Power Domain Constructions

(Potenzbereich-Konstruktionen)

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von

Reinhold Heckmann

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Einleitung

Motivation

Eine Potenz(bereich)-Konstruktion bildet jeden semantischen Bereich **X** aus einer bestimmten Klasse von Bereichen in einen sogenannten Potenzbereich über **X** ab, dessen Punkte Mengen von Punkten des Grundbereichs repräsentieren. Potenzbereich-Konstruktionen wurden ursprünglich entwickelt, um die Semantik nichtdeterministischer Programmiersprachen [Plo76, Smy78, HP79, Mai85] zu modellieren. Andere Motivationen sind die semantische Repräsentation eines Datentyps von Mengen [Hec90c] oder relationaler Datenbanken [BDW88] [Gun89b].

Plotkin [Plo76] schlug die erste Potenzbereich-Konstruktion im Jahre 1976 vor. Weil seine Konstruktion aus der Kategorie der beschränkt vollständigen algebraischen Bereiche hinausführt, schlug er die größere Kategorie der SFP-Bereiche vor, die unter seiner Konstruktion abgeschlossen ist. Kurze Zeit später führte Smyth [Smy78] eine einfachere Konstruktion ein, die obere oder Smyth-Potenzkonstruktion, die beschränkte Vollständigkeit erhält. In [Smy83] kommt eine dritte Potenzbereich-Konstruktion vor, die untere Potenzkonstruktion, die das Trio der klassischen Potenzbereich-Konstruktionen vervollständigt.

Von Problemen in der Datenbanktheorie ausgehend schlugen Buneman et al. [BDW88] vor, den unteren und oberen Potenzbereich zu einem sogenannten Sandwich-Potenzbereich zu kombinieren. Gunter erforschte die Logik der klassischen Potenzbereiche [Gun89a]. Indem er die Logik von Plotkins Bereich auf natürliche Weise ausdehnte, entwickelte er einen sogenannten gemischten Potenzbereich [Gun89b, Gun90]. Plotkins Potenzbereich ist eine Teilmenge des gemischten, welcher wiederum eine Teilmenge des Sandwich-Potenzbereichs ist.

Unabhängig davon fanden wir den Sandwich- und den gemischten Potenzbereich in einer isomorphen Darstellung als großen und kleinen Mengenbereich, als wir Konstruktionen zur Beschreibung der Semantik eines abstrakten Datentyps von Mengen in einer funktionalen Sprache entwickelten (siehe [Hec90c]).

In Anbetracht von mindestens 5 verschiedenen Potenzbereich-Konstruktionen stellt sich die Frage nach dem Wesen dieser Konstruktionen, d.h. nach denjenigen Eigenschaften, die die Anwendung des Begriffs 'Potenzbereich' gestatten. Wir suchen also nach einer Theorie der Potenzbereich-Konstruktionen, die die existierenden beschreibt und die Beantwortung der folgenden Fragen gestattet:

(1) Was sind Potenzbereich-Konstruktionen?

(2) Welche Beziehungen bestehen zwischen den verschiedenen Potenzbereich-Konstruktionen?

(3) Gibt es mehr als die fünf oben aufgezählten Konstruktionen?

(4) Wenn ja, wie sind diese fünf Konstruktionen unter allen anderen ausgezeichnet?

Eine allgemeine Theorie der Potenzkonstruktionen sollte auch — wenn sie nützlich sein soll — allgemeine Sätze enthalten, die auf alle speziellen Potenzbereich-Konstruktionen anwendbar sind.

Die Autoren der frühen Papiere über Potenzbereich-Konstruktionen erwähnen alle einige algebraische Operationen, die in ihren Potenzbereichen möglich sind. Sie geben jedoch weder die algebraischen Eigenschaften dieser Operationen noch die Beziehungen zwischen ihnen an. In [HP79] werden die drei klassischen Potenzbereiche als freie Algebren bezüglich gewisser algebraischer Theorien charakterisiert. Diese Theorien beschreiben kommutative idempotente Halbgruppen mit verschiedenen Zusatzaxiomen, die aber mathematisch nicht gut motiviert sind. Gunter [Gun89b] charakterisiert seine gemischten Potenzbereiche auf etwa dieselbe Weise. Anstelle zusätzlicher Axiome für die Halbgruppe benützt er eine zusätzliche einstellige Operation. In [Hec90c] konnten auch die Sandwich-Potenzbereiche als freie Algebren beschrieben werden bezüglich einer Theorie von Halbgruppen mit einer partiellen Verknüpfung, deren algorithmische Bedeutung im Dunkeln liegt (siehe Kap. 24).

Diese Ergebnisse liefern aber wohl nicht die gewünschte allgemeine Theorie der Potenzbereich-Konstruktionen. Ihre Antwort auf die Frage (1) von oben wäre nämlich, daß Potenzbereiche freie Algebren bezüglich einer algebraischen Theorie von kommutativen idempotenten Halbgruppen mit mehr oder minder seltsamen zusätzlichen Operationen und Axiomen sind. Darüberhinaus hätte diese Theorie den Nachteil, daß sie eher einzelne Potenzbereiche behandelt als die ganze Potenzbereich-Konstruktion, d.h. die Abbildung von Grundbereichen nach Potenzbereichen.

Die algebraische Theorie der Potenzbereich-Konstruktionen

Gunter beschreibt in [Gun90] die Semantik einer nichtdeterministischen Sprache mithilfe einer generischen Potenzbereich-Konstruktion. Diese bietet drei Grundoperationen an, nämlich Einermengenbildung, Vereinigung von zwei Mengen und Ausdehnung einer mengenwertigen Funktion von Punkten auf Mengen. Diese generische Semantik kann instanziiert werden, indem die generische Konstruktion durch eine konkrete ersetzt wird, welche die notwendigen Grundoperationen zur Verfügung stellt.

Wir definieren daher eine Potenzbereich-Konstruktion durch Axiome, die die Existenz einiger Grundoperationen fordern. Dazu kommen Axiome, denen die Grundoperationen genügen müssen. Über die Wahl dieser Axiome könnte man sich streiten. Wir sind jedoch der Ansicht, daß unsere Wahl natürlich ist. Diese Meinung wird dadurch bestärkt, daß unsere Definition zu einer reichen Theorie führt, die in Teil II der Arbeit vorgestellt wird. Die bekannten Konstruktionen werden von dieser Theorie nicht nur erfaßt, sondern auch unter allen anderen möglichen Konstruktionen ausgezeichnet.

Spezifikation der Potenzkonstruktionen

Im folgenden verstehen wir unter Bereichen gerichtet vollständige partiell geordnete Mengen. Wir fordern also weder die Existenz eines kleinsten Elements noch Zusatzeigenschaften wie Algebraizität. Wie in Kap. 9 ausgeführt, bildet eine *Potenz(bereich)-Konstruktion* $\mathcal{P}$ *Grundbereiche* $\mathbf{X}$ in *Potenzbereiche* über $\mathbf{X}$ ab. Die Potenzbereiche müssen den folgenden Axiomen genügen:

Leere Menge: Es gibt ein ausgezeichnetes Element θ in jedem Potenzbereich $\mathcal{P}\mathbf{X}$.

Vereinigung: Es gibt eine stetige Operation $\uplus : [\mathcal{P}\mathbf{X} \times \mathcal{P}\mathbf{X} \to \mathcal{P}\mathbf{X}]$ in jedem Potenzbereich. '$\uplus$' ist kommutativ und assoziativ mit neutralem Element θ.

Einermengen:
Es gibt eine stetige Abbildung $\iota : [\mathbf{X} \to \mathcal{P}\mathbf{X}]$, $x \mapsto \{\!|x|\!\}$ für jeden Grundbereich $\mathbf{X}$.

Erweiterung von Funktionen: Für je zwei Bereiche $\mathbf{X}$ und $\mathbf{Y}$ gibt es eine höhere Funktion $ext : [[\mathbf{X} \to \mathcal{P}\mathbf{Y}] \to [\mathcal{P}\mathbf{X} \to \mathcal{P}\mathbf{Y}]]$, die mengenwertige Funktionen auf Grundbereichelementen in mengenwertige Funktionen auf Mengen abbildet. Sie muß den folgenden Axiomen genügen:

(P1) $ext\, f\, \theta = \theta$
(P2) $ext\, f\, (A \uplus B) = ext\, f\, A \uplus ext\, f\, B$
(P3) $ext\, f\, \{\!|x|\!\} = f x$ oder: $ext\, f \circ \iota = f$
(siehe Abbildung)
(S1) $ext\, (\lambda x.\, \theta)\, A = \theta$

$$\begin{tikzcd} \mathcal{P}\mathbf{X} \arrow[r, "ext\ f"] & \mathcal{P}\mathbf{Y} \\ \mathbf{X} \arrow[u, "\iota"] \arrow[ur, "f"'] & \end{tikzcd}$$

(S2) $ext\, (\lambda x.\, f x \uplus g x)\, A = ext\, f\, A \uplus ext\, g\, A$.
Wenn man '$\uplus$' auf Funktionen hochhebt, kann man kürzer schreiben: $ext\, (f \uplus g) = ext\, f \uplus ext\, g$.

(S3) $ext\, (\lambda x.\, \{\!|x|\!\})\, A = A$ oder: $ext\, \iota = id$.

(S4) Für je zwei Abbildungen $f : [\mathbf{X} \to \mathcal{P}\mathbf{Y}]$ und $g : [\mathbf{Y} \to \mathcal{P}\mathbf{Z}]$ gilt

$$ext\, g\, (ext\, f\, A) = ext\, (\lambda a.\, ext\, g\, (f a))\, A$$

für alle A aus $\mathcal{P}\mathbf{X}$, oder: $ext\, g \circ ext\, f = ext\, (ext\, g \circ f)$
(siehe Abbildung)

$$\begin{tikzcd} \mathcal{P}\mathbf{X} \arrow[r, "ext\ f"] & \mathcal{P}\mathbf{Y} \arrow[r, "ext\ g"] & \mathcal{P}\mathbf{Z} \\ \mathbf{X} \arrow[u, "\iota"] \arrow[ur, "f"'] & \mathbf{Y} \arrow[u, "\iota"] \arrow[ur, "g"'] & \end{tikzcd}$$

Man beachte, daß wir nicht fordern, daß $ext\, f$ durch (P1) bis (P3) eindeutig für jedes f bestimmt ist. Wenn wir diese Eindeutigkeit gefordert hätten, wären die Gleichungen (S1) bis (S4) beweisbar. Deshalb nennen wir sie sekundäre Axiome im Gegensatz zu den primären Axiomen (Pi).

Halbringe und Moduln

Die oben eingeführten Grundoperationen erlauben die Herleitung vieler anderer Operationen mit nützlichen algebraischen Eigenschaften (siehe Kap. 10). In dieser Zusammenfassung erwähnen wir nur die wichtigsten.

Die Erweiterungsfunktion hängt von zwei Grundbereichen $\mathbf{X}$ und $\mathbf{Y}$ ab. Besonders interessante Instanzen der Erweiterungsfunktion entstehen, wenn einer der beiden Bereiche $\mathbf{X}$ und $\mathbf{Y}$ der Einpunkt-Bereich $\mathbf{1} = \{\diamond\}$ ist. Im Falle $\mathbf{X} = \mathbf{1}$ hat die Erweiterungsfunktion die Funktionalität $ext : [[\mathbf{1} \to \mathcal{P}\mathbf{Y}] \to [\mathcal{P}\mathbf{1} \to \mathcal{P}\mathbf{Y}]]$. Weglassen des unnötigen Arguments aus $\mathbf{1}$ sowie Zusammenfassen und Vertauschen der Argumente führt zum 'äußeren Produkt' $\cdot : [\mathcal{P}\mathbf{1} \times \mathcal{P}\mathbf{Y} \to \mathcal{P}\mathbf{Y}]$. Seine Definition ist $b \cdot S = ext\,(\lambda\diamond.S)\,b$. Multiplikation wird zur inneren Verknüpfung auf $\mathcal{P}\mathbf{1}$, wenn man zusätzlich $\mathbf{Y} = \mathbf{1}$ wählt.

Wenn wir die leere Menge θ von $\mathcal{P}\mathbf{1}$ in 0, die Vereinigung '$\uplus$' in '+', und die Einermenge $\{|\diamond|\}$ in 1 umbenennen, erhalten wir aus den Potenzaxiomen die folgenden algebraischen Gesetze für $\mathcal{P}\mathbf{1}$:

$$
\begin{array}{lll}
a + (b + c) = (a + b) + c & a + b = b + a & a + 0 = 0 + a = a \\
a \cdot (b \cdot c) = (a \cdot b) \cdot c & & a \cdot 1 = 1 \cdot a = a \\
a \cdot (b_1 + b_2) = (a \cdot b_1) + (a \cdot b_2) & (a_1 + a_2) \cdot b = (a_1 \cdot b) + (a_2 \cdot b) & a \cdot 0 = 0 \cdot a = 0
\end{array}
$$

Solche algebraischen Strukturen heißen *Halbringe*. Man beachte, daß alle Algebren, die in dieser Arbeit vorkommen, Bereiche als Trägermenge und stetige Funktionen als Operationen haben.

Halbringe verallgemeinern sowohl Ringe als auch distributive Verbände. Man kann sie daher auch logisch interpretieren: 0 als 'falsch', 1 als 'wahr', Addition als Disjunktion und Multiplikation als Konjunktion. Der Halbring $\mathcal{P}\mathbf{1}$ stellt dann die inhärente Logik der Potenzbereich-Konstruktion $\mathcal{P}$ dar. Diese Sicht der Dinge ist besonders sinnvoll, wenn die Halbringaddition idempotent ist. Bei den bekannten Potenzkonstruktionen ist das der Fall.

Wenn wir auch die leere Menge θ von $\mathcal{P}\mathbf{X}$ in 0 und die Vereinigung '$\uplus$' in '+' umbenennen, erhalten wir die folgenden algebraischen Gesetze für $\mathcal{P}\mathbf{X}$:

$$
\begin{array}{lll}
A + (B + C) = (A + B) + C & A + B = B + A & A + 0 = 0 + A = A \\
r \cdot (s \cdot A) = (r \cdot s) \cdot A & & 1 \cdot A = A \\
r \cdot (A_1 + A_2) = (r \cdot A_1) + (r \cdot A_2) & (r_1 + r_2) \cdot A = (r_1 \cdot A) + (r_2 \cdot A) & r \cdot 0 = 0 \cdot A = 0
\end{array}
$$

Hierbei wurden die Elemente von $\mathcal{P}\mathbf{X}$ groß geschrieben im Unterschied zu den Elementen von $\mathcal{P}\mathbf{1}$.

Solche algebraischen Strukturen heißen *$\mathcal{P}\mathbf{1}$-Moduln*. Eine Funktion $f : [M \to M']$ zwischen zwei $\mathcal{P}\mathbf{1}$-Moduln heißt *linear* wenn $f\,(A + B) = fA + fB$ und $f\,(r \cdot A) = r \cdot fA$ gilt.

Die Ergebnisse von Kap. 10 liefern den folgenden Satz:

Satz 11.1.3: Sei $\mathcal{P}$ eine Potenzkonstruktion und seien

$$\begin{array}{ll} + = \uplus : & [\mathcal{P}\mathbf{X} \times \mathcal{P}\mathbf{X} \to \mathcal{P}\mathbf{X}] \\ 0 = \theta : & \mathcal{P}\mathbf{X} \\ \cdot = \lambda(a, S).\, ext\,(\lambda\diamond.\, S)\, a : & [\mathcal{P}\mathbf{1} \times \mathcal{P}\mathbf{X} \to \mathcal{P}\mathbf{X}] \\ 1 = \{\!|\diamond|\!\} : & \mathcal{P}\mathbf{1} \end{array}$$

Dann ist $\mathcal{P}\mathbf{1}$ mit diesen Operationen ein Halbring und $\mathcal{P}\mathbf{X}$ ein $\mathcal{P}\mathbf{1}$-Modul für alle Bereiche $\mathbf{X}$. Für $f : [\mathbf{X} \to \mathcal{P}\mathbf{Y}]$ ist die Erweiterung $ext\, f : [\mathcal{P}\mathbf{X} \to \mathcal{P}\mathbf{Y}]$ linear und $ext\, f \circ \iota = f$ gilt.

Dieses Ergebnis bringt unsere Arbeit in Verbindung mit einem Artikel von Main [Mai85], in dem Potenzbereiche als freie Halbring-Moduln eingeführt werden. Es gibt jedoch einige Unterschiede: unsere Konstruktionen können auch unfreie Moduln erzeugen und unsere Einermengen-Funktion ι braucht nicht strikt zu sein.

Der Halbring $\mathcal{P}\mathbf{1}$ heißt der *charakteristische Halbring* der Potenzkonstruktion $\mathcal{P}$. Größere Allgemeinheit erreicht man, wenn man auch solche Halbringe charakteristisch für die Potenzkonstruktion $\mathcal{P}$ nennt, die zu $\mathcal{P}\mathbf{1}$ isomorph sind. In diesem Falle muß allerdings der Isomorphismus als fest angenommen werden.

Verschiedene Potenzkonstruktionen können durchaus denselben charakteristischen Halbring haben. Umgekehrt ergibt sich aus den Sätzen 14.5.1 und 15.1.1, daß es zu jedem gegebenen Halbring zwei ausgezeichnete Potenzkonstruktionen gibt.

Potenzhomomorphismen

Homomorphismen zwischen algebraischen Strukturen sind Abbildungen, die alle Operationen dieser Strukturen erhalten. Potenz-Konstruktionen können als algebraische Strukturen auf einer höheren Ebene betrachtet werden. Daher ist es möglich und auch sinnvoll, entsprechende Homomorphismen zu definieren (siehe Kap. 12).

Ein Potenzhomomorphismus $H : \mathcal{P} \dot{\to} \mathcal{Q}$ zwischen zwei Potenzkonstruktionen $\mathcal{P}$ und $\mathcal{Q}$ ist eine 'Familie' von Abbildungen $H = (H_\mathbf{X})_\mathbf{X} : [\mathcal{P}\mathbf{X} \to \mathcal{Q}\mathbf{X}]$, die über alle Potenzoperationen kommutieren, d.h.

- Die leere Menge von $\mathcal{P}\mathbf{X}$ wird auf die leere Menge von $\mathcal{Q}\mathbf{X}$ abgebildet: $H\theta = \theta$.
- Das Bild einer Vereinigung ist die Vereinigung der Bilder: $H(A \uplus B) = (HA) \uplus (HB)$.
- Einermengen in $\mathcal{P}\mathbf{X}$ werden zu Einermengen in $\mathcal{Q}\mathbf{X}$: $H\{\!|x|\!\}_\mathcal{P} = \{\!|x|\!\}_\mathcal{Q}$.
- Sei $f : [\mathbf{X} \to \mathcal{P}\mathbf{Y}]$. Dann muß $H \circ f : [\mathbf{X} \to \mathcal{Q}\mathbf{Y}]$ und $H(ext_\mathcal{P}\, f\, A) = ext_\mathcal{Q}\,(H \circ f)\,(HA)$ für alle A in $\mathcal{P}\mathbf{X}$ gelten.

Wenn zwei Konstruktionen $\mathcal{P}$ und $\mathcal{Q}$ denselben charakteristischen Halbring haben, kann man definieren, daß ein Potenzhomomorphismus linear ist, wenn alle Funktionen $H_\mathbf{X}$ linear sind.

Ein Potenzisomorphismus zwischen zwei Konstruktionen $\mathcal{P}$ und $\mathcal{Q}$ ist eine Familie von Isomorphismen $H = H_\mathbf{X} : [\mathcal{P}\mathbf{X} \to \mathcal{Q}\mathbf{X}]$, so daß sowohl $(H_\mathbf{X})_\mathbf{X}$ als auch $(H_\mathbf{X}^{-1})_\mathbf{X}$ Potenzhomomorphismen sind.

Initiale Potenzkonstruktionen

Eine Potenzkonstruktion $\mathcal{P}$ ist *initial* für einen Halbring R, wenn es zu allen Potenzkonstruktionen $\mathcal{Q}$ mit demselben charakteristischen Halbring R genau einen linearen Potenzhomomorphismus $\mathcal{P} \dot{\to} \mathcal{Q}$ gibt. Ein Hauptergebnis der Theorie der Potenzkonstruktionen ist, daß es für alle Halbringe R initiale Konstruktionen gibt.

Initiale Potenzbereiche über $\mathbf{X}$ ergeben sich aus freien R-$\mathbf{X}$-Moduln. $(\mathbf{F}, \eta)$ ist ein freies R-$\mathbf{X}$-Modul, wenn $\mathbf{F}$ ein R-Modul und η eine stetige Abbildung von $\mathbf{X}$ nach $\mathbf{F}$ ist, so daß es für jedes R-Modul M und jede stetige Abbildung $f : \mathbf{X} \to M$ genau eine stetige lineare Erweiterung $\overline{f} : \mathbf{F} \to M$ mit $\overline{f} \circ \eta = f$ gibt.

Die Idee, freie Moduln zu betrachten, geht auf [HP79] zurück. Hoofman [Hoo87] zeigte die Existenz freier Moduln für den Halbring $\{0, 1\}$. Main [Mai85] schlug Potenzkonstruktionen vor, die als freie Moduln für einige ungewöhnliche Halbringe definiert waren. Im Gegensatz zu uns fordert er dabei die Striktheit der Einermengenabbildung, ohne genau zu sagen warum. Die Einermengenabbildungen in die gemischten und Sandwich-Potenzbereiche jedenfalls sind nicht strikt, weswegen sie in unserer allgemeinen Theorie der Potenzkonstruktionen auch nicht strikt sein können.

Freie R-$\mathbf{X}$-Moduln existieren für alle Halbringe R und Bereiche $\mathbf{X}$ und sind bis auf Isomorphie eindeutig bestimmt. Während der Eindeutigkeitsbeweis ziemlich einfach ist, ist der Existenzbeweis recht schwierig. Üblicherweise wird er mit Hilfsmitteln der Kategorientheorie geführt. Für diese Arbeit wurde der Beweis von [Hoo87] abgeändert und so vereinfacht, daß keine Sätze aus der Kategorientheorie mehr benötigt werden (Abschnitt 13.5). Leider wird das freie Modul durch den Beweis nicht explizit gegeben.

Der Hauptsatz über initiale Potenzkonstruktionen wird in Abschnitt 14.5 formuliert.

Satz 14.5.1:

Sei R ein Halbring und sei $\mathcal{P}$ die Konstruktion, die jeden Grundbereich $\mathbf{X}$ in das freie R-$\mathbf{X}$-Modul $(\mathbf{F}, \eta)$ abbildet. Dann ist $\mathcal{P}$ die initiale Potenzkonstruktion für Halbring R.

Die Potenzoperationen Vereinigung und leere Menge sind durch die Addition und ihr neutrales Element in $\mathbf{F}$ gegeben. Die Einermengenabbildung ist η. Für Funktionen $f : [\mathbf{X} \to \mathcal{P}\mathbf{Y}]$ ist $\mathit{ext}\, f$ gegeben durch die eindeutige Erweiterung von f. Das von vornherein gegebene äußere Produkt der Moduln $\mathcal{P}\mathbf{X}$ stimmt mit dem aus den Potenzoperationen abgeleiteten äußeren Produkt überein.

Für jede andere Potenzkonstruktion $\mathcal{Q}$ mit Halbring R erhält man den eindeutigen linearen Potenzhomomorphismus $H : \mathcal{P} \dot{\to} \mathcal{Q}$ als Erweiterung $H_{\mathbf{X}} = \overline{\lambda x. \{\!|x|\!\}_{\mathcal{Q}}}$.

Finale Potenzkonstruktionen

Eine Potenzkonstruktion $\mathcal{P}$ ist *final* für Halbring R, wenn es für jede Potenzkonstruktion $\mathcal{Q}$ mit demselben Halbring R genau einen linearen Potenzhomomorphismus $\mathcal{Q} \dot{\to} \mathcal{P}$ gibt. Finale Konstruktionen werden in Kap. 15 behandelt.

Finale Potenzkonstruktionen wurden in der Literatur bisher nicht vorgeschlagen, wahrscheinlich weil der Begriff des Potenzhomomorphismus fehlte. Im Gegensatz zu initialen

Konstruktionen können finale Potenzkonstruktionen durch Prädikate zweiter Ordnung explizit dargestellt werden. Es ergibt sich, daß die von Smyth [Smy83] eingeführte obere Potenzkonstruktion final ist, während sein ursprünglicher Vorschlag in [Smy78] sich als initial herausstellte.

Wie oben erwähnt, impliziert die Erweiterungsfunktion besonders interessante Operationen, wenn einer der beiden Bereiche, von denen sie abhängt, der Einpunktbereich $\mathbf{1} = \{\diamond\}$ ist. Im Falle $\mathbf{Y} = \mathbf{1}$ erhält man $ext : [[\mathbf{X} \to \mathcal{P}\mathbf{1}] \to [\mathcal{P}\mathbf{X} \to \mathcal{P}\mathbf{1}]]$. Vertauschen der Argumente liefert eine Abbildung $\mathcal{E} : [\mathcal{P}\mathbf{X} \to [[\mathbf{X} \to \mathcal{P}\mathbf{1}] \to \mathcal{P}\mathbf{1}]]$ von formalen Mengen nach Funktionen zweiter Ordnung. Gemäß der logischen Interpretation von $\mathcal{P}\mathbf{1}$ können diese Funktionen als Prädikate zweiter Ordnung aufgefaßt werden. Die intuitive Bedeutung von $\mathcal{E}$ ist existentielle Quantifizierung. Der Wert von $\mathcal{E} A\, p$ zeigt für ein A in $\mathcal{P}\mathbf{X}$ und ein Prädikat $p : [\mathbf{X} \to \mathcal{P}\mathbf{1}]$ an, ob ein Element von A das Prädikat p erfüllt.

Die Axiome der Erweiterungsfunktion liefern die folgenden Eigenschaften:

(P1) $\mathcal{E}\ \theta = \lambda p.\, \theta$

(P2) $\mathcal{E}\ (A \uplus B) = \lambda p.\, (\mathcal{E}\ A\ p) \uplus (\mathcal{E}\ B\ p)$

(P3) $\mathcal{E}\ \{\!|x|\!\} = \lambda p.\, p\, x$

(S4) $\mathcal{E}\ (ext\, f\, A) = \lambda p.\, \mathcal{E} A\, (\lambda a.\, \mathcal{E}\, (f a)\, p)$

Diese Resultate legen nahe, zu einem gegebenen Halbring $\mathcal{P}\mathbf{1}$ eine Potenzkonstruktion durch $\mathcal{P}\mathbf{X} = [[\mathbf{X} \to \mathcal{P}\mathbf{1}] \to \mathcal{P}\mathbf{1}]$ zu definieren. Eine leichte Variante dieser Idee liefert tatsächlich eine Potenzkonstruktion.

Sei R ein gegebener Halbring. Dann können wir ein Produkt $\cdot : [\mathbf{X} \to R] \times R \to [\mathbf{X} \to R]$ durch $f \cdot r = \lambda x. fx \cdot r$ definieren. In ähnlicher Weise kann auch eine Addition als innere Verknüpfung auf $[\mathbf{X} \to R]$ eingeführt werden. Eine Funktion $F : [[\mathbf{X} \to R] \to R]$ ist *rechtslinear*, falls $F\,(f_1 + f_2) = F f_1 + F f_2$ und $F\,(f \cdot r) = F f \cdot r$ gelten. Der Bereich aller rechtslinearen Funktionen von $[\mathbf{X} \to R]$ nach R wird durch $[[\mathbf{X} \to R] \stackrel{rlin}{\to} R]$ bezeichnet.

Satz 15.1.1:

Sei R ein gegebener Halbring. Die finale Potenzkonstruktion für R bildet jeden Grundbereich $\mathbf{X}$ in die Menge der rechtslinearen Prädikate zweiter Ordnung über $\mathbf{X}$ ab: $\mathcal{P}\mathbf{X} = [[\mathbf{X} \to R] \stackrel{rlin}{\to} R]$. Ihre Operationen sind gegeben durch

- $\theta = \lambda g.\, 0$
- $A \uplus B = \lambda g.\, Ag + Bg$
- $\{\!|x|\!\} = \lambda g.\, gx$ für $x \in \mathbf{X}$.
- $ext\, f\, A = \lambda g.\, A\, (\lambda a.\, fag)$ für $f : [\mathbf{X} \to \mathcal{P}\mathbf{Y}]$ und $A \in \mathcal{P}\mathbf{X}$.

Sei $\mathcal{Q}$ eine andere Potenzkonstruktion mit Halbring R. Der eindeutige lineare Potenzhomomorphismus von $\mathcal{Q}$ nach $\mathcal{P}$ ist durch die existentielle Quantifizierung $\mathcal{E} A = \lambda g.\, ext_{\mathcal{Q}}\, g\, A$ für A in $\mathcal{Q}\mathbf{X}$ gegeben.

Die Definition von *ext* sowie die der übrigen Operationen werden durch die oben aufgeführten Eigenschaften von $\mathcal{E}$ motiviert. Man beachte, daß $\mathcal{P}\mathbf{1}$ sich zu $[[\mathbf{1} \to R] \stackrel{rlin}{\to} R]$ ergibt, das nicht gleich sondern nur isomorph zu R ist.

Teilkonstruktionen

Sei $\mathcal{P}$ eine gegebene Potenzkonstruktion. $\mathcal{Q}$ heißt *Teilkonstruktion* von $\mathcal{P}$, falls $\mathcal{Q}$ Grundbereiche $\mathbf{X}$ in Teilmengen von $\mathcal{P}\mathbf{X}$ abbildet, so daß

- $\mathcal{Q}\mathbf{X}$ abgeschlossen ist gegenüber Suprema von gerichteten Mengen,
- $\theta \in \mathcal{Q}\mathbf{X}$,
- wenn A und B in $\mathcal{Q}\mathbf{X}$ sind, dann ist $A \uplus B$ in $\mathcal{Q}\mathbf{X}$,
- $\{\!|x|\!\}$ in $\mathcal{Q}\mathbf{X}$ ist für alle x in $\mathbf{X}$,
- wenn $f : [\mathbf{X} \to \mathcal{Q}\mathbf{Y}]$ und A in $\mathcal{Q}\mathbf{X}$, dann ist $ext\, f\, A$ in $\mathcal{Q}\mathbf{Y}$.

Kurz, $\mathcal{Q}\mathbf{X}$ ist abgeschlossen gegenüber allen Potenzoperationen von $\mathcal{P}$. $\mathcal{Q}$ ist trivialerweise eine Potenzkonstruktion, da die Gültigkeit der Potenzaxiome für $\mathcal{Q}$ von $\mathcal{P}$ geerbt wird. Teilkonstruktionen werden in Kap. 14 behandelt.

Man sieht leicht, daß der Durchschnitt einer Familie von Teilkonstruktionen einer Potenzkonstruktion $\mathcal{P}$ wieder eine Teilkonstruktion von $\mathcal{P}$ ist, wenn wir $(\bigcap_{i\in I} \mathcal{Q}_i)\mathbf{X} = \bigcap_{i\in I}(\mathcal{Q}_i\mathbf{X})$ definieren. Daher bilden die Teilkonstruktionen von $\mathcal{P}$ einen vollständigen Verband.

Sei R ein Halbring. R' ist ein *Teilhalbring* von R, falls R' eine Teilmenge von R ist, die 0 und 1 enthält und abgeschlossen ist gegenüber Suprema von gerichteten Mengen, Addition und Multiplikation. Weil die Operationen im charakteristischen Halbring von den Potenzoperationen abgeleitet sind, ist der Halbring einer Teilkonstruktion $\mathcal{Q}$ von $\mathcal{P}$ ein Teilhalbring des Halbrings von $\mathcal{P}$. Aus denselben Gründen ist der Halbring eines Durchschnitts von Teilkonstruktionen gleich dem Durchschnitt der Halbringe der Teilkonstruktionen.

Wir fanden zwei allgemeine Methoden zur Bildung von Teilkonstruktionen einer gegebenen Konstruktion $\mathcal{P}$ mit Halbring R. Die Methode der Kernbildung ergibt die kleinste Teilkonstruktion von $\mathcal{P}$, die noch den Halbring R hat, während die Methode der existentiellen Einschränkung die größte Teilkonstruktion von $\mathcal{P}$ mit einem gegebenen Teilhalbring R' von R liefert. Der Kern der existentiellen Einschränkung ist dann die kleinste Teilkonstruktion mit dem gegebenen Halbring R'.

Der *Kern* $\mathcal{P}^c\mathbf{X}$ eines Potenzbereichs $\mathcal{P}\mathbf{X}$ ist die kleinste Teilmenge von $\mathcal{P}\mathbf{X}$, die die leere Menge θ und alle Einermengen $\{\!|x|\!\}$ enthält und abgeschlossen ist gegenüber Vereinigung '$\uplus$', Produkt mit Faktoren aus $\mathcal{P}\mathbf{1}$ und gerichteten Suprema. Man kann zeigen, daß die Moduln $\mathcal{P}^c\mathbf{X}$ auch gegenüber der Erweiterungsfunktion von $\mathcal{P}$ abgeschlossen sind. Daher ist $\mathcal{P}^c$ eine Teilkonstruktion von $\mathcal{P}$. Wir nennen sie den *Kern* von $\mathcal{P}$. Er ist die kleinste Teilkonstruktion von $\mathcal{P}$ mit demselben Halbring wie $\mathcal{P}$.

Eine Potenzkonstruktion, die mit ihrem Kern übereinstimmt, heißt *reduziert*. Reduzierte Konstruktionen haben besonders schöne Eigenschaften. Alle initialen Konstruktionen sind reduziert.

Wenn $\mathcal{P}$ eine Potenzkonstruktion mit Halbring R ist und R' ein Teilhalbring von R, dann ist die *existentielle Einschränkung* von $\mathcal{P}$ auf R' definiert durch

$$\mathcal{Q}\mathbf{X} = \mathcal{P}|_{R'}\mathbf{X} = \{A \in \mathcal{P}\mathbf{X} \mid \forall p : [\mathbf{X} \to R'] : \mathcal{E} A\, p \in R'\}.$$

Sie ist eine Teilkonstruktion von $\mathcal{P}$ mit Halbring R' und die größte derartige Teilkonstruktion.

