Bicontinuous Function Spaces

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Abstract

Given a sober space $(X, \mathcal{O}(X))$ and a complete lattice L in its Scott-topology, we study the function space $[X \to L]$ of all continuous maps $f\colon X \to L$, ordered pointwise. We show that this partial order is a bicontinuous lattice (i.e. the lattice and its order dual are continuous) if and only if L is bicontinuous, X is a continuous domain and $\mathcal{O}(X)$ is its Scott-topology. This extends known results on the continuity of the space $[X \to L]$. The techniques are novel in the theory of continuous lattices in that they employ a representation of the dual of $[X \to L]$ as the lattice of maps preserving all suprema of type $\mu\colon \mathcal{O}(X) \to L^{\mathrm{op}}$, where L^{op} is the order dual of L. We specialize these results down to two classes of bicontinuous lattices: linear FS-lattices and completely distributive lattices.

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1 Introduction

For a complete lattice L, a subset U is *Scott open* if it is an upper set that is *inaccessible by directed suprema:* if $D \subseteq L$ is directed and $\bigvee D \in U$, then $D \cap U \neq \emptyset$. This defines a T_0-topology on any complete lattice (the closure of a point is just its lower set), and a map $f\colon L \to M$ between complete

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lattices is *Scott continuous* exactly if it preserves suprema of directed sets. For complete lattices L and M, we write $[L \to M]$ for the complete lattice of all Scott-continuous functions $f: L \to M$ ordered pointwise: $f \leq g$ iff $f(l) \leq g(l)$ for all $l \in L$. The Scott topology of L then is the set

$$\sigma_L = \{f^{-1}(1) \mid f \in [L \to \{0,1\}]\}.$$

One can readily check that $[L \to M]$ viewed as a set contains exactly the continuous maps $f: (L, \sigma_L) \to (M, \sigma_M)$.

One definition of *continuous lattices* [2] can be formulated in terms of *Scott-continuous deflations* – Scott-continuous selfmaps $f: L \to L$ with finite image such that $f \leq \mathsf{id}_L$, where id_L is the identity function on L.

Fact 1 [1] *A complete lattice L is continuous if and only if the set $\mathcal{D}$ of all Scott-continuous deflations on L is directed and its supremum equals id_L in $[L \to L]$. In this case we have $l' \ll l \in L$ if and only if $l' \leq f(l)$ for some $f \in \mathcal{D}$.*

The more traditional approach to defining continuous lattices is given as follows. For a complete lattice L, we say $l \in L$ is *way-below* $l' \in L$ if for all directed sets $D \subseteq L$ with $l \leq \bigvee D$ we have $l' \leq d$ for some $d \in D$; we then write $l' \ll l$. Then continuous lattices are those complete lattices L in which each element $l \in L$ is the (directed) supremum of the elements way-below it.

The mathematical structure of continuous lattices, in particular their closure properties with respect to constructions such as function spaces and retracts, has been studied extensively. However, very little has been said about lattices which are bicontinuous:

Definition 2 *A complete lattice L is* bicontinuous *if L and its order dual L^{op} are continuous lattices.*

About the only mention of bicontinuous lattices in [2] concerns completely distributive lattices L [7, 8], which turn out to be exactly those bicontinuous lattices L which are distributive [2]. Three striking features of completely distributive lattices are that

1. they are closed under retractions preserving all suprema,
2. they are closed under $(M, N) \mapsto (M \multimap N)$, where $(M \multimap N)$ is the complete lattice of maps $f: M \to N$ preserving all suprema, ordered pointwise, and

3. their Scott-topology is the topology induced by upper sets of the form $L \setminus \downarrow x$, where $\downarrow x = \{y \in L \mid y \leq x\}$.

In [4], one finds a generalization of complete distributivity: linear FS-lattices are bicontinuous lattices which share all the features above. These lattices generalize completely distributive lattices in the sense that the distributive linear FS-lattices are exactly those that are completely distributive. The definition of linear FS-lattices is motivated by the work on FS-domains in [5].

