Domain Representations

of Convergence Spaces

Reinhold Heckmann

heckmann@absint.com

AbsInt
Angewandte Informatik GmbH

Convergence Spaces (Overview)

Topological category CONV that includes TOP as a full subcategory.

Advantage:

CONV is cartesian closed,
while TOP is not.

Objects of CONV:

Sets of points
with a convergence relation '$\downarrow$'
between filters and points.

Disadvantage:

Filters are sometimes difficult to work with,
and there are too many of them.

Filters

A filter on a set X is a set $\mathcal{A} \subseteq \mathcal{P}X$ such that

- $\mathcal{A}$ is upper closed:
  $A \in \mathcal{A},\ A \subseteq A' \ \Rightarrow\ A' \in \mathcal{A}$;
- $\mathcal{A}$ is closed under finite intersection.

The filters on X ordered by '$\subseteq$'
form an algebraic lattice ΦX.

Compact elements of ΦX = principal filters
$[A] = \{A' \subseteq X \mid A' \supseteq A\}$ for $A \subseteq X$.

Problem: ΦX for infinite X is
not ω-algebraic (basis $\cong \mathcal{P}X$).

Proper filters ($\emptyset \notin \mathcal{A}$) form an
algebraic domain $\Phi' X$ with basis $\{[A] \mid A \neq \emptyset\}$.

Same problem.

Remark: $\Phi X = \Phi' X \cup \{[\emptyset]\}$
where $[\emptyset]$ is top element.

Convergence Spaces (Definition)

Objects: $(X, \downarrow)$ with set X of points and
convergence relation '$\downarrow$' $\subseteq \Phi X \times X$ satisfying

- $\mathcal{A}' \supseteq \mathcal{A},\ \mathcal{A} \downarrow x \ \Rightarrow\ \mathcal{A}' \downarrow x$
- $\forall x \in X:\ \varphi_X x \downarrow x$
  where $\varphi_X x = [\{x\}] = \{A \subseteq X \mid x \in A\}$.

Morphisms $f : (X, \downarrow_X) \to (Y, \downarrow_Y)$
are continuous functions $f : X \to Y$,
which means $\mathcal{A} \downarrow_X x \ \Rightarrow\ \Phi f \mathcal{A} \downarrow_Y fx$
where $\Phi f \mathcal{A} = \{B \subseteq Y \mid \exists A \in \mathcal{A} : B \supseteq f(A)\}$.

Category CONV.

Alternative definition: ΦX replaced by $\Phi' X$.
This gives an isomorphic category.

Idea:

- Replace ΦX by any continuous lattice,
- or $\Phi' X$ by any continuous domain.

Contexts and Representations

Lattice context: (X, C, γ)

where X is a set (of points),

C is a continuous lattice, and

$\gamma : X \to C$ is a function.

Domain context: (X, C, γ)

where X is a set (of points),

C is a continuous domain, and

$\gamma : X \to C$ is a dense function,

i.e., $\gamma(X)$ is dense in C.

Examples: $(X,\ \Phi X,\ \varphi_X)$ resp. $(X,\ \Phi' X,\ \varphi_X)$.

Lattice (domain) representation: $(X, C, \gamma, \downarrow)$

where (X, C, γ) is a lattice (domain) context,

and '$\downarrow$' $\subseteq C \times X$ satisfies

- $a' \geq a,\ a \downarrow x \ \Rightarrow\ a' \downarrow x$
- $\forall x \in X :\ \gamma x \downarrow x$.

Example: $(X,\ \Phi' X,\ \varphi_X,\ \downarrow)$

for any convergence space $(X, \downarrow)$.

