Characterising FS Domains by means of Power Domains

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Abstract

We present two characterisations of FS domains, using the upper and the lower power domain construction, respectively. These characterisations have a common structure and thus can be generalised from the two power constructions to arbitrary premonads. The resulting classes of domains are considered in general, and some special instances are studied in greater detail. They form classes of domains which lie in between the class of retracts of bifinite domains and FS.

1 Introduction

The category CONT of continuous dcpo's ("domains") and Scott continuous functions is not Cartesian closed, i.e., closed under finite products and function space formation [1, p. 42]. Thus, people looked for full subcategories of CONT which are Cartesian closed, and in particular for maximal ones. Some time ago, the category FC of *finitely continuous* dcpo's (or retracts of bifinite domains) was considered as a candidate [6]. It is Cartesian closed, but its maximality is still an open question. Later, Achim Jung [7] came up with the category FS of *finitely separated domains*, which is a maximal Cartesian closed full subcategory of CONT if one restricts attention to pointed dcpo's (those with $\perp$). FS contains FC, but it is an open question whether FS and FC are different or coincide.

The *probabilistic power domain construction* P, introduced by Jones and Plotkin [4, 5], is an endofunctor of CONT. However, it is not known whether it is an endofunctor of any Cartesian closed full subcategory of CONT. Several

candidate categories are ruled out by explicit counterexamples. The remaining ones are FC and FS, yet it is still unknown whether any of these two is closed under P [12].

The story told above indicates that we still do not know enough about FC, FS, P, and their relationship. The paper at hand adds something to this knowledge by providing several new characterisations of FS domains, equivalent to the original one given by Jung, and by identifying a class of domains in between FC and FS, characterised using the probabilistic powerdomain. Unfortunately, this does not solve any of the open problems stated above, but we hope that it gives useful hints which may lead to a solution in the future.

2 FS Domains Characterised by Lower and Upper Power Domains

After defining FC and FS domains, we shall develop two equivalent characterisations of FS domains, first with upper power domains, then with lower ones.

2.1 FC and FS domains

For the definition of dcpo, directed set, (Scott) continuous function, way-below relation ($\ll$), and continuous dcpo (domain), we refer to [1].

Let D be a dcpo. A continuous function $f\colon D \to D$ is *finitely continuous* or shortly FC if it is below the identity id_D and has finite image $f(D)$. A continuous function $f\colon D \to D$ is *finitely separated* or shortly FS if there is a finite subset E of D such that for every x in D, there is some e_x in E such that $fx \sqsubseteq e_x \sqsubseteq x$. Clearly, every FC function is an FS function by taking E to be the finite image $f(D)$ and $e_x = fx$.

In [7], some elementary properties of FS functions are proved. We formulate them as lemma for reference in later parts of the paper.

Lemma 2.1
Let $f\colon D \to D$ be an FS function witnessed by the finite subset E of D. Then for all x in D, there is some e_x in E such that $f^2x \sqsubseteq e_x \ll x$. In particular, $f^2 = f \circ f : D \to D$ is an FS function again. In $[D \to D]$, $f^2 \ll \mathsf{id}_D$ holds.

A dcpo D is an *FC domain* (*FS domain*) if the identity id_D is a directed join of FC functions (FS functions). FC and FS domains are always continuous dcpo's; thus the name "domain" is justified.

As every FC function is an FS function, every FC domain is an FS domain. Up to now, the converse is an open problem: we do not know whether every FS domain is an FC domain. For an affirmative answer, one should find an

FC function g above every FS function f. A straightforward idea is to let g map into the finite set $E \subseteq D$ used as witness for the FS property of f, defining $gx = e_x$ for all x in D. However, there are often several different e in E satisfying $fx \sqsubseteq e \sqsubseteq x$, and the question is which one to choose. Just choosing any candidate is not sufficient, for g has to be continuous.

In fact, we do not know whether it is possible to find a continuous function $g: D \to E$ above every FS function $f: D \to D$ with witness $E \subseteq_{fin} D$ (otherwise the equality FS $\stackrel{?}{=}$ FC were proved). However, we are able to find a *non-deterministic* continuous function g with finite image above every FS function. Here, non-determinism means that g maps from D into some power domain of D. This idea can be realised best if the upper or Smyth power domain UD is considered, yet the lower or Hoare power domain LD is also possible. In the next section, we consider U, while in Section 2.3, L is considered. After that, we generalise from U or L to an arbitrary premonad.