Definition 3 *We say that $f \in (L \multimap L)$ is* finitely separated [5] *from* id_L *if there exists some finite set $F \subseteq L$ such that for all $x \in L$, there exists some $x_f \in F$ with $f\,x \leq x_f \leq x$. A* linear FS-lattice *is a complete lattice L for which there exists a directed set $\mathcal{D} \subseteq (L \multimap L)$ whose supremum is id_L, and where all $f \in \mathcal{D}$ are finitely separated from id_L.*

The goal of this paper is to derive another principle for constructing bicontinuous lattices, namely that of the function space $[X \to L]$:

Definition 4 *Let $(X, \mathcal{O}(X))$ be a topological space and L a complete lattice with Scott topology σ_L. Then $[X \to L]$ is defined as the partial order of all continuous maps $f\colon (X, \mathcal{O}(X)) \to (L, \sigma_L)$, ordered pointwise.*

Note that this agrees with the ordinary definition of $[M \to N]$ if M and N are complete lattices in their Scott topology. In [2] we find criteria for the partial order $[X \to L]$ to be a continuous lattice:

> **Theorem A** (Theorem 4.7, [2]) Let X be non-empty and L be non-degenerate. Then $[X \to L]$ is a continuous lattice if and only if $\mathcal{O}(X)$ and L are continuous lattices. □

Our goal is a corresponding characterization for the bicontinuity of this space. Then we transfer these results to two special classes of bicontinuous lattices: linear FS-lattices and completely distributive lattices. It turns out that L now has to be bicontinuous. But the condition for X changes as well: it has to be a *continuous space*.

To explain, we first recall some terminology. A *domain* X is a partial order in which each directed subset has a supremum. A domain is *continuous* if each element is the directed supremum of elements way-below it, where $x \ll x'$ is defined in the same way as for complete lattices. The *specialization* *order* on a topological space $(X, \mathcal{O}(X))$ is given by $x \leq_s y$ if and only if $\mathcal{F}_x \subseteq \mathcal{F}_y$, where $\mathcal{F}_x$ denotes the completely prime filter in $\mathcal{O}(X)$ of all $O \in \mathcal{O}(X)$ containing x. Recall that X is *sober* if every completely prime filter $\mathcal{F} \subseteq \mathcal{O}(X)$ is of the form $\mathcal{F}_x$ for a unique $x \in X$.

Definition 5 *A sober space $(X, \mathcal{O}(X))$ is a* continuous space *if:*

- *X is a continuous domain in its specialization order, and*
- *$\mathcal{O}(X)$ is the Scott topology with respect to the specialization order.*

2 Bicontinuity of $[X \to L]$

We begin by assuming that $[X \to L]$ is a bicontinuous lattice. In particular, this assumes that $[X \to L]$ is a complete lattice, which it need not be in general. By Theorem A we know that $\mathcal{O}(X)$ and L must be continuous if $[X \to L]$ is continuous. Using the continuity of $[X \to L]^{\mathrm{op}}$, we claim that L^{op} and X are continuous, too. Both claims are consequences of an alternative representation of $[X \to L]^{\mathrm{op}}$ which was originally shown in [3]. We include a proof here for the sake of completeness, but before doing so, we introduce some notation.

Working with L and L^{op} simulataneously may cause confusion. For $a, b \in L$, $a \leq b$ iff $b \leq^{\mathrm{op}} a$, where $\leq$ is the order on L, and $\leq^{\mathrm{op}}$ is the order on L^{op}. Thus suprema in L correspond to infima in L^{op} and vice versa. Therefore, $\bigvee_{i \in I} a_i = \bigwedge^{\mathrm{op}}_{i \in I} a_i$ and $\bigwedge_{i \in I} a_i = \bigvee^{\mathrm{op}}_{i \in I} a_i$. In other words, moving from a lattice to its opposite simply interchanges suprema and infima. Throughout the following proof, we consistently use $^{\mathrm{op}}$ to denote an operator in L^{op}, while an operator without an $^{\mathrm{op}}$ is meant in L.