Morphisms between Representations

Morphisms $f : (X, C_X, \gamma_X, \downarrow_X) \to (Y, C_Y, \gamma_Y, \downarrow_Y)$

are continuous functions $f : X \to Y$, which means

$\exists$ Scott-continuous $f' : C_X \to C_Y$ such that

- $a \leq \gamma_X\, x \ \Rightarrow\ f'a \leq \gamma_Y(f\, x)$, or equivalently
  $f' \circ \gamma_X \ \leq\ \gamma_Y \circ f$, and
- $a \downarrow_X x \ \Rightarrow\ f'a \downarrow_Y f\, x$.

$$\begin{tikzcd} C_X \arrow[r, "f'"'] & C_Y \\ X \arrow[u, "\gamma_X"] \arrow[r, "f"] & Y \arrow[u, "\gamma_Y"'] \end{tikzcd}$$

Such an f' is called realizer of f.

Facts:

- id_X is realized by id_{C_X}.
- If f, g are realized by f', g',
  then $g \circ f$ is realized by $g' \circ f'$.

Hence we get two categories

- CONV_L (with lattices) and
- CONV_D (with domains).

Alternative Characterization

Theorem:

Let (X, C_X, γ_X) and (Y, C_Y, γ_Y) be

lattice contexts or domain contexts.

Then for every $f : X \to Y$,

there is a greatest Scott-continuous

function $\bar{f} : C_X \to C_Y$ with

the property $\bar{f} \circ \gamma_X \leq \gamma_Y \circ f$.

$$\begin{tikzcd} C_X \arrow[r, "\bar{f}"'] & C_Y \\ X \arrow[u, "\gamma_X"] \arrow[r, "f"] & Y \arrow[u, "\gamma_Y"'] \end{tikzcd}$$

Theorem:

Let $\mathcal{X} \ = \ (X, C_X, \gamma_X, \downarrow_X)$ and

$\mathcal{Y} \ = \ (Y, C_Y, \gamma_Y, \downarrow_Y)$ be

lattice or domain representations.

Then $f : \mathcal{X} \to \mathcal{Y}$ is continuous

if and only if $a \downarrow_X x \ \Rightarrow\ \bar{f}a \downarrow_Y f x$.

In this case, $\bar{f}$ is the greatest realizer of f.

Proof $\Leftarrow$: $\bar{f}$ is a realizer.

Proof $\Rightarrow$: $\bar{f}a \geq f'a \downarrow_Y f x \ \Rightarrow\ \bar{f}a \downarrow_Y f x$.

Equivalence of Categories

Remark: For $f : (X, \Phi X, \varphi_X) \to (Y, \Phi Y, \varphi_Y)$,

$\bar{f} \ = \ \Phi f$ holds.

Hence CONV is full subcategory of CONV_L

(and also of CONV_D).

Theorem:

For every lattice representation $\mathcal{X} = (X, C, \gamma, \downarrow)$,

there is a convergence relation $\downarrow' \ \subseteq\ \Phi X \times X$

such that $\mathcal{X}$ and $(X, \Phi X, \varphi_X, \downarrow')$

are equivalent,

i.e., id_X is continuous in both directions.

Same for domain representations with Φ'.

Corollary:

The categories CONV, CONV_L, CONV_D,

$\mathsf{CONV}_\mathsf{AL}$ (with algebraic lattices) and

$\mathsf{CONV}_\mathsf{AD}$ (with algebraic domains)

are all equivalent.

Topological Spaces

TOP is a full subcategory of **CONV**:

$(X, \tau) \mapsto (X, \downarrow_\tau)$ where $\mathcal{A} \downarrow_\tau x \iff \mathcal{A} \supseteq \mathcal{N}(x)$.

Theorem:

If C is a continuous lattice (domain) and $\gamma : (X, \tau) \to C$ is a (dense) topological pre-embedding, then $(X, \downarrow_\tau)$ is equivalent to $(X, C, \gamma, \downarrow_\gamma)$ where $a \downarrow_\gamma x \iff a \geq \gamma x$.