2.2 Characterising FS Domains by Upper Power Domains

The upper power domain UD of a domain (continuous dcpo) D consists of all compact upper subsets of D, ordered by opposite inclusion '$\supseteq$' [10, 11]. For our purpose, it does not matter whether $\emptyset$ is included in UD or not.

For every domain D, UD is again a domain. There is a continuous function $\mathsf{s} = \mathsf{s}_D : D \to UD$ defined by $\mathsf{s}x = \uparrow_D x$ for every x in D, called the *singleton function*. Actually, it is an order embedding ($x \sqsubseteq y$ iff $\mathsf{s}x \sqsubseteq \mathsf{s}y$), and preserves and reflects the way-below relation ($x \ll y$ iff $\mathsf{s}x \ll \mathsf{s}y$).

The construction U is functorial; for a continuous function $f: D \to D'$, $Uf: UD \to UD'$ with $Uf(K) = \uparrow_{D'} f(K)$ is well-defined and continuous.

Now, let $f: D \to D$ be an FS function witnessed by $E \subseteq_{fin} D$. For every x in D, let $gx = \uparrow_D \{e \in E \mid fx \sqsubseteq e\}$. Since E is finite, gx is the up-closure of a finite set and thus a compact upper subset of D. Hence, g is a function from D to UD. By definition, $gx \subseteq \uparrow fx$ holds, and the FS property of f implies $x \in gx$. Thus, $\uparrow x \subseteq gx \subseteq \uparrow fx$, or $\mathsf{s}x \sqsupseteq gx \sqsupseteq \mathsf{s}(fx)$ holds. Hence, we obtain $\mathsf{s} \circ f \sqsubseteq g \sqsubseteq \mathsf{s} = \mathsf{s} \circ \mathsf{id}$, i.e., g is between f and id modulo the order embedding s.

To prove continuity of g, we split it into functions $\varphi: D \to UE$ and $\psi: UE \to UD$, where $\varphi x = \{e \in E \mid fx \sqsubseteq e\}$ (a finite upper set) and $\psi K = \uparrow_D K$. Function φ is continuous since

$$\begin{aligned} e \in \varphi\left(\bigsqcup_{i\in I} x_i\right) &\iff f\left(\bigsqcup_{i\in I} x_i\right) \sqsubseteq e \\ &\iff \bigsqcup_{i\in I} fx_i \sqsubseteq e \\ &\iff \forall i \in I: fx_i \sqsubseteq e \\ &\iff e \in \bigcap_{i\in I} \varphi x_i = \bigsqcup_{i\in I} \varphi x_i \end{aligned}$$

Function ψ is continuous since it is monotonic and UE is finite. Note that $\psi = U\eta$ where $\eta: E \hookrightarrow D$ is the subset inclusion map.

What has been achieved so far, is taken as the definition of a U-FS function:

Definition 2.2
A continuous function $f: D \to D$ is a U-FS function, if there are a finite subdomain E of D and a continuous function $\varphi: D \to UE$ such that $\mathsf{s} \circ f \sqsubseteq U\eta \circ \varphi \sqsubseteq \mathsf{s}$ holds, where $\eta: E \hookrightarrow D$ is the subset inclusion. A domain D is a U-FS domain if the identity id_D is a directed join of U-FS functions.

The arguments before Definition 2.2 have proved that every FS function is a U-FS function, and so every FS domain is a U-FS domain. The opposite implication is easily shown: Let f be a U-FS function. For every x in D, $\mathsf{s}(fx) \sqsubseteq U\eta(\varphi x) \sqsubseteq \mathsf{s}x$ holds, i.e., $\uparrow_D fx \supseteq \uparrow_D(\varphi x) \supseteq \uparrow_D x$. Hence, there is e in $\varphi x \subseteq E$ such that $x \sqsupseteq e \sqsupseteq fx$. Thus, f is an FS function.

