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Completeness Classes of Equilogical Spaces

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Abstract

Equilogical spaces form a cartesian closed complete category that contains all $\mathcal{T}_0$ spaces. *Assemblies* (on algebraic lattices) are a slight generalisation of an equivalent description of this category. They too form a cartesian closed complete category that even contains all topological spaces. We introduce various notions of completeness for assemblies, some inspired by *Synthetic Domain Theory*, and others inspired by topology, e.g., *sobriety*. Finally, we introduce the class of *ordered assemblies* that still contains all topological spaces, but excludes certain pathologies.

1 Introduction

In December 1996, Dana Scott and his group proposed the category EQU of *equilogical spaces* [6,1,2], a cartesian closed complete category that contains TOP_0, the category of $\mathcal{T}_0$ topological spaces, as a full subcategory.

In [6], the objects of EQU are equivalence relations on topological $\mathcal{T}_0$ spaces. In [1], Scott's group showed that EQU is equivalent to the category MOD of *modest sets* (on algebraic lattices), and also considered the larger category ASSM of *assemblies* (on algebraic lattices). Here, we prefer to work with MOD and ASSM because their morphisms are functions (not equivalence classes of functions), and their function space construction is very simple. The extension to ASSM is useful since it contains all topological spaces, not just the $\mathcal{T}_0$ spaces. Section 2 recalls the main definitions and basic facts about assemblies and modest sets.

Assemblies are too general to support a fixed point theorem for endomorphisms. In *synthetic domain theory* (SDT) [5], a fixed point theorem is obtained by restricting the attention to *complete pointed* objects, where these properties are defined by abstract categorical methods. Although SDT is based on a topos or a constructive type universe and ASSM is neither of this, most ideas and results from SDT can be transferred to ASSM. This is done in Section 3, where we define completeness and weak completeness of assemblies with respect to a morphism, and present the main properties of these

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notions. We also define pointed assemblies and present a fixed point theorem for complete pointed assemblies.

In Section 4, we define some more completeness classes, not by categorical methods this time, but by imposing restrictions on the inner structure of assemblies. The resulting notions of *sober assemblies* and *d-assemblies* generalise the corresponding topological notions: the class of sober assemblies forms a cartesian closed complete subcategory of ASSM which intersects TOP exactly in the class of sober spaces, and analogously for d-assemblies and d-spaces.

Each assembly is equipped with a preorder which is defined in analogy with the specialisation preorder of topological spaces. For general assemblies, this preorder and the SDT notions do not fit well together: a complete assembly may contain an ascending sequence without upper bounds, and an assembly with least element may not be pointed. These and other anomalies are exhibited at a single example in Section 5. They can be ruled out by imposing some restrictions on assemblies. The resulting subcategory of *ordered assemblies* is presented in Section 6. It is again cartesian closed and complete, still contains TOP, and its objects enjoy a very well-behaved specialisation preorder.

2 Assemblies and Modest Sets

2.1 Assemblies

The following definition is taken from [1]. An *assembly* is a triple (X, L, E) where X is a set (the carrier set), L an algebraic lattice, and E a function from X to the powerset of L such that all sets $E(x)$ are non-empty. The elements of $E(x)$ are called *realisers* for x.

Let $\mathcal{X} = (X, L, E)$ and $\mathcal{Y} = (Y, M, F)$ be two assemblies. A Scott continuous function $\varphi : L \to M$ is a *realiser* for a function $f : X \to Y$ if for all x in X and a in $E(x)$, φa is in $F(fx)$. A function $f : X \to Y$ is *realisable* if it has at least one realiser. All constant functions are realisable. Let ASSM be the category of assemblies and realisable functions.

Often, we shall follow the custom known from topology to speak of an element or subset of an assembly if we mean an element or subset of its carrier set.

Two assemblies $\mathcal{X}$ and $\mathcal{Y}$ are *equivalent* ($\mathcal{X} \cong \mathcal{Y}$) if they have the same carrier set, and the set-theoretic identity on this set is realisable in both directions. Two equivalent assemblies are isomorphic in the usual category-theoretic sense, but not vice-versa.

In the definition of equivalence, it is not required that the two realisers compose to be the identities on the two lattices. Hence, the lattices of two equivalent assemblies may be of radically different size and shape. This is quite different from many familiar categories such as topological spaces or groups where equivalent objects would be equal.

An assembly (X, L, E) is *modest* (a *modest set*) if the sets $E(x)$ are pairwise

disjoint, i.e., if $E(x) \cap E(x') \neq \emptyset$ implies $x = x'$. If $\mathcal{X} \cong \mathcal{Y}$ and $\mathcal{X}$ is modest, then so is $\mathcal{Y}$. The full subcategory MOD of modest sets (on algebraic lattices) is equivalent to Scott's category EQU of equilogical spaces [1].

2.2 Initial Assemblies

In [1], it is shown that ASSM has limits, i.e., products and equalisers. Even more can be said:

Proposition 2.1 *In* ASSM*, there are initial structures for all sources. In* MOD*, there are initial structures for jointly injective sources. In* ASSM *and* MOD*, there are final structures for all sinks.*

Proof. Let Y be a set, $(\mathcal{Z}_i)_{i\in I} = (Z_i, N_i, G_i)_{i\in I}$ a family of assemblies, and $(g_i : Y \to Z_i)_{i\in I}$ a family of functions. Then define $\mathcal{Y} = (Y, M, F)$ with $M = \prod_{i\in I} N_i$ and $F(y) = \prod_{i\in I} G_i(g_i y)$ for y in Y. By choice, the sets $F(y)$ are non-empty. The functions $g_i : \mathcal{Y} \to \mathcal{Z}_i$ are realised by the projections $\pi_i : M \to N_i$. If $\mathcal{X} = (X, L, E)$ is another assembly, and $f : \mathcal{X} \to \mathcal{Y}$ is a function such that all $g_i \circ f : \mathcal{X} \to \mathcal{Z}_i$ are realised by $\psi_i : L \to N_i$, then f is realised by the corresponding tuple function $\psi : L \to M$. If all $\mathcal{Z}_i$ are modest, and $g_i y_1 = g_i y_2$ for all i in I implies $y_1 = y_2$ (joint injectivity), then $\mathcal{Y}$ is modest. For the final structures, see [4]. □

Products are special cases of initial structures with $Y = \prod_{i\in I} Z_i$ and $g_i = \pi_i$, and so are *subspaces* ($I = \{1\}$, $Y \subseteq Z_1$, g_1 the inclusion map). The *equaliser* of $f, g : \mathcal{X} \to \mathcal{Y}$ can be obtained as the subspace $\{x \in \mathcal{X} \mid fx = gx\}$ of $\mathcal{X}$. Dually, coproducts and quotients are special cases of final structures.