Wegen ihrer Definition durch existentielle Quantifizierung könnte man glauben, daß die existentielle Einschränkung einer finalen Konstruktion für R eine finale Konstruktion für R' ist. Das ist jedoch nicht der Fall, wie wir in Abschnitt 23.4 sehen werden. Dafür gibt es zwei Gründe. Erstens können zwei verschiedene Prädikate in $[[\mathbf{X} \to R] \stackrel{rlin}{\to} R]$ auf Prädikaten in $[\mathbf{X} \to R']$ übereinstimmen. Sie unterscheiden sich dann noch in der Einschränkung der finalen Konstruktion für R, obwohl sie in $[[\mathbf{X} \to R'] \stackrel{rlin}{\to} R']$ gleich wären. Zweitens könnte $[[\mathbf{X} \to R'] \stackrel{rlin}{\to} R']$ zusätzliche Elemente haben, die sich nicht aus der Einschränkung von Prädikaten in $[[\mathbf{X} \to R] \stackrel{rlin}{\to} R]$ ergeben.

Trotz dieser allgemeinen Resultate finden sich auch Beispiele für Halbringe R und R', wo die existentielle Einschränkung der finalen Konstruktion für R final für R' ist — siehe Satz 23.3.1.

Produkte von Potenzkonstruktionen

In Kap. 16 wird eine weitere Methode zur Bildung neuer Konstruktionen aus bereits existierenden vorgestellt. Zu einer gegebenen Familie $(\mathcal{P}_i)_{i \in I}$ von Potenzkonstruktionen wird eine Produktkonstruktion $\mathcal{P} = \prod_{i \in I} \mathcal{P}_i$ wie folgt gebildet:

- $\mathcal{P}\mathbf{X} = \prod_{i \in I} \mathcal{P}_i \mathbf{X}$ für alle Grundbereiche $\mathbf{X}$
- $\theta = (\theta_i)_{i \in I}$
- $(A_i)_{i \in I} \uplus (B_i)_{i \in I} = (A_i \uplus B_i)_{i \in I}$
- $\{\!|x|\!\} = (\{\!|x|\!\}_i)_{i \in I}$ für alle x in $\mathbf{X}$
- Für $f : [\mathbf{X} \to \mathcal{P}\mathbf{Y}]$ sei $f_i = \pi_i \circ f$. Dann ist $ext\, f\, (A_i)_{i \in I} = (ext_i\, f_i\, A_i)_{i \in I}$, wobei ext_i die Erweiterungsfunktion von $\mathcal{P}_i$ bezeichnet. Hierbei bedeutet π_i Projektion auf die Komponente i.

Die Überprüfung der Potenzaxiome für $\mathcal{P}$ ist einfach, da die Potenzoperationen in den einzelnen Dimensionen unabhängig voneinander arbeiten. Der charakteristische Halbring von $\mathcal{P}$ ist das Produkt der charakteristischen Halbringe der $\mathcal{P}_i$.

Folgende Sätze über das Produkt wurden hergeleitet:

Finalitätssatz 16.3.1:
Wenn $\mathcal{P}_i$ finale Potenzkonstruktionen für die Halbringe R_i sind für alle $i \in I$, dann ist das Produkt $\prod_{i \in I} \mathcal{P}_i$ eine finale Konstruktion für das Produkt $\prod_{i \in I} R_i$.

$$\prod_{i \in I} [[\mathbf{X} \to R_i] \stackrel{rlin}{\to} R_i] \;\cong\; [[\mathbf{X} \to \prod_{i \in I} R_i] \stackrel{rlin}{\to} \prod_{i \in I} R_i]$$

Kernsatz 16.5.1: Der Kern des Produkts zweier Potenzkonstruktionen ist gleich dem Produkt ihrer Kerne: $(\mathcal{P}_1 \times \mathcal{P}_2)^c = \mathcal{P}_1^c \times \mathcal{P}_2^c$

Initialitätssatz 16.5.3: Wenn $\mathcal{P}_1$ initial für R_1 und $\mathcal{P}_2$ initial für R_2 ist, dann ist $\mathcal{P}_1 \times \mathcal{P}_2$ initial für $R_1 \times R_2$.

Faktorisierungssatz 16.6.1:
Sei $\mathcal{P}$ eine Potenzbereich-Konstruktion für den Halbring $R_1 \times R_2$. $\mathcal{P}$ kann faktorisiert werden, d.h. es gibt Potenzkonstruktionen $\mathcal{P}_1$ für R_1 und $\mathcal{P}_2$ für R_2, so daß $\mathcal{P}$ isomorph zu $\mathcal{P}_1 \times \mathcal{P}_2$ ist vermittels eines linearen Potenzisomorphismus genau dann, wenn $ext\,(\lambda x.\, (1, 0) \cdot \{\!|x|\!\})\, B = (1, 0) \cdot B$ für alle B in allen Potenzbereichen $\mathcal{P}\mathbf{X}$ gilt.

Die Bedingung des Faktorisierungssatzes ist für viele Klassen von Potenzkonstruktionen erfüllt. Zum Beispiel kann jede reduzierte Potenzkonstruktion für $R_1 \times R_2$ faktorisiert werden. Die resultierenden Faktoren sind wieder reduziert (vgl. Kor. 16.6.4).

Spezielle Potenzkonstruktionen

Die algebraische Theorie der Potenzbereich-Konstruktionen kann auf die fünf am Anfang erwähnten bekannten Potenzkonstruktionen angewandt werden. In der folgenden Zusammenfassung unserer Ergebnisse beschränken wir uns auf algebraische Grundbereiche.

Die fünf bekannten Potenzkonstruktionen können wie folgt charakterisiert werden:

- Die untere Potenzkonstruktion $\mathcal{L}$ ist sowohl initial als auch final für den Halbring $\mathbf{L} = \{0, 1\}$ mit $0 < 1$ und $1 + 1 = 1$.
- Die obere oder Smyth-Potenzkonstruktion $\mathcal{U}$ ist sowohl initial als auch final für den Halbring $\mathbf{U} = \{0, 1\}$ mit $1 < 0$ und $1 + 1 = 1$.
- Die konvexe oder Plotkin-Potenzkonstruktion $\mathcal{C}$ ist initial für den Halbring $\mathbf{C} = \{0, 1\}$ mit $1 + 1 = 1$ und unvergleichbaren Werten 0 und 1.
- Die gemischte Potenzkonstruktion $\mathcal{M}$ von Gunter ist initial für den Halbring $\mathbf{B} = \{\perp, 0, 1\}$, wobei $\perp$ unter den unvergleichbaren Werten 0 und 1 liegt. In $\mathbf{B}$ ist Addition gegeben durch parallele Disjunktion und Multiplikation durch parallele Konjunktion.
- Die Sandwich-Potenzkonstruktion $\mathcal{S}$ von Buneman et al. ist final für denselben Halbring $\mathbf{B}$.
- Die finale Potenzkonstruktion für den Halbring $\mathbf{C}$ fehlt in dieser Liste. Sie wurde bisher nie betrachtet, weil sie degeneriert ist und ziemlich merkwürdige Eigenschaften hat.

In Kap. 17 studieren wir die Halbringe $\mathbf{L}$, $\mathbf{U}$, $\mathbf{C}$ und $\mathbf{B}$ und die dazugehörenden Moduln. Die algebraischen Strukturen, die in [HP79] zur Charakterisierung von $\mathcal{C}$, $\mathcal{L}$ und $\mathcal{U}$ erwähnt werden, sind nichts anderes als $\mathbf{C}$-, $\mathbf{L}$- und $\mathbf{U}$-Moduln ohne 0, während Gunters Mix-Algebren aus [Gun89b, Gun90] nichts anderes als $\mathbf{B}$-Moduln sind. Gunters zusätzliche einstellige Operation ist gerade Multiplikation mit $\perp$.

Die Theorie dieser speziellen Potenzkonstruktionen wird symmetrischer, wenn man zusätzlich die Halbringe $\mathbf{D}$ und $\overline{\mathbf{B}}$ betrachtet. Der 'doppelte' Halbring $\mathbf{D}$ ist definiert als $\mathbf{L} \times \mathbf{U}$. $\mathbf{D}$ hat 4 Elemente: ein kleinstes Element $\perp = (0, 1)$, ein größtes Element $\top = (1, 0)$ und zwei unvergleichbare Werte $0 = (0, 0)$ und $1 = (1, 1)$ dazwischen. $\overline{\mathbf{B}} = \{0, 1, \top\}$ ist der zu $\mathbf{B}$ ordnungsduale Halbring. $\mathbf{B}$ und $\overline{\mathbf{B}}$ sind Teilhalbringe von $\mathbf{D}$. Der Schnitt von $\mathbf{B}$ und $\overline{\mathbf{B}}$ ist $\mathbf{C}$.

Gemäß der Theorie der Produkte von Potenzkonstruktionen ist das Produkt $\mathcal{D}$ von $\mathcal{L}$ und $\mathcal{U}$ sowohl initial als auch final für Halbring $\mathbf{D}$. Existentielle Einschränkung auf $\mathbf{B}$ führt zu der Sandwich-Konstruktion $\mathcal{S}$, deren Kern wiederum $\mathcal{M}$ ist. Daher ist $\mathcal{S}$ die größte Teilkonstruktion von $\mathcal{D}$ mit Halbring $\mathbf{B}$, während $\mathcal{M}$ die kleinste ist. Wenn man statt $\mathbf{B}$ den dualen Teilhalbring $\overline{\mathbf{B}}$ benützt, erhält man die duale Sandwich-Konstruktion $\overline{\mathcal{S}}$ und die duale gemischte Konstruktion $\overline{\mathcal{M}}$. Der Schnitt $\mathcal{S}\overline{\mathcal{S}}$ von $\mathcal{S}$ und $\overline{\mathcal{S}}$ ist interessant, weil er beschränkte Vollständigkeit erhält (wenn man die leere Menge wegläßt) und denselben Halbring $\mathbf{C}$ wie Plotkins Konstruktion $\mathcal{C}$ hat. Der Schnitt von $\mathcal{M}$ und $\overline{\mathcal{M}}$ ist $\mathcal{C}$ selbst.

Die Sandwich-Bedingung, die $\mathcal{S}$ als Teilkonstruktion von $\mathcal{D} = \mathcal{L} \times \mathcal{U}$ auszeichnet, ist durch existentielle Quantifikation gegeben. In Kap. 21 wird diese Bedingung in verschiedene äquivalente Formen übersetzt. Eine explizte Charakterisierung von $\mathcal{M}$ als Teilkonstruktion von $\mathcal{D}$ oder $\mathcal{S}$ wurde ebenfalls gefunden. Diese Mix-Bedingung wird in Kap. 22 diskutiert. In diesen Kapiteln werden auch die Bedingungen für $\overline{\mathcal{S}}$ und $\overline{\mathcal{M}}$ untersucht.

Teil III enthält noch einige weitere Ergebnisse, die nicht direkt mit der allgemeinen algebraischen Theorie der Potenzkonstruktionen zusammenhängen. In Verallgemeinerung der Ergebnisse von [FM90] wird bewiesen, daß die untere und obere Potenzkonstruktion kommutieren, d.h. $\mathcal{LU}\mathbf{X}$ und $\mathcal{UL}\mathbf{X}$ sind isomorph (Abschnitte 20.5 und 20.6). Die Sandwich-Potenzbereiche $\mathcal{S}\mathbf{X}$ werden als freie Sandwich-Algebren über $\mathbf{X}$ charakterisiert (Kap. 24).

Bereichsklassen

Die Ergebnisse von Teil III werden für möglichst allgemeine Klassen von Bereichen bewiesen. Die Übereinstimmung zwischen der initialen, der finalen und der explizit durch Scott-abgeschlossene Mengen gegebenen unteren Potenzkonstruktion konnte für alle Grundbereiche gezeigt werden. Bei den oberen Potenzkonstruktionen ist die Lage jedoch wesentlich komplizierter. Die Frage der Übereinstimmung zwischen der initialen oberen Konstruktion $\mathcal{U}_i$, der finalen $\mathcal{U}_f$, ihres Kerns $\mathcal{U}_c$ und der explizit durch kompakte obere Mengen gegebenen Konstruktion $\mathcal{U}_K$ konnte nur für bestimmte (wenn auch sehr große) Bereichsklassen geklärt werden.

Zur genaueren Untersuchung der Verhältnisse werden in Teil I der Arbeit topologische Methoden eingeführt. Mit ihrer Hilfe können verschiedene Klassen von Bereichen definiert und untersucht werden, die zum Teil neu sind. Für die oberen Potenzkonstruktionen konnte $\mathcal{U}_i\mathbf{X} \cong \mathcal{U}_f\mathbf{X}$ für stetige Grundbereiche $\mathbf{X}$ gezeigt werden (Kap. 19). $\mathcal{U}_f\mathbf{X}$ und $\mathcal{U}_c\mathbf{X}$ hingegen stimmen auf der größeren Klasse der multi-stetigen Bereiche definitiv überein. Die maximalen Klassen der Übereinstimmung sind dabei jeweils unbekannt. $\mathcal{U}_f\mathbf{X}$ und $\mathcal{U}_K\mathbf{X}$ sind genau auf der Klasse der nüchternen Bereiche isomorph. $\mathcal{U}_K\mathbf{X}$ ist jedoch nicht für alle Bereiche sinnvoll definiert, sondern nur auf der Klasse K-RD, die die Klasse der nüchternen Bereiche enthält.

Neuartige topologische Begriffsbildungen wie starke Kompaktheit (Abschnitt 4.7) erlaubten sogar die Definition weiterer oberer Potenzkonstruktionen (Kap. 20). Diese Konstruktionen sind Teilkonstruktionen von $\mathcal{U}_f$ bzw. $\mathcal{U}_K$ und stimmen mit diesen für multi-stetige Grundbereiche überein. Die Frage der Übereinstimmung auf größeren Klassen blieb offen.

Schlußbemerkungen

Geschichte und Veröffentlichungen

Die algebraische Theorie der Potenzkonstruktionen ist aus dem Versuch entstanden, die semantischen Eigenschaften eines Datentyps von Mengen in einer funktionalen Sprache zu beschreiben. Die so entstandene algebraische Spezifikation der 'Mengenbereich-Konstruktionen' wurde in [Hec90c] veröffentlicht. Zwei spezielle Mengenbereich-Konstruktionen wurden gefunden, die die Spezifikation erfüllen. Die großen Mengenbereiche erwiesen sich später als isomorph zu den Sandwich-Potenzbereichen, während sich die kleinen Mengenbereiche als isomorph zu den gemischten Potenzbereichen herausstellten.

Die Spezifikation der Mengenbereich-Konstruktionen benützt einen von vorneherein gegebenen Bereich logischer Werte, nämlich $\mathbf{B} = \{\perp, 0, 1\}$. Die Untersuchung verschiedener aus der Spezifikation abgeleiteter Operationen ergab — auch inspiriert durch [Mai85] — daß ein Bereich logischer Werte auch im nachhinein aus den mengentheoretischen Operationen der leeren Menge, der Einermengenabbildung, der Vereinigung und der Erweiterungsfunktion gebildet werden kann. Die Spezifikation der Mengenbereich-Konstruktionen ließ sich dann in zwei Teile zerlegen: in eine Spezifikation von allgemeineren Objekten, nämlich Potenzbereich-Konstruktionen, und die Forderung, daß $\mathcal{P}\mathbf{1}$ isomorph zu $\mathbf{B}$ sein solle. Durch Weglassen der jetzt als künstlich empfundenen Einschränkung $\mathcal{P}\mathbf{1} \cong \mathbf{B}$ ergab sich eine erste Version der Spezifikation von Potenzkonstruktionen.

Ein zweiter Ausgangspunkt war die Beobachtung, daß Sandwich-Potenzbereiche $\mathcal{S}\mathbf{X}$ isomorph zu Funktionenräumen $[[\mathbf{X} \to \mathbf{B}] \stackrel{add}{\to} \mathbf{B}]$ sind. Der technische Bericht [Hec90a] enthielt einen ersten ordnungstheoretischen Beweis dieser Isomorphie. In der Folge fanden wir heraus, daß viel allgemeinere Funktionenräume $[[\mathbf{X} \to R] \stackrel{rlin}{\to} R]$ immer Potenzkonstruktionen im Sinne der algebraischen Spezifikation darstellen. Diese Ergebnisse wurden zusammen mit der Beschreibung der gemischten und der Plotkin-Potenzbereiche durch Prädikate zweiter Ordnung auf dem MFPS'90 Workshop in Kingston, Ontario, vorgestellt. Die Finalität dieser Konstruktion wurde erst im Sommer 1990 gefunden. Kern, existentielle Einschränkung und die Theorie der Produkte sind sogar noch jüngere Erkenntnisse.

Die voll entwickelte algebraische Theorie ist in dem Technischen Bericht [Hec90b] enthalten, der zur Veröffentlichung als [Hec91] angenommen wurde. Ein Artikel mit Schwerpunkt auf Prädikaten zweiter Ordnung und den Teilkonstruktionen von $\mathcal{D}$ wurde bei der Zeitschrift TCS eingereicht. Ein dritter Artikel über die Vertauschbarkeit von $\mathcal{L}$ und $\mathcal{U}$ wurde zur MFPS'91-Konferenz geschickt.

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Introduction

Motivation

A power domain construction maps every domain $\mathbf{X}$ of some distinguished class of domains into a so-called power domain over $\mathbf{X}$ whose points represent sets of points of the ground domain. Power domain constructions were originally proposed to model the semantics of non-deterministic programming languages [Plo76, Smy78, HP79, Mai85]. Other motivations are the semantic representation of a set data type [Hec90c], or of relational data bases [BDW88, Gun89b].

In 1976, Plotkin [Plo76] proposed the first power domain construction. Because his construction goes beyond the category of bounded complete algebraic domains, Plotkin proposed the larger category of *SFP-domains* that is closed under his construction. A short time later, Smyth [Smy78] introduced a simpler construction, the upper or Smyth power construction, that respects bounded completeness. In [Smy83], a third power domain construction occurs, the lower power domain, that completes the trio of classical power domain constructions.

Starting from problems in data base theory, Buneman et al. [BDW88] proposed to combine lower and upper power domain to a so-called sandwich power domain. Gunter investigated the logic of the classical power domains [Gun89a]. By extending the logic of Plotkin's domain in a natural way, he developed a so-called mixed power domain [Gun89b, Gun90]. Plotkin's power domain is a subset of the mixed one, and this in turn is a subset of the sandwich power domain.

We independently found the sandwich and mixed power domains in an isomorphic form as big and small set domains when developing domain constructions that would give semantics to an abstract data type of sets in a functional programming language (see [Hec90c]).

Given at least five different power domain constructions, the question arises what is the essence of these constructions, i.e. what are their common features which allow the application of the notion 'power domain'. Thus, we look for a theory of power domain constructions that covers the existing ones and provides answers to the following questions:

(1) What are power domain constructions?

(2) How are different power domain constructions related to each other?

(3) Are there more than the five constructions enumerated above?

(4) If so, how are these five constructions distinguished among all the others?

In addition, a general theory of power constructions provides — if it is to be useful — general theorems that are applicable to all specific power domain constructions.

The authors of the early papers about power domain constructions all mentioned some algebraic operations that are possible in their power domains. They however did not yet indicate the algebraic laws of these operations and the relations among different operations. In [HP79] then, the three classical power domains were characterized as free algebras w.r.t. some algebraic theories. Basically, these theories describe commutative idempotent semigroups, but employ various additional axioms that are mathematically not well motivated. In [Gun89b], Gunter characterized his mixed power domains in roughly the same line. Instead of employing additional axioms for the semigroup, he uses an additional unary operation. In [Hec90c], the sandwich power domains were also characterized as free algebras w.r.t. a theory of semigroups with a partial binary operation whose computational meaning is unclear (see chapter 24).

We felt that these results are not the general theory of power domain constructions. Their answer to question (1) above would be: power domains are free algebras according to algebraic theories of CI semigroups with more or less strange additional operations and axioms. Moreover, this theory has the drawback to deal more with single power domains than the whole power domain construction, i.e. the mapping from ground domains into power domains.

The algebraic theory of power domain constructions

Gunter presents in [Gun90] the semantics of a non-deterministic language in terms of a generic power domain construction using the three basic operators of singleton, binary union, and extending set-valued functions from points to sets. These generic semantics may then be instanciated by choosing a concrete construction instead of the generic one. The concrete construction only has to provide the necessary basic operations.

Thus, we define a power domain construction by axioms concerning the existence of some basic operations. In addition, we specify some axioms that should be satisfied by the basic operations. One might worry about the actual choice of these axioms, but we think that our choice is quite natural. This opinion is strengthened by the fact that our definition leads to a rich theory presented in Part II of the thesis, covers the known power constructions, and allows to characterize them algebraically.

Specification of power constructions

In the sequel, domains are just directed complete partially ordered sets, i.e. we neither require a least element nor algebraicity. As explained in chapter 9, a *power (domain) construction* $\mathcal{P}$ maps *ground domains* $\mathbf{X}$ into *power domains* over $\mathbf{X}$. The power domains have to satisfy the following axioms:

Empty set: There is a distinguished element θ in every power domain $\mathcal{P}\mathbf{X}$.

Binary union: There is a continuous operation $\uplus : [\mathcal{P}\mathbf{X} \times \mathcal{P}\mathbf{X} \to \mathcal{P}\mathbf{X}]$ in every power domain. '$\uplus$' is commutative and associative, and θ is its neutral element.

Singleton sets: There is a continuous mapping $\iota : [\mathbf{X} \to \mathcal{P}\mathbf{X}]$, $x \mapsto \{\!|x|\!\}$ for every ground domain $\mathbf{X}$.

Extension of functions: For every two domains $\mathbf{X}$ and $\mathbf{Y}$, there is a higher order function $ext : [[\mathbf{X} \to \mathcal{P}\mathbf{Y}] \to [\mathcal{P}\mathbf{X} \to \mathcal{P}\mathbf{Y}]]$ mapping set-valued functions on ground domain elements into set-valued functions on sets. It has to satisfy the following axioms:

$$\begin{tikzcd} \mathcal{P}\mathbf{X} \arrow[r, "ext\, f"] & \mathcal{P}\mathbf{Y} \\ \mathbf{X} \arrow[u, "\iota"] \arrow[ur, "f"'] & \end{tikzcd}$$

(P1) $ext\, f\, \theta = \theta$
(P2) $ext\, f\, (A \uplus B) = ext\, f\, A \uplus ext\, f\, B$
(P3) $ext\, f\, \{\!|x|\!\} = f x$ or: $ext\, f \circ \iota = f$
(see the figure to the right)
(S1) $ext\, (\lambda x.\, \theta)\, A = \theta$

(S2) $ext\, (\lambda x.\, f x \uplus g x)\, A = ext\, f\, A \uplus ext\, g\, A$.
Raising '$\uplus$' to functions, one may shortly write $ext\, (f \uplus g) = ext\, f \uplus ext\, g$.

(S3) $ext\, (\lambda x.\, \{\!|x|\!\})\, A = A$ or: $ext\, \iota = id$.

(S4) For every two morphisms $f : [\mathbf{X} \to \mathcal{P}\mathbf{Y}]$ and $g : [\mathbf{Y} \to \mathcal{P}\mathbf{Z}]$,

$$ext\, g\ (ext\, f\, A) = ext\ (\lambda a.\, ext\, g\, (f a))\ A$$

holds for all A in $\mathcal{P}\mathbf{X}$, or: $ext\, g \circ ext\, f = ext\, (ext\, g \circ f)$
(see the figure below)

$$\begin{tikzcd} \mathcal{P}\mathbf{X} \arrow[r, "ext\, f"] & \mathcal{P}\mathbf{Y} \arrow[r, "ext\, g"] & \mathcal{P}\mathbf{Z} \\ \mathbf{X} \arrow[u, "\iota"] \arrow[ur, "f"'] & \mathbf{Y} \arrow[u, "\iota"] \arrow[ur, "g"'] & \end{tikzcd}$$

Note that we do not require $ext\, f$ to be the only morphism satisfying (P1) through (P3) for given f. If we had assumed this uniqueness, (S1) through (S4) would have been provable. That is why we call them secondary axioms in contrast to the primary axioms (Pi).

Semirings and modules

The operations as specified above allow to derive many other operations with useful algebraic properties (see chapter 10). We here include the most important ones only.

Extension depends on two ground domains, $\mathbf{X}$ and $\mathbf{Y}$. Particularly interesting instances of extension are obtained if one of $\mathbf{X}$ and $\mathbf{Y}$ is the one-point domain $\mathbf{1} = \{\diamond\}$. In case $\mathbf{X} = \mathbf{1}$, extension has functionality $ext : [[\mathbf{1} \to \mathcal{P}\mathbf{Y}] \to [\mathcal{P}\mathbf{1} \to \mathcal{P}\mathbf{Y}]]$. Dropping the obsolete argument in $\mathbf{1}$, uncurrying and twisting arguments leads to the 'external product' $\cdot : [\mathcal{P}\mathbf{1} \times \mathcal{P}\mathbf{Y} \to \mathcal{P}\mathbf{Y}]$. The definition is $b \cdot S = ext\, (\lambda\diamond.\, S)\, b$. If we additionally choose $\mathbf{Y} = \mathbf{1}$, then multiplication becomes an inner operation of $\mathcal{P}\mathbf{1}$.

If we rename the empty set θ of $\mathcal{P}\mathbf{1}$ into 0, the union '$\uplus$' into '+', and the singleton $\{\!|\diamond|\!\}$ into 1, then the power axioms imply the following algebraic laws for $\mathcal{P}\mathbf{1}$:

$$\begin{array}{lll} a+(b+c)=(a+b)+c & a+b=b+a & a+0=0+a=a \\ a\cdot(b\cdot c)=(a\cdot b)\cdot c & & a\cdot 1=1\cdot a=a \\ a\cdot(b_1+b_2)=(a\cdot b_1)+(a\cdot b_2) & (a_1+a_2)\cdot b=(a_1\cdot b)+(a_2\cdot b) & a\cdot 0=0\cdot a=0 \end{array}$$

Such algebraic structures are called *semirings*. Note that all algebras occurring in this thesis have domains as carriers and continuous operations.

Semirings are common generalizations of rings as well as of distributive lattices. Thus, they may also be interpreted logically: 0 as 'false', 1 as 'true', addition as disjunction, and multiplication as conjunction. The semiring $\mathcal{P}\mathbf{1}$ then reveals the inherent logic of the power domain construction $\mathcal{P}$. This view makes most sense if the semiring addition is idempotent — a fact that holds for the known power constructions.

If we also rename the empty set θ of $\mathcal{P}\mathbf{X}$ into 0 and its union '$\uplus$' into '+', then we obtain the following algebraic laws where members of $\mathcal{P}\mathbf{X}$ are capitalized in contrast to members of $\mathcal{P}\mathbf{1}$:

$$\begin{array}{lll} A+(B+C)=(A+B)+C & A+B=B+A & A+0=0+A=A \\ r\cdot(s\cdot A)=(r\cdot s)\cdot A & & 1\cdot A=A \\ r\cdot(A_1+A_2)=(r\cdot A_1)+(r\cdot A_2) & (r_1+r_2)\cdot A=(r_1\cdot A)+(r_2\cdot A) & r\cdot 0=0\cdot A=0 \end{array}$$

Such algebraic structures are called *$\mathcal{P}\mathbf{1}$-modules*. A function $f : [M \to M']$ between two $\mathcal{P}\mathbf{1}$-modules is called *linear* iff $f\,(A+B) = f A + f B$ and $f\,(r\cdot A) = r\cdot f A$ hold.

The results of chapter 10 then allow to state:

Theorem 11.1.3: Let $\mathcal{P}$ be a power construction and let

$$\begin{array}{ll} + \;=\; \uplus : & [\mathcal{P}\mathbf{X}\times\mathcal{P}\mathbf{X}\to\mathcal{P}\mathbf{X}] \\ 0 \;=\; \theta : & \mathcal{P}\mathbf{X} \\ \cdot \;=\; \lambda(a,S).\, \mathit{ext}\,(\lambda\diamond.\,S)\,a : & [\mathcal{P}\mathbf{1}\times\mathcal{P}\mathbf{X}\to\mathcal{P}\mathbf{X}] \\ 1 \;=\; \{\!|\diamond|\!\} : & \mathcal{P}\mathbf{1} \end{array}$$

Then $\mathcal{P}\mathbf{1}$ with these operations is a semiring, and $\mathcal{P}\mathbf{X}$ is a $\mathcal{P}\mathbf{1}$-module for all domains $\mathbf{X}$. For $f : [\mathbf{X} \to \mathcal{P}\mathbf{Y}]$, the extension $\mathit{ext}\, f : [\mathcal{P}\mathbf{X} \to \mathcal{P}\mathbf{Y}]$ is linear, and $\mathit{ext}\, f \circ \iota = f$ holds.

This result connects our work with that of Main [Mai85] where power domains are introduced as free semiring modules. There are however some differences: our constructions may create non-free modules, and our singleton function ι need not be strict.

The semiring $\mathcal{P}\mathbf{1}$ is called the *characteristic semiring* of the power construction $\mathcal{P}$. Generalizing a bit, a power construction $\mathcal{P}$ is said to have characteristic semiring R iff $\mathcal{P}\mathbf{1}$ and R are isomorphic semirings where the isomorphism is assumed to be fixed.

Different power constructions may have the same characteristic semiring. Conversely, Th. 14.5.1 and Th. 15.1.1 provide two distinguished power constructions for any given semiring.

Power homomorphisms

Homomorphisms between algebraic structures are mappings preserving all operations of these structures. Power constructions may be considered algebraic structures on a higher level. Thus, it is also possible and useful to define corresponding homomorphisms (see chapter 12).

A power homomorphism $H : \mathcal{P} \dot{\to} \mathcal{Q}$ between two power constructions $\mathcal{P}$ and $\mathcal{Q}$ is a 'family' of morphisms $H = (H_{\mathbf{X}})_{\mathbf{X}} : [\mathcal{P}\mathbf{X} \to \mathcal{Q}\mathbf{X}]$ commuting over all power operations, i.e.

- The empty set in $\mathcal{P}\mathbf{X}$ is mapped to the empty set in $\mathcal{Q}\mathbf{X}$: $H\theta = \theta$.
- The image of a union is the union of the images: $H(A \uplus B) = (HA) \uplus (HB)$.
- Singletons in $\mathcal{P}\mathbf{X}$ are mapped to singletons in $\mathcal{Q}\mathbf{X}$: $H\{\!|x|\!\}_{\mathcal{P}} = \{\!|x|\!\}_{\mathcal{Q}}$.
- Let $f : [\mathbf{X} \to \mathcal{P}\mathbf{Y}]$. Then $H \circ f : [\mathbf{X} \to \mathcal{Q}\mathbf{Y}]$, and $H(ext_{\mathcal{P}}\, f\, A) = ext_{\mathcal{Q}}\,(H \circ f)(HA)$ has to hold for all A in $\mathcal{P}\mathbf{X}$.

If the two constructions $\mathcal{P}$ and $\mathcal{Q}$ share the same characteristic semiring, then one can define: A power homomorphism is linear iff all the functions $H_{\mathbf{X}}$ are linear.

A power isomorphism between two constructions $\mathcal{P}$ and $\mathcal{Q}$ is a family of isomorphisms $H = H_{\mathbf{X}} : [\mathcal{P}\mathbf{X} \to \mathcal{Q}\mathbf{X}]$ such that both $(H_{\mathbf{X}})_{\mathbf{X}}$ and $(H_{\mathbf{X}}^{-1})_{\mathbf{X}}$ are power homomorphisms.

Initial power constructions

A power construction $\mathcal{P}$ is *initial* for semiring R iff for all power constructions $\mathcal{Q}$ with the same characteristic semiring R there is exactly one linear power homomorphism $\mathcal{P} \dot{\to} \mathcal{Q}$. A main result of the theory of power constructions is the existence of such an initial construction for every semiring R.

Initial power domains over $\mathbf{X}$ are obtained as free R-$\mathbf{X}$-modules. $(\mathbf{F}, \eta)$ is a free R-$\mathbf{X}$-module iff $\mathbf{F}$ is an R-module and η is a continuous mapping from $\mathbf{X}$ to $\mathbf{F}$ such that for every R-module M and every continuous $f : \mathbf{X} \to M$, there is exactly one continuous linear $\overline{f} : \mathbf{F} \to M$ with $\overline{f} \circ \eta = f$.

The idea to consider free modules dates back to [HP79]. Hoofman [Hoo87] showed the existence of the free module for semiring $\{0, 1\}$. Main [Mai85] then proposed power constructions defined as free modules for some fancy semirings. In contrast to our work, he requires the singleton mapping to be strict without telling exactly why. The singleton maps of mixed and sandwich power domain are not strict, whence the singleton maps in our general theory of power constructions cannot be strict likewise.

For all semirings R and domains $\mathbf{X}$, free R-$\mathbf{X}$-modules exist and are unique up to isomorphism. Whereas the proof of uniqueness is quite simple, the proof of existence is difficult and usually done by categorical means. In this thesis, we adapted and simplified the proof of [Hoo87] such that categorical theorems are no longer used (section 13.5). Unfortunately, the proof does not provide an explicit representation of the free module.

The main theorem about initial power constructions is formulated in section 14.5.

Theorem 14.5.1:

Let R be a given semiring. Let $\mathcal{P}$ be the construction that maps every ground domain $\mathbf{X}$

into the free R-$\mathbf{X}$-module $(\mathbf{F}, \eta)$. Then $\mathcal{P}$ is the initial power construction for semiring R.

The power operations of union and empty set are given by addition and its neutral element in the free module $\mathbf{F}$, and singleton is η. For functions $f : [\mathbf{X} \to \mathcal{P}\mathbf{Y}]$, $ext\, f$ is given by the unique extension of f. The a priori given external product of the modules $\mathcal{P}\mathbf{X}$ coincides with the external product derived from the power operations.