Theorem 2.3 A function $f: D \to D$ is an FS function iff it is a U-FS function. A dcpo D is an FS domain iff it is a U-FS domain.

2.3 Characterising FS Domains by Lower Power Domains

The lower power domain LD of a domain (continuous dcpo) D consists of all (Scott) closed subsets of D, ordered by inclusion '$\subseteq$'. Like in the case of UD, it does not matter whether $\emptyset$ is included in LD or not.

For every domain D, LD is again a domain. There is a continuous *singleton function* $\mathsf{s} = \mathsf{s}_D : D \to LD$ defined by $\mathsf{s}x = \downarrow_D x$ for every x in D. Like the singleton function of U, it is an order embedding, and preserves and reflects the way-below relation.

The construction L is functorial; for a continuous function $f: D \to D'$, $Lf: LD \to LD'$ with $Lf(C) = \mathsf{cl}_{D'} f(C)$ is well-defined and continuous.

Now, let $f: D \to D$ be an FS function witnessed by $E \subseteq_{fin} D$. A first idea is to define $g: D \to LD$ by $gx = \downarrow_D \{e \in E \mid e \sqsubseteq x\}$. As the lower set of a finite set is closed, this function is well-defined, but unfortunately, it is not continuous in general.

A solution is to use '$\ll$' instead of '$\sqsubseteq$' in the definition of g: let $gx = \downarrow_D \{e \in E \mid e \ll x\}$. Again, this is a well-defined function. To prove its continuity, we split it into functions $\varphi: D \to LE$ and $\psi: LE \to LD$, where $\varphi x = \{e \in E \mid e \ll x\}$ (a finite lower set) and $\psi C = \mathsf{cl}_D C = \downarrow_D C$. Function φ is continuous since

$$\begin{aligned} e \in \varphi\left(\bigsqcup_{i\in I} x_i\right) &\iff e \ll \bigsqcup_{i\in I} x_i \\ &\iff \exists i \in I: e \ll x_i \\ &\iff e \in \bigcup_{i\in I} \varphi x_i = \bigsqcup_{i\in I} \varphi x_i \end{aligned}$$

where the equality in the last line relies on the fact that everything is finite (in general, a least upper bound in a lower power domain is not union, but union followed by closure).

Function ψ is continuous since it is monotonic and LE is finite. Note that $\psi = L\eta$ where $\eta: E \hookrightarrow D$ is the subset inclusion map.

By definition of g, $gx \subseteq \downarrow x$ holds, and so $g \sqsubseteq \mathsf{s}$. However, we cannot show $\mathsf{s} \circ f \sqsubseteq g$, but only $\mathsf{s} \circ f^2 \sqsubseteq g$. This is a consequence of Lemma 2.1 which says $f^2 x \in gx$, or $\mathsf{s}(f^2 x) \subseteq gx$.

We define L-FS functions and domains in complete analogy with U-FS functions and domains; just replace U by L in Definition 2.2.

Above, we have proved that if f is an FS function, then f^2 is an L-FS function. For the opposite direction, assume f is an L-FS function. For every x in D, $\mathsf{s}(fx) \sqsubseteq L\eta(\varphi x) \sqsubseteq \mathsf{s}x$ holds, i.e., $fx \in \mathsf{cl}_D(\varphi x) \subseteq \uparrow_D x$. As $\varphi x \subseteq E$ is finite, $\mathsf{cl}_D(\varphi x)$ equals $\downarrow_D(\varphi x)$. Hence, for every x in D, there is e in $\varphi x \subseteq E$ such that $fx \sqsubseteq e \sqsubseteq x$. Thus, f is an FS function.

Summarising, we have f L-FS $\Rightarrow$ f FS $\Rightarrow$ f^2 L-FS. The function $f \mapsto f^2$ is Scott continuous. Thus, if id_D is a directed join of functions f_i, then it is also a directed join of functions f_i^2. Hence, we obtain:

Theorem 2.4 A dcpo D is an FS domain iff it is an L-FS domain.

