Characterising FS Domains by means of Power Domains

Reinhold Heckmann

FB 14 – Informatik, Universität des Saarlandes,
Postfach 151150, D-66041 Saarbrücken, Germany
e-mail: heckmann@cs.uni-sb.de

Abstract

FS domains can be characterised using the upper or lower power domain construction. In these characterisations, separation by the elements of a finite set is replaced by separation by a continuous non-deterministic function with finite image. This notion of separation can be formalised in many different, but equivalent ways.

1 Introduction

The category CONT of continuous dcpo's ("domains") and Scott continuous functions is not Cartesian closed, i.e., closed under finite products and function space formation [1, p. 42]. Thus, people looked for full subcategories of CONT which are Cartesian closed, and in particular for maximal ones. Some time ago, the category FC of *finitely continuous* dcpo's (or retracts of bifinite domains) was considered as a candidate [6]. It is Cartesian closed, but its maximality is still an open question. Later, Achim Jung [7] came up with the category FS of *finitely separated domains*, which is a maximal Cartesian closed full subcategory of CONT if one restricts attention to pointed dcpo's (those with $\perp$). FS contains FC, but it is an open question whether FS and FC are different or coincide.

The *probabilistic power domain construction* P, introduced by Jones and Plotkin [4,3], is an endofunctor of CONT. However, it is not known whether it is an endofunctor of any Cartesian closed full subcategory of CONT. Several candidate categories are ruled out by explicit counterexamples. The remaining ones are FC and FS, yet it is still unknown whether any of these two is closed under P [10,5].

The story told above indicates that we still do not know enough about FC, FS,

Preprint submitted to Elsevier Preprint *13 April 1999*

and their relationship. The paper at hand adds something to this knowledge by providing several new characterisations of FS domains, equivalent to the original one given by Jung. These characterisations employ separation by continuous non-deterministic functions with finite image. The non-determinism can be of the kind of the upper or lower power domain construction. The exact formalisation of the separation property may vary, yielding several equivalent characterisations for each of the two power domain constructions. Other power domain constructions may be used as well, but we were not able to prove equivalence with FS in those cases. Altogether, we did not solve any of the open problems stated above, but we hope that our characterisations give useful hints which may lead to some solutions in the future.

2 FC and FS domains

In this section, we recapitulate the definitions and basic properties of FC and FS functions and domains.

For the definition of dcpo, directed set, (Scott) continuous function, way-below relation ($\ll$), and continuous dcpo (domain), we refer to [1].

Let D be a dcpo. A continuous function $f: D \to D$ is *finitely continuous* or shortly FC if it is below the identity id_D and has finite image $f(D)$. A continuous function $f: D \to D$ is *finitely separated* or shortly FS if there is a finite subset E of D such that for every x in D, there is some e_x in E such that $fx \sqsubseteq e_x \sqsubseteq x$. Clearly, every FC function is an FS function by taking E to be the finite image $f(D)$ and $e_x = fx$.

In [7], some elementary properties of FS functions are proved. We formulate them as a lemma for reference in later parts of the paper.

Lemma 1 *Let $f: D \to D$ be an FS function witnessed by the finite subset E of D. Then for all x in D, $fx \ll x$ holds, and there is some e_x in E such that $f^2 x \sqsubseteq e_x \ll x$. In particular, $f^2 = f \circ f : D \to D$ is an FS function again. In $[D \to D]$, $f^2 \ll \mathsf{id}_D$ holds.*

A dcpo D is an *FC domain* (*FS domain*) if the identity id_D is a directed join of FC functions (FS functions). FC and FS domains are always continuous dcpo's; thus the name "domain" is justified.

As every FC function is an FS function, every FC domain is an FS domain. Up to now, the converse is an open problem: we do not know whether every FS domain is an FC domain. For an affirmative answer, one should find an FC function g above every FS function f. A straightforward idea is to let

g map into the finite set $E \subseteq D$ used as witness for the FS property of f, defining $gx = e_x$ for all x in D. However, there are often several different e in E satisfying $fx \sqsubseteq e \sqsubseteq x$, and the question is which one to choose. Just choosing any candidate is not sufficient, for g has to be continuous.