For any other power construction $\mathcal{Q}$ with semiring R, the unique linear power homomorphism $H : \mathcal{P} \dot{\to} \mathcal{Q}$ is obtained by $H_{\mathbf{X}} = \overline{\lambda x.\, \{|x|\}_{\mathcal{Q}}}$.

Final power constructions

A power construction $\mathcal{P}$ is *final* for semiring R iff for every power construction $\mathcal{Q}$ with the same semiring R there is exactly one linear power homomorphism $\mathcal{Q} \dot{\to} \mathcal{P}$. Final constructions are dealt with in chapter 15.

Final power constructions were never proposed in the literature, probably because the notion of a power homomorphism was missing. In contrast to initial constructions, the final power constructions may be explicitly presented in terms of second order predicates. Thus, it turns out that the upper power construction of [Smy83] is final, whereas the original proposal in [Smy78] is initial.

As mentioned above, extension yields particularly interesting operations if one of the two domains it depends on is the one-point domain $\mathbf{1} = \{\diamond\}$. In case $\mathbf{Y} = \mathbf{1}$, one obtains $ext : [[\mathbf{X} \to \mathcal{P}\mathbf{1}] \to [\mathcal{P}\mathbf{X} \to \mathcal{P}\mathbf{1}]]$ which can be re-arranged by uncurrying, twisting, and then currying again. The outcome is a morphism $\mathcal{E} : [\mathcal{P}\mathbf{X} \to [[\mathbf{X} \to \mathcal{P}\mathbf{1}] \to \mathcal{P}\mathbf{1}]]$ mapping formal sets into second order functions. In view of the logical interpretation of $\mathcal{P}\mathbf{1}$, these functions may be coined as second order predicates. The intuitive meaning of $\mathcal{E}$ is then existential quantification. Given a formal set A in $\mathcal{P}\mathbf{X}$ and a predicate $p : [\mathbf{X} \to \mathcal{P}\mathbf{1}]$, the value of $\mathcal{E}A\,p$ intuitively tells whether some member of A satisfies p.

Some of the axioms of extension easily translate into the following properties:

(P1) $\mathcal{E}\ \theta = \lambda p.\, \theta$

(P2) $\mathcal{E}\ (A \uplus B) = \lambda p.\, (\mathcal{E}\ A\ p) \uplus (\mathcal{E}\ B\ p)$

(P3) $\mathcal{E}\ \{|x|\} = \lambda p.\, p\ x$

(S4) $\mathcal{E}\ (ext\, f\, A) = \lambda p.\, \mathcal{E} A\, (\lambda a.\, \mathcal{E}\, (f\,a)\, p)$

These results suggest to define a power construction for given semiring $\mathcal{P}\mathbf{1}$ by $\mathcal{P}\mathbf{X} = [[\mathbf{X} \to \mathcal{P}\mathbf{1}] \to \mathcal{P}\mathbf{1}]$. Indeed, a slight variant of this method to obtain power constructions really works out.

Let R be a given semiring. Then we may define a product $\cdot : [\mathbf{X} \to R] \times R \to [\mathbf{X} \to R]$ by $f \cdot r = \lambda x. f x \cdot r$. Similarly, an addition may be defined as an inner operation on $[\mathbf{X} \to R]$. A function $F : [[\mathbf{X} \to R] \to R]$ is *right linear* iff $F\,(f_1 + f_2) = F f_1 + F f_2$ and $F\,(f \cdot r) = F f \cdot r$ hold. The domain of all right linear function from $[\mathbf{X} \to R]$ to R is denoted by $[[\mathbf{X} \to R] \overset{rlin}{\to} R]$.

Theorem 15.1.1: Let R be a given semiring. The final power construction for R is explicitly given by mapping the ground domain $\mathbf{X}$ to the space of right linear second

order predicates over $\mathbf{X}$:
$\mathcal{P}\mathbf{X} = [[\mathbf{X} \to R] \overset{rlin}{\to} R]$. Its operations are defined by

- $\theta = \lambda g.\, 0$
- $A \uplus B = \lambda g.\, Ag + Bg$
- $\{\!|x|\!\} = \lambda g.\, gx$ for $x \in \mathbf{X}$.
- $ext\, f\, A = \lambda g.\, A\,(\lambda a.\, fag)$ for $f : [\mathbf{X} \to \mathcal{P}\mathbf{Y}]$ and $A \in \mathcal{P}\mathbf{X}$.

Let $\mathcal{Q}$ be another power construction with semiring R. The unique linear power homomorphism from $\mathcal{Q}$ to $\mathcal{P}$ is given by existential quantification $\mathcal{E}A = \lambda g.\, ext_{\mathcal{Q}}\, g\, A$ for A in $\mathcal{Q}\mathbf{X}$.

The definition of *ext* as well as those of the remaining operations are motivated by the properties of $\mathcal{E}$ listed above. Note that $\mathcal{P}\mathbf{1}$ is $[[\mathbf{1} \to R] \overset{rlin}{\to} R]$ which does not equal R but is isomorphic to R.

Sub-constructions

Let $\mathcal{P}$ be a given power construction. $\mathcal{Q}$ is called a *sub-construction* of $\mathcal{P}$ iff $\mathcal{Q}$ maps ground domains $\mathbf{X}$ into subsets of $\mathcal{P}\mathbf{X}$ such that

- $\mathcal{Q}\mathbf{X}$ is closed w.r.t. lubs of directed sets,
- $\theta \in \mathcal{Q}\mathbf{X}$,
- If A and B are in $\mathcal{Q}\mathbf{X}$, then $A \uplus B$ is in $\mathcal{Q}\mathbf{X}$,
- $\{\!|x|\!\}$ is in $\mathcal{Q}\mathbf{X}$ for all x in $\mathbf{X}$,
- If $f : [\mathbf{X} \to \mathcal{Q}\mathbf{Y}]$ and A in $\mathcal{Q}\mathbf{X}$, then $ext\, f\, A$ is in $\mathcal{Q}\mathbf{Y}$.

In shorter terms, $\mathcal{Q}\mathbf{X}$ is closed w.r.t. all power operations of $\mathcal{P}$. $\mathcal{Q}$ is obviously a power construction since the validity of the power axioms for $\mathcal{Q}$ is inherited from $\mathcal{P}$. Sub-constructions are handled in chapter 14.

One easily verifies that the intersection of a family of sub-constructions of a power construction $\mathcal{P}$ is again a sub-construction of $\mathcal{P}$, if we define $(\bigcap_{i\in I} \mathcal{Q}_i)\mathbf{X} = \bigcap_{i\in I}(\mathcal{Q}_i\mathbf{X})$. Hence, the sub-constructions of $\mathcal{P}$ form a complete lattice.

Let R be a semiring domain. R' is a *sub-semiring* of R iff R' is a subset of R containing 0 and 1, and being closed w.r.t. lubs of directed sets, addition, and multiplication. Because the operations in the characteristic semiring are derived from the power operations, the semiring of a sub-construction $\mathcal{Q}$ of $\mathcal{P}$ is a sub-semiring of the semiring of $\mathcal{P}$. Similarly, the semiring of an intersection of sub-constructions is the intersection of the semirings of the sub-constructions.

We found two general methods to derive sub-constructions from a given construction $\mathcal{P}$ with semiring R. The method of core formation yields the smallest sub-construction of $\mathcal{P}$ that still has semiring R, whereas the method of existential restriction creates the greatest sub-construction of $\mathcal{P}$ with a given sub-semiring R' of R. The core of the existential restriction is then the least sub-construction with the given semiring R'.

The *core* $\mathcal{P}^c\mathbf{X}$ of a power domain $\mathcal{P}\mathbf{X}$ is the least subset of $\mathcal{P}\mathbf{X}$ that contains the empty set θ and all singletons $\{\!|x|\!\}$, and is closed w.r.t. union '$\uplus$', product by factors in $\mathcal{P}\mathbf{1}$, and directed suprema. One can show that the modules $\mathcal{P}^c\mathbf{X}$ are also closed w.r.t. extension of $\mathcal{P}$ such that $\mathcal{P}^c$ is a sub-construction of $\mathcal{P}$. We call it the *core* of $\mathcal{P}$. It is the least sub-construction with the same semiring as $\mathcal{P}$.

A power construction that coincides with its core is called *reduced*. Reduced constructions enjoy particularly nice properties. All initial constructions are reduced.

If $\mathcal{P}$ is a power construction with semiring R, and R' is a sub-semiring of R, then the *existential restriction* of $\mathcal{P}$ to R' defined by

$$\mathcal{Q}\mathbf{X} = \mathcal{P}|_{R'}\mathbf{X} = \{A \in \mathcal{P}\mathbf{X} \mid \forall p : [\mathbf{X} \to R'] : \mathcal{E} A\, p \in R'\}$$

is a sub-construction of $\mathcal{P}$ with semiring R'. It is the greatest such sub-construction.

Because of its definition in terms of existential quantification, one might believe that the existential restriction of a final construction for R is a final construction for R'. However this is not true as we shall see in section 23.4. There are two reasons for this. First, two distinct second order predicates in $[[\mathbf{X} \to R] \stackrel{rlin}{\to} R]$ may produce equal results for predicates in $[\mathbf{X} \to R']$. They are then still different in the restriction of the final construction for R, but equal in $[[\mathbf{X} \to R'] \stackrel{rlin}{\to} R']$. Second, there may be additional members in $[[\mathbf{X} \to R'] \stackrel{rlin}{\to} R']$ that cannot be obtained by restricting predicates in $[[\mathbf{X} \to R] \stackrel{rlin}{\to} R]$.

Despite of this general result, we also meet examples for semirings R and R' where the existential restriction of the final construction for R is final for R' — see Th. 23.3.1.

Products of power constructions

In chapter 16, another method to build new constructions from existing ones is introduced. Given a family $(\mathcal{P}_i)_{i\in I}$ of power constructions, we may build a product construction $\mathcal{P} = \prod_{i\in I}\mathcal{P}_i$:

- $\mathcal{P}\mathbf{X} = \prod_{i\in I}\mathcal{P}_i\mathbf{X}$ for all ground domains $\mathbf{X}$
- $\theta = (\theta_i)_{i\in I}$
- $(A_i)_{i\in I} \uplus (B_i)_{i\in I} = (A_i \uplus B_i)_{i\in I}$
- $\{\!|x|\!\} = (\{\!|x|\!\}_i)_{i\in I}$ for all x in $\mathbf{X}$
- For $f : [\mathbf{X} \to \mathcal{P}\mathbf{Y}]$ let $f_i = \pi_i \circ f$. Then $ext\, f\,(A_i)_{i\in I} = (ext_i\, f_i\, A_i)_{i\in I}$ where ext_i denotes the extension functional of $\mathcal{P}_i$. Here, π_i denotes projection to component i.

The verification of the power axioms for $\mathcal{P}$ is straightforward since the power operations work independently in all dimensions. The characteristic semiring of $\mathcal{P}$ is the product of the characteristic semirings of the $\mathcal{P}_i$.

We derived the following properties of the product:

Finality theorem 16.3.1: If $\mathcal{P}_i$ are final power constructions for the semirings R_i for all $i \in I$, then the product $\prod_{i\in I}\mathcal{P}_i$ is a final construction for the product $\prod_{i\in I} R_i$.

$$\prod_{i\in I}[[\mathbf{X} \to R_i] \stackrel{rlin}{\to} R_i] \;\cong\; [[\mathbf{X} \to \prod_{i\in I} R_i] \stackrel{rlin}{\to} \prod_{i\in I} R_i]$$

Core theorem 16.5.1: The core of the product of two power constructions equals the product of their cores: $(\mathcal{P}_1 \times \mathcal{P}_2)^c = \mathcal{P}_1^c \times \mathcal{P}_2^c$

Initiality theorem 16.5.3:

If $\mathcal{P}_1$ is initial for R_1 and $\mathcal{P}_2$ initial for R_2, then $\mathcal{P}_1 \times \mathcal{P}_2$ is initial for $R_1 \times R_2$.

Factorization theorem 16.6.1:

Let $\mathcal{P}$ be a power domain construction for semiring $R_1 \times R_2$. $\mathcal{P}$ may be factorized, i.e. there are power constructions $\mathcal{P}_1$ for R_1 and $\mathcal{P}_2$ for R_2 such that $\mathcal{P}$ is isomorphic to $\mathcal{P}_1 \times \mathcal{P}_2$ by a linear power isomorphism, if and only if $ext\,(\lambda x.\,(1,\,0)\cdot\{\!|x|\!\})\,B = (1,\,0)\cdot B$ holds for all B in all power domains $\mathcal{P}\mathbf{X}$.

The condition of the factorization theorem holds for many classes of power constructions. For instance, every reduced power construction for $R_1 \times R_2$ can be factorized. The resulting factors are reduced again (cf. Cor. 16.6.4).

Special power constructions

The algebraic theory of power domain constructions may be applied to the known 5 power constructions mentioned in the beginning. In the following summary of our results, we restrict ourselves to algebraic ground domains.

The five known power constructions may be characterized as follows:

- The lower power construction $\mathcal{L}$ is both initial and final for the semiring $\mathbf{L} = \{0,\,1\}$ with $0 < 1$ and $1 + 1 = 1$.
- The upper or Smyth power construction $\mathcal{U}$ is both initial and final for the semiring $\mathbf{U} = \{0,\,1\}$ with $1 < 0$ and $1 + 1 = 1$.
- The convex or Plotkin power construction $\mathcal{C}$ is initial for the semiring $\mathbf{C} = \{0,\,1\}$ with $1 + 1 = 1$ and uncomparable values 0 and 1.
- The mixed power construction $\mathcal{M}$ of Gunter is initial for the semiring $\mathbf{B} = \{\bot,\,0,\,1\}$ with $\bot$ being below the uncomparable values 0 and 1. In $\mathbf{B}$, addition is given by parallel disjunction and multiplication by parallel conjunction.
- The sandwich power construction $\mathcal{S}$ of Buneman et al. is final for the same semiring $\mathbf{B}$.
- The final power construction for the semiring $\mathbf{C}$ is missing in this list. The reason is that it is degenerated and has quite awkward properties.

In chapter 17, we study these semirings and modules. The algebraic structures mentioned in [HP79] to characterize $\mathcal{C}$, $\mathcal{L}$, and $\mathcal{U}$ are nothing else but $\mathbf{C}$-, $\mathbf{L}$-, and $\mathbf{U}$-modules respectively without 0. Similarly, Gunter's mix algebras of [Gun89b, Gun90] are nothing else but $\mathbf{B}$-modules. Gunter's additional unary operation is just multiplication by $\bot$.

The theory of these special power constructions can be made more symmetric by taking the additional semirings $\mathbf{D}$ and $\overline{\mathbf{B}}$ into consideration. The 'double' semiring $\mathbf{D}$ is defined as $\mathbf{L} \times \mathbf{U}$. $\mathbf{D}$ has 4 elements: a least element $\bot = (0,\,1)$, a greatest element $\top = (1,\,0)$, and the two incomparable values $0 = (0,\,0)$ and $1 = (1,\,1)$ in between. $\overline{\mathbf{B}} = \{0,\,1,\,\top\}$ is the order-dual of $\mathbf{B}$. Both $\mathbf{B}$ and $\overline{\mathbf{B}}$ are sub-semirings of $\mathbf{D}$. Their intersection is $\mathbf{C}$.

According to the theory of products of power constructions, the product $\mathcal{D}$ of $\mathcal{L}$ and $\mathcal{U}$ is both initial and final for semiring $\mathbf{D}$. Existential restriction to $\mathbf{B}$ leads to the sandwich construction $\mathcal{S}$ whose core in turn is $\mathcal{M}$. Thus, $\mathcal{S}$ is the greatest sub-construction of $\mathcal{D}$ with semiring $\mathbf{B}$, and $\mathcal{M}$ is the least one. By employing the dual sub-semiring $\overline{\mathbf{B}}$, we obtain the dual sandwich construction $\overline{\mathcal{S}}$ and the dual mix construction $\overline{\mathcal{M}}$. The intersection $\mathcal{S}\overline{\mathcal{S}}$ of $\mathcal{S}$ and $\overline{\mathcal{S}}$ pays some interest since it preserves bounded completeness and has the same semiring $\mathbf{C}$ as Plotkin's construction $\mathcal{C}$. The intersection of $\mathcal{M}$ and $\overline{\mathcal{M}}$ is $\mathcal{C}$ itself.

The sandwich condition that characterizes $\mathcal{S}$ as sub-construction of $\mathcal{D} = \mathcal{L} \times \mathcal{U}$ is given by existential quantification. In chapter 21, we translate this condition into various equivalent forms. We also found an explicit characterization of $\mathcal{M}$ as a sub-construction of $\mathcal{D}$ or $\mathcal{S}$. This mix condition is discussed in chapter 22. In these chapters, the dual conditions for $\overline{\mathcal{S}}$ and $\overline{\mathcal{M}}$ are also investigated.

Part III also contains some more results that are not directly connected to the general algebraic theory of power constructions. The lower and upper power construction are shown to commute generalizing the results of [FM90], i.e. $\mathcal{L}\mathcal{U}\mathbf{X}$ and $\mathcal{U}\mathcal{L}\mathbf{X}$ are isomorphic (sections 20.5 and 20.6). The sandwich power domains $\mathcal{S}\mathbf{X}$ may be characterized as free sandwich algebras over $\mathbf{X}$ (chapter 24).

Domain classes

The results of Part III are proved for as large classes of domains as possible. The coincidence among the initial and the final lower power construction and the one explicitly given in terms of Scott closed sets could be shown for all ground domains. The situation is however much more complex if upper power constructions are considered. The question of coincidence among the initial upper construction $\mathcal{U}_i$, the final one $\mathcal{U}_f$, its core $\mathcal{U}_c$, and the explicitly given construction in terms of compact upper sets $\mathcal{U}_K$ could be answered for certain (though quite large) classes of domains only.

To analyze the situation precisely, we use topological methods that are introduced in Part I of the thesis. With these methods, various classes of domains can be defined and investigated. Some of these classes are novel. For the upper power constructions, $\mathcal{U}_i\mathbf{X} \cong \mathcal{U}_f\mathbf{X}$ could be shown for continuous ground domains (chapter 19). $\mathcal{U}_f\mathbf{X}$ and $\mathcal{U}_c\mathbf{X}$ however definitively coincide for the larger class of multi-continuous domains. The maximal class of coincidence is unknown in both cases. $\mathcal{U}_f\mathbf{X}$ and $\mathcal{U}_K\mathbf{X}$ are isomorphic iff the ground domain $\mathbf{X}$ is sober. $\mathcal{U}_K\mathbf{X}$ does not make sense for all ground domains. It is defined for the superclass K-RD of the class of sober domains.

Novel topological notions such as strong compactness (section 4.7) allow to define even more upper power constructions (chapter 20). The resulting constructions are sub-constructions of $\mathcal{U}_f$ and $\mathcal{U}_K$ respectively. They coincide with these super-constructions for multi-continuous ground domains. The question of full coincidence remains open.

Final remarks

History and publications

The algebraic theory of power constructions evolved from an attempt to specify the semantic properties of a set data type in a functional language. The resulting algebraic specification of 'set domain constructions' was published in [Hec90c]. Two specific set domain constructions that satisfy the specification were found. The big set domains later turned out to be isomorphic to the sandwich power domains, and the small set domains turned out to be isomorphic to the mixed power domains.

The specification of set domain constructions involves an a priori given domain of logical values, namely $\mathbf{B} = \{\bot, 0, 1\}$. In investigating various derived power operations, we found — also inspired by [Mai85] — that a domain of logical values may be derived a posteriori from the set-theoretic power operations empty set, singleton, union, and extension. The set construction specification then turned out to fall into two parts: a specification of more general entities, namely power domain constructions, and the requirement that $\mathcal{P}\mathbf{1}$ be isomorphic to $\mathbf{B}$. Dropping the now artificial restriction $\mathcal{P}\mathbf{1} \cong \mathbf{B}$ led to a first version of the specification of power constructions.

A second starting point was the observation that sandwich power domains $\mathcal{S}\mathbf{X}$ are isomorphic to function spaces $[[\mathbf{X} \to \mathbf{B}] \stackrel{add}{\to} \mathbf{B}]$. A first order-theoretic proof of this isomorphism is contained in the technical report [Hec90a]. In the sequel, we found that much more general function spaces $[[\mathbf{X} \to R] \stackrel{rlin}{\to} R]$ always define power constructions as defined by the algebraic specification. These results together with the characterization of mixed and Plotkin power domains in terms of second order predicates were presented at the MFPS'90 workshop in Kingston, Ontario. The finality of the construction in terms of second order predicates was found in summer 1990. Core, existential restriction, and the product theory are even more recent results.

The fully evolved algebraic theory is contained in the technical report [Hec90b] and accepted for publication as [Hec91]. A paper with emphasis on second order predicates and the sub-constructions of $\mathcal{D}$ is submitted to the TCS journal, and a third paper about the isomorphism between $\mathcal{LU}\mathbf{X}$ and $\mathcal{UL}\mathbf{X}$ is submitted to the MFPS'91 conference.

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Part I

Foundations

This first part of the thesis contains a collection of definitions and facts from domain theory and topology. Power domain constructions do not occur in this part although its size is more than one third of the whole thesis. The purpose of this foundations part is to serve as a data base for all fundamental facts and technical proofs that are used in the two remaining parts of the thesis when discussing power domain constructions. Experienced readers should skip this part, immediately start reading Part II, and only occasionally look up concepts, definitions, and results from the foundations part.

The material of the first four chapters is standard except the notion of strong compactness introduced in section 4.7. Many of the concepts and ideas are mentioned in [Law88] together with sketches of their proofs. Fully experienced readers will thus not find much interesting in here. Many computer scientists however do not know much of topology, and conversely, many topologists do not know domain theory. Thus, these four chapters serve as a consistent attempt for an introduction of the mathematical aspects of both theories.

The first four chapters handle the more basic aspects of the theory. After introducing some notations in chapter 1, we present the theory of partially ordered sets and monotonic functions in chapter 2. In chapter 3, the theory of directed complete partial orders and continuous functions follows. To be concise, we call the directed complete partial orders *domains* in this thesis. Thus, our domains neither have to possess a least element nor must they be algebraic. In chapter 4, we then introduce the theory of topological spaces and (topologically) continuous functions. In contrast to classical introductions to topology, we neglect the well separated Hausdorff spaces. Instead, we concentrate on the relations between order theory and topology. Several of the lemmas in chapters 3 and 4 will be heavily used in order-theoretic or topological proofs in the remaining two parts of the thesis.

Chapters 5 through 8 introduce classifications of domains. Among these, there are trivial ones such as finiteness, well known ones such as bounded completeness and algebraicity, almost unknown ones such as sobriety and multi-continuity, and novel classes such as SC and S-RD. Chapter 5 is devoted to completeness properties from bounded completeness in section 5.2 to SC in section 5.5. Chapter 6 deals with algebraicity and continuity. Besides all the standard theory, it contains purely topological characterizations of algebraicity (Th. 6.2.5) and continuity (Th. 6.7.9). These characterizations will be used in Part III of the thesis. Chapter 7 treats finitely algebraic (bifinite) and finitely continuous domains, retracts, and the preservation of domain classes by functors. These results directly apply to power domain constructions because they are special functors as shown in chapter 10. In chapter 8, we study multi-algebraicity and multi-continuity (8.1 – 8.4), sobriety (8.5 – 8.8), and the 'Rudin classes' K-RD and S-RD (8.9 – 8.11). The last section 8.12 of Part I contains tables and diagrams summarizing the definitions and relative inclusions of the introduced domain classes.

As the end of this introduction, we include a table of all the sections of particular interest in the foundations part.

• Rudin's Lemma	Section 3.9
• Classes SC and KC	Section 5.5
• Strong compactness	Section 4.7
• Topological criteria for ALG and CONT	Sections 6.2 and 6.7
• Multi-continuity	Sections 8.1 – 8.4
• Sobriety	Sections 8.5 – 8.8
• Rudin classes S-RD and K-RD	Sections 8.9 and 8.10

Chapter 1

Basic notations

This chapter introduces some basic mathematical notations that we shall use throughout the thesis. In the first section, we present our notations for finite subsets, images, inverse images, etc. In the second section, closure formation is considered in more detail.

1.1 Some notations from set theory

Here, we present only those notations that are not standard. For two sets A and B, the notation $A \subseteq_f B$ means A is a finite subset of B. If A is a subset of some universe X, then $\mathsf{co}\, A$ denotes the *complement* of A in X, i.e. $\mathsf{co}\, A = X \setminus A$. The *power set* of a set X is denoted by 2^X.

Let $f : X \to Y$ be a function. We denote the image of a point x in X under f by $f x$, i.e. we mostly omit parentheses in applications. Thus, the multiplication symbol '$\cdot$' will never be omitted, i.e. products are always written $a \cdot b$. Functions are often denoted by λ-expressions. For instance, the function f defined by $f y = x \cdot y$ where x is a constant, may be concisely referred to by $\lambda y.\, x \cdot y$. We assume the reader to be familiar with the elementary rules of α-, β-, and η-conversion for equational reasoning in the λ-calculus.

Usually, applications will group left, i.e. $G f x$ means $(G f)\, x$. In rare circumstances only where the type of the involved terms obviously excludes the meaning $(g f)\, x$, we abbreviate $g\,(f x)$ by $g f x$. As the function occurs to the left of its argument in applications, composition operates from right to left. The composition of $f : X \to Y$ and $g : Y \to Z$ is hence denoted by $g \circ f : X \to Z$, i.e. $(g \circ f)\, x$ equals $g\,(f x)$.

Let $f : X \to Y$ be again a function. The image of a subset A of X under f is denoted by $f[A] = \{f a \mid a \in A\}$. Square brackets are used here to let the reader more safely distinguish between applications to single points x and pseudo-applications to whole sets. Accordingly, the inverse image of a subset B of Y is denoted by $f^{-1}[B] = \{x \in X \mid f x \in B\}$.

1.2 Closure operations

Closure operations are a special form of set operations. Because of their fundamentality, they occur in various different contexts. In the thesis, we shall introduce several distinct closure operations. In this section, we study the generic properties common to all closure operators.

Definition 1.2.1

Let $\mathbf{X}$ be a given non-empty set, and let $\mathcal{C}$ be a set of subsets of $\mathbf{X}$ with

(C1) $\mathbf{X}$ is in $\mathcal{C}$.

(C2) Arbitrary intersections of members of $\mathcal{C}$ are again in $\mathcal{C}$.

Then, for every $A \subseteq \mathbf{X}$, the set $\mathsf{cl}\, A = \bigcap\{B \in \mathcal{C} \mid B \supseteq A\}$ is the *closure* of A w.r.t. $\mathcal{C}$.

Because of the axioms (C1) and (C2), the intersection leading to $\mathsf{cl}\, A$ is not empty and always produces a member of $\mathcal{C}$.

The closure of A is the smallest set in $\mathcal{C}$ that still contains A as a subset. This is formalized in the first three claims of the proposition below. The remaining claims are simple consequences of the first three.

Proposition 1.2.2

(1) $A \subseteq \mathsf{cl}\, A$

(2) $\mathsf{cl}\, A \in \mathcal{C}$

(3) If $A \subseteq B$ and $B \in \mathcal{C}$, then $\mathsf{cl}\, A \subseteq B$

(4) $A \subseteq \mathsf{cl}\, B$ is equivalent to $\mathsf{cl}\, A \subseteq \mathsf{cl}\, B$

(5) $A \subseteq B$ implies $\mathsf{cl}\, A \subseteq \mathsf{cl}\, B$

(6) $\mathsf{cl}\, \mathbf{X} = \mathbf{X}$

(7) A is in $\mathcal{C}$ iff $\mathsf{cl}\, A = A$

(8) $\mathsf{cl}\,(\mathsf{cl}\, A) = \mathsf{cl}\, A$

Proof:

(1) A is a subset of all those sets whose intersection is $\mathsf{cl}\, A$.

(2) Due to axiom (C2).

(3) If $A \subseteq B$ and $B \in \mathcal{C}$, then B is among those sets whose intersection is $\mathsf{cl}\, A$.

(4) By (2), $\mathsf{cl}\, B$ is in $\mathcal{C}$. Using (3), then $A \subseteq \mathsf{cl}\, B$ implies $\mathsf{cl}\, A \subseteq \mathsf{cl}\, B$. For the opposite direction, (1) is used: $A \subseteq \mathsf{cl}\, A \subseteq \mathsf{cl}\, B$.

(5) By (1), $A \subseteq B \subseteq \mathsf{cl}\, B$ holds. Using (4), $\mathsf{cl}\, A \subseteq \mathsf{cl}\, B$ follows.

(6) By (1), $\mathbf{X} \subseteq \mathsf{cl}\, \mathbf{X}$ holds.

(7) If A is in $\mathcal{C}$, then $A \subseteq A$ implies $\mathsf{cl}\, A \subseteq A$ by (3). The opposite inclusion holds by (1). If $\mathsf{cl}\, A$ equals A, then A is in $\mathcal{C}$ by (2).

(8) Follows from (7) and (2). □

The closure operation is uniquely determined by the properties (1) through (3) given above:

Proposition 1.2.3

If $A \mapsto \mathsf{cl}'\, A$ is a set operation with

(1) $A \subseteq \mathsf{cl}'\, A$

(2) $\mathsf{cl}' A \in \mathcal{C}$

(3) If $A \subseteq B$ and $B \in \mathcal{C}$, then $\mathsf{cl}' A \subseteq B$

for all $A, B \subseteq \mathbf{X}$, then $\mathsf{cl}\, A = \mathsf{cl}' A$ holds for all $A \subseteq \mathbf{X}$.

Proof:
From $A \subseteq \mathsf{cl}' A$ and $\mathsf{cl}' A \in \mathcal{C}$, the inclusion $\mathsf{cl}\, A \subseteq \mathsf{cl}' A$ follows by using Prop. 1.2.2 (3). The inverse inclusion is derived analogously by exchanging the roles of the two operations. $\square$

Chapter 2

Partially ordered sets

Semantic domains are partially ordered sets with additional axioms. Their order is to be understood as given by informational content or computational progress. The relation '$x \leq y$' for two semantic values x and y means that x contains less information than y from a static point of view. Considered dynamically, x might be the outcome of a computation at an early state, and might be refined to y as the computation proceeds.

In this chapter, we consider the elementary properties of partially ordered sets. Notions of convergence and other topological properties will be treated later. After having defined partially ordered sets, we investigate their upper, lower, and convex subsets and related closure operations. Finally, we treat upper bounds, monotonic functions, minimal elements, and finitary sets.

Readers who are experienced in poset theory will not find many new results in this chapter. The only non-standard notion is the notion of *finitary sets* introduced in section 2.7. A set is finitary iff its upper closure may be obtained as the upper closure of a finite set. Equivalently, a set is finitary iff there is a minimal element under each of its elements, and the number of minimal elements is finite.

2.1 Definition and examples

The following definition is standard:

Definition 2.1.1 A binary relation '$\leq$' on a set P that satisfies the axioms

(1)	$x \leq x$ for all $x \in P$	(Reflexivity)
(2)	$x \leq y$ and $y \leq x$ implies $x = y$	(Antisymmetry)
(3)	$x \leq y$ and $y \leq z$ implies $x \leq z$	(Transitivity)

is called a *partial order* or shortly an *order*. If it satisfies axioms (1) and (3) only, it is called a *pre-order*.

A *poset* (partially ordered set) $(P, \leq)$ is a non-empty set P together with a partial order '$\leq$'.

A point x is called *least element* of a poset $(P, \leq)$ iff $x \leq p$ for all p in P, and *greatest element* iff $p \leq x$.

We often identify a poset $\mathbf{P} = (P, \leq)$ with its carrier P, writing $\mathbf{P}$ for the carrier. Sometimes, the order is denoted by $\leq_{\mathbf{P}}$ for better distinction.

Least and greatest element are, if they exist, uniquely determined because of antisymmetry. They are often denoted by $\bot$ and $\top$ respectively.

Every pre-order '$\lesssim$' on P induces an equivalence relation '$\approx$' on P by the definition $x \approx y$ iff $x \lesssim y$ and $y \lesssim x$. Then, the pre-order defines a partial order '$\leq$' on the set of equivalence classes of P modulo '$\approx$' by $[x] \leq [y]$ iff $x \lesssim y$ where $[x]$ denotes the equivalence class of the point x.

For a poset $\mathbf{P} = (P, \leq)$, the *dual poset* $\mathbf{P}^d = (P, \geq)$ has the same carrier set P, but the reverse order $x \geq y$ iff $y \leq x$. Many theorems and propositions about posets have a dual version formulated for the dual poset. This dual version need not be explicitly proved.