In ASSM, a morphism is monic if and only if the underlying set-function is *injective*. A morphism $g : \mathcal{Y} \to \mathcal{Z}$ is *initial* if $\mathcal{Y}$ is the initial structure for g, i.e., if for all assemblies $\mathcal{X}$ and $f : \mathcal{X} \to \mathcal{Y}$, the realisability of $g \circ f$ implies the realisability of f. (In TOP, the initial morphisms $g : Y \to Z$ are exactly those continuous functions such that each open of Y is the inverse image of an open of Z.)

An *embedding* is an injective initial morphism. Surjective embeddings are isomorphisms. Each subspace embedding is an embedding, and conversely, if $f : \mathcal{X} \to \mathcal{Y}$ is an embedding, then $\mathcal{X}$ is isomorphic to the subspace $f(\mathcal{X})$ of $\mathcal{Y}$.

2.3 Exponentials

For $\mathcal{X} = (X, L, E)$ and $\mathcal{Y} = (Y, M, F)$, the function space $[\mathcal{X} \to \mathcal{Y}]$ is $(Z, [L \to M], G)$ where Z is the set of realisable functions from $\mathcal{X}$ to $\mathcal{Y}$, $[L \to M]$ is the algebraic lattice of Scott continuous functions from L to M, and $G(f)$ is the set of realisers of f. With this construction, ASSM becomes a cartesian closed category [1].

MOD is closed under exponentiation, i.e., if $\mathcal{X}$ is in ASSM and $\mathcal{Y}$ in MOD, then $[\mathcal{X} \to \mathcal{Y}]$ (formed in ASSM) is in MOD.

2.4 Topological Spaces and Assemblies

The following generalises the embedding of $\mathcal{T}_0$ topological spaces into modest sets from [1].

For a topological space $(X, \mathcal{O})$, we use the powerset $\mathcal{P}\mathcal{O}$ as the algebraic lattice (actually, the powerset of a subbasis would suffice). The function $e : X \to \mathcal{P}\mathcal{O}$ defined by $ex = \{O \in \mathcal{O} \mid x \in O\}$ is continuous and initial (in TOP) if $\mathcal{P}\mathcal{O}$ is endowed with the Scott topology. Using e, the assembly $\mathrm{A_T}(X, \mathcal{O})$ belonging to $(X, \mathcal{O})$ is defined as $(X, \mathcal{P}\mathcal{O}, \$e)$ where $\$e(x) = \{ex\}$.

It is well-known that for any embedding $e : X \to Y$ in TOP, any continuous function from X to a continuous lattice L can be extended to a continuous function from Y to L [3]. It is easy to see that this extension property holds for initial mappings as well. This ensures that any continuous $f : X \to Y$ is realisable from $\mathrm{A_T}X$ to $\mathrm{A_T}Y$. Since continuity of $e_Y \circ f$ implies continuity of f by initiality, the converse also holds; therefore, $\mathrm{A_T}$ is a full and faithful functor from TOP to ASSM.[1] Clearly, $\mathrm{A_T}X$ is modest if and only if X is $\mathcal{T}_0$. Hence, $\mathrm{A_T}$ cuts down to a full and faithful functor from $\mathrm{TOP_0}$ to MOD.

The particular choice of e and $\mathcal{P}\mathcal{O}$ does not matter. If $e_1 : X \to L_1$ and $e_2 : X \to L_2$ are two initial mappings to algebraic lattices L_1 and L_2, then the extension property implies the equivalence of the two assemblies $(X, L_1, \$e_1)$ and $(X, L_2, \$e_2)$. In particular, it follows that for any algebraic lattice L with its Scott topology, $\mathrm{A_T}L$ is equivalent to the assembly $(L, L, x \mapsto \{x\})$.

$\mathrm{A_T}$ may be composed with the functor $\mathrm{T_P}$ from the category PreO of preorders and monotonic functions to TOP, which endows each preorder with its Alexandroff topology (open sets = upper sets), or with $\mathrm{T_D} : \mathrm{DCPO} \to \mathrm{TOP}$ which endows each dcpo with its Scott topology. The resulting compositions are functors $\mathrm{A_P} : \mathrm{PreO} \to \mathrm{ASSM}$ and $\mathrm{A_D} : \mathrm{DCPO} \to \mathrm{ASSM}$.

In the remainder of this paper, we shall often identify X and $\mathrm{A_T}X$ (or something equivalent to $\mathrm{A_T}X$). We use the symbol Ω for the Sierpinski space $\{0, 1\}$ with open sets $\emptyset$, $\{1\}$, and Ω, for the algebraic lattice $\{0, 1\}$ with $0 \leq 1$, and for the assembly $(\Omega, \Omega, x \mapsto \{x\})$. This assembly can be used to define a topology on (the carrier set of) an assembly $\mathcal{X}$ as follows: A subset O of $\mathcal{X}$ is *open* iff its characteristic function $\chi_O : \mathcal{X} \to \Omega$ is realisable. We call this topology the *induced topology* on $\mathcal{X}$. The topological space made from the carrier set of $\mathcal{X}$ and the induced topology is called $\mathrm{T_A}\mathcal{X}$.

If $f : \mathcal{X} \to \mathcal{Y}$ is realisable, then $f : \mathrm{T_A}\mathcal{X} \to \mathrm{T_A}\mathcal{Y}$ is continuous. The resulting functor $\mathrm{T_A} : \mathrm{ASSM} \to \mathrm{TOP}$ is *left adjoint* to $\mathrm{A_T} : \mathrm{TOP} \to \mathrm{ASSM}$. Beware that $\mathrm{T_A}\mathcal{X}$ is not necessarily $\mathcal{T}_0$ if $\mathcal{X}$ is modest, i.e., $\mathrm{T_A}$ does *not* cut down to a functor from MOD to $\mathrm{TOP_0}$.

For any topological space X, $\mathrm{T_A}(\mathrm{A_T}X)$ is X. Thus TOP is equivalent to a reflective subcategory of ASSM, consisting of *topological* assemblies characterised by $\mathcal{X} \cong \mathrm{A_T}(\mathrm{T_A}\mathcal{X})$. An assembly (X, L, E) with the property that

[1] $\mathrm{A_T}$ stands for 'A from T', an abbreviation of 'assembly from topological space'. In this paper, all functors are named in this way.

all sets $E(x)$ are singletons is topological, and any topological assembly is equivalent to an assembly with this property.