3 M-FS Functions and Domains

The notions of L-FS and U-FS functions and domains can easily be generalised. The definition requires an endofunctor M in CONT together with a natural transformation $\mathsf{s}_D: D \to MD$, which means $Mf \circ \mathsf{s}_D = \mathsf{s}_{D'} \circ f$ for $f: D \to D'$. Such a structure is known as a *premonad* (a monad has further data, see Section 4.3).

3.1 Definitions

Given a premonad M, we could define M-FS by replacing U by M in Definition 2.2. However, we prefer a different formulation which looks slightly more general.

Definition 3.1
Let (M, s) be a premonad in CONT. A continuous function $f: D \to D$ is an M-FS function, if there are a finite domain E and continuous functions $\varphi: D \to ME$ and $\eta: E \to D$ such that $\mathsf{s} \circ f \sqsubseteq M\eta \circ \varphi \sqsubseteq \mathsf{s}$ holds. A domain D is an M-FS domain if the identity id_D is a directed join of M-FS functions.

In contrast to Definition 2.2, it is not formally required that E is a subset of D or η is an order embedding. However, these additional properties can easily be achieved if desired: Let E' be the image $\eta(E) \subseteq D$, which is again finite, and split η into $\zeta: E \to E'$ and the inclusion $\eta': E' \hookrightarrow D$. Then $M\eta \circ \varphi = M\eta' \circ \varphi'$ where $\varphi' = M\zeta \circ \varphi : D \to ME'$ is continuous. Therefore, $E \subseteq D$ with η being the inclusion map can be assumed without loss of generality.

3.2 Relationships

Now we may ask how the different notions of M-FS domains are related to each other. We already know L-FS = U-FS = FS. Our goal is to prove a general inclusion statement: if there is a suitable family of functions $H_D: M_1 D \to M_2 D$ between two premonads M_1 and M_2, then every M_1-FS domain is an M_2-FS domain. Here, a suitable family means a family that preserves the premonad structure.

Definition 3.2 A *premonad homomorphism* H from a premonad (M_1, s_1) to a premonad (M_2, s_2) is a natural transformation $H: M_1 \dot{\to} M_2$, i.e., a family of continuous functions $H_D: M_1 D \to M_2 D$ such that $H_{D'} \circ M_1 f = M_2 f \circ H_D$ for every $f: D \to D'$, with the additional property $H \circ \mathsf{s}_1 = \mathsf{s}_2$.

Theorem 3.3 If a premonad homomorphism from M_1 to M_2 exists, then every M_1-FS function (domain) is an M_2-FS function (domain).

Proof: Let f be an M_1-FS function, witnessed by the finite domain E, and continuous functions $\varphi_1: D \to M_1 E$ and $\eta: E \to D$ such that $\mathsf{s}_1 \circ f \sqsubseteq M_1\eta \circ \varphi_1 \sqsubseteq \mathsf{s}_1$ holds. Composition with the monotonic function H_D yields

$$H_D \circ \mathsf{s}_1 \circ f \sqsubseteq H_D \circ M_1\eta \circ \varphi_1 \sqsubseteq H_D \circ \mathsf{s}_1 .$$

The left and right parts can be simplified by $H_D \circ \mathsf{s}_1 = \mathsf{s}_2$. By naturality, the middle part equals $M_2\eta \circ H_E \circ \varphi_1$. With $\varphi_2 = H_E \circ \varphi_1 : D \to M_2 E$, we obtain $\mathsf{s}_2 \circ f \sqsubseteq M_2\eta \circ \varphi_2 \sqsubseteq \mathsf{s}_2$, which shows that f is an M_2-FS function.

The statement about domains is a direct consequence of the statement about functions. □

3.3 The Identity Premonad

There is an especially simple premonad, namely the identity functor I with the identity functions $\mathsf{s}_D = \mathsf{id}_D : D \to ID = D$. There is a (unique) premonad homomorphism from (I, id) to every premonad (M, s), namely s itself. For, its naturality is part of the definition of a premonad, and $\mathsf{s} \circ \mathsf{id} = \mathsf{s}$ is a triviality. By Theorem 3.3, we immediately obtain:

Theorem 3.4
An I-FS domain is also an M-FS domain for every premonad M whatsoever.