Examples for posets:

- The smallest poset has just one point $\diamond$. We denote this poset by $\mathbf{1} = \{\diamond\}$.
- Any set P becomes a poset by defining $x \leq y$ iff $x = y$. This poset $P_{=} = (P, =)$ is called the discrete poset over P.
- For any set S, there is the so-called flat poset $S_\bot$:

  $$S_\bot = (S \cup \{\bot\}, \leq) \quad \text{where} \quad x \leq y \quad \text{iff} \quad x = \bot \quad \text{or} \quad x = y$$

  $\bot$ is the least element of a flat poset. Examples of flat posets include
  - The two-point poset $\mathbf{2} = \{\top\}_\bot$ i.e. $\mathbf{2} = (\{\bot, \top\}, \leq)$ where $\bot \leq \top$. We shortly write in a symbolic notation $\mathbf{2} = \{\bot < \top\}$.
  - The Boolean poset $\mathbf{B} = \{\mathsf{F}, \mathsf{T}\}_\bot$ i.e. $\mathbf{B} = (\{\bot, \mathsf{F}, \mathsf{T}\}, \leq)$ where $\bot \leq \mathsf{F}$ and $\bot \leq \mathsf{T}$. Symbolically, we write $\mathbf{B} = \{\mathsf{F} > \bot < \mathsf{T}\}$ or $\mathbf{B} = \{\bot < [\mathsf{F}, \mathsf{T}]\}$ where the square brackets group a level of uncomparable points.
  - The flat poset of natural numbers $\mathbf{N}_\bot = \{0, 1, 2, \ldots\}_\bot = \{\bot < [0, 1, 2, \ldots]\}$.
- For any set S, the power set 2^S forms a poset with the inclusion order '$\subseteq$'. $\emptyset$ is the least element.
- The set of natural numbers $\mathbf{N}_0 = \{0, 1, 2, \ldots\}$ forms a poset with the usual arithmetic order $0 \leq 1 \leq 2 \leq \cdots$. We symbolically denote it by $\mathbf{N}_0 = \{0 < 1 < 2 < \cdots\}$.
- The poset $\mathbf{N}_0^\infty$ consists of the natural numbers together with a greatest element ∞. The order is $0 \leq 1 \leq 2 \leq \cdots \leq \infty$. We write $\mathbf{N}_0^\infty = \{0 < 1 < 2 < \cdots < \infty\}$.
- Analogously, the poset $\mathbf{N}_0^{\infty+1} = \{0 < 1 < 2 < \cdots < \infty < \infty + 1\}$ consists of the natural numbers together with ∞ and $\infty + 1$.
- Let $\mathbf{N}' = \{0, 1, 2, \ldots, \infty, 0', 1', 2', \ldots\}$ ordered by $0 < 1 < 2 < \cdots < \infty$ and $n < m'$ iff $n \leq m$. Graphically, $\mathbf{N}'$ looks like

  $$\begin{array}{ccccccc} 0' & & 1' & & 2' & & \\ \uparrow & & \uparrow & & \uparrow & & \\ 0 & \rightarrow & 1 & \rightarrow & 2 & \ldots & \infty \end{array}$$

  This is the poset of the so-called 'lazy naturals'. Its points may be understood as constructor terms involving successor, zero, and bottom: the points n correspond to $succ^n \bot$, and the points n' to $succ^n 0$.

Finally, we consider one (of many) methods to construct new posets from given ones.

Definition 2.1.2 Let $(\mathbf{P}_i)_{i\in I}$ be a family of posets where $\mathbf{P}_i = (P_i, \leq_i)$. Then we define their *(Cartesian) product* $\prod_{i\in I} \mathbf{P}_i = (\prod_{i\in I} P_i, \leq)$ where $(x_i)_{i\in I} \leq (y_i)_{i\in I}$ iff $x_i \leq_i y_i$ for all i in I. For k in I, the *projection* to dimension k is the function $\pi_k : \prod_{i\in I} \mathbf{P}_i \to \mathbf{P}_k$ with $\pi_k((x_i)_{i\in I}) = x_k$.

The product $\prod_{i\in I} \mathbf{P}_i$ is again a poset. A special instance is the product of just two posets $\mathbf{P}_1$ and $\mathbf{P}_2$. We write $\mathbf{P}_1 \times \mathbf{P}_2$ instead of $\prod_{i\in\{1,2\}} \mathbf{P}_i$. By definition, $(x_1, x_2) \leq (y_1, y_2)$ holds iff $x_1 \leq_1 y_1$ and $x_2 \leq_2 y_2$.

2.2 Upper and lower sets

In this section, we consider some special classes of subsets of a general poset and their properties. The theory introduced here is not complex but is used a lot in the sequel. Throughout this section, $\mathbf{P}$ always denotes a fixed poset unless explicitly stated.

Definition 2.2.1
A set $A \subseteq \mathbf{P}$ is a *lower set* iff for all $x, y \in P : x \in A$ and $y \leq x$ implies $y \in A$.
Dually, a set $A \subseteq \mathbf{P}$ is an *upper set* iff for all $x, y \in P : x \in A$ and $x \leq y$ implies $y \in A$.

Proposition 2.2.2

(1) $\emptyset$ and the whole poset $\mathbf{P}$ are as well lower as upper sets.

(2) If $(A_i)_{i\in I}$ is a family of lower (upper) sets, then $\bigcup_{i\in I} A_i$ and $\bigcap_{i\in I} A_i$ are lower (upper) sets.

(3) If A_i is a lower (upper) set in $\mathbf{P}_i$ for all i in I, then $\prod_{i\in I} A_i$ is a lower (upper) set in $\prod_{i\in I} \mathbf{P}_i$.

The proof is left to the interested reader.

Lower and upper set are not only dual notions, but also complementary ones:

Proposition 2.2.3 The complement of a lower set is an upper set and vice versa.

Proof:
Let A be a lower set, and let B be its complement. If x is in B and $x \leq y$, then y must be in B since otherwise, it would be in A and so would x. Thus, B is an upper set.
The 'vice versa' follows from duality. □

Definition 2.2.4 For $x \in \mathbf{P}$, let $\downarrow x = \{y \in \mathbf{P} \mid y \leq x\}$ the *lower cone* of x, and $\uparrow x = \{y \in \mathbf{P} \mid y \geq x\}$ the *upper cone*.

Next, we give a list of facts about cones that are easy to verify.

Proposition 2.2.5

(1)	$x \leq y$ iff $x \in \downarrow y$ iff $\downarrow x \subseteq \downarrow y$ iff $\uparrow x \ni y$ iff $\uparrow x \supseteq \uparrow y$	
(2)	$x = y$ iff $\downarrow x = \downarrow y$ iff $\uparrow x = \uparrow y$	
(3)	A is lower iff $\downarrow a \subseteq A$ for all $a \in A$	A is upper iff $\uparrow a \subseteq A$ for all $a \in A$
(4)	$\downarrow x$ is lower	$\uparrow x$ is upper
(5)	$x \in \downarrow x$	$x \in \uparrow x$
(6)	$\downarrow x \cap \uparrow x = \{x\}$	

By Prop. 2.2.2, the set of all lower sets and the set of all upper sets both satisfy the axioms (C1) and (C2) of section 1.2 that are needed to define corresponding closure operations. They are simple enough to be explicitly constructed:

Definition 2.2.6 For $A \subseteq \mathbf{P}$, let $\downarrow A = \{x \in \mathbf{P} \mid \exists a \in A : x \leq a\}$ be the *lower closure* of A, and $\uparrow A = \{x \in \mathbf{P} \mid \exists a \in A : a \leq x\}$ be the *upper closure*.

The operations defined above are really the closure operations for the set of lower sets and the set of upper sets. This may be shown by applying Prop. 1.2.3; the proofs of the three properties required there are straightforward. Hence, all properties of general closure operations as enumerated in Prop. 1.2.2 hold. We repeat them here (in a different order) for later reference.

Proposition 2.2.7

(1)	$\downarrow\emptyset = \emptyset$ and $\downarrow\mathbf{P} = \mathbf{P}$	$\uparrow\emptyset = \emptyset$ and $\uparrow\mathbf{P} = \mathbf{P}$
(2)	$\downarrow A$ is a lower set	$\uparrow A$ is an upper set
(3)	$A \subseteq \downarrow A$	$A \subseteq \uparrow A$
(4)	$A \subseteq B$ and B is lower implies $\downarrow A \subseteq B$	$A \subseteq B$ and B is upper implies $\uparrow A \subseteq B$
(5)	A is lower iff $\downarrow A = A$	A is upper iff $\uparrow A = A$
(6)	$A \subseteq B$ implies $\downarrow A \subseteq \downarrow B$	$A \subseteq B$ implies $\uparrow A \subseteq \uparrow B$
(7)	$A \subseteq \downarrow B$ iff $\downarrow A \subseteq \downarrow B$	$A \subseteq \uparrow B$ iff $\uparrow A \subseteq \uparrow B$
(8)	$\downarrow\downarrow A = \downarrow A$	$\uparrow\uparrow A = \uparrow A$

Next, we consider how the closure operators interfere with set-theoretic operations.

Proposition 2.2.8 Let $(A_i)_{i\in I}$ be a family of subsets of $\mathbf{P}$. Then

$\downarrow(\bigcup_{i\in I} A_i) = \bigcup_{i\in I} \downarrow A_i$ $\qquad$ $\uparrow(\bigcup_{i\in I} A_i) = \bigcup_{i\in I} \uparrow A_i$

$\downarrow(\bigcap_{i\in I} A_i) \subseteq \bigcap_{i\in I} \downarrow A_i$ $\qquad$ $\uparrow(\bigcap_{i\in I} A_i) \subseteq \bigcap_{i\in I} \uparrow A_i$

With intersection, '=' does not generally hold.

Proof:

Union, '$\supseteq$': By monotonicity of '$\downarrow$' (Prop. 2.2.7 (6)), $\downarrow(\bigcup_{i\in I} A_i) \supseteq \downarrow A_i$ holds for each i.

Union, '$\subseteq$': $\bigcup_{i\in I} \downarrow A_i$ is a lower superset of $\bigcup_{i\in I} A_i$. Prop. 2.2.7 (4) above implies the claim.

Intersection, '$\subseteq$': By monotonicity of '$\downarrow$', $\downarrow(\bigcap_{i\in I} A_i) \subseteq \downarrow A_i$ holds for all i. □

We shall give three counterexamples for inequality with intersection: one involving just two sets, and two with a descending sequence.

Example 1: Let $\mathbf{P} = \{\bot < [a, b] < \top\}$ ordered such that $\bot$ is least, $\top$ is greatest, and a and b are incomparable. Then

$$\downarrow(\{a\} \cap \{b\}) = \downarrow\emptyset = \emptyset \qquad \downarrow\{a\} \cap \downarrow\{b\} = \{\bot, a\} \cap \{\bot, b\} = \{\bot\}$$
$$\uparrow(\{a\} \cap \{b\}) = \uparrow\emptyset = \emptyset \qquad \uparrow\{a\} \cap \uparrow\{b\} = \{a, \top\} \cap \{b, \top\} = \{\top\}$$

Example 2: Let $\mathbf{P} = \mathbf{N}_0^\infty = \{0 < 1 < \cdots < \infty\}$ be the poset of natural numbers, and $A_n = \{x \mid n \leq x < \infty\}$. Then

$$\uparrow(\bigcap_{n=1}^{\infty} A_n) = \uparrow\emptyset = \emptyset \qquad \bigcap_{n=1}^{\infty} \uparrow A_n = \bigcap_{n=1}^{\infty} (A_n \cup \{\infty\}) = \{\infty\}$$

Example 3: Let $\mathbf{P} = \mathbf{N}_\bot$ be the flat poset of natural numbers, and $A_n = \{n, n+1, \ldots\}$. Then

$$\downarrow(\bigcap_{n=1}^{\infty} A_n) = \downarrow\emptyset = \emptyset \qquad \bigcap_{n=1}^{\infty} \downarrow A_n = \bigcap_{n=1}^{\infty} (A_n \cup \{\bot\}) = \{\bot\}$$

There is a positive fact about closure and intersection:

Lemma 2.2.9 If L is a lower set, then $\uparrow(L \cap \uparrow A) = \uparrow(L \cap A)$ holds for all sets A.

Proof: The inclusion '$\supseteq$' holds since all involved operations are monotonic w.r.t. set inclusion and $\uparrow A \supseteq A$ holds.

For the opposite inclusion, we have to show $L \cap \uparrow A \subseteq \uparrow(L \cap A)$. If x is a point of $L \cap \uparrow A$, then x is in $\uparrow A$ whence there is a in A below x. a is in L because L is a lower set. Thus, $x \geq a \in L \cap A$ whence $x \in \uparrow(L \cap A)$. □

Finally, we consider Cartesian product.

Proposition 2.2.10 Let $(\mathbf{P}_i)_{i \in I}$ be a family of posets, and let A_i be a subset of $\mathbf{P}_i$ for all i in I. Then $\downarrow(\prod_{i \in I} A_i) = \prod_{i \in I} \downarrow A_i$ and $\uparrow(\prod_{i \in I} A_i) = \prod_{i \in I} \uparrow A_i$ hold.

For the special case of binary products, one obtains $\downarrow(A \times B) = \downarrow A \times \downarrow B$ and $\uparrow(A \times B) = \uparrow A \times \uparrow B$.

Proof: $(x_i)_{i \in I} \in \uparrow(\prod_{i \in I} A_i)$ iff $\exists\, (a_i)_{i \in I} \in \prod_{i \in I} A_i : (a_i)_{i \in I} \leq (x_i)_{i \in I}$ iff $\forall i \in I\, \exists\, a_i \in A_i : a_i \leq x_i$ iff $\forall i \in I\ x_i \in \uparrow A_i$ iff $(x_i)_{i \in I} \in \prod_{i \in I} \uparrow A_i$ □

2.3 Convexity

Convexity is a potential property of subsets of posets. Similar to the properties of being a lower or an upper set, there is an associated closure operation, the convex hull.

A set is convex iff with every two points, all points in between belong to it.

Definition 2.3.1 A set $A \subseteq \mathbf{P}$ is *convex* iff, whenever a and c are in A and $a \leq b \leq c$ holds, then b is also in A.

Proposition 2.3.2

(1) $\emptyset$ and the whole poset $\mathbf{P}$ are convex sets.

(2) If $(A_i)_{i \in I}$ is a family of convex sets, then $\bigcap_{i \in I} A_i$ is convex.

(3) If A_i is a convex set in $\mathbf{P}_i$ for all i in I, then $\prod_{i \in I} A_i$ is a convex set in $\prod_{i \in I} \mathbf{P}_i$.

(4) Every lower set and every upper set is convex.

The proof is left to the interested reader.

By part (1) and (2) of this proposition and the results of section 1.2, every set has a least convex superset, the so-called convex hull. It may be constructed by means of lower and upper closure, as indicated by the following definition.

Definition 2.3.3 For $A \subseteq \mathbf{P}$, let $\updownarrow A = \downarrow A \cap \uparrow A$ be the *convex hull* of A.

The name 'convex hull' for the $\updownarrow$-operator is appropriate, as parts (2) through (4) of the following proposition show.

Proposition 2.3.4

(1) $\updownarrow\emptyset = \emptyset$ and $\updownarrow\mathbf{P} = \mathbf{P}$

(2) $\updownarrow A$ is convex

(3) $A \subseteq \updownarrow A$

(4) If $A \subseteq B$ and B is convex, then $\updownarrow A \subseteq B$

(5) A is convex iff $\updownarrow A = A$

(6) $A \subseteq B$ implies $\updownarrow A \subseteq \updownarrow B$

(7) $A \subseteq \updownarrow B$ iff $\updownarrow A \subseteq \updownarrow B$

(8) $\updownarrow\updownarrow A = \updownarrow A$

(9) $\downarrow\updownarrow A = \downarrow A$ and $\uparrow\updownarrow A = \uparrow A$

(10) $\updownarrow A = \updownarrow B$ iff $\downarrow A = \downarrow B$ and $\uparrow A = \uparrow B$

Proof:

(1) $\updownarrow\emptyset = \downarrow\emptyset \cap \uparrow\emptyset = \emptyset \cap \emptyset = \emptyset$, same with $\mathbf{P}$.

(2) $\downarrow A$ and $\uparrow A$ are convex by Prop. 2.3.2 (4), whence is their intersection $\updownarrow A$ by part (2) of the same proposition.

(3) $A \subseteq \downarrow A$ and $A \subseteq \uparrow A$ by Prop. 2.2.7 (3).

(4) Let y be a point of $\updownarrow A = \downarrow A \cap \uparrow A$. Since $y \in \downarrow A$, there is $z \in A$ with $y \leq z$. Since $y \in \uparrow A$, there is $x \in A$ with $x \leq y$. $A \subseteq B$ implies $x, z \in B$, and convexity of B implies $y \in B$.

Statements (2) through (4) imply that '$\updownarrow$' is indeed the closure operation corresponding to convexity due to Prop. 1.2.3. Statements (5) through (8) are reformulations of parts of Prop. 1.2.2.

(9) $\downarrow\updownarrow A \supseteq \downarrow A$ by (3) and Prop. 2.2.7 (6). $\downarrow\updownarrow A = \downarrow(\downarrow A \cap \uparrow A) \subseteq \downarrow\downarrow A = \downarrow A$ by Prop. 2.2.7 (6) and (8).

(10) $\updownarrow A = \updownarrow B$ implies $\downarrow A = \downarrow B$ and $\uparrow A = \uparrow B$ by (9). The opposite direction follows from the definition of $\updownarrow$. □

2.4 Upper bounds

Definition 2.4.1 Let $\mathbf{P} = (P, \leq)$ be a poset, $S \subseteq P$ and $x \in P$. x is an *upper bound* of S if $x \geq s$ for all $s \in S$. S is called (upwards) *bounded* if it has at least one upper bound.

The *least upper bound (lub)* or *supremum* of S — if it exists — is denoted by $\bigsqcup S$. Hence, the defining properties of the supremum are

(1) $s \leq \bigsqcup S$ for all $s \in S$

(2) $\forall u \in \mathbf{P}:$ if $s \leq u$ for all $s \in S$, then $\bigsqcup S \leq u$

If $\bigsqcup\{a, b\}$ exists, it is also written $a \sqcup b$ using a binary infix notation.

Dual notions are lower bound, greatest lower bound (glb) or infimum, and $a \sqcap b$.

Often, we apply these notions to an indexed family $(x_i)_{i \in I}$ of points instead of a set S. An upper bound of the family is an upper bound of the corresponding set $\{x_i \mid i \in I\}$. We abbreviate the supremum of this set by $\bigsqcup_{i \in I} x_i$, or even by $\bigsqcup x_i$ if the index set is clear from context.

The next three propositions are easy to prove:

Proposition 2.4.2 Each point of $\mathbf{P}$ is an upper bound of the empty set. Thus, $\bigsqcup \emptyset$ exists iff $\mathbf{P}$ has a least element $\perp$, and then, $\bigsqcup \emptyset = \perp$ holds.

Proposition 2.4.3 If a point x is both an upper bound and a member of a set A, then $\bigsqcup A$ exists and equals x.

Proposition 2.4.4 Let A and B be two sets having the same upper bounds. Then $\bigsqcup A$ exists iff $\bigsqcup B$ does, and in this case, both are equal.

The next lemma is not difficult either, but sometimes needed later when one set is replaced by another.

Lemma 2.4.5 Let A and B be two sets with the same lower closure, i.e. $\downarrow A = \downarrow B$ or equivalently $A \subseteq \downarrow B$ and $B \subseteq \downarrow A$. Then A and B have the same upper bounds.

Proof: Let u be an upper bound of B. For each a in A, there is b in B with $a \leq b \leq u$ because of $A \subseteq \downarrow B$. Thus, u is an upper bound of A. The opposite direction holds by symmetry. □

The intersection of a family of upper cones is again an upper cone, if the vertices of the cones have a least upper bound.

Lemma 2.4.6 Let $(x_i)_{i \in I}$ be a family of points in a poset with a least upper bound $\bigsqcup_{i \in I} x_i$. Then $\bigcap_{i \in I} \uparrow x_i = \uparrow \bigsqcup_{i \in I} x_i$.

Proof: $y \in \bigcap_{i \in I} \uparrow x_i$ iff $y \in \uparrow x_i$ for all $i \in I$ iff $y \geq x_i$ for all $i \in I$ iff y is an upper bound of the points x_i iff $y \geq \bigsqcup_{i \in I} x_i$ iff $y \in \uparrow \bigsqcup_{i \in I} x_i$ □

The lub operation is monotonic w.r.t. set inclusion:

Proposition 2.4.7 Let $A \subseteq B$. If $\bigsqcup A$ and $\bigsqcup B$ exist, then $\bigsqcup A \leq \bigsqcup B$ holds.

Proof: $\bigsqcup B$ is an upper bound of B, and hence of A. Thus, the least upper bound $\bigsqcup A$ is below $\bigsqcup B$. □

The next lemma deals with the lub of a set of lubs.

Lemma 2.4.8 Let $(A_i)_{i\in I}$ be a family of sets whose lubs exist. Then the sets $\{\bigsqcup A_i \mid i \in I\}$ and $\bigcup_{i\in I} A_i$ have the same upper bounds. Hence, the lub of one set exists iff the lub of the other one exists, and in this case

$$\bigsqcup\{\bigsqcup A_i \mid i \in I\} = \bigsqcup(\bigcup_{i\in I} A_i)$$

Proof: If u is an upper bound of the set of lubs, then u is an upper bound of all sets A_i, whence it is an upper bound of $\bigcup_{i\in I} A_i$.

Conversely, if u is an upper bound of the union, it is an upper bound of each set A_i and hence of each lub $\bigsqcup A_i$. □

Lub operations may be exchanged:

Proposition 2.4.9 Let $(x_{ij})_{i\in I, j\in J}$ be a doubly indexed family of points in a poset such that $\bigsqcup_{j\in J} x_{ij}$ exist for all i in I, and $\bigsqcup_{i\in I} x_{ij}$ exist for all j in J. Then $\bigsqcup_{i\in I}\bigsqcup_{j\in J} x_{ij}$ exists, iff $\bigsqcup_{j\in J}\bigsqcup_{i\in I} x_{ij}$ exists, iff $\bigsqcup_{(i,j)\in I\times J} x_{ij}$ exists, and all three are equal.

Proof: By comparing the upper bounds of the three sets $\{\bigsqcup_{j\in J} x_{ij} \mid i \in I\}$, $\{\bigsqcup_{i\in I} x_{ij} \mid j \in J\}$, and $\{x_{ij} \mid i \in I,\, j \in J\}$. □

Next we consider upper bounds in the product of posets as introduced in section 2.1.

Proposition 2.4.10 Let $(\mathbf{P}_i)_{i\in I}$ be a family of posets. Let A be a subset of $\prod_{i\in I} \mathbf{P}_i$, and let $A_i = \pi_i[A]$ for all i in I.

(1) $A \subseteq \prod_{i\in I} A_i$

(2) A and $\prod_{i\in I} A_i$ have the same upper bounds.

(3) $\bigsqcup A$ exists iff $\bigsqcup A_i$ exists for all i in I. In this case, $\bigsqcup A = (\bigsqcup A_i)_{i\in I}$ holds.

Proof:

(1) If $(a_i)_{i\in I}$ is in A, then a_i is in A_i for all i in I.

(2) Any upper bound of $\prod_{i\in I} A_i$ is an upper bound of A because of (1).

Let $u = (u_i)_{i\in I}$ be an upper bound of A, and let $(x_i)_{i\in I}$ be a point of $\prod_{i\in I} A_i$. Then for all k in I, $x_k \in A_k$ implies there is a point a in A such that $\pi_k a = x_k$. Hence, $a \le u$ follows whence in particular $x_k = \pi_k a \le u_k$. Thus, $x_k \le u_k$ holds for all k in I, whence $x \le u$.

(3) By (2), $\bigsqcup A$ exists iff $\bigsqcup(\prod_{i\in I} A_i)$ exists, and in this case, both are equal. The remainder of the statement is trivial. □

2.5 Monotonic functions

Definition 2.5.1 Let $\mathbf{P}$ and $\mathbf{Q}$ be two posets. A function f from $\mathbf{P}$ to $\mathbf{Q}$ is *monotonic* iff $x \le_{\mathbf{P}} y$ implies $fx \le_{\mathbf{Q}} fy$ for all $x, y \in \mathbf{P}$.

A function $f : \mathbf{P} \to \mathbf{Q}$ is a *poset embedding* iff for all $x, x' \in \mathbf{P}$ $x \le_{\mathbf{P}} x'$ and $fx \le_{\mathbf{Q}} fx'$ are equivalent.

A *poset isomorphism* is a bijective function that is monotonic in both directions.

Proposition 2.5.2

(1) A poset embedding is always injective.

(2) If $\mathbf{P}$ is a subset of a poset $\mathbf{Q}$ with the inherited order, then the natural inclusion map $i : \mathbf{P} \to \mathbf{Q}$ is a poset embedding.

(3) A surjective poset embedding is a poset isomorphism.

Proof:

(1) $fx = fy$ implies $fx \leq fy$ and $fx \geq fy$, whence $x \leq y$ and $y \leq x$, whence $x = y$ by anti-symmetry.

(2) Because $\mathbf{P}$ inherits the order of $\mathbf{Q}$, $x \leq_{\mathbf{P}} y$ is equivalent to $ix \leq_{\mathbf{Q}} iy$.

(3) By (1), f is bijective. By the definition of embedding, both f and its inverse are monotonic. □

Proposition 2.5.3 A poset isomorphism also preserves all infima and suprema.

With the definition of continuity in the next chapter, a poset isomorphism between two domains is in particular continuous in both directions by this proposition.

Proof: Let f be a poset isomorphism between $\mathbf{P}$ and $\mathbf{Q}$. Let g be its inverse mapping. Then both f and g are monotonic.

Let A be a subset of $\mathbf{P}$ with a supremum $\bigsqcup A$. For all y in $f[A]$, $y = fa$ holds for some a in A, whence $y = fa \leq f(\bigsqcup A)$. Thus, $f(\bigsqcup A)$ is an upper bound of $f[A]$. Let u be another upper bound of $f[A]$. Then for all a in A, $fa \leq u$ holds, whence $a \leq gu$. Thus, we get $\bigsqcup A \leq gu$, whence $f(\bigsqcup A) \leq u$. Together, this proves that $f(\bigsqcup A)$ equals $\bigsqcup(f[A])$.

The proof for infima and for g is analogous. □

Proposition 2.5.4 If $f : \mathbf{X} \to \mathbf{Y}$ is monotonic, then for all $A \subseteq \mathbf{X}$, the inclusions $f[\downarrow A] \subseteq \downarrow f[A]$ and $f[\uparrow A] \subseteq \uparrow f[A]$ hold.

Proof: If y is in $f[\uparrow A]$, then there is $x \in \uparrow A$ with $y = fx$. Further, there is a in A with $a \leq x$, whence $fa \leq fx = y$, i.e. y is in $\uparrow f[A]$.

The proof for '$\downarrow$' is analogous. □

Monotonic functions may also be characterized by means of lower sets.

Proposition 2.5.5 For a function $f : \mathbf{P} \to \mathbf{Q}$ between two posets, the following statements are equivalent:

(1) f is monotonic.

(2) The inverse image of every lower set of $\mathbf{Q}$ is a lower set of $\mathbf{P}$.

(3) The inverse image of every lower cone in $\mathbf{Q}$ is a lower set in $\mathbf{P}$.

Proof:

(1) $\Rightarrow$ (2) Let B be a lower set in $\mathbf{Q}$ and $A = f^{-1}[B]$ its inverse image in $\mathbf{P}$. Let a be a point of A and $a' \leq a$. We have to show $a' \in A$. $a \in A$ implies $fa \in B$. $a' \leq a$ implies $fa' \leq fa$ by monotonicity. Since B is a lower set, fa' is in B whence a' is in A.

(2) $\Rightarrow$ (3) is trivial since lower cones are special lower sets.

(3) $\Rightarrow$ (1) Let x and x' be two points in $\mathbf{P}$ with $x \leq x'$. Let $B = \downarrow fx'$ be the lower cone of fx' in $\mathbf{Q}$, and let A be the inverse image of B by f. $fx' \in B$ implies $x' \in A$ whence $x \in A$ since A is a lower set. Thus, $fx \in B$ holds whence $fx \leq fx'$. □

Now we consider the relations between product and monotonicity. The next proposition is an immediate consequence of the definition of the order in the product.

Proposition 2.5.6
Let $(\mathbf{P}_i)_{i \in I}$ be a family of posets. Then all projections $\pi_k : \prod_{i \in I} \mathbf{P}_i \to \mathbf{P}_k$ are monotonic. A function $f : \mathbf{Q} \to \prod_{i \in I} \mathbf{P}_i$ is monotonic iff all functions $\pi_k \circ f : \mathbf{Q} \to \mathbf{P}_k$ are monotonic.

For functions over pairs, monotonicity may be checked component by component:

Proposition 2.5.7 Let $f : \mathbf{P}_1 \times \mathbf{P}_2 \to \mathbf{Q}$. For $x \in \mathbf{P}_1$ and $y \in \mathbf{P}_2$, let $f^y : \mathbf{P}_1 \to \mathbf{Q}$ and $f_x : \mathbf{P}_2 \to \mathbf{Q}$ be defined by $f^y(x) = f_x(y) = f(x, y)$. Then f is monotonic iff f^y and f_x are monotonic for all $x \in \mathbf{P}_1$ and $y \in \mathbf{P}_2$.

Proof:

'$\Rightarrow$' $x \leq x' \Rightarrow (x, y) \leq (x', y) \Rightarrow f(x, y) \leq f(x', y) \Rightarrow f^y(x) \leq f^y(x')$
Analogously for f_x.

'$\Leftarrow$' Let $(x, y) \leq (x', y')$. Then $x \leq x'$ and $y \leq y'$. Hence $f(x, y) = f^y(x) \leq f^y(x') = f(x', y) = f_{x'}(y) \leq f_{x'}(y') = f(x', y')$. □

2.6 Minimal elements

In this section, we consider the notion of minimal elements of a subset of a poset. The dual notion of maximal elements has analogous properties, but is not treated explicitly.

An element of a set is minimal if there is no smaller element. On the contrary, a least element of a set is smaller than all other elements.

Definition 2.6.1 Let $\mathbf{P} = (P, \leq)$ be a poset and $S \subseteq P$. An element $x \in S$ is called *minimal* in S if for all $s \in S$, $s \leq x$ implies $s = x$. It is a *least element* of S, if $x \leq s$ for all $s \in S$.

The set of all minimal elements of S is denoted by $\mathit{min}\, S$.

Dual notions are *maximal* and *greatest element* and the set $\mathit{max}\, S$.

A set may have zero, one, or many minimal elements, but there is at most one least element by anti-symmetry. If the set S has a least element x, then x is minimal in S and there are no other minimal elements, i.e. $\mathit{min}\, S = \{x\}$.

Proof: x is minimal since $s \leq x$ for s in S implies $s = x$ due to antisymmetry because $x \leq s$ holds since x is the least element of S. Conversely, if m is minimal in S, then $x \leq m$ implies $x = m$. □

The '*min*' operator for sets has simple properties collected in the following proposition.

Proposition 2.6.2

(1) $\mathit{min}\, \emptyset = \emptyset$ and $\mathit{min}\, \{x\} = \{x\}$ for all $x \in \mathbf{P}$.

(2) $min\, A \subseteq A$

(3) If $A \subseteq B$ and $min\, B \subseteq A$, then $min\, B \subseteq min\, A$

(4) $min\, A = min\,(\uparrow A)$

(5) If F is finite, then $F \subseteq \uparrow(min\, F)$ holds.

(6) If $f : \mathbf{P} \to \mathbf{Q}$ is monotonic and $B \subseteq A \subseteq \uparrow B$ holds for two sets A and B of $\mathbf{P}$, then $min\, f[A] \subseteq f[B]$ holds in $\mathbf{Q}$.

(7) If $f : \mathbf{P} \to \mathbf{Q}$ is monotonic and $A \subseteq \uparrow(min\, A)$ holds for a set A of $\mathbf{P}$, then $min\, f[A] \subseteq f[min\, A]$ holds in $\mathbf{Q}$.

(8) $B \subseteq A \subseteq \uparrow B$ implies $min\, A \subseteq B$.

Proof:

(1) and (2) are obvious.

(3) Let $m \in min\, B$. Then m is in A. To show the minimality of m in A, let $a \in A$ with $a \leq m$. Then a is also in B, whence $a = m$ since m is minimal in B.

(4) First we show $min\, A \subseteq min\,(\uparrow A)$. Let m be a point in $min\, A$. Then m is in A and hence in $\uparrow A$. Let $r \in \uparrow A$ with $r \leq m$. By definition of $\uparrow A$, there is $a \in A$ with $a \leq r \leq m$. Since m is minimal in A, a equals m, whence r equals m.

For the opposite direction, we have to show $min\,(\uparrow A) \subseteq A$; then statement (3) can be applied since $A \subseteq \uparrow A$. Let m be a point in $min\,(\uparrow A)$. Then m is in $\uparrow A$, whence there is a point a in A below m. a is in $\uparrow A$ because of $A \subseteq \uparrow A$. Minimality of m in $\uparrow A$ implies $m = a \in A$.

(5) Let x be a point of F. We inductively define a descending chain $x_1 > x_2 > \cdots$.

Let $x_1 = x$. Given x_i, there are two cases: either x_i is in $min\, F$, then we cannot proceed; or x_i is not in $min\, F$, then there is some x_{i+1} in F such that $x_i > x_{i+1}$. Since F is finite, the descending chain cannot go on infinitely. Hence, the first case eventually occurs, i.e. there is some x_k in $min\, F$. $x \geq x_k$ holds, whence $x \in \uparrow(min\, F)$.

(6) Let y be a point of $min\, f[A]$. Then $y = fa$ holds for some a in A. By $A \subseteq \uparrow B$, there is b in $B \subseteq A$ such that $a \geq b$. $y = fa \geq fb$ follows. Since fb is in $f[A]$, $y = fb$ follows by minimality of y.

(7) From (6) by $B = min\, A$.

(8) From (6) by $f = id$. □

2.7 Finitary sets

Finally, we consider sets having a finite number of minimal elements where a minimal element can be found below any arbitrary element. Such sets are called *finitary*. They are needed to characterize M-domains, and also to define stronger forms of compactness.

Proposition 2.7.1 For a set $A \subseteq \mathbf{P}$, the following statements are equivalent:

(1) $A \subseteq \uparrow(min\, A)$ holds and $min\, A$ is finite.

(2) There is a finite subset F of A with $A \subseteq \uparrow F$.

(3) There is a finite set F with $\uparrow A = \uparrow F$.

In case (2) and (3), $min\, A = min\, F$ holds.

Sets with these properties are called *finitary*.

Proof:

$(1) \Rightarrow (2)$: Let $F = min\, A$.

$(2) \Rightarrow (3)$: $A \subseteq \uparrow F$ implies $\uparrow A \subseteq \uparrow F$ by Prop. 2.2.7 (7), and $\uparrow A \supseteq \uparrow F$ follows from $A \supseteq F$ by Prop. 2.2.7 (6).