In fact, we do not know whether it is possible to find a continuous function $g\colon D \to E$ above every FS function $f\colon D \to D$ with witness $E \subseteq_{\mathsf{fin}} D$ (otherwise the equality FS $\stackrel{?}{=}$ FC were proved). However, we are able to find a *non-deterministic* continuous function g with finite image above every FS function. Here, non-determinism means that g maps from D into some power domain of D. This idea can be realised best if the upper or Smyth power domain UD is considered, yet the lower or Hoare power domain LD is also possible. In the next section, we consider U, while in Section 4, L is considered. In Section 5, we consider some variations of the story.

3 Characterising FS Domains by Upper Power Domains

The upper power domain UD of a domain (continuous dcpo) D consists of all compact upper subsets of D, ordered by opposite inclusion '$\supseteq$' [8,9]. For our purpose, it does not matter whether $\emptyset$ is included in UD or not.

For every domain D, UD is again a domain. There is a continuous function $\mathsf{s} = \mathsf{s}_D : D \to UD$ defined by $\mathsf{s}x = \uparrow_D x$ for every x in D, called the *singleton function*. Actually, it is an order embedding ($x \sqsubseteq y$ iff $\mathsf{s}x \sqsubseteq \mathsf{s}y$), and preserves and reflects the way-below relation ($x \ll y$ iff $\mathsf{s}x \ll \mathsf{s}y$).

The construction U is functorial; for a continuous function $f\colon D \to D'$, the function $Uf\colon UD \to UD'$ with $Uf\,(K) = \uparrow_{D'} f(K)$ is well-defined and continuous.

Now, let $f\colon D \to D$ be an FS function witnessed by $E \subseteq_{\mathsf{fin}} D$. For every x in D, let $gx = \uparrow_D \{e \in E \mid fx \sqsubseteq e\}$. Since E is finite, gx is the up-closure of a finite set and thus a compact upper subset of D. Hence, g is a function from D to UD. By definition, $gx \subseteq \uparrow fx$ holds, and the FS property of f implies $x \in gx$. Thus, $\uparrow x \subseteq gx \subseteq \uparrow fx$, or $\mathsf{s}x \sqsupseteq gx \sqsupseteq \mathsf{s}(fx)$ holds. Hence, we obtain $\mathsf{s} \circ f \sqsubseteq g \sqsubseteq \mathsf{s} = \mathsf{s} \circ \mathsf{id}$, i.e., g is between f and id modulo the order embedding s.

To prove continuity of g, we split it into functions $\phi\colon D \to UE$ and $\psi\colon UE \to UD$, where $\phi x = \{e \in E \mid fx \sqsubseteq e\}$ (a finite upper set) and $\psi K = \uparrow_D K$. The function ϕ is continuous since for all directed families $(x_i)_{i \in I}$

$$e \in \phi\,(\textstyle\bigsqcup_{i \in I} x_i) \iff f\,(\textstyle\bigsqcup_{i \in I} x_i) \sqsubseteq e$$

$$\iff \bigsqcup_{i\in I} fx_i \sqsubseteq e$$
$$\iff \forall i \in I\colon fx_i \sqsubseteq e$$
$$\iff e \in \bigcap_{i\in I} \phi x_i = \bigsqcup_{i\in I} \phi x_i$$

The function ψ is continuous since it is monotonic and UE is finite. Note that $\psi = U\eta$ where $\eta\colon E \hookrightarrow D$ is the subset inclusion map.

What has been achieved so far, is taken as the definition of a U-FS function:

Definition 2 *A continuous function $f\colon D \to D$ is a U-FS function, if there exist a finite subdomain E of D and a continuous function $\phi\colon D \to UE$ such that $\mathsf{s} \circ f \sqsubseteq U\eta \circ \phi \sqsubseteq \mathsf{s}$ holds, where $\eta\colon E \hookrightarrow D$ is the subset inclusion. A domain D is a U-FS domain if the identity id_D is a directed join of U-FS functions.*

The arguments before Definition 2 have proved that every FS function is a U-FS function, and so every FS domain is a U-FS domain. The opposite implication is easily shown: Let f be a U-FS function. For every x in D, $\mathsf{s}(fx) \sqsubseteq U\eta(\phi x) \sqsubseteq \mathsf{s}x$ holds, i.e., $\uparrow_D fx \supseteq \uparrow_D(\phi x) \supseteq \uparrow_D x$. Hence, there is e in $\phi x \subseteq E$ such that $x \sqsupseteq e \sqsupseteq fx$. Thus, f is an FS function.