Theorem 6 *Let L be a continuous lattice and $(X, \mathcal{O}(X))$ a topological space. Then $[X \to L]$ is a complete lattice and its order dual is isomorphic to the function space $(\mathcal{O}(X) \multimap L^{\mathrm{op}})$. The isomorphism is induced by the following two order-reversing functions:*

$$\Psi\colon [X \to L] \to (\mathcal{O}(X) \multimap L^{\mathrm{op}}) \text{ by } \Psi\, f\, O = \textstyle\bigvee^{\mathrm{op}}_{x \in O} f\, x, \text{ and}$$
$$\Phi\colon (\mathcal{O}(X) \multimap L^{\mathrm{op}}) \to [X \to L] \text{ by } \Phi\, \mu\, x = \textstyle\bigvee^{\uparrow}_{O \in \mathcal{F}_x} \mu\, O.$$

Remark: Note that for $f \in [X \to L]$ and $x \in O \in \mathcal{O}(X)$, $f\,x$ is in L, but the supremum for $\Psi\, f\, O$ is taken in L^{op}, which is to say it is the infimum in L. Likewise, for $\mu \in (\mathcal{O}(X) \multimap L^{\mathrm{op}})$ and $O \in \mathcal{F}_x, x \in X$, $\mu\, O$ is in L^{op}, but the supremum for $\Phi\, \mu\, x$ is taken in L. This supremum is directed because $\mathcal{F}_x$ is a filter and $\mu : \mathcal{O}(X) \to L^{\mathrm{op}}$ is monotonic.

Proof. Given $f \in [X \to L]$, the function $\Psi\, f$ preserves all suprema by the general associativity of suprema. Therefore Ψ is well-defined as a set-theoretic map. To see that Ψ is order-reversing, note that the order in $[X \to L]$ is pointwise. Thus, $f \leq g$ in $[X \to L]$ means $f\,x \leq g\,x$ in L for all $x \in X$, which readily entails $\bigvee^{\mathrm{op}}_{x \in O} g\,x \leq^{\mathrm{op}} \bigvee^{\mathrm{op}}_{x \in O} f\,x$ in L^{op} for all $O \in \mathcal{O}(X)$, whence $\Psi\, g \leq \Psi\, f$ in $(\mathcal{O}(X) \multimap L^{\mathrm{op}})$.

Next, we show that Φ is well-defined. Given $\mu \in (\mathcal{O}(X) \multimap L^{\mathrm{op}})$ we need to show that f, defined as $\Phi\, \mu$, is a continuous map between X and L in its Scott topology. To this end let V be a Scott-open subset of L. We are done if $f^{-1}(V)$ is open in X. But $x \in f^{-1}(V)$ means that $\bigvee^{\uparrow}_{O \in \mathcal{F}_x} \mu\, O \in V$ where the supremum is directed because $\mathcal{F}_x$ is a filter. Since V is Scott open, we conclude that there must be some $O \in \mathcal{F}_x$ such that $\mu\, O \in V$. Since $x \in O$ it suffices to show $O \subseteq f^{-1}(V)$. Given any $x' \in O$ we have $O \in \mathcal{F}_{x'}$ so $f\,x' = \bigvee^{\uparrow}_{O' \in \mathcal{F}_{x'}} \mu\, O' \geq \mu\, O$. Since $\mu\, O \in V$ we get $f\,x' \in V$, since V is an upper set in L.

To show that Φ is order-reversing, let $\mu \leq \nu$ in $(\mathcal{O}(X) \multimap L^{\mathrm{op}})$. Then $\mu\, O \leq^{\mathrm{op}} \nu\, O$ in L^{op} for all $O \in \mathcal{O}(X)$ implies that $\bigvee^{\uparrow}_{O \in \mathcal{F}_x} \mu\, O \geq \bigvee^{\uparrow}_{O \in \mathcal{F}_x} \nu\, O$ in L, so $\Phi\, \mu \geq \Phi\, \nu$ holds in $[X \to L]$.