Specialising Definition 3.1 to the identity premonad yields: a function $f: D \to D$ is an I-FS function if there are a finite domain E and continuous functions $\varphi: D \to E$ and $\eta: E \to D$ such that $f \sqsubseteq \varphi \circ \eta \sqsubseteq \mathsf{id}_D$. Recalling that w.l.o.g. η can be assumed to be the inclusion function of $E \subseteq_{fin} D$, we can formulate this even simpler: a continuous function $f: D \to D$ is an I-FS function if there is a continuous function $g: D \to D$ with finite image such that $f \sqsubseteq g \sqsubseteq \mathsf{id}$, or even shorter, if there is an FC function above it.

Using this characterisation, it is obvious that every FC function f is an I-FS function; take $g = f$. While the opposite is not true for functions, it holds for domains.

Theorem 3.5 The classes of FC domains and of I-FS domains coincide.

Proof: From the above, it is obvious that every FC domain is an I-FS domain. For the opposite direction, let D be an I-FS domain witnessed by a directed set $\mathcal{F}$ of I-FS functions with $\bigsqcup \mathcal{F} = \mathsf{id}_D$. Let $\mathcal{F}^2 = \{f^2 \mid f \in \mathcal{F}\}$, where f^2 means $f \circ f$. By continuity of $f \mapsto f^2$, $\mathcal{F}^2$ is again directed with join id_D.

Let $\mathcal{G}$ be the set of all FC functions of D, and let $\mathcal{G}^2 = \{g^2 \mid g \in \mathcal{G}\}$. Above every element of $\mathcal{F}$, there is an element of $\mathcal{G}$, and so above every element of $\mathcal{F}^2$, there is an element of $\mathcal{G}^2$, i.e., $\mathcal{F}^2 \subseteq {\downarrow}\mathcal{G}^2$. In particular, $\mathcal{G}^2$ is not empty. Clearly, all functions in $\mathcal{G}^2$ are FC functions. We shall show that $\mathcal{G}^2$ is directed with join id_D. This will prove that D is an FC domain.

Let $G_1 = g_1^2$ and $G_2 = g_2^2$ be two elements of $\mathcal{G}^2$. Every FC function is an FS function. Thus, Lemma 2.1 applies, and $G_1, G_2 \ll \mathsf{id}_D$ holds. Since id_D is the directed join of $\mathcal{F}^2$, we find F in $\mathcal{F}^2$ such that $G_1, G_2 \sqsubseteq F$. By $\mathcal{F}^2 \subseteq {\downarrow}\mathcal{G}^2$, there is G in $\mathcal{G}^2$ such that $G_1, G_2 \sqsubseteq F \sqsubseteq G$. This shows that $\mathcal{G}^2$ is directed. Because of $\mathcal{F}^2 \subseteq {\downarrow}\mathcal{G}^2$ and $\mathcal{G}^2 \subseteq {\downarrow}\mathsf{id}_D$ and $\bigsqcup \mathcal{F}^2 = \mathsf{id}_D$, the join of $\mathcal{G}^2$ is id_D. □

Combining Theorems 3.4 and 3.5, we obtain:

Corollary 3.6
An FC domain is also an M-FS domain for every premonad M whatsoever.

3.4 Some Classes in between

By choosing suitable premonads, some domain classes in between FC and FS can be established. We shall consider *Plotkin power domain*, *probabilistic power domain*, and *lower bag domain*.

The *probabilistic power domain* [4, 5] of a domain D can be defined in terms of valuations on D. A *valuation* on D is a strict modular Scott continuous function from the frame ΩD of Scott open sets of D to $\overline{\mathbf{R}_+} = [0, \infty]$, the extended non-negative reals. Here, strictness of ν means $\nu(\emptyset) = 0$, while modularity means

$$\nu(U \cup V) + \nu(U \cap V) = \nu U + \nu V$$

for all open sets U and V. The probabilistic power domain PD of D in the proper sense consists of valuations ν with $\nu D \leq 1$, while the extended probabilistic power domain $P'D$ consists of all of them. We shall soon see that this distinction is immaterial in the context of the paper at hand. The *lower bag domain* BD of D [2] consists of all integer-valued valuations on D, i.e., those valuations which assume only values in $\overline{\mathbf{N}_0}$.