The *specialisation preorder* of a topological space defines a functor from TOP to PreO. By composition with $\mathrm{T_A}$, we obtain a functor $\mathrm{P_A} : \mathrm{ASSM} \to \mathrm{PreO}$ which equips each assembly $\mathcal{X}$ with the specialisation preorder of $\mathrm{T_A}\mathcal{X}$, called the *induced preorder* of $\mathcal{X}$.

3 Completeness Classes from Synthetic Domain Theory

People working in Synthetic Domain Theory (SDT) generally consider a set-like category (a topos or a universe of constructive type theory), and try to find objects with domain-like properties in it. Most of the SDT ideas apply to the category ASSM as well although it is neither a topos nor a universe of constructive type theory. Moreover, many SDT theorems hold for ASSM, some of them with their original proof that uses only cartesian closedness, some with slightly adapted proofs. In particular, this holds for the theorems presented in this section. We have checked the validity of all these theorems in ASSM, but do not include any proofs. Instead, the reader may consult [5], which is our main reference, or any other text on SDT. The only exception is Theorem 3.1, which the author could not find in SDT literature.

3.1 Completeness

Let $\overline{\omega}$ denote the algebraic lattice $0 \leq 1 \leq 2 \leq \cdots \leq \infty$, and also the corresponding modest set $(\overline{\omega}, \overline{\omega}, n \mapsto \{n\})$. Let ω be $\overline{\omega}$ without ∞, considered as subspace of $\overline{\omega}$, i.e., $\omega = (\omega, \overline{\omega}, n \mapsto \{n\})$. The inclusion map $\omega : \omega \hookrightarrow \overline{\omega}$ is used to define a certain notion of completeness in SDT. This notion may be generalised by considering other functions instead of ω, as it is done in the following presentation.

Let $c : \mathcal{U} \to \mathcal{V}$ be a fixed realisable function. (The most prominent choice of c is the above-mentioned inclusion map ω.) An assembly $\mathcal{X}$ is *weakly c-complete* if for any realisable $f : \mathcal{U} \to \mathcal{X}$, there is a unique realisable $g : \mathcal{V} \to \mathcal{X}$ such that $g \circ c = f$, or put differently, if $g \mapsto g \circ c : [\mathcal{V} \to \mathcal{X}] \to [\mathcal{U} \to \mathcal{X}]$ is a bijection. The assembly $\mathcal{X}$ is *c-complete* if this function is even an isomorphism, i.e., if the assignment $f \mapsto g$ from the definition of weakly c-complete is a *realisable* function from $[\mathcal{U} \to \mathcal{X}]$ to $[\mathcal{V} \to \mathcal{X}]$.

It is useful to single out the uniqueness part of the definition of weak c-completeness: $\mathcal{X}$ is *c-separated* if for all $g, g' : \mathcal{V} \to \mathcal{X}$, $g \circ c = g' \circ c$ implies $g = g'$. The class of c-separated assemblies is closed under product, subspace, and exponentiation. The class of weakly c-complete assemblies is closed under products and contains all equalisers of $f, f' : \mathcal{X} \to \mathcal{Y}$ with weakly c-complete $\mathcal{X}$ and c-separated $\mathcal{Y}$. The class of c-complete assemblies has the same properties and is moreover closed under exponentiation (if $\mathcal{Y}$ is c-complete, then so is

$[\mathcal{X} \to \mathcal{Y}]$ for all $\mathcal{X}$).

In SDT, the difference between weak completeness and completeness appears as the difference between an external and an internal definition. Usually, SDT researchers consider only one of these two notions. Their relationship is as follows:

Theorem 3.1 *Let $c : \mathcal{U} \to \mathcal{V}$ be a fixed realisable function. For an assembly $\mathcal{Y}$, the following are equivalent:*

(i) *$\mathcal{Y}$ is c-complete.*

(ii) *For all assemblies $\mathcal{X}$, the function space $[\mathcal{X} \to \mathcal{Y}]$ is weakly c-complete.*

(iii) *$\mathcal{Y}$ is c-separated, and $[[\mathcal{U} \to \mathcal{Y}] \to \mathcal{Y}]$ is weakly c-complete.*

Proof. Since the class of c-complete assemblies is closed under exponentiation, (i) $\Rightarrow$ (ii) follows. The implication (ii) $\Rightarrow$ (iii) is trivial. For (iii) $\Rightarrow$ (i), one has to show that $C = \lambda g.\, g \circ c : [\mathcal{V} \to \mathcal{Y}] \to [\mathcal{U} \to \mathcal{Y}]$ is an isomorphism. We start with $s = \lambda u.\, \lambda p.\, pu : \mathcal{U} \to [[\mathcal{U} \to \mathcal{Y}] \to \mathcal{Y}]$ which is realisable because it is defined by a lambda expression in a cartesian closed category. Since the codomain of s is weakly c-complete by hypothesis, we get a realisable function $\bar{s} : \mathcal{V} \to [[\mathcal{U} \to \mathcal{Y}] \to \mathcal{Y}]$. Swapping arguments leads to the realisable function $\tilde{s} : [\mathcal{U} \to \mathcal{Y}] \to [\mathcal{V} \to \mathcal{Y}]$. By reasoning in the lambda calculus, it can be shown that $C \circ \tilde{s}$ is the identity on $[\mathcal{U} \to \mathcal{Y}]$. This equality implies $\tilde{s} \circ C = \mathsf{id}_{[\mathcal{U} \to \mathcal{Y}]}$ because C is injective as $\mathcal{Y}$ is c-separated. Therefore C is an isomorphism. □

3.2 Lifting

For an assembly $\mathcal{X} = (X, L, E)$, let $\mathcal{X}_\perp = (X_\perp, L_\perp, E_\perp)$, where $X_\perp$ is X with an additional element $\perp$, and $L_\perp$ is L with an additional least element $\perp$, with $E_\perp(\perp) = \{\perp\}$ and $E_\perp(x) = E(x)$ for $x \in X$. Note that $L_\perp$ is an algebraic lattice again.

The lift $\mathcal{X}_\perp$ is modest iff $\mathcal{X}$ is modest. The assignment $\mathcal{X} \mapsto \mathcal{X}_\perp$ is the object part of a functor which is defined in the obvious way. The inclusion $\ell_\mathcal{X}$ of $\mathcal{X}$ into $\mathcal{X}_\perp$ is realisable and defines a natural transformation. An assembly $\mathcal{X}$ is *pointed* (also called *focal lift algebra* in SDT) if there is a realisable function $\alpha : \mathcal{X}_\perp \to \mathcal{X}$ such that $\alpha \circ \ell_\mathcal{X} = \mathsf{id}_\mathcal{X}$. In this case, $\alpha(\perp)$ is the *bottom element* of $\mathcal{X}$.