We still need to show that these two maps compose to the respective identities. Let ν be the map $\Psi(\Phi\, \mu)$. For any $O \in \mathcal{O}(X)$ we have $\nu\, O = \bigvee^{\mathrm{op}}_{x \in O} \Phi\, \mu\, x = \bigvee^{\mathrm{op}}_{x \in O} \bigvee^{\uparrow}_{U \in \mathcal{F}_x} \mu\, U = \bigvee^{\mathrm{op}}_{x \in O} \bigwedge^{\mathrm{op}}_{U \in \mathcal{F}_x} \mu\, U$ in L^{op}. For all $x \in O$ we obtain $\bigwedge^{\mathrm{op}}_{U \in \mathcal{F}_x} \mu\, U \leq^{\mathrm{op}} \mu\, O$, which shows $\nu\, O \leq^{\mathrm{op}} \mu\, O$ in L^{op}.

To establish the reverse inequality, we show $\nu\, O \leq \mu\, O$ in L. Since L is continuous it suffices to prove that $l \ll \nu\, O$ implies $l \leq \mu\, O$. Note that $\nu\, O = \bigwedge_{x \in O} \bigvee^{\uparrow}_{U \in \mathcal{F}_x} \mu\, U$ holds in L. Given $l \ll \nu\, O$ we have in particular that $l \ll \bigvee^{\uparrow}_{U \in \mathcal{F}_x} \mu\, U$ for all $x \in O$. Since this supremum is directed, we know that for every $x \in O$, there is some $U_x \in \mathcal{F}_x$ such that $l \leq \mu\, U_x$. Using that μ preserves suprema we conclude $l \leq \bigwedge_{x \in O} \mu\, U_x = \bigvee^{\mathrm{op}}_{x \in O} \mu\, U_x = \mu\left(\bigcup_{x \in O} U_x\right) \leq \mu\, O$ where the last relation holds since $x \in U_x$ entails $O \subseteq \bigcup_{x \in O} U_x$ and μ is monotone. By continuity of L this proves $\nu\, O \leq \mu\, O$ in L.

Second, for $f \in [X \to L]$ we define g to be $\Phi(\Psi\, f)$. Then for any $x \in X$ we have $g\,x = \bigvee^{\uparrow}_{O \in \mathcal{F}_x} \Phi\, f\, O = \bigvee^{\uparrow}_{O \in \mathcal{F}_x} \bigvee^{\mathrm{op}}_{y \in O} f\,y = \bigvee^{\uparrow}_{O \in \mathcal{F}_x} \bigwedge_{y \in O} f\,y$. This shows

$g\,x \leq f\,x$ in L since for all $O \in \mathcal{F}_x$ we have $\bigwedge_{y \in O} f\,y \leq f\,x$.

Again, we use the continuity of L to show the reverse inequality $f\,x \leq g\,x$. So let $l \ll f\,x$ in L be given. We are done if $l \leq g\,x$ holds in L. The set $V = \Uparrow l = \{a \in L \mid l \ll a\}$ is Scott-open in L as L is continuous, so $f^{-1}(V)$ is open in X since $f\colon X \to L$ is continuous. For all $y \in f^{-1}(V)$ we have $f\,y \in V$ and in particular $l \leq f\,y$ in L. Thus, $l \leq \bigwedge_{y \in f^{-1}(V)} f\,y$. But since $l \ll f\,x$ we have $x \in f^{-1}(V)$, i.e. $f^{-1}(V) \in \mathcal{F}_x$ and therefore $l \leq \bigwedge_{y \in f^{-1}(V)} f\,y \leq g\,x$ in L. □

The representation theorem above is the key to almost all the results presented in this paper. We use it to make inferences from the continuity of the dual of $[X \to L]$. Since the (bi)continuity of $[X \to L]$ implies the continuity of L, we may apply the theorem above and think of the order dual of $[X \to L]$ as $(\mathcal{O}(X) \multimap L^{\mathrm{op}})$. Thus, this function space is continuous. What makes this worthwhile is that continuity of this space enforces continuity of its arguments in the respective polarities given by the hom-functor $(\ \multimap\ )$. To prove this and some additional variants involving FS-lattices and completely distributive lattices, we first note a simple symmetry property of the linear function space [4]:

Proposition 7 *For all complete lattices M and N, $(M \multimap N)$ and $(N^{\mathrm{op}} \multimap M^{\mathrm{op}})$ are isomorphic.*