$(3) \Rightarrow (1)$: $min\, A = min\,(\uparrow A) = min\,(\uparrow F) = min\, F$ holds by Prop. 2.6.2 (4). Hence, $min\, A \subseteq F$ is finite.
Applying Prop. 2.6.2 (5), we obtain $A \subseteq \uparrow A = \uparrow F \subseteq \uparrow(min\, F) = \uparrow(min\, A)$. □

The basic properties of finitary sets are given in the following proposition:

Proposition 2.7.2

(1) A is finitary iff $\uparrow A$ is.

(2) Every finite set is finitary.

(3) The union of two finitary sets is again so.

(4) The intersection of a lower set and a finitary set is finitary.

(5) Monotonic images of finitary sets are finitary. More precisely: If $f : \mathbf{P} \to \mathbf{Q}$ is monotonic and A is finitary in $\mathbf{P}$, then $f[A]$ is finitary in $\mathbf{Q}$.

Proof:
In the proofs, we use the condition given in part (3) of the proposition above. Only for part (5), condition (2) is used.

(1) $\uparrow\uparrow A = \uparrow A$ holds. Hence, $\uparrow A = \uparrow F$ and $\uparrow\uparrow A = \uparrow F$ are equivalent.

(2) Let $F = A$. Then F is finite with $\uparrow A = \uparrow F$.

(3) Given two finitary sets A_1 and A_2, let F_i be finite sets with $\uparrow A_i = \uparrow F_i$. Then $F_1 \cup F_2$ is a finite set with $\uparrow(A_1 \cup A_2) = \uparrow A_1 \cup \uparrow A_2 = \uparrow F_1 \cup \uparrow F_2 = \uparrow(F_1 \cup F_2)$.

(4) Let L be the lower set and A the finitary set. Let F be a finite set with $\uparrow A = \uparrow F$. By Lemma 2.2.9, $\uparrow(L \cap A) = \uparrow(L \cap \uparrow A) = \uparrow(L \cap \uparrow F) = \uparrow(L \cap F)$. $L \cap F$ is finite.

(5) Since A is finitary, there is some finite set F with $A \subseteq \uparrow F$. Then $f[A] \subseteq f[\uparrow F] \subseteq \uparrow f[F]$ where Prop. 2.5.4 was used for the second inclusion. The set $f[F]$ is finite. □

Chapter 3

Directed sets and domains

Directed sets are a useful generalization of ascending sequences. By their nature, sequences are always countable whereas directed sets may have any cardinality. Thus, Zorn's Lemma may be applied if every directed set has an upper bound.

The chapter at hand is organized as follows: In section 3.1, directed sets and their basic properties are introduced. Section 3.2 handles directed lubs and the notion of a domain. Domains are posets where every directed set has a supremum. We neither require a least element nor algebraicity.

Next, we define continuous functions (section 3.3) by means of directed sets. In section 3.4, the domain of continuous functions between two domains is introduced. Next, we introduce directed closed sets (section 3.5) and the Scott topology (section 3.6) whose notion of continuity coincides with that of directed sets. The domain of Scott open sets of $\mathbf{X}$ and the function domain from $\mathbf{X}$ to $\mathbf{2} = \{\perp < \top\}$ are isomorphic as pointed out in section 3.7. In section 3.8, we present a Lemma telling that Scott open sets remain open if some of their minimal points are removed. The final section 3.9 presents Rudin's Lemma.

Readers who are experienced in domain theory will not find much new in this chapter. It however collects a host of statements that are heavily used throughout the thesis.

3.1 Directed sets

Definition 3.1.1 A subset D of a poset is (upwards) *directed* iff all finite subsets of D have an upper bound in D.

The notion of directed set is biased because it involves upper bounds, not lower bounds. However, the dual notion of downwards directed sets will not occur much in this paper. In case of doubt, we shall explicitly indicate the order. Directed sets as defined above are $\leq$-directed sets, and downwards directed sets would be $\geq$-directed. Sometimes, $\subseteq$- or $\supseteq$-directed sets of sets will occur.

We often also speak of *directed families* of points. A family $(d_i)_{i \in I}$ of points is directed iff the set $\{d_i \mid i \in I\}$ is directed.

The following proposition is an immediate consequence of the definition:

Proposition 3.1.2 Every finite directed set contains a greatest element which is also the supremum of the set.

There is a characterization of directed sets involving only two points instead of an arbitrary finite number.

Proposition 3.1.3
A set D is directed iff it is not empty and for all x, y in D there is a point z in D such that $z \geq x$ and $z \geq y$, i.e. any two points of D have a common upper bound in D.

Proof: A directed set contains upper bounds for all its finite subsets. Hence, it contains at least an upper bound of the empty set, i.e. it is not empty. If x and y are in D, then $\{x, y\}$ is a finite subset of D, whence there is an upper bound z of x and y in D.

For the opposite direction, let F be a finite subset of D. If F is empty, any arbitrary point is an upper bound of F. Since D is not empty, there is such a point in D. If F is not empty, let $F = \{x_1, \ldots, x_n\}$. Then we recursively construct points $y_1, \ldots, y_n$ in D: Let $y_1 = x_1$, and y_{i+1} be an upper bound of y_i and x_{i+1} in D. Finally, point y_n is an upper bound of the whole set F and contained in D. □

Monotonic images of directed sets are directed.

Proposition 3.1.4 If $f : \mathbf{P} \to \mathbf{Q}$ is a monotonic function between two posets and D is a directed set of $\mathbf{P}$, then $f[D]$ is a directed set of $\mathbf{Q}$.

Proof: We apply criterion 3.1.3. $f[D]$ is not empty since D is not empty. For y_1 and y_2 in $f[D]$, there are x_1 and x_2 in D with $y_i = fx_i$. x_1 and x_2 have a common upper bound x in D. Then fx is a common upper bound of y_1 and y_2 in $f[D]$. □

A directed union of directed sets is directed.

Lemma 3.1.5 Let $\mathcal{D}$ be a $\subseteq$-directed set of directed sets of a fixed poset $\mathbf{P}$. Then $\bigcup \mathcal{D}$ is again a directed set in $\mathbf{P}$.

Proof: Let $D = \bigcup \mathcal{D}$. D is not empty, because $\mathcal{D}$ is not empty, and thus contains at least one non-empty set.

Let x_1 and x_2 be two points of D. Then there are two members D_1 and D_2 of $\mathcal{D}$ with $x_i \in D_i$. Since $\mathcal{D}$ is $\subseteq$-directed, there is a set D' in $\mathcal{D}$ with $D_i \subseteq D'$, whence $x_i \in D'$. Because D' is directed, there is a point $x' \in D' \subseteq D$ with $x_i \leq x$. □

The property of being directed carries over among sets with the same lower closure:

Proposition 3.1.6 If $\downarrow A = \downarrow B$ holds, then A is directed iff B is.

Corollary: A is directed iff $\downarrow A$ is.

Proof: The corollary is easily proved, because $\downarrow A = \downarrow\downarrow A$ holds.

Let $\downarrow A = \downarrow B$ and assume A is directed. We apply Prop. 3.1.3 to show that B is directed. B is non-empty, since otherwise $A \subseteq \downarrow A = \downarrow B = \downarrow \emptyset = \emptyset$ would imply that A is empty.

Let b_1 and b_2 be two points of B. By $B \subseteq \downarrow B = \downarrow A$, there are points a_1 and a_2 in A with $b_i \leq a_i$. Since A is directed, there is $a' \in A$ with $a_i \leq a'$. By $A \subseteq \downarrow A \subseteq \downarrow B$, there is a point $b' \in B$ with $a' \leq b'$. Thus, we have $b_i \leq a_i \leq a' \leq b'$, i.e. b' is an upper bound of b_1 and b_2 in B. □

The proposition and its corollary indicate that instead of considering general directed sets, one could restrict oneself to directed lower sets.

Definition 3.1.7
Directed lower sets are called *ideals*. Two sets with the same lower closure are *cofinal*. If A is a subset of B and A and B are cofinal, one says A is cofinal *in* B.

A is cofinal in B iff $A \subseteq B \subseteq \downarrow A$ holds. Remember that cofinal sets have the same upper bounds by Lemma 2.4.5. By Prop. 3.1.6, two cofinal sets are either both directed or both not directed. Summarizing, one obtains

Proposition 3.1.8 If D is directed with supremum $\bigsqcup D$ and A is cofinal in D, then A is also directed and $\bigsqcup A = \bigsqcup D$ holds.

In some sense, a directed set cannot be partitioned into two smaller sets.

Lemma 3.1.9 If D is directed and $D = A \cup B$ holds, then at least one of the two sets A and B is cofinal in D (and hence directed).

Proof: Assume A is not cofinal in D. Since A is a subset of D, D cannot be a subset of $\downarrow A$. Thus, there is a point d_0 in D with $d_0 \notin \downarrow A$.

Since $B \subseteq D$ holds anyway, we only have to show $D \subseteq \downarrow B$. Let d be any point of D. Since D is directed, there is a common upper bound d' of d and d_0 in D. If d' were in A, d_0 would be in $\downarrow A$. Thus, d' is in B, whence d is in $\downarrow B$. □

Finally, we consider directed sets in a product of two posets.

Proposition 3.1.10 Let $\mathbf{P}$ and $\mathbf{Q}$ be two posets, and let D be a directed set in their product $\mathbf{P} \times \mathbf{Q}$. Then $D_P = \{x \mid (x, y) \in D\}$ and $D_Q = \{y \mid (x, y) \in D\}$ are both directed, their product $D_P \times D_Q$ is directed, and D is cofinal in $D_P \times D_Q$.

The cofinality statement cannot be shown in case of infinite products.

Proof: Since projections are monotonic, the sets $D_P = \pi_1[D]$ and $D_Q = \pi_2[D]$ are directed by Prop. 3.1.4.

Once we have shown the cofinality of D in $D_P \times D_Q$, the latter set is directed by Prop. 3.1.8.

If (x, y) is in D, then x is in D_P and y is in D_Q. Thus, D is a subset of $D_P \times D_Q$. Conversely, if (x, y) is a point in $D_P \times D_Q$, then there are points x' and y' such that (x, y') and (x', y) are in D. Since D is directed, there is a point (x'', y'') in D above both (x, y') and (x', y). Hence, $x'' \geq x$ and $y'' \geq y$ hold, i.e. (x'', y'') is above (x, y), whence the latter point is in $\downarrow D$. □

3.2 Directed lubs and directed completeness

Posets where all directed sets have a lub are particularly important.

Proposition 3.2.1 For a poset $\mathbf{P}$, the following statements are equivalent:

(1) Every directed set of $\mathbf{P}$ has a lub.

(2) Every ideal of $\mathbf{P}$ has a lub.

Posets with these properties are called *directed complete* posets or *domains*. Directed lubs are also called *(directed) limits*.

Note that for domains, the existence of a least element is not required.

Proof: (1) implies (2) since every ideal is a directed set. (2) implies (1), since by Prop. 3.1.6, every directed set D is cofinal in the ideal $\downarrow D$. By Lemma 2.4.5, the lub of $\downarrow D$ is also the lub of D. $\square$

The notion of directed completeness is not self-dual: the poset $\mathbf{N}_0$ is itself directed, but has no upper bound; thus it is not directed complete. But it is downwards directed complete since every non-empty subset has a least element.

Directed completeness admits the application of the non-constructive Lemma of Zorn since every totally ordered subset ('chain') of a poset is trivially directed. Hence, we obtain:

Lemma 3.2.2 (Zorn's Lemma for domains)
In a domain, there is a maximal point above any given point. Formally:

$$\forall x \in \mathbf{P}\ \exists m \in \mathbf{P} \text{ such that } m \geq x \text{ and } \forall z \in \mathbf{P} : (z \geq m \text{ implies } z = m)$$

Proposition 3.2.3 The product of a family of domains is a domain again.

Proof: If D is a directed set in the product, then the projections $D_i = \pi_i[D]$ are directed by Prop. 3.1.4. Prop. 2.4.10 then states that $\bigsqcup D$ exists and equals $(\bigsqcup D_i)_{i \in I}$. $\square$

Examples for domains:

- Every finite poset is a domain by Prop. 3.1.2.
- Discrete and flat posets are directed complete.
- Power sets are domains.
- $\mathbf{N}_0$ with the arithmetic order is not directed complete as mentioned above.
- $\mathbf{N}_0^\infty$ is directed complete. $\mathbf{N}'$ is also directed complete.
- For a poset $\mathbf{P}$, the set of directed subsets of $\mathbf{P}$ ordered by '$\subseteq$' is a domain by Lemma 3.1.5. Hence, there is a maximal directed superset for every directed set by Lemma 3.2.2.

3.3 Continuous functions

Whereas monotonic functions may be defined between two arbitrary posets, we define continuous functions for domains only.

Definition 3.3.1 Let $\mathbf{X}$ and $\mathbf{Y}$ be domains. $f : \mathbf{X} \to \mathbf{Y}$ is (order) *continuous* if for all directed sets D in $\mathbf{X}$, $\bigsqcup(f[D])$ exists and equals $f(\bigsqcup D)$. Continuous functions are sometimes called *morphisms*.

Later, there will be other definitions of continuity. The notion of continuity introduced here is then coined 'order continuity'.

Often, we shall apply the following criterion for continuity.

Proposition 3.3.2 Let $\mathbf{X}$ and $\mathbf{Y}$ be domains.

(1) Every continuous $f : \mathbf{X} \to \mathbf{Y}$ is monotonic.

(2) A monotonic $f : \mathbf{X} \to \mathbf{Y}$ is continuous iff for all directed sets D of $\mathbf{X}$, $\bigsqcup(f[D]) \geq f(\bigsqcup D)$ holds.

$\bigsqcup(f[D])$ exists for all monotonic f and directed D.

Proof:

(1) Let $x \leq y$ in $\mathbf{X}$. Then $\{x, y\}$ is a directed set of $\mathbf{X}$ with supremum y. By continuity, $\bigsqcup\{fx, fy\}$ exists and equals fy. Hence, $fx \leq fy$ holds.

(2) A continuous function obviously satisfies the criterion since it weakens '$=$' to '$\geq$'.

For the opposite direction, we have to show $\bigsqcup(f[D]) \leq f(\bigsqcup D)$. For all d in D, $d \leq \bigsqcup D$ holds, thus $fd \leq f(\bigsqcup D)$ by monotonicity, whence $\bigsqcup(f[D]) \leq f(\bigsqcup D)$.

Monotonic functions f map directed sets D to directed sets $f[D]$. Hence, $\bigsqcup(f[D])$ exists since $\mathbf{Y}$ is a domain. □

Even a bijective continuous map may have a non-monotonic inverse which is then more than ever non-continuous. An example is given by $f : \{1, 2\} \to \{\bot, \top\}$ with $f1 = \bot$ and $f2 = \top$ where 1 and 2 are incomparable, but $\bot \leq \top$ holds.

A monotonic function from a finite poset is always continuous:

Proposition 3.3.3 Let $\mathbf{P}$ be a finite poset and $\mathbf{Y}$ a domain. A function $f : \mathbf{P} \to \mathbf{Y}$ is monotonic iff it is continuous.

Proof: Let D be a directed set in $\mathbf{P}$. By Prop. 3.1.2, D has a greatest element x which is the lub of D. By monotonicity, fx is the greatest element of $f[D]$, whence it is the lub of $f[D]$. Thus, $f(\bigsqcup D) = fx = \bigsqcup(f[D])$ holds. □

Continuous functions may be used as the arrows of a category:

Proposition 3.3.4 The identity $id : \mathbf{X} \to \mathbf{X}$ is continuous for every domain $\mathbf{X}$. If $\mathbf{X}$, $\mathbf{Y}$, and $\mathbf{Z}$ are domains and $f : \mathbf{X} \to \mathbf{Y}$ and $g : \mathbf{Y} \to \mathbf{Z}$ are continuous, then the composition $g \circ f : \mathbf{X} \to \mathbf{Z}$ is continuous.

Proof: Obviously, $\bigsqcup(id[D]) = \bigsqcup D = id(\bigsqcup D)$ holds. Because continuous functions are monotonic, the set $f[D]$ is directed in $\mathbf{Y}$. Hence, $\bigsqcup((g \circ f)[D]) = \bigsqcup g[f[D]] = g(\bigsqcup f[D]) = g(f(\bigsqcup D)) = (g \circ f)(\bigsqcup D)$ holds. □

Next, we consider the relations between continuity and product.

Proposition 3.3.5 Let $(\mathbf{Y}_i)_{i \in I}$ be a family of domains and $\mathbf{Y} = \prod_{i \in I} \mathbf{Y}_i$ their product. The projections $\pi_k : \mathbf{Y} \to \mathbf{Y}_k$ are continuous for all k in I. A function $f : \mathbf{X} \to \mathbf{Y}$ is continuous iff $f_k = \pi_k \circ f : \mathbf{X} \to \mathbf{Y}_k$ is continuous for all k in I.

Proof: Let D be a directed set of $\mathbf{Y}$. By Prop. 2.4.10, we know $\bigsqcup D = (\bigsqcup \pi_i[D])_{i \in I}$, whence $\pi_k(\bigsqcup D) = \bigsqcup \pi_k[D]$ follows for all k in I.

If f is continuous, then all functions $\pi_k \circ f$ are continuous by composition. Conversely, assume all functions f_k are continuous. Then for all k in I, $\pi_k(\bigsqcup f[D]) = \bigsqcup(\pi_k[f[D]]) = \bigsqcup(f_k[D]) = f_k(\bigsqcup D) = \pi_k(f(\bigsqcup D))$ holds where the first equality is due to the continuity of π_k and the third one due to the continuity of f_k. Thus, all components of $\bigsqcup f[D]$ and $f(\bigsqcup D)$ are equal. □

Similar to monotonicity, continuity may be checked component by component. This would however not be possible for infinite products.

Proposition 3.3.6 Let $f : \mathbf{X} \times \mathbf{Y} \to \mathbf{Z}$ where all posets are domains. Let f^y and f_x be defined as in Prop. 2.5.7. f is continuous iff f^y and f_x are continuous for all $x \in \mathbf{X}$ and $y \in \mathbf{Y}$.

Proof: By Prop. 2.5.7, f is monotonic iff all f^y and f_x are monotonic. Hence, all the suprema in the following proof exist since they are suprema of directed sets.

Assume $f : \mathbf{X} \times \mathbf{Y} \to \mathbf{Z}$ is continuous. Fix x in $\mathbf{X}$ and let D be a directed set of $\mathbf{Y}$. Then

$$\begin{array}{rcll} \bigsqcup(f_x[D]) & = & \bigsqcup(f\,[\{x\} \times D]) & \\ & = & f\,(\bigsqcup(\{x\} \times D)) & \text{by continuity of } f \text{ since } \{x\} \times D \text{ is directed} \\ & = & f\,(\bigsqcup\{x\}, \bigsqcup D) & \text{by Prop. 2.4.10} \\ & = & f\,(x, \bigsqcup D) \; = \; f_x(\bigsqcup D) & \end{array}$$

Conversely, let f_x and f^y all be continuous. For a directed set $D \subseteq \mathbf{X} \times \mathbf{Y}$ let D_X and D_Y be the projections to $\mathbf{X}$ and $\mathbf{Y}$ as in Prop. 3.1.10. By this Proposition, D is cofinal in $D_X \times D_Y$, i.e. $D \subseteq D_X \times D_Y \subseteq \downarrow D$. Then $f[D] \subseteq f[D_X \times D_Y] \subseteq f[\downarrow D] \subseteq \downarrow f[D]$ where the last inclusion is due to Prop. 2.5.4, i.e. $f[D]$ is cofinal in $f[D_X \times D_Y]$. Thus, we get

$$\begin{array}{rcll} \bigsqcup(f[D]) & = & \bigsqcup(f\,[D_X \times D_Y]) & \text{by cofinality} \\ & = & \bigsqcup(f\,[\bigcup_{x \in D_X}\{x\} \times D_Y]) & \\ & = & \bigsqcup \bigcup_{x \in D_X}(f[\{x\} \times D_Y]) & \\ & = & \bigsqcup_{x \in D_X} \bigsqcup(f_x[D_Y]) & \text{by Prop. 2.4.8} \\ & = & \bigsqcup_{x \in D_X} f_x(\bigsqcup D_Y) & \text{by continuity of } f_x \\ & = & \bigsqcup\{f(x, \bigsqcup D_Y) \mid x \in D_X\} & \\ & = & \bigsqcup f^{\bigsqcup D_Y}[D_X] & \\ & = & f^{\bigsqcup D_Y}(\bigsqcup D_X) & \text{by continuity of } f^{\bigsqcup D_Y} \\ & = & f(\bigsqcup D_X, \bigsqcup D_Y) & \\ & = & f(\bigsqcup D) & \text{by Prop. 3.1.10} \qquad \Box \end{array}$$

In section 2.5, we defined poset embeddings to be functions $f : \mathbf{P} \to \mathbf{Q}$ with $x \leq_P x'$ iff $fx \leq_Q fx'$. Such embeddings need not be continuous. Consider as an example the function $f : \mathrm{N}_0^\infty \to \mathrm{N}_0^{\infty+1}$ with $fn = n$ for all $n \in \mathrm{N}_0$ and $f(\infty) = \infty + 1$. Here, the order in the domains is given by $\mathrm{N}_0^\infty = \{0 < 1 < \cdots < \infty\}$ and $\mathrm{N}_0^{\infty+1} = \{0 < 1 < \cdots < \infty < \infty + 1\}$.

In contrast to embeddings, poset isomorphisms are always continuous in both directions because of Prop. 2.5.3.

3.4 The function domain

Since we are mainly interested in continuous functions, we define

Definition 3.4.1 For two domains $\mathbf{X}$ and $\mathbf{Y}$, let $[\mathbf{X} \to \mathbf{Y}]$ be the set of all continuous functions from $\mathbf{X}$ to $\mathbf{Y}$. $[\mathbf{X} \to \mathbf{Y}]$ is ordered pointwise, i.e. $f \leq g$ iff $fx \leq gx$ holds for all x in $\mathbf{X}$.

By this definition, $[\mathbf{X} \to \mathbf{Y}]$ becomes a poset. We shall soon see that it is moreover a domain. First, we however establish some conventions. Whereas the notation $f : \mathbf{X} \to \mathbf{Y}$ simply means, f is a function from $\mathbf{X}$ to $\mathbf{Y}$, the notation $f : [\mathbf{X} \to \mathbf{Y}]$ will immediately imply continuity of f such that this continuity need not be explicitly stated.

Lemma 3.4.2 For two domains $\mathbf{X}$ and $\mathbf{Y}$, let $\mathcal{F} \subseteq [\mathbf{X} \to \mathbf{Y}]$ be a set of continuous functions such that for every x in $\mathbf{X}$, the set $\{g\, x \mid g \in \mathcal{F}\}$ has a supremum. Then the function f defined by $f x = \bigsqcup_{g \in \mathcal{F}} g\, x$ is continuous and represents the supremum of $\mathcal{F}$.

Proof: For continuity of f, we have to show $\bigsqcup f[D] = f(\bigsqcup D)$ for all directed sets D.

$$\begin{aligned}
\textstyle\bigsqcup f[D] &= \textstyle\bigsqcup_{d \in D} f d \\
&= \textstyle\bigsqcup_{d \in D} \bigsqcup_{g \in \mathcal{F}} g\, d \\
&= \textstyle\bigsqcup_{g \in \mathcal{F}} \bigsqcup_{d \in D} g\, d && \text{by Prop. 2.4.9} \\
&= \textstyle\bigsqcup_{g \in \mathcal{F}} \bigsqcup g[D] \\
&= \textstyle\bigsqcup_{g \in \mathcal{F}} g\,(\bigsqcup D) && \text{by continuity of } g \\
&= f\,(\textstyle\bigsqcup D)
\end{aligned}$$

Once f is shown to be continuous, it is the supremum of $\mathcal{F}$ because of the pointwise order of functions. □

From this Lemma, one concludes:

Proposition 3.4.3 If $\mathbf{X}$ and $\mathbf{Y}$ are domains, then $[\mathbf{X} \to \mathbf{Y}]$ is a domain.

Proof: Let $\mathcal{F}$ be a directed set of functions. Then for all x in $\mathbf{X}$, the set $\{f x \mid f \in \mathcal{F}\}$ is directed. Thus, the function $\lambda x.\, \bigsqcup_{f \in \mathcal{F}} f x$ exists, is continuous, and represents the supremum of $\mathcal{F}$. □

Next, we establish some continuous functions around the function domain.

Proposition 3.4.4
Application $A : [\mathbf{X} \to \mathbf{Y}] \times \mathbf{X} \to \mathbf{Y}$ defined by $A\,(f,\, x) = f x$ is continuous.

Proof: By Prop. 3.3.6, we have to show the continuity in each component separately. For fixed f, $A_f = \lambda x.\, A\,(f,\, x) = \lambda x.\, f x = f$ holds, whence it is continuous.

To show the continuity of A^x for fixed x, let $\mathcal{F}$ be a directed set of morphisms. Then $\bigsqcup A^x[\mathcal{F}] = \bigsqcup_{f \in \mathcal{F}} (f x) = (\bigsqcup \mathcal{F})\, x = A^x(\bigsqcup \mathcal{F})$ holds because of Lemma 3.4.2. □

Proposition 3.4.5 For every $f : [\mathbf{X} \times \mathbf{Y} \to \mathbf{Z}]$, there is a curried version
$f' = \lambda x.\, (\lambda y.\, f\,(x,\, y)) :\ [\mathbf{X} \to [\mathbf{Y} \to \mathbf{Z}]]$.

Proof: We have to show the continuity of $f' x$ for every x in $\mathbf{X}$ (the inner square brackets), and the continuity of f' itself (the outer square brackets).

For every x in $\mathbf{X}$, $f' x = \lambda y.\, f\,(x,\, y) = f_x$ is continuous by Prop. 3.3.6. For continuity of f', let D be a directed set of $\mathbf{X}$. Then for every y in $\mathbf{Y}$

$$\begin{aligned}
(\textstyle\bigsqcup f'[D])\, y &= (\textstyle\bigsqcup_{d \in D} f_d)\, y \\
&= \textstyle\bigsqcup_{d \in D} (f_d\, y) && \text{by Prop. 3.4.2} \\
&= \textstyle\bigsqcup_{d \in D} f\,(d,\, y) \\
&= \textstyle\bigsqcup f^y[D] \ = \ f^y\,(\bigsqcup D) && \text{by Prop. 3.3.6} \\
&= f\,(\textstyle\bigsqcup D,\, y) \ = \ (f'\,(\bigsqcup D))\, y
\end{aligned}$$

By extensionality, $\bigsqcup f'[D] = f'\left(\bigsqcup D\right)$ follows. □

Because of the propositions 3.2.3 (product of domains is a domain), 3.3.5, 3.4.3 (function domain), 3.4.4 (application), and 3.4.5 (currying), domains and continuous functions form a Cartesian closed category.

Besides application and currying, there are some more useful functional combinators.

Proposition 3.4.6 If $\mathbf{X}$, $\mathbf{Y}$, and $\mathbf{Z}$ are domains, then functional composition $\circ : [[\mathbf{Y} \to \mathbf{Z}] \times [\mathbf{X} \to \mathbf{Y}] \to [\mathbf{X} \to \mathbf{Z}]]$ is continuous.

The proof is a straightforward application of Prop. 3.3.6, Prop. 3.4.2, and continuity of g.

Proposition 3.4.7 If $\mathbf{X}_1$, $\mathbf{X}_2$, $\mathbf{Y}_1$, and $\mathbf{Y}_2$ are domains, then the Cartesian combinator $\times : [[\mathbf{X}_1 \to \mathbf{Y}_1] \times [\mathbf{X}_2 \to \mathbf{Y}_2] \to [\mathbf{X}_1 \times \mathbf{X}_2 \to \mathbf{Y}_1 \times \mathbf{Y}_2]]$ defined by $(f_1 \times f_2)(x_1, x_2) = (f_1 x_1, f_2 x_2)$ is continuous.

The proof is also straightforward by using the same auxiliary facts as in the proof of the previous proposition.

3.5 Directed closed sets

In this section, we investigate those sets which are closed w.r.t. suprema of directed subsets. In the next section, they are restricted to the more important Scott closed sets.

Definition 3.5.1 Let $\mathbf{P}$ be a domain. A subset A of $\mathbf{P}$ is *directed closed* or shortly *d-closed* iff the suprema w.r.t. $\mathbf{P}$ of all directed subsets of A are again in A.

Directed closed sets A of $\mathbf{P}$ are also called *sub-domains* of $\mathbf{P}$ because they are domains in the order inherited from $\mathbf{P}$.

In the sequel, we study the basic properties of d-closed sets.

Proposition 3.5.2

(1) $\emptyset$ and $\mathbf{P}$ are d-closed.

(2) The intersection of a family of d-closed sets is d-closed.

(3) The union of two d-closed sets is d-closed.

Proof:

(1) $\emptyset$ has no directed subset since directed sets are not empty. Hence, the condition for being d-closed is vacuously satisfied.

The whole domain $\mathbf{P}$ contains the lubs of all its subsets.

(2) Let $(A_i)_{i \in I}$ be a family of d-closed sets, and A its intersection. If D is a directed subset of A, then it is a subset of all sets A_i whence its lub is in all sets A_i whence it is in A.

(3) Let A and B be two d-closed sets, and let $D \subseteq A \cup B$. Then $D = D \cap (A \cup B) = (D \cap A) \cup (D \cap B)$ holds. By Lemma 3.1.9, one of these two sets — say $D \cap A$ — is cofinal in D. By Prop. 3.1.8, $D \cap A$ is directed and $\bigsqcup(D \cap A) = \bigsqcup D$ holds. Since A is d-closed and $D \cap A$ is a directed subset of A, this implies $\bigsqcup D \in A \subseteq A \cup B$ as required. □

Next, we establish some important classes of d-closed sets.

Proposition 3.5.3

(1) All upper sets are d-closed.

(2) All finite sets are d-closed.

(3) If F is finite, then $\downarrow F$ is d-closed.

Proof:

(1) Let A be an upper set, $D \subseteq A$ a directed set and x the lub of D. Since D is not empty, there is some member $d \in D \subseteq A$. Then x is in A, too, since A is an upper set and $d \leq x$ holds.

(2) We show that singletons are d-closed and apply Prop. 3.5.2 (3).
If $A = \{x\}$ is a singleton, then the only directed subset of A is A itself, whose lub x is contained in A.

(3) We show that cones are d-closed and apply Prop. 3.5.2 (3). If $A = \downarrow x$ is a cone, and D is a directed subset of A, then all elements of D are below x, whence their lub is so. □

Considering part (2) and (3) of the definition above, one might believe that for all sets A, A is d-closed iff $\downarrow A$ is. This belief is strengthened by the intuition that being d-closed is a property of the upper margin of a set only. It is however completely wrong as two examples show:

Example 1: Let $\mathbf{N}'$ be the poset of 'lazy natural' numbers as introduced in chapter 2. Let $A = \{0, s0, s^2 0, \ldots\}$. This is an upper set, and thus d-closed. $\downarrow A = \mathbf{N}' - \{s^\infty \bot\}$ however is not d-closed.

Example 2: Let $\mathbf{P} = \mathbf{N}_0^{\infty+1}$ and $A = \mathbf{P} - \{\infty\}$. Then A is not d-closed, but $\downarrow A = \mathbf{P}$ is.

For convex sets, at least one implication holds:

Proposition 3.5.4 If A is convex and $\downarrow A$ is d-closed, then A is d-closed.

Proof: Let D be a directed subset of A. Then $D \subseteq A \subseteq \downarrow A$ holds, whence $\bigsqcup D \in \downarrow A$ since $\downarrow A$ is d-closed. This implies there is $a \in A$ such that $\bigsqcup D \leq a$. Since D is not empty, there is $d \in D \subseteq A$. $d \leq \bigsqcup D \leq a$ implies $\bigsqcup D \in A$ by convexity of A. □

Now, we consider the complementary notion of d-open sets.

Proposition 3.5.5

For a subset A of a domain $\mathbf{P}$, the following statements are equivalent:

(1) For all directed sets $D \subseteq \mathbf{P}$ with $\bigsqcup D \in A$, $D \cap A$ is cofinal in D.

(2) For all directed sets $D \subseteq \mathbf{P}$ with $\bigsqcup D \in A$, $D \cap A$ is not empty.

(3) A is *d-open*, i.e. the complement of a d-closed set.

Proof:

(1) $\Rightarrow$ (2) By Prop. 3.1.8, $D \cap A$ is directed since it is cofinal in the directed set D. Directed sets are not empty.

(2) $\Rightarrow$ (3) Let B be the complement of A, and D be a directed subset of B. If $\bigsqcup D$ were not in B, it would be in A, whence some element of D would be in A by (2) in contradiction to $D \subseteq B$.

(3) $\Rightarrow$ (1) Let B be again the complement of A. By (3), B is d-closed. $D = (D \cap A) \cup (D \cap B)$ holds, whence one of these two sets is cofinal in D by Prop. 3.1.9. Assume $D' = D \cap B$ were cofinal in D. By Prop. 3.1.8, D' is directed and $\bigsqcup D' = \bigsqcup D$ holds. Then $D' \subseteq B$ would follow, but $\bigsqcup D'$ is in A in contradiction to B being d-closed. □

Now, we investigate the relationship between continuous functions and d-closed sets.

Proposition 3.5.6 Let $f : \mathbf{P} \to \mathbf{Q}$ be a continuous map between two domains. Then the inverse image of every d-closed set in $\mathbf{Q}$ is d-closed in $\mathbf{P}$.

Proof: Let B be a d-closed subset of $\mathbf{Q}$ and let $A = f^{-1}[B]$ be the inverse image of B. Let D be a directed subset of A, and let x be the lub of D. Then fx is the lub of the directed set $f[D]$ by continuity of f. Since $f[D]$ is a subset of B and B is d-closed, fx is in B, whence x is in A. □

The converse implication does not hold. The mapping $f : \{\bot, \top\} \to \{\bot, \top\}$ with $f\bot = \top$ and $f\top = \bot$ for instance is not continuous since it is not even monotonic. However, it backwards maps d-closed sets to d-closed sets since every subset of $\{\bot, \top\}$ is d-closed because of Prop. 3.5.3 (2).