Examples:

- Bounded complete domain environments ($X \cong \max C$ for some Scott domain C).
- X discrete: Domain $X_\perp$ Lattice $X_\perp^\top$
- $X = [0, 1]$ (unit interval)
  Domain: closed intervals ordered by '$\supseteq$'
  Lattice: same plus $\emptyset$
  Both cases: $\gamma x = [x, x]$

Further Examples

- X locally compact Hausdorff:
  compact sets plus X ordered by '$\supseteq$'
  with $\gamma x = \{x\}$ and $K \downarrow x \iff K \subseteq \{x\}$.
- X stably locally compact sober:
  compact upper sets plus X ordered by '$\supseteq$'
  with $\gamma x = \uparrow x$ and $K \downarrow x \iff K \subseteq \uparrow x$.
- X with basis $\mathcal{B}$ closed under finite intersection:
  $\Phi_\mathcal{B} X$ = filters of elements of $\mathcal{B}$ ordered by '$\subseteq$'
  with $\gamma x = \varphi_\mathcal{B} x = \{B \in \mathcal{B} \mid x \in B\}$.

Note: X second countable $\Rightarrow$
X has an ω-algebraic representation.

Theorem for X in **TOP**:
X has an ω-continuous representation $\Rightarrow$
X is second countable.

Product

$\mathcal{X}_i = (X_i, C_i, \gamma_i, \downarrow_i) \Rightarrow \prod_{i \in I} \mathcal{X}_i = (X, C, \gamma, \downarrow)$

where $X = \prod_{i \in I} X_i$, $C = \prod_{i \in I} C_i$, $\gamma = \prod_{i \in I} \gamma_i$,

and $a \downarrow x \iff \forall i \in I : a_i \downarrow_i x_i$.

- Projections $X \to X_i$ are realized by projections $C \to C_i$.
- Consider $(Y, C_Y, \gamma_Y, \downarrow_Y)$.
  If $f_i : Y \to X_i$ are realized by $f_i' : C_Y \to C_i$,
  then $\langle f_i \rangle_{i \in I} : Y \to X$ is realized
  by $\langle f_i' \rangle_{i \in I} : C_Y \to C$.

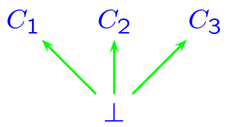
All this holds for lattices and for domains.
Domains need proof that γ is dense (uses axiom of choice).

Disjoint Sum

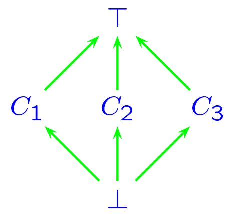
$\mathcal{X}_i = (X_i, C_i, \gamma_i, \downarrow_i) \Rightarrow \sum_{i \in I} \mathcal{X}_i = (X, C, \gamma, \downarrow)$

where $X = \sum_{i \in I} X_i$, $C \approx \sum_{i \in I} C_i$, $\gamma = \sum_{i \in I} \gamma_i$

Domains $(I \neq \emptyset)$:



Lattices:



Convergence relation:

$\top \downarrow (j, x)$ always

$(i, a) \downarrow (j, x) \iff i = j \ \& \ a \downarrow_i x$

$\perp \downarrow (j, x)$ never

(Generalized) Subspace

Given $\mathcal{X} = (X, C, \gamma, \downarrow)$ and $e : X' \to X$,

find $\mathcal{X}' = (X', C', \gamma', \downarrow')$ such that

$e : \mathcal{X}' \to \mathcal{X}$ is continuous and

for $f : \mathcal{Y} \to \mathcal{X}'$: $e \circ f$ continuous $\Rightarrow f$ continuous.

Subspace: $X' \subseteq X$, e inclusion.

Lattices:

$C' = C, \quad \gamma' = \gamma \circ e,$

$a \downarrow' x \iff a \downarrow e\,x.$

$$\begin{tikzcd}
C_Y \arrow[rr, bend left, "g"] & C' \arrow[r, equal] & C \\
Y \arrow[u, "\gamma_Y"'] \arrow[r, "f"] & X' \arrow[u, "\gamma'"'] \arrow[r, "e"] & X \arrow[u, "\gamma"']
\end{tikzcd}$$

- id_C realizes e.
- If g realizes $e \circ f$, then g also realizes f.