The *Plotkin* or *convex power domain* CD of D [9] admits a concrete representation similar to that of P and B [3]. In this description, it consists of *abstract valuations*, i.e., Scott continuous functions from ΩD to the three point chain $\{0 \sqsubset \mathsf{M} \sqsubset 1\}$, with the properties $\alpha(\emptyset) = 0$, $\alpha(D) = 1$, and the 0-1-laws: if $\alpha U = 0$, then $\alpha(U \cup V) = \alpha V$ for all opens V, and if $\alpha U = 1$, then $\alpha(U \cap V) = \alpha V$ for all opens V. The advantage of this description is that we can use uniform formulae for all four constructions considered in this section.

Every M in $\{P, P', B, C\}$ becomes a functor by defining $Mf\,(\nu)(V) = \nu(f^{-1}(V))$ for $f\colon D \to D'$, ν in MD and V in $\Omega D'$. It becomes a premonad by defining $\mathsf{s}x\,(U) = 1$ if x is in U, and 0 otherwise.

From all the four premonads M, there is a premonad homomorphism H to the lower power domain construction L. It maps every ν in MD to its *support* $H(\nu)$, defined by $x \in H(\nu)$ if for every open U containing x, $\nu U \neq 0$ holds. Thus, by combining Corollary 3.6 and Theorems 3.3 and 2.4, we see that FC $\subseteq$ M-FS $\subseteq$ FS for all premonads M considered in this section.

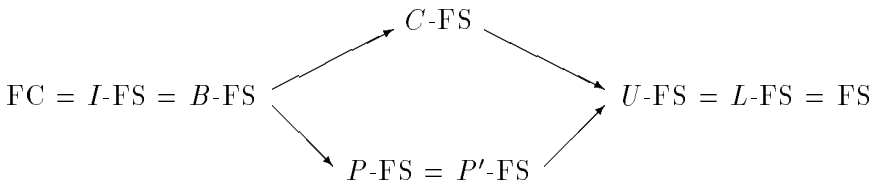
Now, we prove that P-FS and P'-FS actually coincide, and so do B-FS and I-FS = FC. There is an obvious homomorphism from P (consisting of valuations bounded by 1) to P' (consisting of unbounded valuations), given by subset inclusions. For the opposite direction, let $f\colon D \to D$ be an P'-FS function. Hence, there are a finite domain E and continuous functions $\varphi\colon D \to P'E$ and $\eta\colon E \to D$ such that $\mathsf{s} \circ f \sqsubseteq g \sqsubseteq \mathsf{s}$, where $g = P'\eta \circ \varphi$. Then for all x in D, $1 = \mathsf{s}(fx)(D) \leq gx(D) \leq \mathsf{s}x(D) = 1$, whence $gx(D) = 1$, i.e., g actually maps to PD. Now, $\varphi x(E) = \varphi x(\eta^{-1}(D)) = P'\eta(\varphi x)(D) = gx(D) = 1$, and so, φ maps to PE. Thus, f is actually a P-FS function.

Now, we compare I-FS and B-FS. Let $f\colon D \to D$ be an B-FS function witnessed by E, φ, and η. By arguments as above, $\varphi x(E) = 1$ holds for every x in D. An integer valued valuation ν on E with $\nu(E) = 1$ assumes exactly two values, 0 and 1. By theorems in [8], there is a unique e in E such that $\nu = \mathsf{s}e$. Thus, φ actually maps from D to $\mathsf{s}(E) \subseteq BE$. Since $B\eta(\mathsf{s}e) = \mathsf{s}(\eta e)$, $B\eta$ maps from

$\mathsf{s}(E)$ to $\mathsf{s}(D)$. All mappings $\mathsf{s}: X \to BX$ are order embeddings. Therefore, $\mathsf{s}(E) \subseteq E$ is isomorphic to E, and likewise for D, and we may cancel out s in $\mathsf{s} \circ f \sqsubseteq \ldots \sqsubseteq \mathsf{s}$, proving that f is an I-FS mapping.

3.5 A Map of M-FS Classes

All M-FS classes introduced so far and their mutual relationships are depicted in the following diagram, where arrows mean inclusions.