Towards the end of the next section, we shall see that the converse implication however holds if the function is monotonic.

Because of Prop. 3.5.2 (1) and (2), there is a least d-closed superset to every set in a domain.

Definition 3.5.7

Let $\mathbf{X}$ be a domain and $A \subseteq \mathbf{X}$. The least d-closed superset of A is denoted by $\overline{A}$.

A continuous function on $\overline{A}$ is uniquely determined by its values on A.

Proposition 3.5.8

Let $\mathbf{X}$ and $\mathbf{Y}$ be domains, $A \subseteq \mathbf{X}$, and $f, g : \overline{A} \to \mathbf{Y}$ be two continuous functions. If $fa = ga$ holds for all a in A, then $fx = gx$ also holds for all x in $\overline{A}$.

Proof: Let $B = \{x \in \overline{A} \mid fx = gx\}$. By the pre-condition, $A \subseteq B$ holds. Let D be a directed subset of B with lub x. Then $fx = f(\bigsqcup D) = \bigsqcup f[D] = \bigsqcup g[D] = g(\bigsqcup D) = gx$ holds by continuity of f and g and since both functions coincide on B. Thus, B is d-closed. From $A \subseteq B$, $\overline{A} \subseteq B$ follows. □

We now provide an upper bound for the cardinality of the d-closure of a set.

Proposition 3.5.9 Let A be a subset of a domain $\mathbf{X}$. Then $|\overline{A}| \leq 2^{|A|}$ holds.

Proof: Let $B = \{\bigsqcup S \mid S \subseteq A, \bigsqcup S \text{ exists}\}$. Then $A \subseteq B$ holds since $a = \bigsqcup\{a\}$ holds for all $a \in A$. We show that B is d-closed. If D is a directed subset of B, then for all d in D there is a subset S_d of A such that $d = \bigsqcup S_d$. Then $\bigsqcup D = \bigsqcup_{d \in D}(\bigsqcup S_d) = \bigsqcup(\bigcup_{d \in D} S_d)$ by Prop. 2.4.8. Since $\bigcup_{d \in D} S_d$ is a subset of A, $\bigsqcup D$ is a member of B.

Since B is a d-closed superset of A, $\overline{A} \subseteq B$ follows, whence $|\overline{A}| \leq |B| \leq |2^A| = 2^{|A|}$. □

Note that the set B in this proof contained the lubs of *all* subsets of A that exist, not only the lubs of the directed subsets of A. One might believe that the set $\widehat{A}$ of all lubs of *directed* subsets of A equals $\overline{A}$. This belief is however wrong; in general, $\widehat{A}$ does not contain the lubs of directed sets of lubs of directed sets of A.

3.6 Scott closed sets

In this section, we define the important notions of Scott closed and Scott open sets. Scott closed sets may be used to characterize continuous functions.

Definition 3.6.1 Let $\mathbf{P}$ be a domain. A subset A of $\mathbf{P}$ is called *Scott closed* iff it is a d-closed lower set. The complements of Scott closed sets are called *Scott open*.

By Prop. 2.2.3, Scott open sets are upper d-open sets. For the properties of d-open sets see Prop. 3.5.5.

Proposition 3.6.2

(1) $\emptyset$ and $\mathbf{P}$ are Scott closed.

(2) Arbitrary intersections of Scott closed sets are Scott closed.

(3) Finite unions of Scott closed sets are Scott closed.

Proof: Just apply Prop. 2.2.2 for lower sets and Prop. 3.5.2 for d-closed sets. □

By part (1) and (2) of this proposition, every set A has a least Scott closed superset $\mathsf{cl}_S\, A$.

Proposition 3.6.3 The Scott closure operator 'cl_S' has the following properties:

(1) $\mathsf{cl}_S\, A \supseteq \downarrow A$ for all $A \subseteq \mathbf{P}$.

(2) $\mathsf{cl}_S\, A = \downarrow A$ for all finite sets $A \subseteq \mathbf{P}$.

Proof:

(1) $\downarrow A$ is the least lower superset of A. $\mathsf{cl}_S\, A$ is also a lower superset of A, since it is the least Scott closed one.

(2) '$\supseteq$' holds by (1). $\downarrow A$ is a Scott closed superset of A, since it is d-closed by Prop. 3.5.3 (3). Hence, $\mathsf{cl}_S\, A \subseteq \downarrow A$ holds since $\mathsf{cl}_S\, A$ is the least Scott closed superset of A. □

In contrast to the d-closure, the cardinality of the Scott closure of a set A is not bounded by the cardinality of A. Let for instance S be a set, and let the domain $\mathbf{X}$ be the powerset of S ordered by inclusion. Since S is the greatest element of $\mathbf{X}$, the Scott closure of the singleton set $\{S\}$ is the whole domain $\mathbf{X}$. There are also examples where a continuous function on $\mathsf{cl}_S\, A$ is not uniquely determined by its values on A.

The order continuous functions of section 3.3 may be characterized in terms of Scott closed sets:

Proposition 3.6.4 For a function $f : \mathbf{P} \to \mathbf{Q}$ between two domains, the following statements are equivalent:

(1) f is order continuous.

(2) f is monotonic, and the inverse image of every d-closed set of $\mathbf{Q}$ is a d-closed set of $\mathbf{P}$.

(3) The inverse image of every Scott closed set of $\mathbf{Q}$ is a Scott closed set of $\mathbf{P}$.

(4) The inverse image of every lower cone in $\mathbf{Q}$ is a Scott closed set in $\mathbf{P}$.

(5) The inverse image of every Scott open set of $\mathbf{Q}$ is Scott open in $\mathbf{P}$.

Proof:

(1) $\Rightarrow$ (2) by Prop. 3.3.2 and Prop. 3.5.6.

(2) $\Rightarrow$ (3) Let B be a Scott closed set in $\mathbf{Q}$ and A its inverse image by f. By Prop. 2.5.5, A is a lower set by monotonicity of f, and it is d-closed by (2).

(3) $\Rightarrow$ (4) is trivial since every lower cone is Scott closed by Prop. 3.6.3 (2).

(3) $\Leftrightarrow$ (5) since the complement of an inverse image of a set A is the inverse image of $\mathsf{co}\, A$.

(4) $\Rightarrow$ (1) The inverse image of every lower cone is a lower set whence f is monotonic by Prop. 2.5.5. Let D be a directed set in $\mathbf{P}$ with lub x. By monotonicity, $f[D]$ is a directed set in $\mathbf{Q}$ with lub y, and $fx \geq y$ holds. We have to show $fx \leq y$. Let $B = \downarrow y$ be the lower cone of y in $\mathbf{Q}$, and let A be its inverse image in $\mathbf{P}$, which is Scott closed by (4). Since y is an upper bound of $f[D]$, $f[D] \subseteq B$ holds, whence $D \subseteq A$. Since A is d-closed, it contains x, whence $fx \in B$, i.e. $fx \leq y$. □

3.7 Open sets and continuous functions

Let $\mathbf{X}$ be a domain. In this section, we compare the domain $\Omega\mathbf{X}$ of (Scott) open sets of $\mathbf{X}$ with the domain of continuous functions from $\mathbf{X}$ to $\mathbf{2} = \{\bot < \top\}$. The open sets form a domain when ordered by inclusion '$\subseteq$' because arbitrary unions of open sets are open.

If $f : [\mathbf{X} \to \mathbf{2}]$ is a morphism, then $f^{-1}[\{\top\}]$ is open in $\mathbf{X}$ because $\{\top\}$ is open in $\mathbf{2}$. Conversely, if an open set O of $\mathbf{X}$ is given, define a function $f : \mathbf{X} \to \mathbf{2}$ by $fx = \top$ for x in O and $fx = \bot$ otherwise. Continuity of this function is asserted by the following Lemma that deals with a more general situation.

Lemma 3.7.1 Let $\mathbf{X}$ and $\mathbf{Y}$ be two domains, O an open set of $\mathbf{X}$, and u and v two points of $\mathbf{Y}$ with $u \leq v$. Then the function $f = \lambda x. \begin{cases} v & \text{if } x \in O \\ u & \text{otherwise} \end{cases}$ is continuous.

Proof: Applying Prop. 3.6.4, we have to show that the inverse image of every open set of $\mathbf{Y}$ is open in $\mathbf{X}$. Let O' be an open set of $\mathbf{Y}$. Concerning u and v, there are three cases:

(1) Both u and v are in O'. Then $f^{-1}[O'] = \mathbf{X}$.

(2) Neither u nor v is in O'. Then $f^{-1}[O'] = \emptyset$.

(3) v is in O', but u is not. Then $f^{-1}[O'] = O$.

In any case, $f^{-1}[O']$ is open. The case where u is in O' but v is not, is impossible since $u \leq v$ and O' is upper. □

Thus, we get a one-to-one correspondence between $\Omega\mathbf{X}$ and $[\mathbf{X} \to \mathbf{2}]$ by means of $O \mapsto f_O = \lambda x. \begin{cases} \top & \text{if } x \in O \\ \bot & \text{otherwise} \end{cases}$ and $f \mapsto f^{-1}[\{\top\}]$. This correspondence is an order isomorphism since

$$\begin{array}{lll} f \le g & \text{iff} & fx \le gx \text{ for all } x \in \mathbf{X} \\ & \text{iff} & fx = \top \text{ implies } gx = \top \text{ for all } x \in \mathbf{X} \\ & \text{iff} & x \in f^{-1}[\{\top\}] \text{ implies } x \in g^{-1}[\{\top\}] \text{ for all } x \in \mathbf{X} \\ & \text{iff} & f^{-1}[\{\top\}] \subseteq g^{-1}[\{\top\}] \end{array}$$

Order isomorphisms preserve all suprema and infima. Hence, union in $\Omega\mathbf{X}$ corresponds to pointwise lub in $[\mathbf{X} \to \mathbf{2}]$, and binary intersection in $\Omega\mathbf{X}$ corresponds to binary infimum in $[\mathbf{X} \to \mathbf{2}]$. Furthermore, $\emptyset$ and $\mathbf{X}$ in $\Omega\mathbf{X}$ are associated with $\lambda x.\bot$ and $\lambda x.\top$ in $[\mathbf{X} \to \mathbf{2}]$ respectively.

3.8 Removing minimal points of open sets

In this section, we state and prove a technical Lemma that informally reads as follows: if one removes a part of the minimal points of a Scott open set O, then the set remains open.

Lemma 3.8.1 Let O be a Scott open set in a domain $\mathbf{X}$, and let M be a subset of $min\, O$. Then $O \setminus M$ is Scott open.

Proof:
Let $O' = O \setminus M$ and $O'' = O \setminus min\, O$. Because of $M \subseteq min\, O$, $O'' \subseteq O' \subseteq O$ holds.

O' is upper: Let x in O' and $x \le y$. Since O is upper, $x \in O$ implies $y \in O$. If y is not in $min\, O$, then $y \in O'' \subseteq O'$ follows. If y is in $min\, O$, then $x \le y$ implies $y = x \in O'$.

O' is d-open: Let D be a directed set with $\bigsqcup D$ in O'. We have to show that D meets O'. Assume it does not. Since $\bigsqcup D \in O' \subseteq O$, there is some x in $D \cap O$. Let d be an arbitrary point of D. The two points d and x have a common upper bound y in D. x in O and O being upper implies y in O. By assumption, y is not in O', whence it is in $M \subseteq min\, O$. $x \le y$ in O implies $x = y \ge d$. Thus, x is an upper bound of D. $\bigsqcup D \le x$ follows, whence x is in O' as $\bigsqcup D$ is. This contradicts the assumption that D does not meet O'. □

3.9 Rudin's Lemma and its consequences

In [GLS83] and [Jun88], an interesting Lemma is stated and proved that in its original weaker form dates back to [Rud80]. Although the Lemma is concerned with arbitrary posets, we present it in this chapter since we derive some consequences from it that only hold in domains.

In the version of [Jun88], the Lemma reads as follows:

Lemma 3.9.1 Let $\mathbf{X}$ be a poset and $\mathcal{E}$ a set of *non-empty finite* subsets of $\mathbf{X}$ such that $\uparrow[\mathcal{E}] = \{\uparrow E \mid E \in \mathcal{E}\}$ is $\supseteq$-*directed*. Then there is a *directed* subset D of $\bigcup \mathcal{E}$ that *meets* all members of $\mathcal{E}$.

For the proof of this Lemma, we refer to [Jun88], or to [GLS83] for a proof of a slightly different version. In the sequel, we derive some consequences of the Lemma.

Lemma 3.9.2 Let $\mathbf{X}$ be a *domain* and $\mathcal{E}$ a set of *non-empty finite* subsets of $\mathbf{X}$ such that $\uparrow[\mathcal{E}] = \{\uparrow E \mid E \in \mathcal{E}\}$ is $\supseteq$*-directed*. Then there is a point x that is both in $\overline{\bigcup \mathcal{E}}$ and in $\bigcap \uparrow[\mathcal{E}]$.

Proof: Lemma 3.9.1 gives us a directed subset D of $\bigcup \mathcal{E}$ that meets all sets E in $\mathcal{E}$, i.e. there is a point d_E in $D \cap E$ for all E in $\mathcal{E}$. Since $\mathbf{X}$ is a domain, the supremum x of D exists. For every E in $\mathcal{E}$, $x \geq d_E \in E$ implies $x \in \uparrow E$. $D \subseteq \bigcup \mathcal{E}$ implies $x \in \overline{\bigcup \mathcal{E}}$. □

The next Lemma is concerned with a (Scott) closed set.

Lemma 3.9.3 Let $\mathbf{X}$ be a *domain* and $\mathcal{F}$ a $\supseteq$*-directed* set of *finitary upper* sets of $\mathbf{X}$. If all members of $\mathcal{F}$ meet a closed set C, then $\bigcap \mathcal{F}$ meets C.

Proof: By Prop. 2.7.1, for every F in $\mathcal{F}$, $min\, F$ is finite and $F = \uparrow(min\, F)$ holds. Let $E_F = C \cap min\, F$. We show some properties of the sets E_F.

(1) The sets E_F are finite since $E_F \subseteq min\, F$.

(2) The sets E_F are not empty.
Proof: F meets C, whence there is a point u in $C \cap F$. By $F = \uparrow(min\, F)$, there is a point $v \in min\, F$ below u. Since C is lower, v also is in C.

(3) $F \supseteq F'$ implies $\uparrow E_F \supseteq \uparrow E_{F'}$.
Proof: Let u be a point in $E_{F'} = C \cap min\, F'$. Then u is in $min\, F' \subseteq F' \subseteq F = \uparrow(min\, F)$, whence there is a point $v \in min\, F$ below u. Since C is lower, v is also in C. Thus, $v \in C \cap min\, F = E_F$, whence $u \in \uparrow E_F$.

Now let $\mathcal{E} = \{E_F \mid F \in \mathcal{F}\}$. By (1) and (2), $\mathcal{E}$ is a set of non-empty finite sets. By (3), $\uparrow[\mathcal{E}]$ is directed since $\mathcal{F}$ is directed. Hence, Lemma 3.9.2 applies, and there is a point x in both $\overline{\bigcup \mathcal{E}}$ and $\bigcap \uparrow[\mathcal{E}]$. By the first containment, $x \in \overline{\bigcup \mathcal{E}} \subseteq \overline{C} = C$ holds since all members of $\mathcal{E}$ are subsets of C by definition. By the second containment, $x \in \uparrow E_F \subseteq \uparrow(min\, F) = F$ holds for all F in $\mathcal{F}$. □

The last consequence involves a (Scott) open set.

Lemma 3.9.4 Let $\mathbf{X}$ be a *domain*, $\mathcal{F}$ a $\supseteq$*-directed* set of *finitary upper* subsets of $\mathbf{X}$, and O an open set of $\mathbf{X}$ such that $\bigcap \mathcal{F} \subseteq O$ holds. Then $F \subseteq O$ holds for some $F \in \mathcal{F}$.

Proof: Assume $F \not\subseteq O$ for all $F \in \mathcal{F}$. Then $F \cap \mathsf{co}\, O \neq \emptyset$ follows for all $F \in \mathcal{F}$. By Lemma 3.9.3, $\bigcap \mathcal{F} \cap \mathsf{co}\, O \neq \emptyset$ follows, whence $\bigcap \mathcal{F} \not\subseteq O$ — a contradiction. □

Chapter 4

Set-theoretic topology

This chapter presents general notions from topology such as closed and open sets, closure, compact sets, and continuous functions. Besides these well known notions, the novel concept of *strong compactness* is introduced in section 4.7. As the name suggests, strongly compact sets have properties analogous to those of compact sets.

In contrast to usual introductions to topology, we are especially concerned with the connections between order theory and topology. The previous chapter showed that every domain is equipped with a standard topology, the Scott topology. In this chapter, we show that every topological space induces a pre-order on its carrier that becomes an order under some mild assumption.

In section 4.1, we define topological spaces and their closed and open sets. Section 4.2 handles topological closure. The specialization pre-order is introduced in section 4.3. The separation properties (T0) and (T1) are defined in terms of this pre-order. In section 4.4, we investigate the relations between order-theoretic and topological notions.

In section 4.5, topological continuity and its properties are studied. Section 4.6 is devoted to the theory of compact sets, and the subsequent section 4.7 introduces strongly compact sets. Section 4.8 investigates product domains topologically, and the final section 4.9 deals with strongly compact sets in products.

Readers who are experienced in domain-oriented topology will not find much new in this chapter. An exception are the strongly compact sets of section 4.7. Strong compactness seems to be a novel concept. It is used in the theory of the upper power domain in chapter 20.

Prop. 4.3.1 and Lemma 4.4.4 (2) deserve particular attention. They allow to prove inequalities and inclusions respectively by topological means and will be used heavily throughout the thesis.

4.1 Closed and open sets

The Propositions 2.2.2, 3.5.2, and 3.6.2 show some common properties of lower sets, d-closed sets, and Scott closed sets respectively. By abstraction, we obtain the following definition:

Definition 4.1.1 (Topological space)

A topological space $\mathbf{X}$ is a non-empty set X together with a set Γ of subsets of X such that

(C1) $\emptyset$ and X are in Γ.

(C2) Arbitrary intersections of members of Γ are again in Γ.

(C3) Finite unions of members of Γ are again in Γ.

The sets in Γ are *closed sets*, their complements are called *open*.

A set R is an *environment* of a point x or a set S if there is an open set O such that $x \in O \subseteq R$ or $S \subseteq O \subseteq R$ respectively hold.

The set of all open environments of a set S is denoted by $\mathcal{O}(S)$. $\mathcal{O}(\{x\})$ is often abbreviated to $\mathcal{O}(x)$.

As with posets, we often identify the topological space $\mathbf{X}$ and its carrier X.

From the definition, one may immediately deduce that $\emptyset$ and $\mathbf{X}$ are open, and arbitrary union and finite intersection of open sets are open. Usually, topological spaces are defined by their open sets.

As indicated in the beginning, every domain implies (at least) three topological spaces: the *Alexandroff space* where the closed sets are the lower sets, the *d-space* with the d-closed sets, and the *Scott space* with the Scott closed sets. From these, the Scott space is the most important. The Scott space of a domain $\mathbf{X}$ is denoted by $\Sigma\mathbf{X}$, and the Alexandroff space by $\mathrm{A}\mathbf{X}$. Mostly, we shall identify $\mathbf{X}$ and $\Sigma\mathbf{X}$.

4.2 Topological closure

The set Γ of closed sets oversupplies the axioms of section 1.2 needed to establish closure operations. Hence, there is a topological closure $\mathsf{cl}\, A = \bigcap\{B \in \Gamma \mid B \supseteq A\}$. The closure of a singleton set $\mathsf{cl}\,\{x\}$ will often be abbreviated by $\mathsf{cl}\, x$. The topological closure enjoys all properties given in Prop. 1.2.2. In addition, it has some further properties:

Proposition 4.2.1

(1) $\mathsf{cl}\,\emptyset = \emptyset$

(2) $\mathsf{cl}\,(A \cup B) = \mathsf{cl}\, A \cup \mathsf{cl}\, B$

Proof:

(1) By Prop. 1.2.2 (8) and $\emptyset \in \Gamma$.

(2) $A \cup B \subseteq \mathsf{cl}\, A \cup \mathsf{cl}\, B$ holds by Prop. 1.2.2 (1). Since the latter set is closed by axiom (C3), Prop. 1.2.2 (4) implies $\mathsf{cl}\,(A \cup B) \subseteq \mathsf{cl}\, A \cup \mathsf{cl}\, B$.

The opposite inclusion follows from $\mathsf{cl}\, A \subseteq \mathsf{cl}\,(A \cup B)$ and $\mathsf{cl}\, B \subseteq \mathsf{cl}\,(A \cup B)$, which in turn follow from $A \subseteq A \cup B$ and $B \subseteq A \cup B$ by Prop. 1.2.2 (5). $\square$

As introduced above, we denote the generic topological closure by 'cl'. The Scott closure is denoted for the moment by 'cl_S', whereas the d-closure is denoted by overlining. The Alexandroff closure is already known as lower closure '$\downarrow$'.

The topological closure may also be characterized in terms of environments:

Proposition 4.2.2 For all $x \in \mathbf{X}$ and $S \subseteq \mathbf{X}$, x is in $\mathsf{cl}\, S$, iff all environments of x meet S, iff all *open* environments of x meet S.

Proof: Let the three statements be numbered by (1) through (3).

$(1) \Rightarrow (2):$ Let x be in $\mathsf{cl}\, S$ and let R be an environment of x by the open set O with $x \in O \subseteq R$. Then the complement of O is a closed set C not containing x. If S were a subset of C, then $\mathsf{cl}\, S \subseteq C$ would hold, whence x would be in C. Hence S is not a subset of C, i.e. $S \cap O \neq \emptyset$, whence more than ever $S \cap R \neq \emptyset$.

$(2) \Rightarrow (3):$ Open environments are special environments.

$(3) \Rightarrow (1):$ Let x be a point such that all open environments of x meet S. If x were not in $\mathsf{cl}\, S$, then the complement of $\mathsf{cl}\, S$ would be an open environment O of x, whence $S \cap O \neq \emptyset$ would hold, i.e. S would not be a subset of $\mathsf{cl}\, S$. □

From this proposition, we may deduce the next one:

Proposition 4.2.3
Let A be an arbitrary set. If an open set O meets $\mathsf{cl}\, A$, then it also meets A.

Proof: Let x be a point of $O \cap \mathsf{cl}\, A$. Then x is in $\mathsf{cl}\, A$, and O is an environment of x. By Prop. 4.2.2, O meets A. □

4.3 The specialization pre-order

A domain may be turned into a topological space by the Alexandroff, Scott, or d-topology. In this section, we consider the opposite direction. Given a topological space $\mathbf{X}$, we define a pre-order on the carrier of $\mathbf{X}$. Later, we establish conditions when this pre-ordered set is a poset or a domain.

Proposition 4.3.1 Let $\mathbf{X}$ be a topological space. For points x and y of $\mathbf{X}$, the following statements are equivalent:

(1) $x \in \mathsf{cl}\, y$

(2) $\mathsf{cl}\, x \subseteq \mathsf{cl}\, y$

(3) $\mathcal{O}(x) \subseteq \mathcal{O}(y)$

We denote this situation by $x \lesssim y$.

Proof: (1) and (2) are equivalent because of Prop. 1.2.2 (4).

By Prop. 4.2.2, $x \in \mathsf{cl}\, y$ is equivalent to the fact that every open environment of x meets $\{y\}$. This in turn is equivalent to the fact that every member of $\mathcal{O}(x)$ contains y, i.e. is in $\mathcal{O}(y)$.□

Proposition 4.3.2 For every space $\mathbf{X}$, the relation '$\lesssim$' is reflexive and transitive.

Proof: Since $x \lesssim y$ iff $\mathsf{cl}\, x \subseteq \mathsf{cl}\, y$. □

Applying Prop. 4.3.1, it is not difficult to characterize those spaces whose pre-order is an order.

Proposition 4.3.3 ((T0)-spaces)
For a space $\mathbf{X}$, the following three statements are equivalent:

(1) '$\lesssim$' is an order, i.e. $x \lesssim y$ and $y \lesssim x$ implies $x = y$.

(2) For all $x, y \in \mathbf{X}$, $\mathsf{cl}\, x = \mathsf{cl}\, y$ implies $x = y$.

(3) For all $x, y \in \mathbf{X}$, $\mathcal{O}(x) = \mathcal{O}(y)$ implies $x = y$.

A topological space with these properties is called *(T0)-space.*

In the sequel, we shall mostly consider (T0)-spaces. The poset belonging to a (T0)-space $\mathbf{X}$ is denoted by $\Delta\mathbf{X}$. As we always did in chapter 2, we denote the order by '$\leq$'.

If we start with a domain $\mathbf{X}$, then form the corresponding Alexandroff space $\mathrm{A}\mathbf{X}$ or Scott space $\Sigma\mathbf{X}$, and finally form the pre-order of these spaces, then we get back the original domain $\mathbf{X}$.

Proposition 4.3.4 For all domains $\mathbf{X}$, $\Delta(\mathrm{A}\mathbf{X})$ and $\Delta(\Sigma\mathbf{X})$ are isomorphic to $\mathbf{X}$.

Proof: Let '$\leq$' be the order in $\mathbf{X}$. $x \leq y$ holds iff $x \in \downarrow y$. By Prop. 3.6.3, $\downarrow y = \mathsf{cl}_S\, y$ holds. Hence, '$\leq$' is also the order of $\Delta(\mathrm{A}\mathbf{X})$ and $\Delta(\Sigma\mathbf{X})$. $\square$

The proposition does not hold for the d-space. By Prop. 3.5.3 (2), all singleton sets are d-closed, whence the order induced by the d-space of a domain degenerates to equality (cf. Prop. 4.3.5 below).

The converse of the proposition is not true. From the poset belonging to a (T0)-space, the space, i.e. the closed sets, cannot be recovered. It is possible that different (T0)-spaces induce the same poset.

The poset belonging to a (T0)-space becomes uninteresting if the order degenerates to equality. Such spaces are known as (T1)-spaces.

Proposition 4.3.5 ((T1)-spaces)
For a space $\mathbf{X}$, the following seven statements are equivalent:

(1) For all $x, y \in \mathbf{X}$, $x \lesssim y$ iff $x = y$.

(2) For all $x, y \in \mathbf{X}$, $\mathcal{O}(x) \subseteq \mathcal{O}(y)$ implies $x = y$.

(3) For all $x, y \in \mathbf{X}$, $\mathsf{cl}\, x \subseteq \mathsf{cl}\, y$ implies $x = y$.

(4) For all x in $\mathbf{X}$, $\mathsf{cl}\, x = \{x\}$ holds.

(5) Every singleton set is closed.

(6) Every finite set is closed.

(7) For all $x, y \in \mathbf{X}$ with $x \neq y$, there is an open set O such that $x \in O$ and $y \notin O$.

Topological spaces with these properties are called (T1)-spaces.

Proof:
(1) through (3) are equivalent by Prop. 4.3.1. (3) implies (4) since $a \in \mathsf{cl}\, x$ implies $\mathsf{cl}\, a \subseteq \mathsf{cl}\, x$ whence $a = x$. (4) implies (5) since all closures are closed. (5) implies (6) since the empty set is closed (C1), and finite unions of closed sets are closed (C3).

(6) $\Rightarrow$ (7): Let O be the complement of $\{y\}$. Then x is in O, but y is not. O is open as complement of the finite set $\{y\}$.

(7) $\Rightarrow$ (3): Let $\mathsf{cl}\, x$ be a subset of $\mathsf{cl}\, y$ and assume $x \neq y$. By (7), there is an open set O with $x \in O$ and $y \notin O$. Let C be the complement of O, i.e. $x \notin C$ and $y \in C$. Then $x \in \mathsf{cl}\, x \subseteq \mathsf{cl}\, y \subseteq C$ holds in contradiction to $x \notin C$. □

Statement (7) in the definition of (T1)-spaces may be strengthened. This results in the defining condition of (T2)-spaces.

Definition 4.3.6 ((T2)-space) A space $\mathbf{X}$ is a (T2)-space iff for all points x and y with $x \neq y$ there are disjoint open sets O_x and O_y with $x \in O_x$ and $y \in O_y$.

Classical topologists mainly investigate (T2)-spaces. They are usually not interested in (T1)-spaces and do not consider (T0)-spaces at all. The Scott space of a domain however satisfies (T0), but not in general (T1).

4.4 The structure of (T0)-spaces

Throughout this section, we state and prove some technical properties concerning the relations between closed and lower sets, and between open and upper sets in (T0)-spaces. The notions of lower and upper sets and the operators '$\downarrow$' and '$\uparrow$' refer to the poset induced by the (T0)-space. The results of this section in particular hold for the Scott spaces of domains.

Proposition 4.4.1 Let $\mathbf{X}$ be a (T0)-space. All closed sets of $\mathbf{X}$ are lower sets, and all open sets are upper sets.

Proof: Let C be a closed set, and let $x \in C$ and $x' \leq x$. Since C is closed, $x \in C$ implies $\mathsf{cl}\, x \subseteq C$, whence by definition of '$\leq$', $x' \in \mathsf{cl}\, x \subseteq C$ follows.

Open sets are upper sets, because closed sets are lower, and complements of lower sets are upper sets (Prop. 2.2.3). □

Some properties that we already proved for Scott spaces also hold in this more general setting. The following Proposition generalizes Prop. 3.6.3.

Proposition 4.4.2 Let $\mathbf{X}$ be a (T0)-space. For all subsets A of $\mathbf{X}$, $\mathsf{cl}\, A \supseteq \downarrow A$ holds. If F is a finite subset of $\mathbf{X}$, then $\mathsf{cl}\, F = \downarrow F$ holds.

Proof: Since A is lower and $A \subseteq \mathsf{cl}\, A$ holds, $\downarrow A \subseteq \mathsf{cl}\, A$ follows.

Because $x \in \mathsf{cl}\,\{y\}$ is equivalent to $x \leq y$, we obtain $\mathsf{cl}\,\{y\} = \downarrow\{y\}$. By means of $\mathsf{cl}\,\emptyset = \downarrow\emptyset = \emptyset$, $\downarrow(A \cup B) = \downarrow A \cup \downarrow B$, and $\mathsf{cl}\,(A \cup B) = \mathsf{cl}\, A \cup \mathsf{cl}\, B$, we may deduce $\mathsf{cl}\, F = \downarrow F$ for all finite sets F. □

Lower and upper sets may be obtained as unions of closed sets and intersections of open sets respectively.

Lemma 4.4.3

(1) Every lower set A is the union of all its closed subsets:

$$A = \bigcup\{C \mid C \text{ closed}, C \subseteq A\}$$

(2) Every upper set B is the intersection of all its open supersets

$$B = \bigcap\{O \mid O \text{ open}, O \supseteq B\} = \bigcap \mathcal{O}(B)$$

Proof:

(1) '$\supseteq$' is trivial. Let $a \in A$. Then $a \in \downarrow a \subseteq A$, since A is a lower set. $\downarrow a$ is a closed subset of A because of Prop. 4.4.2.

(2) follows from (1) by complementing all sets. □

Consequently, upper closures are obtained as the intersection of all open environments, and may be compared by comparing the environments:

Lemma 4.4.4

(1) For all sets A, $\uparrow A = \bigcap \mathcal{O}(A)$ holds.

(2) For two sets A and B, $\uparrow A \supseteq \uparrow B$ is equivalent to $\mathcal{O}(A) \subseteq \mathcal{O}(B)$.

Proof:

(1) Since open sets are upper, $\mathcal{O}(A) = \mathcal{O}(\uparrow A)$ holds. By Prop. 4.4.3 (2), $\bigcap \mathcal{O}(\uparrow A)$ is $\uparrow A$.

(2) If $\uparrow A \supseteq \uparrow B$, then $A \subseteq O$ where O is open implies $B \subseteq \uparrow A \subseteq O$.

Conversely assume $\mathcal{O}(A) \subseteq \mathcal{O}(B)$. If x is in $\uparrow B = \bigcap \mathcal{O}(B)$, then it is in all members of $\mathcal{O}(B)$. Thus, it is more than ever in all members of $\mathcal{O}(A)$, whence it is in $\uparrow A = \bigcap \mathcal{O}(A)$. □

Part (2) of the Lemma above will be heavily used to prove inclusions of upper sets. If A and B are upper sets, one may prove $A \subseteq B$ by showing that $B \subseteq O$ for some open set O implies $A \subseteq O$.

4.5 Continuity

In section 3.3, we defined the notion of order continuity for functions between domains. By Prop. 3.6.4, a function is order continuous iff the inverse images of all Scott closed sets are Scott closed. Generalizing from Scott spaces to arbitrary spaces, we define a function between topological spaces to be continuous iff it backwards maps closed sets to closed sets. By the above-mentioned Prop. 3.6.4, order continuity and Scott continuity coincide. Furthermore, monotonicity coincides with Alexandroff continuity by Prop. 2.5.5, and all order continuous functions are d-continuous by Prop. 3.5.6.

The definition of continuity comes in several equivalent shapes:

Theorem 4.5.1 (Continuity) For a function $f : \mathbf{X} \to \mathbf{Y}$ between two topological spaces, the following statements are equivalent:

(1) The inverse image of every closed set of $\mathbf{Y}$ is closed in $\mathbf{X}$.

(2) The inverse image of every open set of $\mathbf{Y}$ is open in $\mathbf{X}$.

(3) For every x in $\mathbf{X}$ and every environment R of fx, there is an environment R' of x such that $f[R'] \subseteq R$.

(4) For all subsets A of $\mathbf{X}$, $f[\mathsf{cl}\, A] \subseteq \mathsf{cl}\, f[A]$ holds.

A function satisfying these properties is called *continuous*.

Statement (3) corresponds to the definition of continuity known from real calculus.