Theorem 3 *A function $f\colon D \to D$ is an FS function iff it is a U-FS function. A dcpo D is an FS domain iff it is a U-FS domain.*

4 Characterising FS Domains by Lower Power Domains

The lower power domain LD of a domain (continuous dcpo) D consists of all (Scott) closed subsets of D, ordered by inclusion '$\subseteq$'. Like in the case of UD, it does not matter whether $\emptyset$ is included in LD or not.

For every domain D, LD is again a domain. There is a continuous *singleton function* $\mathsf{s} = \mathsf{s}_D : D \to LD$ defined by $\mathsf{s}x = \downarrow_D x$ for every x in D. Like the singleton function of U, it is an order embedding, and preserves and reflects the way-below relation.

The construction L is functorial; for a continuous function $f\colon D \to D'$, the function $Lf\colon LD \to LD'$ with $Lf(C) = \mathsf{cl}_{D'} f(C)$ is well-defined and continuous.

Now, let $f\colon D \to D$ be an FS function witnessed by $E \subseteq_{\mathrm{fin}} D$. A first idea is to define $g\colon D \to LD$ by $gx = \downarrow_D \{e \in E \mid e \sqsubseteq x\}$. As the lower set of a finite set is closed, this function is well-defined, but unfortunately, it is not continuous in general.

A solution is to use '$\ll$' instead of '$\sqsubseteq$' in the definition of g: let $gx = \downarrow_D \{e \in E \mid e \ll x\}$. Again, this is a well-defined function. To prove its continuity, we split it into functions $\phi\colon D \to LE$ and $\psi\colon LE \to LD$, where $\phi x = \{e \in E \mid e \ll x\}$ (a finite lower set) and $\psi C = \mathsf{cl}_D\, C = \downarrow_D C$. The function ϕ is continuous since for all directed families $(x_i)_{i \in I}$

$$\begin{aligned} e \in \phi\left(\bigsqcup_{i\in I} x_i\right) &\Longleftrightarrow e \ll \bigsqcup_{i\in I} x_i \\ &\Longleftrightarrow \exists i \in I\colon e \ll x_i \\ &\Longleftrightarrow e \in \bigcup_{i\in I} \phi x_i = \bigsqcup_{i\in I} \phi x_i \end{aligned}$$

where the equality in the last line relies on the fact that everything is finite (in general, a least upper bound in a lower power domain is not union, but union followed by closure).

The function ψ is continuous since it is monotonic and LE is finite. Note that $\psi = L\eta$ where $\eta\colon E \hookrightarrow D$ is the subset inclusion map.

By definition of g, $gx \subseteq \downarrow x$ holds, and so $g \sqsubseteq \mathsf{s}$. However, we cannot show $\mathsf{s} \circ f \sqsubseteq g$, but only $\mathsf{s} \circ f^2 \sqsubseteq g$. This is a consequence of Lemma 1 which says $f^2 x \in gx$, or $\mathsf{s}(f^2 x) \subseteq gx$.

We define L-FS functions and domains in complete analogy with U-FS functions and domains; just replace U by L in Definition 2.

Above, we have proved that if f is an FS function, then f^2 is an L-FS function. For the opposite direction, assume f is an L-FS function. For every x in D, $\mathsf{s}(fx) \sqsubseteq L\eta(\phi x) \sqsubseteq \mathsf{s}x$ holds, i.e., $fx \in \mathsf{cl}_D\,(\phi x) \subseteq \downarrow_D x$. As $\phi x \subseteq E$ is finite, $\mathsf{cl}_D\,(\phi x)$ equals $\downarrow_D (\phi x)$. Hence, for every x in D, there is e in $\phi x \subseteq E$ such that $fx \sqsubseteq e \sqsubseteq x$. Thus, f is an FS function.