Proof. Every map $f \in (M \multimap N)$ preserving all suprema has an upper adjoint $g : N \to M$ that preserves all infima. If we turn both lattices around, g becomes a map from N^{op} to M^{op} preserving all suprema. The correspondence between lower and upper adjoints is order-reversing, so we obtain a monotonic correspondence between $(M \multimap N)$ and $(N^{\mathrm{op}} \multimap M^{\mathrm{op}})$. Since adjoints determine each other uniquely, this is an isomorphism. □

As announced above, we use this symmetry to show how properties of $(M \multimap N)$ infer properties of M^{op} and N.

Proposition 8 *Let M and N be non-degenerate complete lattices. Then M^{op} and N are sup-closures of $(M \multimap N)$. Thus:*

1. *If $(M \multimap N)$ is continuous, then so are M^{op} and N.*
2. *If $(M \multimap N)$ is a linear FS-lattice, then so are M^{op} and N.*
3. *If $(M \multimap N)$ is completely distributive, then so are M^{op} and N.*

Proof. Because of $(M \multimap N) \cong (N^{\mathrm{op}} \multimap M^{\mathrm{op}})$, it suffices to show the claims for N. We may think of an element n of N as a strict constant function: let $l_n\colon M \to N$ map 0_M to 0_N and all other elements to n. Then define $G\colon N \to (M \multimap N)$ as $Gn = l_n$. Clearly, this is well-defined and monotone. Mapping $(M \multimap N)$ to N is easy since we can evaluate functions; for our purposes we need to evaluate them at 1_M. So $F\colon (M \multimap N) \to N$ maps f to $f(1_M)$. Since suprema in $(M \multimap N)$ are computed pointwise, both F and G preserve suprema. For $m \neq 0_M$, we have $G(F(f))(m) = F(f) = f(1_M) \geq f\,m$, and both f and $G(F(f))$ map 0_M to 0_N. Thus, we have $f \leq G(F(f))$.

Conversely, since M has more than one element, we see that $F(G(n)) = n$ for all $n \in N$. Thus, $G \circ F$ is a closure operator preserving all suprema on $(M \multimap N)$ whose image is isomorphic to N.

Finally, we provide arguments for the three numbered statements.

1. Continuous lattices are closed under Scott-continuous retracts.

2. Linear FS-lattices are closed under sup-retracts [4].

3. Completely distributive lattices are closed under sup-retracts [2]. □

The results above allow us to characterize the circumstances under which the function space $[X \to L]$ is bicontinuous. Since we rely on the framework of Stone duality we need to impose on X the very mild restriction of being a sober space.

Theorem 9 *Let $(X, \mathcal{O}(X))$ be a non-empty sober space and L a non-degenerate complete lattice for which $[X \to L]$ (in the pointwise order) is a bicontinuous lattice. Then L is bicontinuous and X is a continuous space.*

Proof. Since $[X \to L]$ is continuous we know that $\mathcal{O}(X)$ and L are continuous as well (Theorem A). By Theorem 6, we infer that the order dual of $[X \to L]$ is isomorphic to $(\mathcal{O}(X) \multimap L^{\mathrm{op}})$. Thus, the function space $(\mathcal{O}(X) \multimap L^{\mathrm{op}})$ is continuous. By Proposition 8, we conclude that $\mathcal{O}(X)^{\mathrm{op}}$ and L^{op} are continuous. Therefore, both lattices are bicontinuous. Since $\mathcal{O}(X)$ is also distributive as a lattice of opens, it is completely distributive. Because $(X, \mathcal{O}(X))$ is sober we may use Stone duality to infer that X is a continuous space [6, 1]. □

We conclude this section with a proof of the converse of Theorem 9. So let X be a continuous space and L bicontinuous. Then $\mathcal{O}(X)$ is certainly continuous, so $[X \to L]$ is continuous by Theorem A. As L is continuous we may use Theorem 6, so it suffices to prove that $(\mathcal{O}(X) \multimap L^{\mathrm{op}})$ is continuous.