4 Closure Properties of the M-FS Classes

In this section, we look at the closure properties of M-FS classes for arbitrary premonads M.

4.1 Closure under Retracts

A domain X is a retract of a domain Y if there are continuous functions $e: X \to Y$ and $r: Y \to X$ such that $r \circ e$ is the identity on X.

Lemma 4.1 Let $e: X \to Y$ and $r: Y \to X$ such that $r \circ e = \mathsf{id}_X$. If $f: Y \to Y$ is an M-FS map, then so is $r \circ f \circ e : X \to X$.

Proof: Let E be a finite domain, $\varphi: Y \to ME$, and $\eta: E \to Y$ such that $\mathsf{s}_Y \circ f \sqsubseteq M\eta \circ \varphi \sqsubseteq \mathsf{s}_Y$, which is a relation in $[Y \to MY]$. By composing with Mr on the left and e on the right, we obtain

$$Mr \circ \mathsf{s}_Y \circ f \circ e \;\sqsubseteq\; Mr \circ M\eta \circ \varphi \circ e \;\sqsubseteq\; Mr \circ \mathsf{s}_Y \circ e.$$

By applying the equality $Mr \circ \mathsf{s}_Y = \mathsf{s}_X \circ r$ twice, followed by using $r \circ e = \mathsf{id}_X$, this relation becomes

$$\mathsf{s}_X \circ r \circ f \circ e \;\sqsubseteq\; Mr \circ M\eta \circ \varphi \circ e \;\sqsubseteq\; \mathsf{s}_X.$$

This shows that $r \circ f \circ e$ is an M-FS function with $\varphi' = \varphi \circ e : X \to ME$ and $\eta' = r \circ \eta : E \to X$. □

Theorem 4.2 All retracts of M-FS domains are M-FS domains again.

Proof: Let X be an retract of Y by virtue of e and r. Since Y is an M-FS domain, there is a directed family $(f_i)_{i\in I}$ of M-FS functions with $\mathsf{id}_Y = \bigsqcup_{i\in I} f_i$. Hence, $\mathsf{id}_X = r \circ \mathsf{id}_Y \circ e = r \circ (\bigsqcup_{i\in I} f_i) \circ e = \bigsqcup_{i\in I}(r \circ f_i \circ e)$ together with Lemma 4.1 implies that X is an M-FS domain. □

4.2 Closure under a Functor

It is tempting to believe that M-FS is closed under M. Yet, by applying M to the relation $\mathsf{s}_D \circ f \sqsubseteq M\eta \circ \varphi \sqsubseteq \mathsf{s}_D$, one obtains a relation involving $M\mathsf{s}_D$ instead of s_{MD}, as it should be. To overcome this problem, additional features are needed. The situation is less confusing, if instead of M, an arbitrary functor F is considered.

Thus, let $f{:}D \to D$ be an M-FS function, and F a functor on domains. We look for conditions sufficient to guarantee that $Ff{:}FD \to FD$ is an M-FS function again.

First, F should operate monotonically on function spaces, i.e., $f \sqsubseteq f'$ in $[D \to D']$ should imply $Ff \sqsubseteq Ff'$ in $[FD \to FD']$. Applying such an F to the usual relation $\mathsf{s}_D \circ f \sqsubseteq M\eta \circ \varphi \sqsubseteq \mathsf{s}_D$ yields

$$F\mathsf{s}_D \circ Ff \sqsubseteq FM\eta \circ F\varphi \sqsubseteq F\mathsf{s}_D.$$

This relation falls short of proving the M-FS property of Ff in several ways: it involves $F\mathsf{s}_D$ instead of s_{FD}, $F\varphi$ maps from FD to FME instead of MFE, and it contains $FM\eta$ instead of $MF\eta$. The latter two observations give a hint how the situation can be mastered: We need a natural transformation from FM to MF, i.e., a family of continuous functions $C_D{:}FMD \to MFD$ with $C_Y \circ FMg = MFg \circ C_X$ for $g{:}X \to Y$, with the additional property $C_X \circ F\mathsf{s}_X = \mathsf{s}_{FX}$ for all X.