Summarising, we have f L-FS $\Rightarrow$ f FS $\Rightarrow$ f^2 L-FS. The function $f \mapsto f^2$ is Scott continuous. Thus, if id_D is a directed join of functions f_i, then it is also a directed join of functions f_i^2. Hence, we obtain:

Theorem 4 *A dcpo D is an FS domain iff it is an L-FS domain.*

5 Other Characterisations using Lower and Upper Power Domains

So far, we have seen that a dcpo is an FS domain iff it is a U-FS domain, iff it is an L-FS domain. The definitions of U-FS and L-FS domain admit some variations that do not destroy the equivalence with FS domains. To avoid double work, we consider these variations for L and U together.

In the sequel, we use P ("power domain") as a symbol standing for L or U. Note that P is a *monad* or *Kleisli triple*. We have already seen a functorial action that maps $f: D \to D'$ to $Pf: PD \to PD'$, and a natural transformation $\mathsf{s}_D: D \to PD$. In addition, each continuous function $f: D \to PD'$ has a *Kleisli extension* $\overline{P}f: PD \to PD'$ given by $\overline{P}fA = \bigcup_{a \in A} fa$ for $P = U$, and by $\overline{P}fA = \mathsf{cl}_{D'}\left(\bigcup_{a \in A} fa\right)$ for $P = L$. This is a continuous function $\overline{P} : [D \to PD'] \to [PD \to PD']$. We have $\overline{P}f \circ \mathsf{s}_D = f$ for continuous $f: D \to PD'$, $\overline{P}\mathsf{s}_D = \mathsf{id}_{PD}$, $\overline{P}(\mathsf{s}_{D'} \circ f) = Pf$ for continuous $f: D \to D'$, and $\overline{P}(g \circ f) = \overline{P}g \circ Pf$ for continuous $f: D \to D'$ and $g: D' \to PD''$. We also use that P preserves finiteness, i.e., that PD is finite if D is finite.

Recall Definition 2: A continuous function $f: D \to D$ is a P-FS function if

(1) there exist a finite subdomain E of D and a continuous function $\phi: D \to PE$ such that $\mathsf{s}_D \circ f \sqsubseteq P\eta \circ \phi \sqsubseteq \mathsf{s}_D$ holds, where $\eta: E \hookrightarrow D$ is the subset inclusion.

We take the property in this definition as the starting point of a sequence of properties where each property implies the next one. The last property in this sequence will be that f^2 is an FS function. Hence, any of these properties can be used equivalently in the characterisation of an FS domain as a domain where the identity is a directed join of functions with special properties.

First we drop the condition that the finite domain E is a subdomain of D:

(2) There exist a finite domain E and continuous functions $\phi: D \to PE$ and $\eta: E \to D$ such that $\mathsf{s}_D \circ f \sqsubseteq P\eta \circ \phi \sqsubseteq \mathsf{s}_D$ holds.

Since E is finite, PE is finite, and moreover, any element of PE is a finite union of singletons ("finitely generated"). This property is preserved by $P\eta$, and so we get with $g = P\eta \circ \phi: D \to PD$:

(3) There is a continuous function $g: D \to PD$ with a finite image consisting of finitely generated sets such that $\mathsf{s}_D \circ f \sqsubseteq g \sqsubseteq \mathsf{s}_D$ holds.

Next, we drop the requirement that g produces finitely generated sets:

(4) There is a continuous function $g: D \to PD$ with finite image such that $\mathsf{s}_D \circ f \sqsubseteq g \sqsubseteq \mathsf{s}_D$ holds.

Let $E \subseteq PD$ be the finite image of g. After splitting g into continuous functions $\phi: D \to E$ and $\psi: E \to PD$, we apply Kleisli extension and obtain $\overline{P}g = \overline{P}(\psi \circ \phi) = \overline{P}\psi \circ P\phi : PD \to PE \to PD$. As PE is finite, this shows that $\overline{P}g: PD \to PD$ has finite image.

Extension $\overline{P}$ is monotonic. Thus, (4) implies $\overline{P}(\mathsf{s}_D \circ f) \sqsubseteq \overline{P}g \sqsubseteq \overline{P}\mathsf{s}_D$. Recall

$\overline{P}(\mathsf{s}_D \circ f) = Pf$ and $\overline{P}\mathsf{s}_D = \mathsf{id}_{PD}$. Hence, we obtain with $G = \overline{P}g$:

(5) There is a continuous function $G: PD \to PD$ with finite image such that $Pf \sqsubseteq G \sqsubseteq \mathsf{id}_{PD}$ holds.