The Sierpinski space Ω is pointed. It is obtained as the lift of the terminal assembly. Together with suitable structure maps, ω forms the initial algebra and $\overline{\omega}$ the final coalgebra for the lift functor.

3.3 Fixed Point Theorems

(Weakly) ω-complete assemblies, i.e., assemblies which are (weakly) complete for the inclusion $\omega : \omega \hookrightarrow \overline{\omega}$, admit the following fixed point theorems:

Theorem 3.2 *If $\mathcal{X}$ is a* pointed weakly ω-complete *assembly, then every realisable function $f : \mathcal{X} \to \mathcal{X}$ has a fixed point.*

The fixed point is obtained almost in the usual way: starting from the bottom element $\bot$ of $\mathcal{X}$, the sequence $(f^n(\bot))_{n\in\omega}$ is built and turned into a realisable function from ω to $\mathcal{X}$. The extension of this function to $\overline{\omega}$ is evaluated at ∞, which yields a fixed point of f.

If $\mathcal{X}$ is even ω-complete, a realisable fixed point function can be obtained:

Theorem 3.3 *If $\mathcal{X}$ is a* pointed ω-complete *assembly, then there is a* realisable *function $F : [\mathcal{X} \to \mathcal{X}] \to \mathcal{X}$ such that for every realisable f in $[\mathcal{X} \to \mathcal{X}]$, $F(f)$ is a fixed point of f.*

4 Structural Completeness Classes

The notions and results of the previous section are surely useful, but they are also quite abstract and do not convey much information on the inner structure of assemblies. Conversely, c-completeness is hard to verify for any concretely given assembly.

These problems are the motivation to define some classes of assemblies from the structure of the realisability relation. These classes are contained in the class of ω-complete assemblies, and also extend classifications of topological spaces to assemblies.

4.1 Irreducible Sets and Sobriety

We first recall some topological notions which form the basis for the subsequent definitions for assemblies. A subset A of a topological space $(X, \mathcal{O})$ is *irreducible* if the set $\{O \in \mathcal{O} \mid A \cap O \neq \emptyset\}$ is closed under finite intersection. Sets which are directed in the specialisation preorder are irreducible; in particular, all singleton sets are irreducible. If A is irreducible in X and $f : X \to Y$ is continuous, then $f(A)$ is irreducible in Y. If f is even a topological embedding, then the opposite implication also holds, i.e., A is irreducible in X if and only if $f(A)$ is irreducible in Y.

The topological space X is *sober* iff for every irreducible subset A of X, there is a unique point x of X such that $\mathsf{cl}A = \mathsf{cl}\{x\}$. We abbreviate the latter equality by $A \uparrow x$. If f is continuous, then $A \uparrow x$ implies $f(A) \uparrow fx$; equivalence holds for topological embeddings f. If $A \uparrow x$, then x is the least upper bound of A in the specialisation preorder. Thus, uniqueness of x in $A \uparrow x$ is equivalent to $\mathcal{T}_0$-separation while existence is the real restriction.

All these facts apply in particular to algebraic lattices L with their Scott topology. Since these spaces are sober, $A \uparrow \bigvee A$ holds for every irreducible subset A of L. Thus, Scott continuous functions between algebraic lattices preserve least upper bounds of irreducible sets.

4.2 Realising Sets

Before we generalise the notion of sobriety to assemblies, we introduce some auxiliary notation. For an assembly $\mathcal{X} = (X, L, E)$, E is extended to subsets S of X by defining $E(S) = \bigcup_{x \in S} E(x)$. A subset A of L *realises* a subset S of X if $A \subseteq E(S)$ and $A \cap E(x) \neq \emptyset$ for all x in S.

Any subset S is realised by $E(S)$, but there are smaller realising sets in general. Conversely, if $A \subseteq E(X)$, then A realises $\{x \in X \mid E(x) \cap A \neq \emptyset\}$.

If A realises S and f is a realisable function, then $f(A)$ realises $f(S)$. If subsets A_i realise S_i in $\mathcal{X}_i$, then $\prod_{i \in I} A_i$ realises $\prod_{i \in I} S_i$ in $\prod_{i \in I} \mathcal{X}_i$.

In case you wonder about the nature of the assembly $(\mathcal{P}X, \mathcal{P}L, \ldots)$ defined by this notion of realisation, it is simply equivalent to $\mathcal{P}X$ in its Scott topology. Nevertheless, the notion is useful.

Lemma 4.1 *Let $\mathcal{X} = (X, L, E)$ be an assembly. If an irreducible set $A \subseteq L$ realises $S \subseteq X$ and $\bigvee A$ is in $E(y)$, then $S \uparrow y$ holds in the induced topological space $\mathrm{T_A}\mathcal{X}$.*

Proof. The open sets of the induced topology correspond to the realisable functions $u : \mathcal{X} \to \Omega$. Let $u : X \to \Omega$ be realised by the continuous function $v : L \to \Omega$.

Let x be in S with $u(x) = 1$. Since A realises S, there is a in $A \cap E(x)$. Since v realises u, $v(a) = 1$ follows. Conversely, if $v(a) = 1$ for some a in $A \subseteq E(S)$, there is x in S with $u(x) = 1$.

Hence, S meets $u^{-1}(1)$ iff A meets $v^{-1}(1)$. Because of $A \uparrow \bigvee A$, i.e., $\mathsf{cl}A = \mathsf{cl}\{\bigvee A\}$, the latter is equivalent with $v(\bigvee A) = 1$. Because of $\bigvee A \in E(y)$, this is equivalent to $u(y) = 1$. Altogether, $\mathsf{cl}S = \mathsf{cl}\{y\}$ follows. □

4.3 Sober Assemblies

Definition 4.2 An assembly $\mathcal{X} = (X, L, E)$ is *directly sober* if its induced topology is $\mathcal{T}_0$, and for all irreducible subsets A of $E(X)$, $\bigvee A$ is in $E(X)$ again. The assembly $\mathcal{X}$ is *sober* if it is equivalent to a directly sober assembly.

The difference between 'directly sober' and 'sober' is caused by the unfortunate fact that direct sobriety is not preserved under equivalence.

First we show that the notion of sobriety for assemblies extends the corresponding notion for topological spaces.

Proposition 4.3 *For a topological space X, the following are equivalent:*

(i) *X is sober (in the topological sense).*

(ii) *$\mathrm{A_T}X$ is a directly sober assembly.*

(iii) *$\mathrm{A_T}X$ is a sober assembly.*

Proof. Recall that $\mathrm{A_T}X = (X, L, E)$ with $E(x) = \{ex\}$ where $e : X \hookrightarrow L$ is an embedding if X is $\mathcal{T}_0$.