Composing the above relation with C_D from the left and applying the equations of C_D yields

$$\mathsf{s}_{FD} \circ Ff \sqsubseteq MF\eta \circ C_E \circ F\varphi \sqsubseteq \mathsf{s}_{FD}.$$

Introducing $\varphi' = C_E \circ F\varphi : FD \to MFE$, this nearly proves that Ff is an M-FS map; the only missing information is finiteness of FE. Therefore, we add preservation of finiteness to the list of properties required for F.

So far, we have shown that under certain conditions, F maps M-FS functions to M-FS functions. To conclude the analogous property for domains, one further thing is needed: from $\mathsf{id}_D = \bigsqcup_{i\in I} f_i$ for a directed family $(f_i)_{i\in I}$, $\mathsf{id}_{FD} = F\mathsf{id}_D = F(\bigsqcup_{i\in I} f_i)$ follows, yet $\mathsf{id}_{FD} = \bigsqcup_{i\in I} Ff_i$ is required. Therefore, F must be continuous on function spaces (locally continuous).

The properties required for F are summarised in the following definition:

Definition 4.3 Let F be a functor of domains.

- F *preserves finiteness* if FE is finite whenever E is finite.
- F is *locally continuous* if $F{:}[X \to Y] \to [FX \to FY]$ is continuous for all domains X and Y.
- F commutes with a premonad (M, s) if there is a natural transformation $C_D{:}\, FMD \to MFD$ with $C_D \circ F\mathsf{s}_D = \mathsf{s}_{FD}$.

These notions are used in the following theorem:

Theorem 4.4 Let (M, s) be a premonad and let F be a locally continuous functor that preserves finiteness and commutes with M. Then F preserves M-FS, i.e., maps M-FS domains to M-FS domains.

4.3 Some Special Cases

Our general theorem about preservation of M-FS by F has several interesting special cases. One special case is M being the identity premonad I. Every functor trivially commutes with I by taking $C_D = \mathsf{id}_D$. Hence, we obtain—using FC = I-FS—the following theorem (which was known previously):

Theorem 4.5 If F is a locally continuous functor preserving finiteness, then F preserves FC.

Another special case is $F = M$. For a premonad to commute with itself, we need a natural transformation $C_D{:}\, MMD \to MMD$ with $C_D \circ M\mathsf{s}_D = \mathsf{s}_{MD}$. Because of the latter property, C_D cannot be id_D except for some trivial cases such as the identity premonad.

If M actually happens to be a monad—and all premonads considered in this paper are in fact monads—then C_D can be derived from the monad structure. Every monad comes with a natural transformation, the so-called multiplication $\mathsf{m}_D{:}\, MMD \to MD$, which—among other properties—satisfies the equation $\mathsf{m}_D \circ M\mathsf{s}_D = \mathsf{id}_{MD}$. Thus, choosing $C_D = \mathsf{s}_{MD} \circ \mathsf{m}_D$ yields a natural transformation satisfying the required equation $C_D \circ M\mathsf{s}_D = \mathsf{s}_{MD}$. Hence, we obtain:

Theorem 4.6 If M is a locally continuous monad preserving finiteness, then M preserves M-FS.

As already mentioned, all premonads occurring in this paper actually are locally continuous monads. The monads L, U, and C preserve finiteness, while P, P', and B do not. Thus, L, U, and C preserve FC, and L and U preserve FS because of FS = L-FS = U-FS. (In fact, L does much better; it maps any domain whatsoever to a continuous lattice, which is in FC and so in all M-FS classes.)

Since P, P', and B do not preserve finiteness, the theorems in this paper do not apply to them. In fact, it is not known whether P or P' preserve FC or FS. From [2], it is known, however, that B does *not* preserve FC, nor FS.

5 Conclusion

In this paper, we have derived equivalent characterisations of FS domains using lower or upper power domains. In generalising these descriptions, we have defined a whole bunch of domain classes, one for every premonad. The more interesting of these classes lie in between FC and FS, partially bridging the gap between these two classes.

Unfortunately, we were not able to settle any of the big open questions, namely, whether FC and FS actually coincide, and whether P or P' preserve FC or FS. However, we hope that the results of this paper may provide valuable hints in a future, more successful treatment of these questions.

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