Proposition 10 *Let X be a continuous space and L a continuous lattice. Then $(\mathcal{O}(X) \multimap L)$ is a Scott-continuous projection of $[\mathcal{O}(X) \to L]$. In particular, $(\mathcal{O}(X) \multimap L)$ is continuous and its way-below relation is induced by the one on $[\mathcal{O}(X) \to L]$.*

Proof. We define $P: [\mathcal{O}(X) \to L] \to [\mathcal{O}(X) \to L]$ by $P\mu\, O = \bigvee_{x \in O} \mu(\Uparrow x)$.

1. For μ in $[\mathcal{O}(X) \to L]$, $P\mu$ preserves all suprema by the general associativity of suprema. Thus, the image of P is contained in $(\mathcal{O}(X) \multimap L)$.

2. P is obviously monotonic. Since directed suprema are computed pointwise in $[\mathcal{O}(X) \to L]$, P is Scott continuous by the general associativity of suprema.

3. Since $x \in O$ implies $\Uparrow x \subseteq O$ we infer that $P\mu\, O \leq \mu\, O$. Thus $P\mu \leq \mu$, and so $P \leq \mathsf{id}_{[\mathcal{O}(X) \to L]}$.

4. Since the way-below relation on the image of a Scott-continuous projection is induced by that of the global space, it remains to show only that P is idempotent. So let $\mu \in (\mathcal{O}(X) \multimap L)$ and $O \in \mathcal{O}(X)$. Then

$$P\mu\, O = \bigvee_{x \in O} \mu(\Uparrow x) = \mu\left(\bigcup_{x \in O} \Uparrow x\right) = \mu\, O$$

since X is continuous. □

Now we can summarize.

Theorem 11 *Let $(X, \mathcal{O}(X))$ be a non-empty sober space and L a non-degenerate complete lattice. Then the following are equivalent:*

1. *The partial order $[X \to L]$ is a bicontinuous lattice.*

2. *The lattice L is bicontinuous and X is a continuous space.*

Proof. By Theorem 9, we only need to show that 2) implies 1). By Proposition 10, we know that $(\mathcal{O}(X) \multimap L^{\mathrm{op}})$ is continuous, and Theorem 6 implies this lattice is isomorphic to the order dual of $[X \to L]$. In particular, $[X \to L]$ is a complete lattice. But $[X \to L]$ also is continuous by Theorem A since $\mathcal{O}(X)$ and L are continuous. □

3 Linear FS-lattices of Type $[X \to L]$

We now show how these results transfer to a special class of bicontinuous lattices: linear FS-lattices.

Theorem 12 *Let $(X, \mathcal{O}(X))$ be a non-empty sober space and L a non-degenerate complete lattice. If the partial order $[X \to L]$ is a linear FS-lattice, then X is a continuous space and L a linear FS-lattice.*

Proof. Since linear FS-lattices are bicontinuous [4] we already have established the facts about X. This also guarantees that L is continuous. Since the order dual of a linear FS-lattice is a linear FS-lattice [4], we see that the order dual of $[X \to L]$ is a linear FS-lattice. Using the continuity of L and Theorem 6, we see that $(\mathcal{O}(X) \multimap L^{\mathrm{op}})$ is a linear FS-lattice. By Proposition 8, L^{op} is a linear FS-lattice, too. $\square$

As for the converse, we are dealing with a linear FS-lattice L and a continuous space X, and we need to show that $[X \to L]$ is a linear FS-lattice. Again, L is continuous and linear FS-lattices are closed under order duals, so it suffices to show that $(\mathcal{O}(X) \multimap L^{\mathrm{op}})$ is a linear FS-lattice by Theorem 6. Since X is a continuous space, the lattice $\mathcal{O}(X)$ is completely distributive and therefore a linear FS-lattice [4]. Thus, $\mathcal{O}(X)$ and L^{op} are linear FS-lattices. But linear FS-lattices are closed under $(M, N) \mapsto (M \multimap N)$ [4].