First, let X be sober. The induced topological space of $\mathrm{A_T}X$ is the $\mathcal{T}_0$ space X. In $\mathrm{A_T}X$, $E(X)$ is $e(X)$, the image of X under e. If $A \subseteq e(X)$ is irreducible, then $A = e(B)$ for some irreducible $B \subseteq X$. Since X is sober, there is some x in X such that $B \uparrow x$, whence $A = e(B) \uparrow e(x)$. On the other hand, $A \uparrow \bigvee A$, and thus $\bigvee A = e(x)$ is in $E(X)$. This shows that $\mathrm{A_T}X$ is directly sober.

Any directly sober assembly is trivially sober.

Finally, consider a topological space X such that $\mathrm{A_T}X = (X, L, E)$ is sober, i.e., is equivalent to a directly sober assembly (X, M, F). Let the equivalence be realised by the continuous functions $\varphi : L \to M$ and $\psi : M \to L$. The defining property for realisers implies $\psi \circ \varphi \circ e = e$. If $A \subseteq X$ is irreducible, then $\varphi(e(A)) \subseteq \varphi(E(X)) \subseteq F(X)$ is irreducible. Since (X, M, F) is directly sober, there is some b in $F(X)$ such that $\varphi(e(A)) \uparrow b$, whence $e(A) \uparrow \psi(b)$ follows by applying ψ on both sides. Since b is in $F(X)$, $\psi(b)$ is in $E(X)$, i.e., $\psi(b) = e(x)$ for some x in X. Since e is an embedding, $e(A) \uparrow e(x)$ implies $A \uparrow x$. □

It is not difficult to see that the class of assemblies with $\mathcal{T}_0$ topology is closed under equivalence, product, subspace, and exponentiation [4]. Hence, we may concentrate on the existence part of the definition of direct sobriety when proving the following theorem. Since algebraic lattices are sober, this property can be stated in the following way: if $A \subseteq E(X)$ and $A \uparrow a$ in L, then a is in $E(X)$.

Theorem 4.4 *The class of sober assemblies is closed under product, exponentiation, and equalisers. To be more specific:*

(i) *If $(\mathcal{X}_i)_{i\in I}$ is a family of sober assemblies, then $\prod_{i\in I} \mathcal{X}_i$ is sober again.*

(ii) *If $\mathcal{Y}$ is sober, then $[\mathcal{X} \to \mathcal{Y}]$ is sober (even if $\mathcal{X}$ is not sober).*

(iii) *If $\mathcal{X}$ is sober, $\mathcal{Y}$'s induced topology is $\mathcal{T}_0$, and $f, g : \mathcal{X} \to \mathcal{Y}$ are realisable functions, then $\{x \in \mathcal{X} \mid fx = gx\}$ is a sober subspace of $\mathcal{X}$.*

Proof. Since all sober assemblies may be replaced by equivalent directly sober ones, the proof is about direct sobriety.

(i) If $\mathcal{X}_i = (X_i, L_i, E_i)$, then $\prod_{i\in I} \mathcal{X}_i = (X, L, E)$ with $X = \prod_{i\in I} X_i$, $L = \prod_{i\in I} L_i$, and $E((x_i)_{i\in I}) = \prod_{i\in I} E_i(x_i)$. If $A \subseteq E(X)$ and $A \uparrow a$, then for all i in I, $\pi_i(A) \subseteq E(X_i)$ and $\pi_i A \uparrow \pi_i a$, whence $\pi_i a \in E(X_i)$ and thus a in $E(X)$.

(ii) For $\mathcal{X} = (X, L, E)$ and $\mathcal{Y} = (Y, M, F)$, let $[\mathcal{X} \to \mathcal{Y}] = (Z, N, G)$. Assume $C \subseteq G(Z)$ and $C \uparrow \psi$. Set C realises $S = \{f \in Z \mid G(f) \cap C \neq \emptyset\}$. For every x in X and a in $E(x)$, $C(a) \subseteq F(Y)$ realises $S(x)$ and $C(a) \uparrow \psi(a)$ where $C(a) = \{\varphi(a) \mid \varphi \in C\}$ and $S(x) = \{f(x) \mid f \in S\}$. Since $\mathcal{Y}$ is directly sober, $\psi(a)$ is in $F(y)$ for some y in Y. By Lemma 4.1, $S(x) \uparrow y$ holds in the induced topology. The $\mathcal{T}_0$ property of this topology guarantees that y only depends on x, not on the choice of a in $E(x)$.

The mapping $x \mapsto y$ is a function f from X to Y which is realised by ψ. Hence, ψ is in $G(f) \subseteq G(Z)$.

(iii) For $\mathcal{X} = (X, L, E)$ directly sober, $\mathcal{Y} = (Y, M, F)$ $\mathcal{T}_0$ and $f, g : \mathcal{X} \to \mathcal{Y}$ realisable, let $Z = \{x \in X \mid fx = gx\}$ so that $\mathcal{Z} = (Z, L, E)$ is the equaliser of f and g. Let $A \subseteq E(Z)$ with $A \uparrow a$ in L. By direct sobriety of $\mathcal{X}$, a is in $E(x)$ for some x in X. Define $S = \{z \in Z \mid E(z) \cap A \neq \emptyset\}$ so that A realises S. By Lemma 4.1, $S \uparrow x$ holds in $\mathrm{T_A}\mathcal{X}$. Since the realisable functions f and g are continuous from $\mathrm{T_A}\mathcal{X}$ to $\mathrm{T_A}\mathcal{Y}$, $f(S) \uparrow fx$ and $g(S) \uparrow gx$ holds in $\mathrm{T_A}\mathcal{Y}$. Since S is a subset of Z, $f(S) = g(S)$ holds. The $\mathcal{T}_0$ property of $\mathrm{T_A}\mathcal{Y}$ yields $fx = gx$, whence x in Z and thus a in $E(Z)$. □

4.4 D-Assemblies

(Topological) *d-spaces* are $\mathrm{TOP_0}$-spaces which are dcpo's in their specialisation preorder with the additional property that all open sets are Scott open (these spaces are called *monotone convergence spaces* in [3, Def. II.3.9]). Equivalently, d-spaces X can be characterised by the property that for every directed subset D of X, there is a unique point x of X such that $\mathsf{cl}D = \mathsf{cl}\{x\}$. This characterisation differs from the definition of sobriety only by the replacement of irreducible sets by directed sets. Consequently, the definition of d-assemblies results from Def. 4.2 by the same modification.