Domains:

Similar, but $C' = \mathsf{cl}\,\gamma'(X')$ in C.

Function Space

Let $\mathcal{Y} = (Y, C_Y, \gamma_Y, \downarrow_Y)$ and $\mathcal{Z} = (Z, C_Z, \gamma_Z, \downarrow_Z)$.

Lattices:

$[\mathcal{Y} \to \mathcal{Z}] = ([Y \to Z], [C_Y \to C_Z], \gamma, \downarrow)$

where $\gamma\, f$ = greatest realizer of f,

$h' \downarrow h \iff (b \downarrow_Y y \Rightarrow h'b \downarrow_Z h\,y).$

- Application $[Y \to Z] \times Y \to Z$ is realized by application $[C_Y \to C_Z] \times C_Y \to C_Z$.
- Consider $(X, C_X, \gamma_X, \downarrow_X)$.

  If $f : X \times Y \to Z$ is realized

  by $f' : C_X \times C_Y \to C_Z$,

  then curry $f : X \to [Y \to Z]$ is realized

  by curry $f' : C_X \to [C_Y \to C_Z]$.

Domains:

Same basic idea, but γ is not always dense.

$\longrightarrow$ Replace $[C_Y \to C_Z]$ by $\mathsf{cl}\,\gamma([Y \to Z])$.

Quotient

Given $\mathcal{X} = (X, C, \gamma, \downarrow)$ and

a partition X' of X into disjoint classes,

inducing $q : X \to X'$,

find $\mathcal{X}' = (X', C', \gamma', \downarrow')$ such that

$q : \mathcal{X} \to \mathcal{X}'$ is continuous and

for $f : \mathcal{X}' \to \mathcal{Y}$: $f \circ q$ continuous $\Rightarrow f$ continuous.

Lattices:

$C' = \mathcal{P}_{\mathsf{L}}C, \quad \gamma' x' = \mathsf{cl}\,\{\gamma\, x \mid x \in x'\},$

$A \downarrow' x' \iff \exists a \in A, x \in x' : a \downarrow x.$

$$\begin{tikzcd}
C \arrow[r, "s"] & \mathcal{P}_{\mathsf{L}}C \\
X \arrow[u, "\gamma"'] \arrow[r, "q"] & X' \arrow[u, "\gamma'"']
\end{tikzcd}$$

- Unit $s : C \to \mathcal{P}_{\mathsf{L}}C$, $s\,a = {\downarrow} a$ realizes q.
- If $g : C \to C_Y$ realizes $f \circ q : X \to Y$, then its Kleisli extension $G : \mathcal{P}_{\mathsf{L}}C \to C_Y$ realizes f.

Domains: Same basic idea,

but restrict $\mathcal{P}_{\mathsf{L}}C$ to $\mathsf{cl}\,\gamma'(X')$.

Conclusion

- Convergence spaces can be described without filters.
- Scott-domain environments are special domain representations.

  Similar ones exist for all topological spaces.
- Given representations $\mathcal{X}_i$ of convergence spaces X_i, we get representations
  - $\prod_{i \in I} \mathcal{X}_i$ of $\prod_{i \in I} X_i$,
  - $\sum_{i \in I} \mathcal{X}_i$ of $\sum_{i \in I} X_i$,
  - $[\mathcal{X}_1 \to \mathcal{X}_2]$ of $[X_1 \to X_2]$, etc.
- ω-CONV (convergence spaces with an ω-continuous representation)

  contains all second-countable topological spaces and is closed under countable sum and product, subspace, quotient, and function space.

http://www.cs.uni-sb.de/~heckmann/spaces.html