Next, we forget about continuity of G and only note that for any A in PD, there is an element GA in the finite set $G(PD)$ such that $PfA \sqsubseteq GA \sqsubseteq A$, or shortly

(6) $Pf: PD \to PD$ is an FS function.

By the FS property of Pf, we have $PfB \ll B$ for all B in PD. Since the finitely generated sets form a basis of the continuous domain PD, there is a finitely generated set in between PfB and B, i.e., there is some finite subset E_B of D such that $PfB \sqsubseteq \bigcup_{e \in E_B} \mathsf{s}_D e \sqsubseteq B$. In particular, this holds for all elements B of the finite set $\mathcal{E} \subseteq_{\mathsf{fin}} PD$ that witnesses the FS property of Pf. Let $E = \bigcup_{B \in \mathcal{E}} E_B$, which is a finite subset of D. We claim that $f^2: D \to D$ is FS, witnessed by E. For x in D, there is B in $\mathcal{E}$ such that $Pf(\mathsf{s}_D x) \sqsubseteq B \sqsubseteq \mathsf{s}_D x$ by the FS property of Pf. Hence, $Pf(Pf(\mathsf{s}_D x)) \sqsubseteq PfB \sqsubseteq \bigcup_{e \in E_B} \mathsf{s}_D e \sqsubseteq B \sqsubseteq \mathsf{s}_D x$. By naturality of f, $\mathsf{s}_D(f^2 x) \sqsubseteq \bigcup_{e \in E_B} \mathsf{s}_D e \sqsubseteq \mathsf{s}_D x$ follows. For the next step, we must distinguish between lower and upper power domain. For $P = L$, we have $f^2 x \in {\downarrow} E_B \subseteq {\downarrow} x$, and for $P = U$, ${\uparrow} f^2 x \supseteq {\uparrow} E_B \ni x$ holds. In both cases, there exists $e \in E_B \subseteq E$ such that $f^2 x \sqsubseteq e \sqsubseteq x$. This shows that f^2 is an FS function.

Summarising, we obtain:

Theorem 5 *A continuous dcpo D is an FS domain if and only if id_D is a directed join of functions $f: D \to D$ satisfying property (n). Here, (n) is any of the properties (1) – (7) listed above, with P being the upper or the lower power domain construction.*

6 Further Generalisation

One may wonder whether lower and upper power domain may be replaced by other constructions. A definition in the spirit of Def. 2 only requires an endofunctor M in CONT together with a natural transformation $\mathsf{s}_D: D \to MD$, which means $Mf \circ \mathsf{s}_D = \mathsf{s}_{D'} \circ f$ for $f: D \to D'$. Such a structure is known as a *premonad*.

6.1 M-FS Functions and Domains

Definition 6 *Let (M, s) be a premonad in CONT. A continuous function $f\colon D \to D$ is an M-FS function, if there exist a finite domain E and continuous functions $\phi\colon D \to ME$ and $\eta\colon E \to D$ such that $\mathsf{s}_D \circ f \sqsubseteq M\eta \circ \phi \sqsubseteq \mathsf{s}_D$ holds. A domain D is an M-FS domain if the identity id_D is a directed join of M-FS functions.*

As in property (2) of Section 5, it is not required that E is a subset of D or η is an order embedding. However, these additional properties can easily be achieved if desired: Let E' be the image $\eta(E) \subseteq D$, which is again finite, and split η into $\zeta\colon E \to E'$ and the inclusion $\eta'\colon E' \hookrightarrow D$. Then $M\eta \circ \phi = M\eta' \circ \phi'$ where $\phi' = M\zeta \circ \phi : D \to ME'$ is continuous. Therefore, $E \subseteq D$ with η being the inclusion map can be assumed without loss of generality.

6.2 Relationships

Now we may ask how the different notions of M-FS domains are related to each other. Our goal is to prove a general inclusion statement: if there is a suitable family of functions $H_D\colon M_1 D \to M_2 D$ between two premonads M_1 and M_2, then every M_1-FS domain is an M_2-FS domain. Here, a suitable family means a family that preserves the premonad structure.