Definition 4.5 An assembly $\mathcal{X} = (X, L, E)$ is a *direct d-assembly* if $\mathrm{T_A}\mathcal{X}$ is $\mathcal{T}_0$ and $E(X)$ is closed under directed joins. The assembly $\mathcal{X}$ is a *d-assembly* if it is equivalent to a direct d-assembly.

Since every directed set is irreducible, we immediately obtain:

Proposition 4.6 *Every (directly) sober assembly is a (direct) d-assembly.*

The properties of d-assemblies (and their proofs) are analogous to those of sober assemblies.

Proposition 4.7 *A topological space X is a d-space, iff $\mathrm{A_T}X$ is a direct d-assembly, iff $\mathrm{A_T}X$ is a d-assembly.*

Theorem 4.8 *The class of d-assemblies is closed under product, exponentiation, and equalisers.*

Now, we present a link between the structural completeness classes of this section and the SDT completeness classes of Section 3. The *ideal completion* IP of a poset P is the set of ideals (directed lower subsets) of P ordered by inclusion. If P is endowed with its Alexandroff topology and IP with its Scott topology, the function $i_P : P \to IP$ which maps x in P to $\downarrow x$ in IP becomes a topological embedding. These topological spaces P and IP may be considered as assemblies via the inclusion functor $\mathrm{A_T}$. This is done in the following theorem:

Theorem 4.9 *Every d-assembly is i_P-complete for all posets P* (d-complete).

Proof. We show that every direct d-assembly is *weakly* i_P-complete. Since the class of d-assemblies is closed under exponentiation, Theorem 3.1 admits the step to i_P-completeness itself.

Let $\mathrm{A_T}P = (P, L, \$e)$ and $\mathrm{A_T}(IP) = (IP, L', \$e')$ with suitable embeddings $e : P \hookrightarrow L$ and $e' : IP \hookrightarrow L'$. Let $\mathcal{Y} = (Y, M, F)$ be a direct d-assembly and $f : P \to Y$ be a function realised by $\varphi : L \to M$. Then $\varphi \circ e$ is a monotonic function from P to the subset $F(Y)$ of M. Since $\mathcal{Y}$ is a direct d-assembly, $F(Y)$ is closed under directed joins. By the universal property of ideal completions, $\varphi \circ e$ extends to a continuous function $\varphi' : IP \to F(Y) \subseteq M$ with $\varphi' \circ i_P = \varphi \circ e$. Now, φ' extends to a continuous $\psi : L' \to M$ with $\psi \circ e' = \varphi'$. This function realises $g = q \circ \varphi' : IP \to Y$ where $q : F(Y) \to Y$ maps each b in $F(Y)$ to the element y of Y with $b \in F(y)$. (This y is unique since d-assemblies are modest.) Finally note that $g \circ i_P = q \circ \varphi' \circ i_P = q \circ \varphi \circ e = f$ since $\varphi(ex) \in F(fx)$ for all x in P. Thus, we have shown that every realisable $f : P \to Y$ has a realisable extension $g : IP \to Y$.

If $g_1, g_2 : IP \to Y$ are two realisable extensions of f, then g_1 and g_2 are two continuous functions from IP to the $\mathcal{T}_0$ space $\mathrm{T_A}\mathcal{Y}$ which coincide on the basis $i_P(P)$ of IP, whence $g_1 = g_2$ follows. □

Since $\overline{\omega}$ is isomorphic to $I\omega$ in a way that identifies the embedding $\omega : \omega \hookrightarrow \overline{\omega}$ with $i_\omega : \omega \hookrightarrow I\omega$, we obtain the following corollary:

Theorem 4.10 *Every d-assembly is ω-complete. Hence, every pointed d-assembly admits a realisable fixed point operator.*

5 An Example

Each assembly is equipped with the specialisation preorder of its induced topological space. Since realisable functions are continuous with respect to the induced topologies, they are also monotonic with respect to the specialisation preorders. Hence, the bottom element of a pointed assembly (Section 3.2) is the least element in the specialisation preorder, and for every realisable $f : \omega \to \mathcal{Y}$, the sequence of image points $f(0)$, $f(1)$, etc. forms an ascending sequence in $\mathcal{Y}$. Moreover, with the continuity of realisable functions, it easily follows that for realisable $g : \overline{\omega} \to \mathcal{Y}$, $g(\infty)$ is the least upper bound of the chain $(g(n))_{n \in \omega}$. Hence, in a weakly ω-complete assembly, each ascending chain which is the realisable image of ω has a least upper bound.

Unfortunately, the opposites of the first two remarks are wrong. There are assemblies with least element which are not pointed, and ascending sequences which are not the image of ω by a realisable function. Such sequences need not have a least upper bound in an ω-complete assembly, and consequently, the induced topological space of a d-assembly is not necessarily a d-space. There is a single example which shows all these deficiencies.

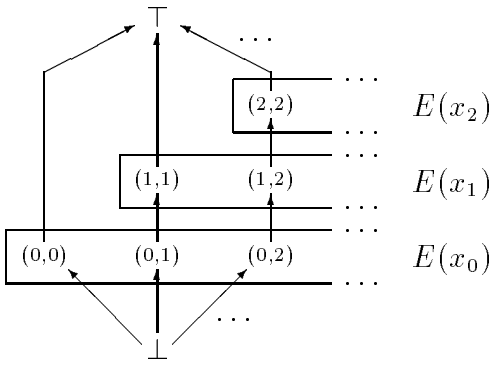

Fig. 1. Structure of the example

Let $\mathcal{X} = (X, L, E)$ be given by $X = \{x_n \mid n \in \omega\}$ and $L = \{\bot, \top\} \cup \{(n, m) \in \omega^2 \mid n \leq m\}$ with the ordering $(n, m) \leq (n', m')$ if $n \leq n'$ and $m = m'$, and let $E(x_n) = \{(n, m) \in L\}$. Thus, L without $\bot$ and $\top$ forms an infinite collection of incomparable *finite* ascending sequences $(0, m) \leq (1, m) \leq \cdots \leq (m, m)$, and each $E(x_n)$ is an infinite set of incomparable lattice elements which forms a horizontal level in L (see Figure 1).

There are no infinite ascending sequences in L. It is easy to see that $E(X) = L \setminus \{\bot, \top\}$ is closed under joins of irreducible sets. Thus, $\mathcal{X}$ is a directly sober assembly. By Prop. 4.6, it is a direct d-assembly, and therefore it is ω-complete by Theorem 4.10. Yet the induced topological space $\mathrm{T_A}\mathcal{X}$ is isomorphic ω, i.e., it is not sober, not even a d-space.

