A Non-Topological View of Dcpo's as Convergence Spaces

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Problems with Function Spaces

Category TOP of topological spaces and continuous functions

- has products and subspaces, hence equalisers,
- but no proper function spaces.

In general, there is no topology τ on $(Y \to Z) = \{f : Y \to Z \text{ continuous}\}$ such that

$f : X \times Y \to Z$ continuous $\iff$
$\tilde{f} : X \to (Y \to Z)_\tau$ continuous,

where $f(x, y) = \tilde{f}\, x\, y$.

Problems with Dcpo's

Dcpo $D \mapsto$ topological space D_{S} with Scott topology.

$f : D \to E$ Scott continuous, i.e. preserves directed joins
$\iff$ $f : D_{\mathsf{S}} \to E_{\mathsf{S}}$ continuous.

Hence $(-)_{\mathsf{S}} : \mathsf{DCPO} \hookrightarrow \mathsf{TOP}$.

- X topological space, $f_i : X \to D_{\mathsf{S}}$ continuous, directed $\Rightarrow \bigsqcup_{i \in I} f_i = (x \mapsto \bigsqcup_{i \in I} f_i x)$ continuous.
- $(D \times E)_{\mathsf{S}} \neq D_{\mathsf{S}} \times E_{\mathsf{S}}$ in general.
- For a complete lattice L, $\vee : L_{\mathsf{S}} \times L_{\mathsf{S}} \to L_{\mathsf{S}}$ is not always continuous.
- $f, g : X \to L_{\mathsf{S}}$ continuous $\not\Rightarrow$ $f \vee g = (x \mapsto fx \vee gx)$ continuous.

Yet continuous dcpo's are well-behaved.

Solutions

- Extend TOP to larger category $\mathsf{CONV} \supseteq \mathsf{TOP}$ which has function spaces.
- Define new embedding $(-)_{\mathsf{C}} : \mathsf{DCPO} \hookrightarrow \mathsf{CONV}$ which is well-behaved for all dcpo's.
  (D_{C} "cotopological dcpo",
  L_{C} "cotopological lattice")
- X topological, Y cotopological lattice $\Rightarrow [X \to Y]$ cotopological lattice.
- X cotopological dcpo, Y topological $\Rightarrow [X \to Y]$ topological.

Convergence in Topological Spaces

For topological space $(X, \mathcal{O})$,
filter $\mathcal{A}$ on X, and $x \in X$, define:

$$\mathcal{A} \downarrow_{\mathcal{O}} x \iff \forall \text{open } O \ni x\colon O \in \mathcal{A}$$
$$\iff \mathcal{A} \supseteq \mathcal{N}(x)\,.$$

Theorem:

$f : (X, \mathcal{O}) \to (Y, \mathcal{O}')$ continuous

$\iff (\mathcal{A} \downarrow_{\mathcal{O}} x \;\Rightarrow\; f^{+}\mathcal{A} \downarrow_{\mathcal{O}'} fx)$

where

$$f^{+}\mathcal{A} = \{B \subseteq Y \mid \exists A \in \mathcal{A} : f^{+}A \subseteq B\}$$
$$= \{B \subseteq Y \mid f^{-}B \in \mathcal{A}\}\,.$$

Convergence Spaces

$(X, \downarrow)$ where '$\downarrow$' is a relation
between filters and points, with

- $[x] \downarrow x \quad \forall x \in X$
- $\mathcal{A} \downarrow x,\ \mathcal{A}' \supseteq \mathcal{A} \;\Rightarrow\; \mathcal{A}' \downarrow x$.

A function $f : (X, \downarrow) \to (Y, \downarrow')$ is continuous
$\iff (\mathcal{A} \downarrow x \;\Rightarrow\; f^{+}\mathcal{A} \downarrow' fx)\,.$

Category CONV $\supseteq$ TOP

- has products and subspaces,
  which are compatible with those in TOP,
- and function spaces $[Y \to Z]$ such that

  $f : X \times Y \to Z$ continuous $\iff$
  $\tilde{f} : X \to [Y \to Z]$ continuous.

