Using this equation, it suffices to show $f(y,r) \leq F(\mu_1) \vee F(\mu_2)$ for $r \ll (\mu_1 \vee \mu_2)(\Uparrow x)$ and $y \ll x$. From Lemma 16 we infer that $\mathcal{F}_x \searrow r \ll \mu_1 \vee \mu_2$. Let $s_i = \mu_i(\Uparrow y)$ $(i = 1,2)$. If $U \in \mathcal{F}_y$ then $s_i = \mu_i(\Uparrow y) \leq \mu_i(U)$. Therefore, $\mathcal{F}_y \searrow s_i \leq \mu_i$ holds for $i = 1,2$. By parts 2 and 4, $f(y, s_i) = F(\mathcal{F}_y \searrow s_i) \leq F(\mu_i)$ follows for $i = 1,2$. But $r = (\mathcal{F}_x \searrow r)(\Uparrow y) \leq (\mu_1 \vee \mu_2)(\Uparrow y) = s_1 \vee s_2$ and the fact that $r \mapsto f(y,r) : L \to L$ is a sup-map entail $f(y,r) \leq f(y, s_1 \vee s_2) = f(y,s_1) \vee f(y,s_2) \leq F(\mu_1) \vee F(\mu_2)$. Thus, F is a sup-map. □

10 Related Work

Recently, Jimmie Lawson kindly pointed us toward the article "Random Upper Semicontinuous Functions and Extremal Processes" by Wim Vervaat [14]. In loc. cit. the author considers L to be a closed interval of $[-\infty, +\infty]$ and proves the lattice isomorphism between $[X \to \hat{L}]$ and $\mathcal{O}(X) \multimap L$ (in our notation). Here, $\hat{L}$ is the lattice L equipped with the Scott-topology of L^{op}, so that $[X \to \hat{L}]$ is order-isomorphic to our $[X \to L^{\mathrm{op}}]^{\mathrm{op}}$. Vervaat further shows that $[X \to \hat{L}]$ is isomorphic to $\Gamma(X \times L')$, the lattice of closed subsets of the product space $X \times L'$, where L' is $L \setminus \{0_L\}$ endowed with the Scott-topology. The latter isomorphism results from considering hypographs of $f \in [X \to \hat{L}]$. These isomorphisms allow the transfer of a number of topologies to study the convergence of extremal processes [14]. Although our work has been done independently, it should be clear that our results concerning the first isomorphism are much more general: we have basically no restrictions on X or L, we reveal a duality (which would be hard to spot in the case of [14] since $L \cong L^{\mathrm{op}}$ always holds therein), and we apply this isomorphism to obtain structural insights about the space $[X \to L]$

for continuous X and co-continuous L—its universal property, its co-continuity, and its inf-irreducible elements.

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