Proof:

(1) $\Rightarrow$ (2) : If O is open in $\mathbf{Y}$, then $\mathsf{co}\, O$ is closed, whence $\mathsf{co}\,(f^{-1}[O]) = f^{-1}[\mathsf{co}\, O]$ is closed in $\mathbf{X}$ by (1), whence $f^{-1}[O]$ is open.

(2) $\Rightarrow$ (3) : Let x in $\mathbf{X}$, and let R be an environment of fx by the open set O with $fx \in O \subseteq R$. Then $fx \in O$ holds, whence $x \in f^{-1}[O]$. The latter set is open by (2). Hence, it provides an (open) environment $R' = f^{-1}[O]$ of x. $O \subseteq R$ implies $f[R'] \subseteq R$.

(3) $\Rightarrow$ (4) : Let x be a point of $\mathsf{cl}\, A$. We have to show $fx \in \mathsf{cl}\, f[A]$. Applying Prop. 4.2.2, let R be an environment of fx. By (3), there is an environment R' of x such that $f[R'] \subseteq R$. Since x is in $\mathsf{cl}\, A$, R' meets A, i.e. there is a point p in $R' \cap A$. Then $fp \in f[R'] \cap f[A] \subseteq R \cap f[A]$ follows, i.e. R meets $f[A]$.

(4) $\Rightarrow$ (1) : Let B be closed in $\mathbf{Y}$, and let $A = f^{-1}[B]$. To show that A is closed, we prove $\mathsf{cl}\, A \subseteq A$. Applying (4), $x \in \mathsf{cl}\, A$ implies $fx \in f[\mathsf{cl}\, A] \subseteq \mathsf{cl}\, f[A] \subseteq \mathsf{cl}\, B = B$, whence $x \in f^{-1}[B] = A$. □

Obviously, the identity function and the composition of continuous functions is continuous. The inverse of a bijective continuous function however need not be continuous. Let $\mathbf{X}$ be the topological space with carrier set $\{1, 2\}$ where all sets are closed, and let $\mathbf{Y}$ be the space with the same carrier set where only $\emptyset$ and $\{1, 2\}$ are closed. The function mapping 1 to 1 and 2 to 2 is obviously bijective and continuous, whereas its inverse is not because $\{1\}$ is not closed in $\mathbf{Y}$.

With the definition of topological continuity, the Propositions 2.5.5 and 3.6.4 may be reformulated:

Proposition 4.5.2

(1) A function between two posets is monotonic iff it is Alexandroff continuous.

(2) A function between two domains is order continuous, iff it is Scott continuous, iff it is monotonic and d-continuous.

By this proposition and part (4) of Prop. 4.5.1, we obtain

Proposition 4.5.3 If f is an order continuous function between two domains $\mathbf{X}$ and $\mathbf{Y}$, then for all $A \subseteq \mathbf{X}$, the inclusions $f[\overline{A}] \subseteq \overline{f[A]}$ and $f[\mathsf{cl}_S\, A] \subseteq \mathsf{cl}_S\, f[A]$ hold.

Remember that overlining denotes the d-closure, and 'cl_S' is the Scott closure.

Not only the Scott continuous functions are monotonic, but all continuous functions.

Proposition 4.5.4 Let $\mathbf{X}$ and $\mathbf{Y}$ be (T0)-spaces. Every continuous $f : \mathbf{X} \to \mathbf{Y}$ is monotonic in the induced orders.

Proof: By Prop. 2.5.5, a function is monotonic iff the inverse image of every lower cone in $\mathbf{Y}$ is a lower set in $\mathbf{X}$. Now, lower cones in $\mathbf{Y}$ are closed in $\mathbf{Y}$ since $\downarrow\{x\} = \mathsf{cl}\,\{x\}$, whence their inverse image is closed in $\mathbf{X}$ by continuity. Closed sets are lower sets by Prop. 4.4.1. □

4.6 Compact sets

Besides open and closed sets, there is another important class of sets in a topological space: the compact sets. They provide a generalization of finite sets in some sense.

There are several equivalent definitions of compact sets where (1) is the standard definition:

Proposition 4.6.1 (Compact sets)

For a subset K of a topological space $\mathbf{X}$, the following statements are equivalent:

(1) For every set $\mathcal{O}$ of open sets with $K \subseteq \bigcup \mathcal{O}$, there is a finite subcollection $\mathcal{F}$ of $\mathcal{O}$ with $K \subseteq \bigcup \mathcal{F}$.

(2) For every $\subseteq$-directed set $\mathcal{O}$ of open sets with $K \subseteq \bigcup \mathcal{O}$, there is a member O of $\mathcal{O}$ with $K \subseteq O$.

(3) If $\mathcal{C}$ is a $\supseteq$-directed set of closed sets that all meet K, then the intersection $\bigcap \mathcal{C}$ meets K.

In this case, the set K is called *compact*.

Proof:

$(1) \Rightarrow (2)$: Let $\mathcal{O}$ be a $\subseteq$-directed set of open sets with $K \subseteq \bigcup \mathcal{O}$. By (1), there is a finite subset $\mathcal{F}$ of $\mathcal{O}$ such that $K \subseteq \bigcup \mathcal{F}$. Since $\mathcal{O}$ is directed, there is an upper bound O of $\mathcal{F}$ in $\mathcal{O}$. $K \subseteq \bigcup \mathcal{F}$ implies $K \subseteq O$.

$(2) \Rightarrow (1)$: For given $\mathcal{O}$, let $\mathcal{O}' = \{\bigcup \mathcal{F} \mid \mathcal{F} \subseteq_f \mathcal{O}\}$. Then $\bigcup \mathcal{O} = \bigcup \mathcal{O}'$ holds, and $\mathcal{O}'$ is $\subseteq$-directed. By (2), $K \subseteq \bigcup \mathcal{O} = \bigcup \mathcal{O}'$ implies there is $O \in \mathcal{O}'$ such that $K \subseteq O$. Then $K \subseteq \bigcup \mathcal{F}$ for some finite subset $\mathcal{F}$ of $\mathcal{O}$ as required.

$(2) \Rightarrow (3)$: For given $\supseteq$-directed $\mathcal{C}$, let $\mathcal{O} = \{\mathsf{co}\, C \mid C \in \mathcal{C}\}$. $\mathcal{O}$ is a $\subseteq$-directed set of open sets. If $K \cap \bigcap \mathcal{C}$ were empty, then $K \subseteq \mathsf{co} \bigcap \mathcal{C} = \bigcup \mathcal{O}$ would hold, whence by (2), there would be a member O of $\mathcal{O}$ such that $K \subseteq O$. Thus, there would be a member C of $\mathcal{C}$, namely $\mathsf{co}\, O$, such that $K \cap C = \emptyset$ in contradiction to the precondition.

$(3) \Rightarrow (2)$: follows the same lines. □

Topologists often define compactness only for the case $K = \mathbf{X}$. Classically, they then require the additional property (T2), and speak of quasi-compactness if (T2) is not satisfied.

The properties of compact sets are analogous to those of finitary sets as expressed in Prop. 2.7.2. Indeed, the finitary sets of a poset are just the compact sets of its Alexandroff space. We now present the properties of compact sets as a sequence of propositions.

Proposition 4.6.2 A set A is compact iff $\uparrow A$ is compact.

Proof: A is a subset of a union of open sets iff $\uparrow A$ is. This holds because open sets are upper sets. □

Proposition 4.6.3 The union of two compact sets is compact.

Proof: Let K_1 and K_2 be the compact sets, and $K = K_1 \cup K_2$. If $K \subseteq \bigcup \mathcal{O}$ holds, then $K_i \subseteq \bigcup \mathcal{O}$ follows for all $i = 1, 2$. Hence, there are finite collections $\mathcal{F}_i \subseteq \mathcal{O}$ with $K_i \subseteq \bigcup \mathcal{F}_i$. Then, $\mathcal{F} = \mathcal{F}_1 \cup \mathcal{F}_2$ is a finite subset of $\mathcal{O}$ with $K \subseteq \bigcup \mathcal{F}$. □

There is also some relationship to closed sets:

Proposition 4.6.4 If C is closed and K is compact, then $C \cap K$ is compact.

Proof: Let O be the complement of C. If $C \cap K \subseteq \bigcup \mathcal{O}$ for some collection $\mathcal{O}$ of open sets, then $K = (C \cap K) \cup (O \cap K) \subseteq (C \cap K) \cup O \subseteq \bigcup(\mathcal{O} \cup \{O\})$. Since K is compact, there is a finite collection $\mathcal{F} \subseteq \mathcal{O} \cup \{O\}$ with $K \subseteq \bigcup \mathcal{F}$. Let $\mathcal{F}' = \mathcal{F} \cap \mathcal{O}$. Then $K \subseteq O \cup \bigcup \mathcal{F}'$ holds. This implies $C \cap K \subseteq (C \cap O) \cup (C \cap \bigcup \mathcal{F}') \subseteq \bigcup \mathcal{F}'$. □

Proposition 4.6.5
If $f : \mathbf{X} \to \mathbf{Y}$ is continuous and $K \subseteq \mathbf{X}$ is compact, then $f[K] \subseteq \mathbf{Y}$ is compact.

Proof: Let $\mathcal{O}$ be a collection of open sets of $\mathbf{Y}$ with $f[K] \subseteq \bigcup \mathcal{O}$. Then $K \subseteq \bigcup\{f^{-1}[O] \mid O \in \mathcal{O}\}$ follows. By continuity of f, this is a collection of open sets whence there is a finite subset $\mathcal{F}$ of $\mathcal{O}$ with $K \subseteq \bigcup\{f^{-1}[O] \mid O \in \mathcal{F}\}$. From this, one finally obtains $f[K] \subseteq \bigcup \mathcal{F}$. □

In a general (T0)-space, compact sets need not be closed. However, they possess a closure property w.r.t. *descending* directed sets.

Proposition 4.6.6 Let K be a compact set in a (T0)-space $\mathbf{X}$, and let D be a $\geq$-directed set in K. Then D has a lower bound in K.

Proof: Let $\mathcal{C} = \{\downarrow d \mid d \in D\}$. By Prop. 4.4.2, the lower cones $\downarrow d = \mathsf{cl}\, d$ are closed. Since $d' \leq d$ implies $\downarrow d' \subseteq \downarrow d$, the set $\mathcal{C}$ is $\supseteq$-directed. Because of $D \subseteq K$, all members of $\mathcal{C}$ meet K. By Prop. 4.6.1 (3), the intersection $\bigcap \mathcal{C}$ meets K. Let x be a point of $\bigcap \mathcal{C} \cap K$. Then for all d in D, x is in $\downarrow d$, i.e. $x \leq d$ holds. Thus, x is a lower bound of D in K. □

Applying Zorn's Lemma (for the dual order), one obtains that there is a minimal element of K below every point of K. Thus,

Proposition 4.6.7 Let K be a compact set in a (T0)-space $\mathbf{X}$. Then $K \subseteq \uparrow(min\, K)$ holds. If K is a compact upper set, then $K = \uparrow(min\, K)$ follows.

Combining these results with Lemma 3.8.1, one obtains

Proposition 4.6.8
Let $\mathbf{X}$ be a *domain*. Then every compact open set in the Scott space of $\mathbf{X}$ is finitary.

Proof: Let K be the compact open set. By Prop. 4.6.7, $K = \uparrow(min\, K)$ holds. We have to show that $min\, K$ is finite.

For the following, let $K' = K \setminus min\, K$. For every element m of $min\, K$, the set $M = min\, K \setminus \{m\}$ is a subset of $min\, K$, whence $O_m = K \setminus M = K' \cup \{m\}$ is open by Lemma 3.8.1. The sets O_m cover K because of $K = K' \cup min\, K = \bigcup_{m \in min\, K}(K' \cup \{m\}) = \bigcup_{m \in min\, K} O_m$. Since K is compact, there is a finite subset E of $min\, K$ such that $K' \cup min\, K = K = \bigcup_{e \in E} O_e = K' \cup E$. $min\, K = E$ follows. □

The proposition above was shown for Scott spaces only because Lemma 3.8.1 was proved by means of directed sets.

4.7 Strongly compact sets

In [Smy78], Smyth used finitely generable sets to define the first upper power construction. A set S is *finitely generable* iff it is obtained as set of all labels of the infinite border of a finitarily branching tree with infinite branches only. The nodes of this tree are labeled by domain elements such that the labels of parents are smaller than the labels of their children. Intuitively, S is approximated by the sequence of horizontal tree cuts. All these cuts are finite. This suggests the following definition of the more abstract notion of strong compactness.

Definition 4.7.1 A set S in a topological space $\mathbf{X}$ is *strongly compact* iff for all open sets O with $S \subseteq O$ there is a finitary set F with $S \subseteq F \subseteq O$.

Without restriction, one may assume the finitary set F to be an upper set, i.e. to be obtained as upper closure of a finite set.

The name 'strong compactness' was chosen since the properties of this notion are analogous to those of compactness.

Proposition 4.7.2
Every finite set is finitary, every finitary set is strongly compact, and every strongly compact set is compact.

Proof: The first two claims are trivial.

Let S be strongly compact, and $S \subseteq \bigcup \mathcal{O}$ where $\mathcal{O}$ is a set of open sets. Because the union is open, too, there is a finite set E such that $S \subseteq \uparrow E \subseteq \bigcup \mathcal{O}$. Every e in E is in some open set $O_e \in \mathcal{O}$. Hence, $S \subseteq \bigcup_{e \in E} \uparrow e \subseteq \bigcup_{e \in E} O_e$. $\square$

Remark: In (T1)-spaces where the pre-order degenerates to equality, a set is strongly compact iff it is finite. Thus, there are compact sets on the real line that are not strongly compact. The topology of the real line however is not the Scott topology of some domain.

Problem 1 Are there domains where not every Scott compact set is strongly compact?

Next, we present the properties of strongly compact sets. They are analogous to those of compact sets.

Proposition 4.7.3 A is strongly compact iff $\uparrow A$ is so.

Proof: Since open sets are upper sets, and $A \subseteq \uparrow E$ iff $\uparrow A \subseteq \uparrow E$. $\square$

Proposition 4.7.4 If A and B are strongly compact, then so is $A \cup B$.

Proof: $A \cup B \subseteq O$ open implies $A \subseteq \uparrow E \subseteq O$ and $B \subseteq \uparrow F \subseteq O$ for some finite sets E and F, whence $A \cup B \subseteq \uparrow(E \cup F) \subseteq O$. $\square$

Proposition 4.7.5
If C is closed and K is strongly compact, then $C \cap K$ is strongly compact.

Proof: $C \cap K \subseteq O$ where O is open implies $K \subseteq \mathsf{co}\, C \cup O$. By strong compactness, there is a finitary set F such that $K \subseteq F \subseteq \mathsf{co}\, C \cup O$. Intersecting by C yields $C \cap K \subseteq C \cap F \subseteq C \cap O \subseteq O$. The set $C \cap F$ is finitary by Prop. 2.7.2 since C is lower. $\square$

Proposition 4.7.6 If A is strongly compact in $\mathbf{X}$ and $f : [\mathbf{X} \to \mathbf{Y}]$ is continuous, then $f[A]$ is strongly compact in $\mathbf{Y}$.

Proof: $f[A] \subseteq O$ implies $A \subseteq f^{-1}[O]$ whence $A \subseteq F \subseteq f^{-1}[O]$ for some finitary F. This implies $f[A] \subseteq f[F] \subseteq O$. The set $f[F]$ is finitary as a monotonic image of the finitary set F by Prop. 2.7.2. □

Prop. 4.6.8 claimed that compact open sets in *Scott spaces* are finitary. With strongly compact sets, we can do better:

Proposition 4.7.7 Every strongly compact open set in a (T0)-space $\mathbf{X}$ is finitary.

Proof: If S is strongly compact and open, then $S \subseteq S$ implies there is a finitary set F such that $S \subseteq F \subseteq S$. □

By Prop. 4.7.2, every finitary set is compact. In Scott spaces, moreover directed intersections of finitary upper sets result in strongly compact sets.

Proposition 4.7.8 Let $\mathbf{X}$ be a domain, and $\mathcal{F}$ a $\supseteq$-directed set of finitary upper sets. Then $\bigcap \mathcal{F}$ is strongly compact in the Scott topology.

Proof: Let O be an open set such that $\bigcap \mathcal{F} \subseteq O$. By Lemma 3.9.4, there is some F in $\mathcal{F}$ such that $F \subseteq O$. $\bigcap \mathcal{F} \subseteq F$ holds anyway. □

Note that there might also be strongly compact sets which are not obtained as $\supseteq$-directed intersections of finite upper sets.

4.8 The product domain in topological view

In this section, we assume two given domains $\mathbf{X}$ and $\mathbf{Y}$, and investigate the topological properties of their product domain $\mathbf{X} \times \mathbf{Y}$. In doing so, we are mainly concerned with the Scott topologies of $\mathbf{X}$, $\mathbf{Y}$, and $\mathbf{X} \times \mathbf{Y}$. Thus, we simply write 'cl' for the Scott closure instead of 'cl_S', say Scott closed instead of closed etc. As usual, the d-closure is denoted by overlining.

First, we collect some facts we already know:

- The projections $\pi_1 : \mathbf{X} \times \mathbf{Y} \to \mathbf{X}$ and $\pi_2 : \mathbf{X} \times \mathbf{Y} \to \mathbf{Y}$ are continuous.
- A function $f : \mathbf{X} \times \mathbf{Y} \to \mathbf{Z}$ is continuous iff for all x in $\mathbf{X}$ and y in $\mathbf{Y}$, the functions $\lambda x.\, f(x, y) : \mathbf{X} \to \mathbf{Z}$ and $\lambda y.\, f(x, y) : \mathbf{Y} \to \mathbf{Z}$ are continuous.
- For every y in $\mathbf{Y}$, the function $\sigma^y = \lambda x.\,(x, y) : \mathbf{X} \to \mathbf{X} \times \mathbf{Y}$ is continuous. For every x in $\mathbf{X}$, the function $\sigma_x = \lambda y.\,(x, y) : \mathbf{Y} \to \mathbf{X} \times \mathbf{Y}$ is continuous.

The first fact is Prop. 3.3.5, and the second one is Prop. 3.3.6. The third fact is derived from the second one by assuming $f = id : \mathbf{X} \times \mathbf{Y} \to \mathbf{X} \times \mathbf{Y}$.

Next, we consider the effect of Cartesian product to various kinds of sets.

Proposition 4.8.1 If A in $\mathbf{X}$ and B in $\mathbf{Y}$ are both lower / upper / finitary / d-closed / closed / open, then their product $A \times B$ has the same property in $\mathbf{X} \times \mathbf{Y}$.

Proof: If A and B are lower sets, then $\downarrow(A \times B) = \downarrow A \times \downarrow B = A \times B$ holds by Prop. 2.2.10, whence $A \times B$ is lower. The proof for upper sets is analogous.

If A and B are finitary, then there are finite subsets E of A and F of B such that $A \subseteq \uparrow E$ and $B \subseteq \uparrow F$. Then $E \times F$ is a finite subset of $A \times B$, and $A \times B \subseteq \uparrow E \times \uparrow F = \uparrow(E \times F)$ holds by Prop. 2.2.10.

Next, we consider closed sets. Since π_1 is continuous, the inverse image $\pi_1^{-1}[A] = A \times \mathbf{Y}$ is closed as A is. Similarly, $\mathbf{X} \times B$ is closed. Thus, their intersection $A \times \mathbf{Y} \cap \mathbf{X} \times B = A \times B$ is also closed.

The proof for open and d-closed sets is analogous; continuous functions are also d-continuous.□

Despite the topological Theorem of Tychonoff, we cannot prove the compactness of a product of compact sets since the Scott space of the product domain in general is not the topological product of the Scott spaces of the factors (see also Def. 4.8.6 below). In contrast to compactness, we are able to show the strong compactness of a product of two strongly compact sets (Prop. 4.9.1). This is the first in a long sequence of results we could show for strongly compact sets, but not for compact sets.

First however we consider how a property of the product of two sets carries over to the factors.

Proposition 4.8.2 Let $\mathbf{X}$ and $\mathbf{Y}$ be two domains, and let $A \subseteq \mathbf{X}$ and $B \subseteq \mathbf{Y}$ be two *non-empty* sets. If $A \times B$ is lower / upper / finitary / d-closed / closed / open / compact / strongly compact in $\mathbf{X} \times \mathbf{Y}$, then both factors A and B have the corresponding property.

Proof: Let b be a member of B. Then $a \in A$ holds iff $\sigma^b a = (a, b)$ is in $A \times B$, whence $A = {\sigma^b}^{-1}[A \times B]$. Since σ^b is continuous, this proves the claim for lower, upper, d-closed, closed, and open sets.

For all a in A, $a = \pi_1(a, b)$ holds. Hence, $A = \pi_1[A \times B]$ is a continuous image of $A \times B$. This proves the claim for finitary, strongly compact, and compact sets. □

The next propositions concerns Scott and d-closure in the product.

Proposition 4.8.3 Let $\mathbf{X}$ and $\mathbf{Y}$ be two domains, and $A \subseteq \mathbf{X}$ and $B \subseteq \mathbf{Y}$. Then $\overline{A \times B} = \overline{A} \times \overline{B}$ and $\mathsf{cl}\,(A \times B) = \mathsf{cl}\,A \times \mathsf{cl}\,B$ hold.

Proof: We show the statement for the Scott closure. The other proof is fully analogous.

$\mathsf{cl}\,A$ and $\mathsf{cl}\,B$ are closed, whence their product is closed by Prop. 4.8.1. Thus, $A \times B \subseteq \mathsf{cl}\,A \times \mathsf{cl}\,B$ implies $\mathsf{cl}\,(A \times B) \subseteq \mathsf{cl}\,A \times \mathsf{cl}\,B$.

For the opposite direction, we use Prop. 4.5.1: for continuous f, $f[\mathsf{cl}\,S] \subseteq \mathsf{cl}\,f[S]$ holds for all S. Using $\sigma_u = \lambda y.\,(u, y)$, one obtains for arbitrary sets $U \subseteq \mathbf{X}$ and $V \subseteq \mathbf{Y}$ the inclusion $U \times \mathsf{cl}\,V = \bigcup_{u \in U} \sigma_u[\mathsf{cl}\,V] \subseteq \bigcup_{u \in U} \mathsf{cl}\,(\sigma_u[V]) \subseteq \bigcup_{u \in U} \mathsf{cl}\,(U \times V) = \mathsf{cl}\,(U \times V)$. Analogously, one may show $\mathsf{cl}\,U \times V \subseteq \mathsf{cl}\,(U \times V)$. Combining both inclusions, one finally obtains $\mathsf{cl}\,A \times \mathsf{cl}\,B \subseteq \mathsf{cl}\,(A \times \mathsf{cl}\,B) \subseteq \mathsf{cl}\,(\mathsf{cl}\,(A \times B)) = \mathsf{cl}\,(A \times B)$. □

The Proposition above allows to prove two statements about closure properties of sets w.r.t. continuous operations.

Proposition 4.8.4 Let $\mathbf{X}$ be a domain and A a subset of $\mathbf{X}$.

(1) If A is closed w.r.t. a continuous unary operation $f : [\mathbf{X} \to \mathbf{X}]$, i.e. $f[A] \subseteq A$, then $\overline{A}$ and $\mathsf{cl}\, A$ are also closed w.r.t. f.

(2) If A is closed w.r.t. a continuous binary operation $g : [\mathbf{X} \times \mathbf{X} \to \mathbf{X}]$, i.e. $g[A \times A] \subseteq A$, then $\overline{A}$ and $\mathsf{cl}\, A$ are also closed w.r.t. g.

Proof: We show the statements for the Scott closure operator 'cl'. The statements for the d-closure operator '$^{-}$' are shown analogously; remember that all Scott continuous functions are d-continuous.

(1) By Th. 4.5.1 (4), $f[\mathsf{cl}\, A] \subseteq \mathsf{cl}\, f[A] \subseteq \mathsf{cl}\, A$ holds.

(2) By the same statement and Prop. 4.8.3, $g[\mathsf{cl}\, A \times \mathsf{cl}\, A] = g[\mathsf{cl}\,(A \times A)] \subseteq \mathsf{cl}\, g[A \times A] \subseteq \mathsf{cl}\, A$ holds. □

Next, we investigate the general structure of open sets in the product.

Lemma 4.8.5 Let $\mathbf{X}$ and $\mathbf{Y}$ be two domains. Let O be an open set of $\mathbf{X} \times \mathbf{Y}$ and (x, y) a point of O. Then there are open sets O_1 in $\mathbf{X}$ and O_2 in $\mathbf{Y}$ such that $(x, y) \in O_1 \times \{y\} \subseteq O$ and $(x, y) \in \{x\} \times O_2 \subseteq O$.

Proof: For $\sigma^y = \lambda u.\,(u, y) : [\mathbf{X} \to \mathbf{X} \times \mathbf{Y}]$, let $O_1 = \sigma^{y-1}[O]$. By continuity, this is an open set in $\mathbf{X}$. By definition of O_1 and $(x, y) \in O$, $(x, y) \in O_1 \times \{y\} \subseteq O$ follows as required. □

Instead of the statement of this Lemma, one often would like to be able to conclude $(x, y) \in O_1 \times O_2 \subseteq O$. This would characterize the topological product of spaces as topologists define it. An example in the introductory chapter of [Bar80] however shows that such a strong statement does not generally hold. Therefore, we define

Definition 4.8.6 A pair of domains $\mathbf{X}$ and $\mathbf{Y}$ is $\times$-*conform* iff for every open set O of $\mathbf{X} \times \mathbf{Y}$ and every point (x, y) of O, there are open sets O_1 of $\mathbf{X}$ and O_2 of $\mathbf{Y}$ such that $(x, y) \in O_1 \times O_2 \subseteq O$.

If a pair of domains is $\times$-conform, then the topological Theorem of Tychonoff applies and one may conclude

Proposition 4.8.7 If $\mathbf{X}$ and $\mathbf{Y}$ form a $\times$-conform pair of domains, then the product of a compact set of $\mathbf{X}$ and a compact set of $\mathbf{Y}$ is compact in $\mathbf{X} \times \mathbf{Y}$.

We do not include the proof here.

Problem 2 Is the product of compact sets compact again even if the ground domains are not $\times$-conform?

4.9 The product of strongly compact sets

In contrast to compact sets, we are able to show that strong compactness is preserved by binary products.

Theorem 4.9.1 The product of strongly compact sets of $\mathbf{X}$ and $\mathbf{Y}$ is strongly compact in $\mathbf{X} \times \mathbf{Y}$ (no matter whether $\mathbf{X}$ and $\mathbf{Y}$ are $\times$-conform or not).

The theorem is proved by means of some auxiliary lemmas. The first Lemma generalizes Lemma 4.8.5 from single points y to finitary sets.

Lemma 4.9.2 Let x be a point of $\mathbf{X}$, F a finitary set of $\mathbf{Y}$, and O an open set of $\mathbf{X} \times \mathbf{Y}$ with $\{x\} \times F \subseteq O$. Then there is an open set O' of $\mathbf{X}$ such that $x \in O'$ and $O' \times F \subseteq O$.

Proof: As F is finitary, there is a finite subset E of F such that $F \subseteq \uparrow E$. For all e in E, $(x, e) \in O$ holds. By Lemma 4.8.5, there is an open set O_e for every e in E such that $(x, e) \in O_e \times \{e\} \subseteq O$. Let $O' = \bigcap_{e \in E} O_e$ (and $O' = \mathbf{X}$ if E is empty). O' is open as *finite* intersection of open sets. x is in O' since it is in all sets O_e.

Let (u, v) be a point of $O' \times F$. $v \in F \subseteq \uparrow E$ implies there is e in E such that $v \geq e$. $u \in O' \subseteq O_e$ implies $(u, e) \in O$, whence also $(u, v) \in O$ because $(u, v) \geq (u, e)$ and O is upper. □

In the next step, the finitary set is generalized to a strongly compact one.

Lemma 4.9.3 Let x be a point of $\mathbf{X}$, S a strongly compact set of $\mathbf{Y}$, and O an open set of $\mathbf{X} \times \mathbf{Y}$ with $\{x\} \times S \subseteq O$. Then there are an open set O' of $\mathbf{X}$ and a finitary set F of $\mathbf{Y}$ such that $\{x\} \times S \subseteq O' \times F \subseteq O$.

Proof: Using the continuous function $\sigma_x = \lambda y.\,(x, y)$, the inclusion $\{x\} \times S \subseteq O$ implies $S \subseteq \sigma_x^{-1}[O]$ where the latter set is open. By strong compactness, there is a finitary set F such that $S \subseteq F \subseteq \sigma_x^{-1}[O]$. Applying σ_x, one obtains $\{x\} \times S \subseteq \{x\} \times F \subseteq O$. Lemma 4.9.2 yields an open set O' of $\mathbf{X}$ such that $x \in O'$ and $O' \times F \subseteq O$ whence the claim follows. □

Now we are able to prove the theorem itself.

Proof: Let A and B be the two strongly compact sets, and assume $A \times B \subseteq O$ where O is open. Then for every a in A, $\{a\} \times B \subseteq O$ holds, whence by Lemma 4.9.3 there are open sets O_a of $\mathbf{X}$ and finitary sets F_a of $\mathbf{Y}$ such that $\{a\} \times B \subseteq O_a \times F_a \subseteq O$.

Thus, $A \subseteq \bigcup_{a \in A} O_a$ holds whence by strong compactness, there is a finitary set G such that $A \subseteq G \subseteq \bigcup_{a \in A} O_a$. Let E be a finite subset of G with $G \subseteq \uparrow E$. Because of $E \subseteq G \subseteq \bigcup_{a \in A} O_a$, for every e in E there is $a(e)$ in A such that $e \in O_{a(e)}$. Let $H = \bigcup_{e \in E}(\uparrow e \times F_{a(e)})$. H is finitary since it is a finite union of a product of finitary sets. We claim $A \times B \subseteq H \subseteq O$.

Let (u, v) be a point of $A \times B$. Because of $A \subseteq G \subseteq \uparrow E$, there is e in E below u, i.e. $u \in \uparrow e$. In addition, $\{a(e)\} \times B \subseteq O_{a(e)} \times F_{a(e)}$ holds, whence $v \in B \subseteq F_{a(e)}$. Thus, we obtain $(u, v) \in \uparrow e \times F_{a(e)}$ for some e in E.

If p is a point of H, then p is above a point (e, v) where e in E and $v \in F_{a(e)}$. By $e \in O_{a(e)}$, we get $(e, v) \in O_{a(e)} \times F_{a(e)} \subseteq O$. Since O is upper, p is in O, too. □

Chapter 5

Completeness properties

In the next four chapters, we present various classifications of domains. Domains belonging to small classes usually have useful additional properties that are not satisfied by arbitrary domains. If a certain domain class is given, we are also interested in whether it is closed w.r.t. finite or even infinite products, and w.r.t. function domain formation. A class that is closed w.r.t. finite products and function domains is called *Cartesian closed* because it forms a Cartesian closed full sub-category of the Cartesian closed category of domains and continuous functions. If C and D are two classes of domains, we denote their intersection by C & D.

Throughout these chapters, we shall identify a domain with its Scott space. Thus, we shall apply topological notions directly to a domain. If we refer to closed, open, or compact sets of a domain, we always mean Scott closed, Scott open, or Scott compact sets. Consequently, the Scott closure 'cl_S' is abbreviated to 'cl'.

After having considered some trivial classes in section 5.1, we continue by introducing some classes characterized by completeness properties. Let P be a possible property of sets, e.g. finitary or compact. Then a domain $\mathbf{X}$ is P-complete if the whole domain $\mathbf{X}$ has property P, and for every two points x and y of $\mathbf{X}$, the set of common upper bounds $\uparrow x \cap \uparrow y$ has property P. Some of these completeness properties have already established names which we also use. In particular, we consider the following classifications:

- (Cone) Complete (CC): P is the property to be an upper cone (section 5.3).
- Bounded complete (BC): P means either being empty or an upper cone (section 5.2).
- Finitarily complete (FC) = property M: P means finitary (section 5.4).
- Strongly compactly complete (SC): P means strongly compact (section 5.5).
- Compactly complete (KC): P means compact (section 5.5).

By their definition, these domain classes form an increasing hierarchy. The first three classifications apply to posets since no directed lubs or topological notions are involved in their definition. The latter two classifications will be applied to domains only. In the respective sections of this chapter, we present equivalent definitions of these classes and various examples.

The defining property of class FC is also known as property M [Jun88]. By Th. 6.4.4, we shall see that algebraic domains are in SC, iff they are in KC, iff their base has property M.

5.1 Some trivial classes

The class of all domains is denoted by DOM. As we have already seen, it allows to form arbitrary products and function domains.

Finite posets are characterized by the finiteness of their carrier set. All finite posets are domains. Finite products of finite domains are finite, and the function domain of two finite domains is finite. Hence, the class FIN of finite domains is Cartesian closed.

Discrete posets are characterized by the degeneration of their order: $a \leq b$ holds iff a equals b. They consist of a set of uncomparable points. All discrete posets are domains. Arbitrary products of discrete domains are discrete. If $\mathbf{X}$ is an arbitrary domain and $\mathbf{Y}$ is discrete, then $[\mathbf{X} \to \mathbf{Y}]$ is discrete. Thus, the class DIS of discrete domains is Cartesian closed. By Prop. 4.3.5, there is an equivalent topological characterization of discreteness: a domain is discrete iff its Scott space is (T1).

The class of domains with least element is denoted by $\mathrm{DOM}_\perp$. Similarly, we denote the class of all domains belonging to some class C and having a least element by $\mathrm{C}_\perp$. The class $\mathrm{DOM}_\perp$ is preserved by arbitrary products and function domains. If $(\mathbf{X}_i)_{i\in I}$ is a family of domains with least elements $\perp_i$, then $\prod_{i\in I}\mathbf{X}_i$ has the least element $(\perp_i)_{i\in I}$. If $\mathbf{X}$ is an arbitrary domain and $\mathbf{Y}$ is a domain with least element $\perp$, then $[\mathbf{X} \to \mathbf{Y}]$ has a least element, namely $\lambda x.\perp$. Domains $\mathbf{D}$ with least element satisfy the famous Kleene fixed point theorem: Every continuous function $f : [\mathbf{D} \to \mathbf{D}]$ has a unique least fixed point.