Induced Topology

For a convergence space X, define

$$O \subseteq X \text{ open} \iff \chi_O : X \to \Omega \text{ continuous}$$
$$\iff (\mathcal{A} \downarrow x \in O \;\Rightarrow\; O \in \mathcal{A}).$$

This gives a topological space $\mathsf{T}X$.

TOP $\subseteq$ CONV is reflective subcategory,
T : CONV $\to$ TOP is reflection.
X topological $\iff$ $\mathsf{T}X = X$.

Induced preorder on X =
specialisation preorder of $\mathsf{T}X$:

$$x_1 \sqsubseteq x_2 \iff u\,x_1 \sqsubseteq u\,x_2 \quad \forall u \in [X \to \Omega].$$

X and $\mathsf{T}X$ have the same induced preorder.

Classes of Nice Spaces

These are classes N of convergence spaces
with

- TOP $\subseteq N \subseteq$ CONV,
- N is closed under product and subspace,
- $Y \in N \;\Rightarrow\; [X \to Y] \in N$.

Some such classes:

- "Limit spaces":

  $\mathcal{A} \downarrow x,\ \mathcal{B} \downarrow x \;\Rightarrow\; (\mathcal{A} \cap \mathcal{B}) \downarrow x$.

- Up-nice spaces:

  $\mathcal{A} \downarrow x \;\Rightarrow\; (\uparrow\!\mathcal{A}) \downarrow x$
  where $\uparrow\!\mathcal{A} = \{B \mid \exists A \in \mathcal{A} : \uparrow\!A \subseteq B\}$.

- Down-nice spaces:

  $\mathcal{A} \downarrow x,\ x \sqsupseteq x' \;\Rightarrow\; \mathcal{A} \downarrow x'$.

D-Spaces

X is a d-space if

- X is up-nice and down-nice,
- its induced preorder forms a dcpo,
- the limit sets $\{x \in X \mid \mathcal{A} \downarrow x\}$ of all filters $\mathcal{A}$ are closed under directed joins.

Categories D-CONV and D-TOP.

Properties:

- Continuous functions between d-spaces are Scott continuous
  $\Rightarrow$ functors $\mathsf{D} : \text{D-CONV} \to \text{DCPO}$
  $\mathsf{D} : \text{D-TOP} \to \text{DCPO}$
- D-CONV and D-TOP are closed under products and equalisers;
  D preserves these.
- $Y \in \text{D-CONV} \;\Rightarrow\; [X \to Y] \in \text{D-CONV}$.

Strongest D-Spaces

Given a dcpo D, what is the strongest '$\downarrow$' such that $(D, \downarrow)$ is a d-space with induced dcpo D?

- The strongest topological one is '$\downarrow_{\mathsf{S}}$', derived from the Scott topology:

  $\mathcal{A} \downarrow_{\mathsf{S}} x \iff \forall$ Scott open $O \ni x$: $O \in \mathcal{A}$

- The strongest of all is '$\downarrow_{\mathsf{C}}$', defined by

  $\mathcal{A} \downarrow_{\mathsf{C}} x \iff \forall$ Scott open $O \ni x$: $\exists x' \in O$: $\uparrow x' \in \mathcal{A}$

Define $D_{\mathsf{S}} = (D, \downarrow_{\mathsf{S}})$ and $D_{\mathsf{C}} = (D, \downarrow_{\mathsf{C}})$.

Cotopological Dcpo's

- $\mathsf{T} D_{\mathsf{C}} = D_{\mathsf{S}}$ (open = Scott open).
- $f : D_{\mathsf{C}} \to E_{\mathsf{C}}$ continuous
  $\iff f : D_{\mathsf{S}} \to E_{\mathsf{S}}$ continuous
  $\iff f : D \to E$ Scott continuous.
- $(-)_{\mathsf{S}}$ and $(-)_{\mathsf{C}}$ embed DCPO into CONV.
- $(-)_{\mathsf{S}}$ is left adjoint to $\mathsf{D} : \text{D-TOP} \to \text{DCPO}$,
  $(-)_{\mathsf{C}}$ is left adjoint to $\mathsf{D} : \text{D-CONV} \to \text{DCPO}$.
- $D_{\mathsf{C}} = D_{\mathsf{S}}$
  $\iff D_{\mathsf{C}}$ topological
  $\iff D$ continuous dcpo.