In the induced preorder of $\mathcal{X}$, the sequence $(x_n)_{n\in\omega}$ is ascending, but the function $n \mapsto x_n : \omega \to \mathcal{X}$ is *not* realisable (and the sequence $(x_n)_{n\in\omega}$ lacks a least upper bound). The point x_0 is a least element of $\mathcal{X}$, but $\mathcal{X}$ is *not* pointed in the sense of Section 3.2. The function $f : \mathcal{X}_\bot \to \mathcal{X}$ with $f(x_n) = x_n$ and $f(\bot) = x_0$ is not realisable by any continuous $\varphi : L_\bot \to L$ since there is no possible value for the attached bottom element of $L_\bot$.

The fixed point theorem 3.2 fails for $\mathcal{X}$ not because it is not weakly ω-complete, but because it is not pointed. An example for a realisable function without fixed point is $x_n \mapsto x_{n+1}$, realised by the function that maps $\bot$ to $\bot$, (n, m) to $(n + 1, m + 1)$, and $\top$ to $\top$.

6 Ordered Assemblies

In the previous section, we presented an example that exhibits some pathologies. Now, we define a class of more well-behaved assemblies which still contains all topological spaces. Though the definition looks quite ad hoc, we claim that it is useful because it admits strong conclusions.

6.1 The Definition of Ordered Assemblies

Definition 6.1 An assembly (X, L, E) is *directly ordered* if for all $x \sqsubseteq x'$ (in the induced preorder) and all sets $A \subseteq L$ which realise $\{x, x'\}$, the join $\bigvee A$

is in $E(x')$. An assembly is *ordered* if it is isomorphic to a directly ordered assembly.

The distinction between ordered and directly ordered comes from the unfortunate fact that of two isomorphic assemblies, one may be directly ordered, but the other one not. We do not know how to characterise ordered assemblies in a way that is invariant under isomorphism. In the sequel, our results will be formulated for ordered assemblies, but proved by considering directly ordered assemblies.

Another problem is the name "ordered assembly" itself. In topology, an ordered space refers to a (Hausdorff) space with an extra ordering that is different from the specialisation preorder in general. Here we mean an assembly with a well-behaved specialisation preorder.

The defining property of directly ordered assemblies can be split into two properties:

(i) All sets $E(x)$ are closed under non-empty joins.

(ii) If $x \sqsubseteq x'$, then $a \in E(x)$ and $b \in E(x')$ implies $a \vee b \in E(x')$.

Here, (i) holds because of $x \sqsubseteq x$, and (ii) is the defining property for $A = \{a, b\}$. Conversely, (i) and (ii) imply the defining property since $\bigvee A = \bigvee(A \cap E(x)) \vee \bigvee(A \cap E(x'))$.

By property (i), all sets $E(x)$ have a greatest element $\hat{E}(x) := \bigvee E(x) \in E(x)$. The specialisation preorder can be characterised in terms of this greatest element:

Proposition 6.2 *For x and x' in a directly ordered assembly, $x \sqsubseteq x'$ and $\hat{E}(x) \leq \hat{E}(x')$ are equivalent.*

Proof. First assume $\hat{E}(x) \leq \hat{E}(x')$. For every function $u : X \to \Omega$ realised by $v : L \to \Omega$, $ux = 1$ implies $v(\hat{E}(x)) = 1$, whence $v(\hat{E}(x')) = 1$, and thus $ux' = 1$. Therefore, $x \sqsubseteq x'$ holds. Conversely, if $x \sqsubseteq x'$, then $\hat{E}(x) \vee \hat{E}(x')$ is in $E(x')$ by (ii), hence $\hat{E}(x) \leq \hat{E}(x) \vee \hat{E}(x') \leq \hat{E}(x')$ follows. □

6.2 Properties of Ordered Assemblies

All assemblies with a $\mathcal{T}_0$ induced topology are modest, but the opposite is not true; there are modest sets whose induced topology is not $\mathcal{T}_0$. For ordered assemblies however, there is no distinction between $\mathcal{T}_0$ and modest.

Proposition 6.3 *An ordered assembly is modest if and only if its induced topology is $\mathcal{T}_0$.*

Proof. Let $\mathcal{X} = (X, L, E)$ be a directly ordered modest set. By Prop. 6.2, $x \sqsubseteq x'$ and $x' \sqsubseteq x$ together imply $\hat{E}(x) = \hat{E}(x')$. Hence, $\hat{E}(x)$ is in $E(x) \cap E(x')$, and the disjointness condition of modest sets implies $x = x'$. □

In the example assembly $\mathcal{X}$ from Section 5, there is an ascending sequence which is not a realisable image of ω, or put differently, there is a monotonic

function from ω to $\mathcal{X}$ which is not realisable. This cannot happen for ordered assemblies.

Theorem 6.4 *Let P be a preorder and $\mathrm{A_P}P$ the assembly that results from considering P as a topological space with its Alexandroff topology. Furthermore, let $\mathcal{Y}$ be an ordered assembly and $\mathrm{P_A}\mathcal{Y}$ the induced preorder. Then $f : \mathrm{A_P}P \to \mathcal{Y}$ is realisable if and only if $f : P \to \mathrm{P_A}\mathcal{Y}$ is monotonic. (Or more concisely: $\mathrm{A_P}$ is left adjoint to $\mathrm{P_A}$ restricted to ordered assemblies.)*

Proof. Let $\mathrm{A_P}P = (P, L, \$e)$ with an initial map $e : P \to L$, and let $\mathcal{Y} = (Y, M, E)$. All realisable functions are monotonic with respect to the specialisation preorders. For the opposite direction, consider a monotonic function $f : P \to Y$. By Prop. 6.2, $g = \hat{E} \circ f : P \to M$ is monotonic as well, hence continuous if P is endowed with its Alexandroff topology. By injectivity of L, there is a continuous $\varphi : L \to M$ with $\varphi \circ e = g$, which realises f. □

Corollary 6.5 *For each ascending sequence $(x_n)_{n\in\omega}$ in an ordered assembly $\mathcal{X}$, the function $n \mapsto x_n : \omega \to \mathcal{X}$ is realisable.*

The example from Section 5 is sober, hence a d-assembly and d-complete by Theorem 4.9, yet its induced topological space is not a d-space. With ordered assemblies, this cannot happen.

Theorem 6.6 *If $\mathcal{X}$ is an ordered weakly d-complete assembly, then its induced topological space $\mathrm{T_A}\mathcal{X}$ is a d-space.*

Proof. Let $\mathcal{X}$ be weakly d-complete, i.e., weakly complete for all ideal completions i_P. First, $\mathcal{X}$ is weakly ω-complete and thus modest because a non-modest assembly cannot satisfy the uniqueness condition of weak ω-completeness. By Prop. 6.3, it follows that $\mathrm{T_A}\mathcal{X}$ is $\mathcal{T}_0$.