Definition 7 *A* premonad homomorphism *H from a premonad (M_1, s_1) to a premonad (M_2, s_2) is a natural transformation $H\colon M_1 \dot{\to} M_2$, i.e., a family of continuous functions $H_D\colon M_1 D \to M_2 D$ such that $H_{D'} \circ M_1 f = M_2 f \circ H_D$ for every $f\colon D \to D'$, with the additional property $H \circ \mathsf{s}_1 = \mathsf{s}_2$.*

Theorem 8 *If a premonad homomorphism from M_1 to M_2 exists, then every M_1-FS function (domain) is an M_2-FS function (domain).*

PROOF. Let f be an M_1-FS function, witnessed by the finite domain E, and continuous functions $\phi_1\colon D \to M_1 E$ and $\eta\colon E \to D$ such that $\mathsf{s}_1 \circ f \sqsubseteq M_1\eta \circ \phi_1 \sqsubseteq \mathsf{s}_1$ holds. Composition with the monotonic function H_D yields

$$H_D \circ \mathsf{s}_1 \circ f \sqsubseteq H_D \circ M_1\eta \circ \phi_1 \sqsubseteq H_D \circ \mathsf{s}_1.$$

The left and right parts can be simplified by $H_D \circ \mathsf{s}_1 = \mathsf{s}_2$. By naturality, the middle part equals $M_2\eta \circ H_E \circ \phi_1$. With $\phi_2 = H_E \circ \phi_1 : D \to M_2 E$, we obtain $\mathsf{s}_2 \circ f \sqsubseteq M_2\eta \circ \phi_2 \sqsubseteq \mathsf{s}_2$, which shows that f is an M_2-FS function.

The statement about domains is a direct consequence of the statement about functions. $\Box$

6.3 FC and the Identity Premonad

There is an especially simple premonad, namely the identity functor I with the identity functions $\mathsf{s}_D = \mathsf{id}_D : D \to ID = D$. There is a (unique) premonad homomorphism from (I, id) to every premonad (M, s), namely s itself. For, its naturality is part of the definition of a premonad, and $\mathsf{s} \circ \mathsf{id} = \mathsf{s}$ is a triviality. By Theorem 8, we immediately obtain:

Theorem 9 *An I-FS domain is also an M-FS domain for every premonad M whatsoever.*

Specialising Definition 6 to the identity premonad yields: a function $f: D \to D$ is an I-FS function if there are a finite domain E and continuous functions $\phi: D \to E$ and $\eta: E \to D$ such that $f \sqsubseteq \phi \circ \eta \sqsubseteq \mathsf{id}_D$. Recalling that w.l.o.g. η can be assumed to be the inclusion function of $E \subseteq_{\mathsf{fin}} D$, we can formulate this even simpler: a continuous function $f: D \to D$ is an I-FS function if there is a continuous function $g: D \to D$ with finite image such that $f \sqsubseteq g \sqsubseteq \mathsf{id}$, or even shorter, if there is an FC function above it.

Using this characterisation, it is obvious that every FC function f is an I-FS function; take $g = f$. While the opposite is not true for functions, it holds for domains.

Theorem 10 *The classes of FC domains and of I-FS domains coincide.*

PROOF. From the above, it is obvious that every FC domain is an I-FS domain. For the opposite direction, let D be an I-FS domain witnessed by a directed set $\mathcal{F}$ of I-FS functions with $\bigsqcup \mathcal{F} = \mathsf{id}_D$. Let $\mathcal{F}^2 = \{f^2 \mid f \in \mathcal{F}\}$, where f^2 means $f \circ f$. By continuity of $f \mapsto f^2$, $\mathcal{F}^2$ is again directed with join id_D.

Let $\mathcal{G}$ be the set of all FC functions of D, and let $\mathcal{G}^2 = \{g^2 \mid g \in \mathcal{G}\}$. Above every element of $\mathcal{F}$, there is an element of $\mathcal{G}$, and so above every element of $\mathcal{F}^2$, there is an element of $\mathcal{G}^2$, i.e., $\mathcal{F}^2 \subseteq {\downarrow}\mathcal{G}^2$. In particular, $\mathcal{G}^2$ is not empty. Clearly, all functions in $\mathcal{G}^2$ are FC functions. We shall show that $\mathcal{G}^2$ is directed with join id_D. This will prove that D is an FC domain.