Theorem 13 *Let $(X, \mathcal{O}(X))$ be a non-empty sober space and L a non-trivial complete lattice. Then the partial order $[X \to L]$ is a linear FS-lattice if and only if L is a linear FS-lattice and X is a continuous space.* $\square$

4 Complete Distributivity of $[X \to L]$

Completely distributive lattices are known to be closed under retracts that preserve all suprema. Furthermore, in [4] it has been observed that linear FS-lattices are distributive exactly if they are completely distributive. Thus, we may transfer one half of Theorem 13 to the realm of completely distributive lattices:

Proposition 14 *Let $[X \to L]$ be a completely distributive lattice where X is non-empty sober and L is non-degenerate. Then X is a continuous space and L is completely distributive.*

Proof. By Theorem 13, we only need to show that L is completely distributive. But this follows along similar lines as above since L^{op} is a sup-closure of $(\mathcal{O}(X) \multimap L^{\mathrm{op}})$. Since the order dual of a completely distributive lattice is again completely distributive, this means that L is completely distributive. $\Box$

For the converse, we have a continuous space X and a completely distributive lattice L. By Theorem 13, we know that $[X \to L]$ is a linear FS-lattice. Therefore, it suffices to show that the lattice $[X \to L]$ is distributive. The binary supremum operation in $[X \to L]$ is computed pointwise [1], and since L is continuous, we know that the infimum map $\wedge : L \times L \to L$ is Scott-continuous. A standard argument then shows $(f \wedge g)\, x = f\, x \wedge g\, x$. Therefore, $[X \to L]$ is distributive since L is distributive.

Theorem 15 *Let $(X, \mathcal{O}(X))$ be a sober space and L a non-degenerate complete lattice. Then $[X \to L]$ is a completely distributive lattice in the pointwise order if and only if L is completely distributive and X a continuous space.* $\Box$

5 Consequences

If M and N are continuous lattices then $(M \multimap N)$ need not be continuous again. In fact, if $N = M$ and if the Scott topology of M is the upper topology then the continuity of $(M \multimap N)$ is equivalent to M being a linear FS-lattice [4]. However, things work out better if M is completely distributive and N is continuous.

Theorem 16 *Let M be a completely distributive and N a continuous lattice. Then the lattice $(M \multimap N)$ is continuous.*

Proof. By Stone duality, M is the Scott topology $\mathcal{O}(X)$ of a continuous space X. By Proposition 10, we see that $(M \multimap N) \cong (\mathcal{O}(X) \multimap N)$ is continuous. $\Box$

Theorem 17 *Let X be a continuous space and L a complete lattice such that L^{op} is continuous. Then the order dual of $[X \to L]$ is a continuous lattice.*

Proof. In [3], it has been shown that the isomorphism of Theorem 6 holds without the continuity of L provided that X be a continuous space. Thus, the order dual of $[X \to L]$ is isomorphic to $(\mathcal{O}(X) \multimap L^{\mathrm{op}})$ which is continuous by Proposition 10. $\square$

Theorem 18 *Let $\mathcal{O}(X)$ be a continuous lattice with X sober. Then $(\mathcal{O}(X) \multimap L)$ is continuous if and only if L is continuous and X is a continuous space.*

Proof. The "if" part has been shown in Proposition 10. For the "only if" part, we know that L is continuous from Proposition 8. The same proposition also tells us that $\mathcal{O}(X)^{\mathrm{op}}$ is continuous. Thus, $\mathcal{O}(X)$ is completely distributive, and so X is a continuous space by Stone duality. $\square$

Note that for the "only if" part to fail, it must be the case that $\mathcal{O}(X)$ is not continuous. In particular, X could not be a locally compact space [2]. One might wonder whether this result can be rephrased for $[X \to L^{\mathrm{op}}]^{\mathrm{op}}$? Again, the "if" part is not a problem because of Theorem 17.

If M and N are bicontinuous lattices then M is certainly a continuous space. By Theorem 11, the lattice $[M \to N]$ is bicontinuous as well. Since bicontinuous lattices are closed under finite products we see that bicontinuous lattices form a full cartesian closed subcategory of TOP.

Corollary 19 *The category of bicontinuous lattices and Scott-continuous maps is cartesian closed.* $\square$

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