(continuous = topological + cotopological)

Cotopological Dcpo's and Lattices

- D, E dcpo's: $(D \times E)_{\mathsf{C}} = D_{\mathsf{C}} \times E_{\mathsf{C}}$
- D_i dcpo's, almost all pointed:
  $(\prod_{i \in I} D_i)_{\mathsf{C}} = \prod_{i \in I} (D_i)_{\mathsf{C}}$
- $[X \to D_{\mathsf{C}}]$ is d-space,
  directed joins are pointwise.
- L complete lattice:
  $\vee : L_{\mathsf{C}} \times L_{\mathsf{C}} \to L_{\mathsf{C}}$ is always continuous.
- $\wedge : L_{\mathsf{C}} \times L_{\mathsf{C}} \to L_{\mathsf{C}}$ is continuous
  $\iff L_{\mathsf{C}}$ is "limit space".
- $[X \to L_{\mathsf{C}}]$ is complete lattice,
  all joins are pointwise.
- $[X \to L_{\mathsf{C}}]$ is cotopological
  if X is topological.

Cotopological Lattices

- In a complete lattice, we have
  $\mathcal{A} \downarrow_{\mathsf{C}} x \iff x \leq \bigvee_{A \in \mathcal{A}} \bigwedge A.$

- For topological X:
  $f : X \to L_{\mathsf{C}}$ continuous $\iff$
  $fx = \bigvee_{U \in \mathcal{O}(x)} \bigwedge f^{+}U \quad \forall x \in X$
  where $\mathcal{O}(x) = \{U \text{ open} \mid U \ni x\}.$

- Injectivity:

$$\begin{tikzcd} Y \arrow[r, "\widehat{f}"] & L_{\mathsf{C}} \\ X \arrow[u, hook, "e"] \arrow[ur, "f"'] & \end{tikzcd}$$

L complete lattice
X, Y topological spaces
$e : X \hookrightarrow Y$ embedding
f continuous

$\Rightarrow$ $\exists$ continuous $\widehat{f}$ such that $\widehat{f} \circ e = f$;
$\widehat{f}y = \bigvee_{V \in \mathcal{O}(y)} \bigwedge f^{+}(e^{-}V).$

Topological Function Spaces

Theorem:
$X = D_{\mathsf{C}}$ cotopological dcpo, Y topological
$\Rightarrow$ $[X \to Y]$ topological with subbasis
$\{\langle x \to V \rangle \mid x \in X, V \in \Omega Y\}$
where $\langle x \to V \rangle = \{f \mid fx \in V\}$
(point-open topology).

Theorem: For any space X:
$\Omega X = [X \to \Omega]$ is topological
$\iff$ $[X \to Y]$ is topological $\forall$ topological Y.

Recall: X topological $\Rightarrow$ ΩX cotopological.

Corollary: For topological space X:
ΩX is continuous lattice
$\iff$ $[X \to Y]$ is topological $\forall$ topological Y.

Here, $[X \to Y]$ has the Isbell topology.

Spaces of Valuations

X topological
$\Rightarrow$ $\Omega X = [X \to \Omega]$ cotopological
$\Rightarrow$ $[\Omega X \to \overline{\mathsf{R}}_{+}]$ topological.

Points: Scott continuous functions;
subbasis $\{\langle O > r \rangle \mid O \in \Omega X, r \in \overline{\mathsf{R}}_{+}\}$
where $\langle O > r \rangle = \{\nu \mid \nu O > r\}.$

Subspace of strict and modular functions gives space VX of valuations as defined earlier.

Conclusion:
Natural power space constructions in CONV cut down to familiar constructions in TOP justifying the chosen topologies.

Conclusion

- Extension of TOP to CONV gives function spaces,
- and allows new way to look at dcpo's and complete lattices,
- avoiding some problems of the Scott topology view.
- Extension of properties of continuous lattices to all complete lattices.
- Point-open topology is the right one in certain situations.
- New proofs of old theorems.