Let $G_1 = g_1^2$ and $G_2 = g_2^2$ be two elements of $\mathcal{G}^2$. Every FC function is an FS function. Thus, Lemma 1 applies, and $G_1, G_2 \ll \mathsf{id}_D$ holds. Since id_D is the directed join of $\mathcal{F}^2$, we find F in $\mathcal{F}^2$ such that $G_1, G_2 \sqsubseteq F$. By $\mathcal{F}^2 \subseteq {\downarrow}\mathcal{G}^2$, there is G in $\mathcal{G}^2$ such that $G_1, G_2 \sqsubseteq F \sqsubseteq G$. This shows that $\mathcal{G}^2$ is directed. Because of $\mathcal{F}^2 \subseteq \downarrow\mathcal{G}^2$ and $\mathcal{G}^2 \subseteq \downarrow\mathsf{id}_D$ and $\bigsqcup \mathcal{F}^2 = \mathsf{id}_D$, the join of $\mathcal{G}^2$ is id_D. $\square$

Combining Theorems 9 and 10, we obtain:

Corollary 11 *An FC domain is also an M-FS domain for every premonad M whatsoever.*

6.4 Other Power Domain Construction

In addition to L, U, and I, we looked at the cases where M is the Plotkin power construction, the probabilistic power construction, or the lower bag construction [2]. From all three constructions, there is a premonad homomorphism to the lower power construction L. Hence, all three M-FS classes are between FC and FS by Cor. 11 and Theorem 8. The M-FS class of the lower bag domain is FC, but we were not able to show whether the other two classes coincide with FC or FS.

We also tried to prove that the class of M-FS domains is closed under M. From [2], it is known that the lower bag construction does *not* preserve FC, nor FS. This means that there cannot be a general positive result about closure of M-FS under M. Indeed, we were successful in proving closure only in the case of a locally continuous monad that preserves finiteness, a result that neither applies to the lower bag construction nor to the probabilistic power construction. Thus, we do not have any new information about the question whether the probabilistic construction preserves any of FC, FS, or its own M-FS class.

We did not explore the generalisations of statements (3) – (6) from Section 5 to other power domain constructions. Since preservation of finiteness was used several times in Section 5, the classes of domains that result from these variations may well be different for the probabilistic or lower bag construction.

References

[1] S. Abramsky and A. Jung. Domain theory. In S. Abramsky, D. M. Gabbay, and T. S. E. Maibaum, editors, *Handbook of Logic in Computer Science, Vol. III.* Oxford University Press, 1994.

[2] R. Heckmann. Lower bag domains. *Fundamenta Informaticae*, 24(3):259–281, November 1995.

[3] C. J. Jones and G. D. Plotkin. A probabilistic powerdomain of evaluations. In *Logic in Computer Science LICS '89*, pages 186–195. IEEE Computer Society Press, 1989.

[4] C. J. Jones. *Probabilistic Non-Determinism.* PhD thesis, University of Edinburgh, 1990.

[5] A. Jung and R. Tix. The troublesome probabilistic powerdomain. In A. Edalat, A. Jung, K. Keimel, and M. Kwiatkowska, editors, *Proceedings of the Third Workshop on Computation and Approximation*, volume 13 of *Electronic Notes in Theoretical Computer Science*. Elsevier, 1998. URL: `http://www.elsevier.nl/locate/entcs/volume13.html`.

[6] A. Jung. *Cartesian Closed Categories of Domains.* PhD thesis, FB Mathematik, Technische Hochschule Darmstadt, 1988.

[7] A. Jung. The classification of continuous domains. In *Logic in Computer Science LICS '90*, pages 35–40. IEEE Computer Society Press, 1990.

[8] M. B. Smyth. Power domains. *Journal of Computer and System Sciences*, 16:23–36, 1978.

[9] M. B. Smyth. Power domains and predicate transformers: A topological view. In J. Diaz, editor, *ICALP '83*, pages 662–676. Lecture Notes in Computer Science 154, Springer-Verlag, 1983.

[10] R. Tix. Stetige Bewertungen auf topologischen Räumen. Master's thesis, Technische Hochschule Darmstadt, 1995.