Let D be a directed set of $\mathrm{T_A}\mathcal{X}$ (in the specialisation preorder). Let P be D considered as a poset on its own with the Alexandroff topology, and let IP be its ideal completion endowed with the Scott topology. The inclusion function from P into $\mathrm{T_A}\mathcal{X}$ is monotonic. By Theorem 6.4, it forms a realisable function f from P considered as assembly to $\mathcal{X}$. Since $\mathcal{X}$ is weakly d-complete, f has a (unique) extension to a realisable function $g : IP \to \mathcal{X}$ with $g \circ i_P = f$. This function g also forms a continuous function from IP to $\mathrm{T_A}\mathcal{X}$.

The whole poset P forms an ideal in P which is the top element of IP. In IP, $i_P(P) \uparrow P$ holds. By continuity of g, $D = f(P) = g(i_P(P)) \uparrow g(P)$ holds in $\mathrm{T_A}\mathcal{X}$. Thus, we found an x in $\mathrm{T_A}\mathcal{X}$ with $D \uparrow x$. It is unique because $\mathrm{T_A}\mathcal{X}$ is $\mathcal{T}_0$. □

The example from Section 5 has a least element, but is not pointed. This cannot happen with ordered assemblies.

Theorem 6.7 *Every ordered assembly with a least element is pointed.*

Proof. Let $\mathcal{X} = (X, L, E)$ be an assembly with least element x_0. Since $\mathcal{X}$ is ordered, $E(x_0)$ has a greatest element $a_0 = \hat{E}(x_0)$. For $\mathcal{X}$ to be pointed, the

function $f : \mathcal{X}_\perp \to \mathcal{X}$ which maps $\perp$ to x_0 and each x in X to itself must be realisable. Consider $\varphi : L_\perp \to L$ with $\varphi a = a_0 \vee a$. This function is Scott continuous and realises f since for a in $E_\perp(\perp)$, a is $\perp$ and so $\varphi a = a_0 \in E(x_0)$, and for a in $E(x)$, $\varphi a = a_0 \vee a$ is in $E(x)$ again because $x_0 \sqsubseteq x$ and $\mathcal{X}$ is ordered. □

The next property shows that there are not too many ordered assemblies.

Theorem 6.8 *All finite ordered assemblies are topological.*

Proof. Finite means finite carrier, a property preserved by isomorphism. Thus, we may assume a directly ordered assembly $\mathcal{X} = (X, L, E)$ with finite X. Let $\mathcal{Y} = (X, L, E')$ where $E'(x) = \{\hat{E}(x)\}$. We show $\mathcal{X} \cong \mathcal{Y}$. The set-theoretic identity from $\mathcal{Y}$ to $\mathcal{X}$ is realised by the identity on L. The realiser for the opposite direction is given by $\varphi a = \bigwedge\{\hat{E}(x) \mid \hat{E}(x) \geq a\}$. This function is Scott continuous by finiteness of X. It realises the identity since $a \in E(x)$ implies $\varphi a = \hat{E}(x)$; for '$\geq$', property (ii) of Section 6.1 is needed.

The set X is endowed with the initial topology induced by $\hat{E} : X \to L$, where L has the Scott topology. Then $\mathcal{Y} \cong \mathrm{A_T}X$ holds by the remarks in Section 2.4. □

6.3 Closure Properties of Ordered Assemblies

Before we come to the closure properties of the class of ordered assemblies, we present an auxiliary lemma.

Lemma 6.9 *Let $\mathcal{X} = (X, L, E)$ be any assembly and $\mathcal{Y} = (Y, M, F)$ a directly ordered assembly. Construct $[\mathcal{X} \to \mathcal{Y}] = (Z, N, G)$. If $f, g : \mathcal{X} \to \mathcal{Y}$ are two realisable functions with $fx \sqsubseteq gx$ for all x in $\mathcal{X}$, then $\varphi \in G(f)$ and $\psi \in G(g)$ implies $\varphi \vee \psi \in G(g)$; therefore, $f \sqsubseteq g$ follows.*

Proof. First, $\varphi \vee \psi$ is Scott continuous as the (pointwise) join of two Scott continuous functions. For any x in X and a in $E(x)$, φa is in $F(fx)$ and ψa in $F(gx)$. Since $fx \sqsubseteq gx$ by hypothesis, property (ii) of Section 6.1 implies $\varphi a \vee \psi a$ in $F(gx)$. Thus, $\varphi \vee \psi$ realises g.

Since $G(f)$ and $G(g)$ are not empty, we find φ in $G(f)$ and ψ in $G(g)$. Since φ in $G(f)$ is below $\varphi \vee \psi$ in $G(g)$, the additional statement $f \sqsubseteq g$ follows. □

Theorem 6.10 *The class of ordered assemblies contains all topological spaces, and is closed under subspace, product, and exponentiation, i.e., if $\mathcal{Y}$ is ordered, then so is $[\mathcal{X} \to \mathcal{Y}]$ for any $\mathcal{X}$. Moreover, the order on realisable functions is pointwise.*

Proof. The theorem is best proved by considering the properties (i) and (ii) of Section 6.1 independently. Property (ii) for exponentiation essentially is Lemma 6.9 which gives also the fact about the pointwise ordering. □

6.4 Consequences

Finally, we consider some consequences of our results. Let $\Omega\mathcal{X}$ be an abbreviation for the function space $[\mathcal{X} \to \Omega]$. By the closure theorems 4.4 and 6.10, $\Omega\mathcal{X}$ is a sober ordered assembly for arbitrary assemblies $\mathcal{X}$. By Theorem 6.6, the induced topological space $\mathrm{T_A}(\Omega\mathcal{X})$ is a d-space, and therefore, all open sets of (the induced topology of) $\Omega\mathcal{X}$ are Scott open.

An assembly $\mathcal{X}$ is Ω-*embedded* if it can be embedded into $\Omega\mathcal{Y}$ for some assembly $\mathcal{Y}$, or equivalently, if $\lambda x.\,\lambda u.\,ux : \mathcal{X} \to \Omega(\Omega\mathcal{X})$ is an embedding (see [4,5]). By Theorem 6.10, all Ω-embedded assemblies are ordered. If you dislike the notion of ordered assemblies because of its seemingly ad-hoc definition, you may consider it just as a technical means to show that Ω-embedded assemblies enjoy all the good properties listed in Section 6.2.

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