5.2 Bounded completeness

We start the presentation of the completeness classes by BC instead of the smallest class CC since some facts about CC may be proved using facts about BC.

The class BC of bounded complete domains was investigated early. In connection with algebraicity, bounded completeness delivers a Cartesian closed class of domains with particularly nice properties. Bounded complete algebraic domains may also be characterized alternatively in terms of information systems.

There is some confusion around whether bounded complete domains should contain a least element or not. (Should *every* bounded subset have a supremum, or should every *non-empty* bounded subset have a supremum?) We decided that bounded complete domains should possess a least element.

The definition of bounded completeness is presented in form of a proposition claiming several conditions to be equivalent. It defines bounded complete posets; bounded complete domains are considered later.

Proposition 5.2.1 (Bounded complete posets)

For a *poset* $\mathbf{P}$, the following statements are equivalent:

(1) $\mathbf{P}$ is an upper cone, and for every two points x and y, the set $\uparrow x \cap \uparrow y$ is empty or an upper cone.

(2) $\mathbf{P}$ has a least element, and every two points with a common upper bound have a common least upper bound.

(3) The set of upper bounds of a finite set is either empty or an upper cone.

(4) Every finite bounded subset of $\mathbf{P}$ has a supremum.

Posets with these properties are called *bounded complete*.

Proof:

$(1) \Rightarrow (2)$: $\uparrow x \cap \uparrow y$ is empty iff x and y have no common upper bound. Every upper cone has a least element.

$(2) \Rightarrow (4)$: The proof is performed by induction on the cardinality of the finite set. The empty set is bounded since $\mathbf{P}$ is not empty. Thus, it has a lub, namely the least element of $\mathbf{P}$. Singletons trivially have a lub.

If A is a finite set with at least two points and upper bound u, let a be a point of A and $A' = A \setminus \{a\}$. A' is also bounded by u, whence it has a lub by induction hypothesis. Both $\bigsqcup A'$ and a are below u, whence they have a common upper bound. By (2), $a \sqcup \bigsqcup A' = \bigsqcup\{\bigsqcup\{a\}, \bigsqcup A'\}$ exists. By Prop. 2.4.8, $\bigsqcup(\{a\} \cup A') = \bigsqcup A$ also exists.

$(4) \Rightarrow (3)$: The finite set is either bounded, then it has a lub x, and the set of its upper bounds is $\uparrow x$. Or, the finite set is not bounded, then the set of its upper bounds is empty.

$(3) \Rightarrow (1)$: $\mathbf{P}$ is the set of upper bounds of $\emptyset$. Hence, it is an upper cone. Two points with a common upper bound form a finite set with a non-empty set of upper bounds. Hence, this set is an upper cone. □

By the definition above, every *finite* bounded set has a supremum in a bounded complete poset. The stronger property that every bounded set has a supremum comes in two equivalent statements.

Proposition 5.2.2 For a *poset* $\mathbf{P}$, the following statements are equivalent:

(1) Every bounded subset of $\mathbf{P}$ has a supremum.

(2) Every non-empty subset of $\mathbf{P}$ has an infimum.

Proof:

$(1) \Rightarrow (2)$: Let A be a non-empty subset of $\mathbf{X}$, and let B be the set of lower bounds of A. B is bounded by the elements of A, whence it possesses a supremum x by (1). Every member of A is an upper bound of B, whence it is above the least upper bound x. Thus, x is a lower bound of A. Since it is above all members of B, it is the greatest lower bound.

$(2) \Rightarrow (1)$: Let A be a bounded subset of $\mathbf{X}$, and let B be the set of upper bounds of A. B is not empty since A is bounded, whence it possesses an infimum x by (2). By similar arguments as above, x is shown to be the supremum of A. □

Posets satisfying the criteria of Prop. 5.2.2 are not necessarily domains. For instance, in the poset $\mathbf{N}_0 = \{0 < 1 < 2 < \cdots\}$, every non-empty subset has an infimum. It is however not a domain since ∞ is missing.

There are bounded complete posets that however do not satisfy the stronger condition of Prop. 5.2.2. An example is given by $\mathbf{P} = \{0 < 1 < 2 < \cdots < [a, b]\}$, i.e. an infinite ascending chain with two incomparable upper bounds. **P** is bounded complete, but the chain $\{0, 1, \ldots\}$ is bounded without having a lub, and correspondingly, the non-empty set $\{a, b\}$ has no infimum.

For domains however, the two groups of conditions are equivalent.

Proposition 5.2.3 (Bounded complete domains)
For a *domain* $\mathbf{X}$, the statements of Prop. 5.2.1 and Prop. 5.2.2 are equivalent.

The class of bounded complete domains is denoted by BC.

Proof: Let S be an arbitrary set bounded by some point u. Let $\mathcal{F}$ be the set of finite subsets of S, and let $D = \{\bigsqcup F \mid F \in \mathcal{F}\}$ where all lubs exist since all subsets of S are bounded by u. Set D is directed, since $\bigsqcup \emptyset$ is in D and $\bigsqcup(F_1 \cup F_2)$ is an upper bound of $\bigsqcup F_1$ and $\bigsqcup F_2$ in D by Prop. 2.4.7. Since $\mathbf{X}$ is a domain, $\bigsqcup D$ exists. By Prop. 2.4.8, $\bigsqcup(\bigcup \mathcal{F})$ exists. The union of all finite subsets of S equals S, whence $\bigsqcup S$ exists. □

Examples for bounded complete domains are provided towards the end of the next section. We now turn to the preservation of BC by products and function domains.

Proposition 5.2.4
Arbitrary products of bounded complete domains are bounded complete.

Proof: Let A be a subset of $\prod_{i \in I} \mathbf{X}_i$ bounded by a point $u = (u_i)_{i \in I}$. Let $A_i = \pi_i[A]$ for all i in I. A_i is bounded by u_i, whence $\bigsqcup A_i$ exists for all i in I. By Prop. 2.4.10, $\bigsqcup A$ exists. □

Proposition 5.2.5 If $\mathbf{X}$ and $\mathbf{Y}$ are domains, then bounded completeness of $\mathbf{Y}$ implies bounded completeness of $[\mathbf{X} \to \mathbf{Y}]$.

Proof: Let A be a subset of $[\mathbf{X} \to \mathbf{Y}]$ bounded by a function g. For all x in $\mathbf{X}$, the set $\{f x \mid f \in A\}$ is bounded by $g\,x$. Hence, all these sets possess a supremum. By Lemma 3.4.2, $\bigsqcup A$ exists. □

5.3 Complete domains

In analogy to the previous section, we first present complete posets and then complete domains. Because it seems inappropriate to coin posets without lubs of directed sets complete, we speak of finitely complete posets instead of complete posets.

Proposition 5.3.1 (Finitely complete posets)
For a *poset* $\mathbf{P}$, the following statements are equivalent:

(1) $\mathbf{P}$ is an upper cone, and for every two points x and y, the set of upper cones $\uparrow x \cap \uparrow y$ is an upper cone.

(2) $\mathbf{P}$ has a least element, and every two points have a common least upper bound.

(3) The set of upper bounds of a finite set is an upper cone.

(4) Every finite subset of $\mathbf{P}$ has a supremum.

Posets with these properties are called *finitely complete*.

Proof: The proof is a simplified version of that of Prop. 5.2.1. □

In analogy to the conditions of Prop. 5.2.2, we now consider those posets where every subset has a supremum. In contrast to the situation in the previous section, such posets are always domains.

Proposition 5.3.2 (Complete domains)
For a *poset* $\mathbf{X}$, the following statements are equivalent:

(1) $\mathbf{X}$ is a finitely complete domain.

(2) All finite and all directed subsets of $\mathbf{X}$ have suprema.

(3) Every subset of $\mathbf{X}$ has a supremum.

(4) Every subset of $\mathbf{X}$ has an infimum.

(5) $\mathbf{X}$ is a bounded complete domain with a greatest element.

We call these posets complete domains. The class of complete domains is denoted by CC. Complete domains are also known as complete lattices.

Proof:

(1) ⇒ (5) : If every finite set has a supremum, then more than ever every bounded finite set has a supremum.

Since $\mathbf{X}$ has a least element, and every two points of $\mathbf{X}$ have an upper bound, the whole domain $\mathbf{X}$ is directed. By the definition of domains, $\mathbf{X}$ has an upper bound that represents the greatest element of $\mathbf{X}$.

(5) ⇒ (4) : In a bounded complete domain, every non-empty set has an infimum. The empty set also possesses an infimum, namely the greatest element.

(4) ⇒ (3) : First, the empty set has an infimum. This represents the greatest element $\top$ of $\mathbf{X}$. Second, every non-empty set has an infimum, whence by Prop. 5.2.2, every bounded set has a supremum. All sets are bounded by $\top$.

(3) ⇒ (2) is trivial.

(2) ⇒ (1) : Finitely complete means all finite sets have suprema, and domain means all directed sets have suprema. □

The class CC has the same preservation properties as BC.

Proposition 5.3.3 Arbitrary products of complete domains are complete.

Proposition 5.3.4
If $\mathbf{X}$ and $\mathbf{Y}$ are domains, then completeness of $\mathbf{Y}$ implies completeness of $[\mathbf{X} \to \mathbf{Y}]$.

The proofs are simpler variants of those for the class BC.

In a complete domain, every two points x and y have a supremum $x \sqcup y$ and an infimum $x \sqcap y$. The operation '$\sqcup$' is always continuous, whereas '$\sqcap$' is generally not continuous (see example 7 below).

Proposition 5.3.5 In a complete domain $\mathbf{X}$, the operation $\sqcup : \mathbf{X} \times \mathbf{X} \to \mathbf{X}$ is continuous.

Proof: By Prop. 3.3.6 and commutativity of '$\sqcup$', we only have to show continuity in the second argument. Let x be a point and D a directed set of $\mathbf{X}$, and let $f_x = \lambda y.\, x \sqcup y$. Then $\bigsqcup f_x[D] = \bigsqcup\{x \sqcup d \mid d \in D\} = \bigsqcup(\bigcup_{d\in D}\{x,\, d\}) = \bigsqcup(\{x\} \cup D) = x \sqcup \bigsqcup D = f_x(\bigsqcup D)$ holds by applying Prop. 2.4.8 twice. □

Examples:

(1) For any set X, the powerset 2^X ordered by '$\subseteq$' is a complete domain.

(2) $\mathbf{N}_0^\infty = \{0 < 1 < 2 < \cdots < \infty\}$ is a complete domain.

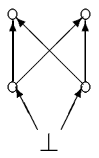
(3) The domain $\mathbf{N}'$ of lazy naturals is bounded complete, but not complete.

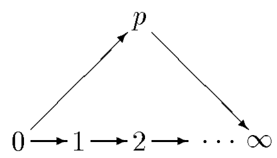
(4) $\{\mathsf{T} > \perp < \mathsf{F}\}$ is a bounded complete domain. It is not complete.

(5) The only discrete domain that is bounded complete is the one-point domain $\mathbf{1}$. $\mathbf{1}$ is even complete. The other discrete domains have no least element.

(6) Finite domains need not be bounded complete. The smallest domain with least element that is not bounded complete is depicted below to the left.

(7) Let $\mathbf{X} = \{p,\, 0,\, 1,\, \ldots,\, \infty\}$ where $0 < 1 < \cdots < \infty$ and $0 < p < \infty$ holds. $\mathbf{X}$ is depicted below to the right. It is a complete domain where the operation '$\sqcap$' is not continuous. The directed set $D = \{0,\, 1,\, \ldots\}$ has lub ∞. $\bigsqcup\{p \sqcap d \mid d \in D\} = \bigsqcup\{0\} = 0$ holds, whereas $p \sqcap \bigsqcup D = p \sqcap \infty = p$.





5.4 Finitary completeness or property M

In bounded complete posets, the set of upper bounds of two points is either empty or has a least element. As a generalization one may require the set of upper bounds to be finitary. This property is known as property M [Jun88]. We also use the term 'finitarily complete'.

As usual, there are several equivalent definitions collected in the following proposition.

Proposition 5.4.1 (Finitarily complete)

For a *poset* $\mathbf{P}$, the following statements are equivalent:

(1) The set of upper bounds of every finite set is finitary.

(2) $\mathbf{P}$ is finitary, and for every two points x and y, the set $\uparrow x \cap \uparrow y$ is finitary.

(3) $\mathbf{P}$ is finitary, and the intersection of two finitary upper sets is finitary.

(4) Finite intersections of finitary sets are finitary.

A poset satisfying these conditions is said to have property M. We also say it is *finitarily complete*. The class of all *domains* with property M is called FC.

Note that finite intersections also include empty intersections. Empty intersections always result in the whole carrier $\mathbf{P}$.

Proof:

(1) $\Rightarrow$ (2) : $\mathbf{P}$ is the set of upper bounds of $\emptyset$, and $\uparrow x \cap \uparrow y$ is the set of upper bounds of $\{x, y\}$.

(2) $\Rightarrow$ (3) : Let A and B be two finitary upper sets. Then there are finite sets E and F such that $A = \uparrow E$ and $B = \uparrow F$. Hence, $A \cap B = \uparrow E \cap \uparrow F = \bigcup_{e \in E, f \in F} \uparrow e \cap \uparrow f$ is a finite union of finitary sets by (2). Thus, it is finitary.

(3) $\Rightarrow$ (4) : By induction. The empty intersection is $\mathbf{P}$.

(4) $\Rightarrow$ (1) : If E is finite, then the set of upper bounds of E is $\bigcap_{e \in E} \uparrow e$. Upper cones are finitary, whence this is a finite intersection of finitary sets. $\square$

Proposition 5.4.2

(1) Every finite poset is finitarily complete.

(2) Every bounded complete poset is finitarily complete.

Proof:

(1) All finite sets are finitary.

(2) By Prop. 5.2.1 (1), the set of upper bounds of a finite set is either empty or an upper cone, whence finitary in any case. $\square$

The preservation properties of the class FC are worse than those of CC and BC.

Proposition 5.4.3 If $\mathbf{X}$ and $\mathbf{Y}$ are in FC, then so is $\mathbf{X} \times \mathbf{Y}$.

Proof: Products of finitary sets are finitary by Prop. 4.8.1. Hence, $\mathbf{X} \times \mathbf{Y}$ is finitary. Furthermore, $\uparrow(x, y) \cap \uparrow(x', y') = (\uparrow x \times \uparrow y) \cap (\uparrow x' \times \uparrow y') = (\uparrow x \cap \uparrow x') \times (\uparrow y \cap \uparrow y')$ holds. The final outcome is finitary as product of finitary sets. $\square$

Infinite products and function domains of FC domains are generally not FC.

Example 1: Let $\mathbf{X} = \{a, b\}$ where a and b are incomparable. If I is an infinite index set, then $\mathbf{Y} = \prod_{i \in I} \mathbf{X}$ is infinite and discrete, whence it cannot be finitary.

Example 2: Let $\mathbf{X} = \{[a_1, b_1] < [a_2, b_2] < \cdots < \infty\}$:

$$\begin{tikzcd}
a_1 \arrow[r] \arrow[dr] & a_2 \arrow[r] \arrow[dr] & a_3 & \\
b_1 \arrow[r] \arrow[ur] & b_2 \arrow[r] \arrow[ur] & b_3 & \cdots \quad \infty
\end{tikzcd}$$

The domain $\mathbf{X}$ is in FC, but not in BC. For every n in $\mathbf{N}$, the function $f : \mathbf{X} \to \mathbf{X}$ defined by $f(a_i) = a_i$ and $f(b_i) = b_i$ for $i < n$, and $f(a_i) = f(b_i) = a_n$ for $i \geq n$, is minimal in $[\mathbf{X} \to \mathbf{X}]$. Hence, $[\mathbf{X} \to \mathbf{X}]$ has an infinite number of minimal points and thus cannot be finitary.

There are two typical examples of domains not in FC: $\{[a, b] < \cdots < 3 < 2 < 1\}$ and $\{[a, b] < [1, 2, 3, \ldots]\}$. In both cases, the set $\uparrow a \cap \uparrow b$ is not finitary.

In connection with Rudin's Lemma, we are able to prove further properties of finitarily complete domains.

Proposition 5.4.4 Let $\mathbf{X}$ be a domain in FC. A subset S of $\mathbf{X}$ is a strongly compact upper set iff it is the intersection of some $\supseteq$-directed set of finitary upper sets. One such set is the set of all finitary upper supersets of S.

Proof: One direction is provided by Prop. 4.7.8. For the opposite direction, let S be a strongly compact upper set, and let $\mathcal{F}$ be the set of all finitary upper supersets of S. $\mathcal{F}$ is $\supseteq$-directed by the FC condition. $S \subseteq \bigcap \mathcal{F}$ obviously holds. For the opposite inclusion, we apply Lemma 4.4.4. If $S \subseteq O$ where O is open, then $F \subseteq O$ holds for some F in $\mathcal{F}$ by strong compactness. $\bigcap \mathcal{F} \subseteq F \subseteq O$ follows. □

5.5 The classes SC and KC

The class FC may be further generalized by replacing the finitary sets by strongly compact sets in its definition. Since strong compactness is a topological notion, we only consider domains, not general posets.

Proposition 5.5.1 (SC) For a domain $\mathbf{X}$, the following two statements are equivalent:

(1) $\mathbf{X}$ is strongly compact, and for every two points x and y, the set $\uparrow x \cap \uparrow y$ is strongly compact.

(2) $\mathbf{X}$ is finitary, and the intersection of two finitary upper sets is strongly compact.

Domains with these properties are called *strongly compactly complete*. The class of such domains is denoted by SC.

Proof: Since $\mathbf{X}$ is open, $\mathbf{X}$ being strongly compact and being finitary is equivalent by Prop. 4.7.7.

Upper cones are finitary. Conversely, if A and B are finitary upper sets, then there are finite sets E and F such that $A = \uparrow E$ and $B = \uparrow F$. $A \cap B = \bigcup_{e \in E, f \in F} \uparrow e \cap \uparrow f$ is strongly compact as a finite union of strongly compact sets. □

In contrast to finitary completeness, we are neither able to derive from SC that the set of upper bounds of every finite set is strongly compact, nor that the intersection of two strongly compact upper sets is strongly compact.

The class of compactly complete domains is defined analogously to SC.

Proposition 5.5.2 (KC) For a domain $\mathbf{X}$, the following two statements are equivalent:

(1) $\mathbf{X}$ is compact, and for every two points x and y, the set $\uparrow x \cap \uparrow y$ is compact.

(2) $\mathbf{X}$ is finitary, and the intersection of two finitary upper sets is compact.

Domains with these properties are called *compactly complete*. The class of such domains is denoted by KC.

Proof: Analogous to that of Prop. 5.5.1 using Prop. 4.6.8 instead of Prop. 4.7.7. □

The class FC is a subclass of SC, and SC is in turn a subclass of KC. The examples of section 5.4 provide domains in class FC whose infinite product and function domain are not contained in FC since their whole carrier is not finitary. By part (2) of Prop. 5.5.2, they are in KC neither. Thus, neither SC nor KC are closed w.r.t. infinite products and function domains.

Class SC is however closed w.r.t. finite products:

Proposition 5.5.3 If $\mathbf{X}$ and $\mathbf{Y}$ are in SC, then their product $\mathbf{X} \times \mathbf{Y}$ is in SC.

Proof: Analogous to the proof of Prop. 5.4.3 using the fact that the product of two strongly compact sets is strongly compact (Prop. 4.9.1). □

We cannot show an analogous property for KC because the analogue of Prop. 4.9.1 is missing.

Problem 3 Is the class KC closed w.r.t. finite products?

Chapter 6

Algebraic and continuous domains

In the previous chapter, we presented a hierarchy of properties defined by a common scheme: from CC to KC. In this chapter, we present two well-known domain classes: algebraic and continuous domains. We start by introducing the smaller, but more well-known class of algebraic domains and then consider continuous domains.

The classical definitions of these notions will be shown to be equivalent to purely topological definitions that could be applied to all (T0)-spaces (Theorems 6.2.5 and 6.7.9). These results do not occur in the literature as far as I know it. On the other hand, they are fundamental and easy to prove such that it is unlikely that they are really new.

Sections 6.1 through 6.5 handle algebraicity. In section 6.1, isolated points are studied in arbitrary domains. Section 6.2 starts by the usual order-theoretic definition in terms of isolated points, and derives the equivalent topological definition. In section 6.3, some examples for algebraic domains are presented. In section 6.4 we show how the completeness properties of an algebraic domain may be characterized in terms of the base. Section 6.5 then treats the relations between an algebraic domain and its base: monotonic functions defined on the base can be uniquely extended to continuous functions on the whole domain (Th. 6.5.1). Also, the whole domain can be recovered from the base by ideal completion (Th. 6.5.3).

Sections 6.6 through 6.8 handle continuity of domains. Concerning the theory of continuous domains, I owe much to the paper [Law88]. In section 6.6, the way-below relation is introduced. Section 6.7 first defines continuous domains classically by means of the way-below relation and then derives the equivalent topological characterization: in every environment of every point, there is an environment that is an upper cone. In section 6.8, we present some further properties of continuous domains, and provide some examples for continuous and non-continuous domains.

6.1 Isolated points

Algebraic domains are characterized by a base from which all domain points can be generated by directed lubs. The points of this base are required to have a special property that is investigated in this section.

The domain $\mathbf{X} = \{p, 0, 1, \ldots, \infty\}$ where $0 < 1 < \cdots < \infty$ and $0 < p < \infty$ was introduced in section 5.3 as an example for a complete domain where the operation '$\sqcap$' is not continuous. It suffers from the fact that the point p is below the limit of the directed set $D = \{0, 1, \ldots\}$, but not below any of the members of D.

Definition 6.1.1 A point a in a domain $\mathbf{X}$ is *isolated* (or: finite, or: compact) iff for all directed sets $D \subseteq \mathbf{X}$, the relation $a \leq \bigsqcup D$ implies $a \in \downarrow D$, i.e. there is an element d in D such that $a \leq d$.

The set of all isolated points of $\mathbf{X}$ is denoted by $\mathbf{X}^0$. Finite sets of isolated points are called *iso-finite*.

We adopt the name 'isolated' in order to avoid the name conflict with finite or compact sets. The name 'finite' is inspired by powersets where the isolated 'points' are just the finite sets (cf. section 6.3). The name 'compact' probably was chosen because of the characterization of compact sets by part (2) of Prop. 4.6.1. As the next proposition shows, there is also some motivation to call the isolated points 'open'.

Proposition 6.1.2 A point x is isolated iff the upper cone $\uparrow x$ is Scott open.

Proof: A Scott open set is a d-open upper set. The definition of x being isolated coincides with statement (3) in Prop. 3.5.5, i.e. x is isolated iff $\uparrow x$ is d-open. □

The upper cone being open implies a further topological property of isolated points.

Proposition 6.1.3 A point x is isolated iff for all sets S, $x \in \mathsf{cl}\, S$ implies $x \in \downarrow S$.

Proof: Let x be isolated and $x \in \mathsf{cl}\, S$. By Prop. 6.1.2, $\uparrow x$ is an open environment of x. By Prop. 4.2.2, it meets S by a point y. $y \in \uparrow x$ implies $x \leq y$, whence x is in $\downarrow S$.

Conversely, let x be a point such that $x \in \mathsf{cl}\, S$ implies $x \in \downarrow S$ for all sets S. Let $A = \mathsf{co} \uparrow x$. We show A is closed, whence its complement $\uparrow x$ is open and x is isolated by Prop. 6.1.2. Let y be a point of $\mathsf{cl}\, A$. Assume y is not in A. Then it is in $\uparrow x$, i.e. $y \geq x$ holds. Since $\mathsf{cl}\, A$ is lower, $x \in \mathsf{cl}\, A$ follows, whence $x \in \downarrow A = A$ by precondition. $x \in A$ means $x \not\geq x$ — a contradiction. Thus, $\mathsf{cl}\, A = A$ holds. □

After having considered the topological properties of isolated points, we now turn to their order-theoretic properties. The following proposition is simple, but sometimes useful.

Proposition 6.1.4 Let D be a directed set whose lub a is isolated. Then a is in D.

Proof: Since a is isolated and below the lub of D by reflexivity, there is some d in D such that $a \leq d$ holds. $d \leq a$ holds since a is an upper bound of D. □

Proposition 6.1.5 If it exists, the lub of an iso-finite set is isolated.

Corollary: If $\mathbf{X}$ has a least element $\perp$, then $\perp$ is isolated.

Proof: Let E be an iso-finite set with lub $x = \bigsqcup E$. By Lemma 2.4.6, $\uparrow x = \bigcap_{e \in E} \uparrow e$ holds. The claim then follows from Prop. 6.1.2 and the fact that the intersection of a finite number of Scott open sets is Scott open. Even the intersection of zero open sets is open, since it is the whole domain.

The corollary holds, since $\perp$ is the lub of the empty set. A more direct argument is that $\uparrow \perp = \mathbf{X}$ is Scott open. □

The isolated points of a product are easily characterized.

Proposition 6.1.6 Let $\mathbf{X}$ and $\mathbf{Y}$ be two domains. (x, y) is isolated in $\mathbf{X} \times \mathbf{Y}$ iff x is isolated in $\mathbf{X}$ and y is isolated in $\mathbf{Y}$. Hence, $(\mathbf{X} \times \mathbf{Y})^0 = \mathbf{X}^0 \times \mathbf{Y}^0$.

Proof: We apply Prop. 6.1.2. By Prop. 2.2.10, $\uparrow(x, y) = \uparrow x \times \uparrow y$ holds. By Prop. 4.8.1 and 4.8.2, $\uparrow(x, y)$ is open iff both $\uparrow x$ and $\uparrow y$ are open. □

We now define a variant of the lower closure that only contains isolated points.

Definition 6.1.7 For points x in a domain $\mathbf{X}$, let $\downarrow^0 x = \{a \in \mathbf{X} \mid a \text{ isolated and } \leq x\} = \downarrow x \cap \mathbf{X}^0$ be the set of isolated points below x.

Obviously, $x \leq y$ implies $\downarrow^0 x \subseteq \downarrow^0 y$. The inverse implication is not true in general domains.

6.2 Algebraic domains

An important class of domains is characterized by the fact that every point of theirs may be reached as a directed lub of isolated points.

Definition 6.2.1 A domain $\mathbf{X}$ is *algebraic* iff for all x in $\mathbf{X}$, there is a directed set of isolated points with lub x. The set $\mathbf{X}^0$ of isolated points of an algebraic domain is called its *base*. The class of all algebraic domains is denoted by ALG.

An equivalent definition is given by the following proposition:

Proposition 6.2.2 A domain $\mathbf{X}$ is algebraic iff for all x in $\mathbf{X}$, the set $\downarrow^0 x$ of isolated points below x is directed with lub x.

Proof: The direction from right to left is trivial. For the opposite direction, assume $\mathbf{X}$ is algebraic, and let $x \in \mathbf{X}$. Then there is a directed set D of isolated points with lub x. D obviously is a subset of $\downarrow^0 x$. We show D is cofinal in $\downarrow^0 x$, whence by Prop. 3.1.8, the set $\downarrow^0 x$ is directed with lub x.

We have to show $\downarrow^0 x \subseteq \downarrow D$. Let a be in $\downarrow^0 x$. Then a is isolated and below x i.e. below the lub of D. By definition of isolated, a is below some member of D, i.e. $a \in \downarrow D$. □

The proposition above implies some conclusions about the '$\downarrow^0$' operator in algebraic domains.

Proposition 6.2.3 For all points x and y in an algebraic domain, $\downarrow^0 x \subseteq \downarrow^0 y$ is equivalent to $x \leq y$. Hence, $\downarrow^0 x = \downarrow^0 y$ is equivalent to $x = y$.

Proof: The second claim follows from the first one be anti-symmetry.

Let $x \leq y$. Then every isolated point below x is below y by transitivity. Conversely, let $\downarrow^0 x \subseteq \downarrow^0 y$. Then $x = \bigsqcup \downarrow^0 x \leq \bigsqcup \downarrow^0 y = y$ holds by Prop. 2.4.7. □

In Prop. 6.1.2, we saw that in a domain, an upper cone is open iff its vertex is isolated. In algebraic domains, arbitrary open sets may be characterized.

Proposition 6.2.4 In an algebraic domain, a set O is open iff there is a set S of isolated points such that $O = \uparrow S$.

Proof: Let O be an open set in the algebraic domain $\mathbf{X}$, and let $S = O \cap \mathbf{X}^0$. S is a set of isolated points. $S \subseteq O$ implies $\uparrow S \subseteq \uparrow O = O$. For the opposite inclusion, let x be a point of O. x is the lub of the directed set $\downarrow^0 x$. Since O is open, there is a in $\downarrow^0 x$ such that a in O. a is in S and below x, whence x is in $\uparrow S$.

Conversely, let $O = \uparrow S = \uparrow \bigcup_{s \in S}\{s\} = \bigcup_{s \in S} \uparrow s$. By Prop. 6.1.2, the cones $\uparrow s$ are open, whence their union is. □

The propositions above prepare a characterization of algebraicity in topological terms that might be applied to arbitrary topological spaces.

Theorem 6.2.5 A domain $\mathbf{X}$ is algebraic iff it has the 'local open upper cone' property: for all points x in all open sets O, there is an open upper cone $\uparrow y$ such that $x \in \uparrow y \subseteq O$.

Proof: First, let $\mathbf{X}$ be algebraic, and let x be a point in an open set O. x is the lub of the directed set $\downarrow^0 x$, whence some member y of $\downarrow^0 x$ is in O. $y \leq x$ implies $x \in \uparrow y$, and $y \in O$ implies $\uparrow y \subseteq O$. y being isolated implies $\uparrow y$ is open.

For the opposite direction, assume $\mathbf{X}$ satisfies the topological property. Let x be a point of $\mathbf{X}$. We show $\downarrow^0 x$ is directed with lub x. If a and b are in $\downarrow^0 x$, then x is contained in the open sets $\uparrow a$ and $\uparrow b$. Hence, x is in the open intersection $\uparrow a \cap \uparrow b$. By the precondition, there is an open upper cone $\uparrow c$ such that $x \in \uparrow c \subseteq \uparrow a \cap \uparrow b$. Then c is isolated and below x, whence c is in $\downarrow^0 x$. By $\uparrow c \subseteq \uparrow a \cap \uparrow b$, $c \geq a, b$ holds.

Since all points of $\downarrow^0 x$ are below x, $\bigsqcup \downarrow^0 x \leq x$ holds. To show $x \leq \bigsqcup \downarrow^0 x$, we apply Prop. 4.3.1. Assuming $x \in O$ open, we have to show $\bigsqcup \downarrow^0 x \in O$. By the precondition, $x \in O$ implies there is an open upper cone $\uparrow y$ such that $x \in \uparrow y \subseteq O$. Then y is in $\downarrow^0 x$ and y is in O, whence $\bigsqcup \downarrow^0 x$ is in O because O is upper. □

Next, we present a proposition which allows to efficiently prove algebraicity and determine the base.

Proposition 6.2.6 Let $\mathbf{X}$ be a domain and B a subset of $\mathbf{X}$ such that

(1) all members of B are isolated,

(2) for all x in $\mathbf{X}$, there is a directed subset of B with lub x.

Then $\mathbf{X}$ is algebraic and B its base, i.e. B contains *all* isolated points of $\mathbf{P}$.

Proof: The definition of algebraicity is trivially satisfied. By (1), $B \subseteq \mathbf{X}^0$ holds. We have to show $\mathbf{X}^0 \subseteq B$. Let a be an isolated point of $\mathbf{X}$. By (2), there is a directed set $D \subseteq B$ with lub a. By Prop. 6.1.4, $a \in D \subseteq B$ holds. □

In section 8.4, we shall see that binary products of algebraic domains are algebraic. The function domain of two algebraic domains is however not generally algebraic again. An example is provided is section 6.8.

6.3 Examples for algebraic domains

We start by some classes of examples introduced by several propositions.

Proposition 6.3.1 Every finite poset $\mathbf{P}$ is an algebraic domain whose base is $\mathbf{P}$ itself.

Proof: By Prop. 3.1.2, every directed set in **P** contains a greatest element which is the lub of **P**. Hence, all directed sets have a lub, i.e. **P** is a domain. Furthermore, for all directed sets D, $\bigsqcup D \in D$ holds, whence all members of **P** are isolated. By $x = \bigsqcup\{x\}$, every point of **P** is the lub of a directed set. Applying Prop. 6.2.6 to all these facts, **P** is shown to be an algebraic domain with base **P**. □

Proposition 6.3.2 Every discrete poset **P** is an algebraic domain whose base is **P** itself.

Proof: Since two distinct points in **P** have no common upper bound, all directed sets of **P** are singletons. All singletons $\{x\}$ have lub x, whence **P** is a domain and all its points are isolated. □

Proposition 6.3.3 If X is any set, then the powerset of X forms an algebraic domain when ordered by inclusion '$\subseteq$'. The isolated points are just the finite sets.

Proof: The powerset is a domain since every set of sets has a lub, namely their union.

Let E be a finite set, and let $\mathcal{D}$ be a directed set of sets such that $E \subseteq \bigcup \mathcal{D}$. For every e in E, there is S_e in $\mathcal{D}$ such that $e \in S_e$. Let S be the upper bound in $\mathcal{D}$ of the finite set $\{S_e \mid e \in E\}$. Then $E \subseteq S$ and $S \in \mathcal{D}$ holds.

Every set S is the union of all its finite subsets. The set of finite subsets of S is directed. Hence, every set is a lub of a directed set of isolated sets. Applying Prop. 6.2.6, we obtain that the powerset is algebraic and all isolated sets are finite. □

- The domain $\mathbf{N}_0^\infty = \{0 < 1 < 2 < \cdots < \infty\}$ is algebraic. All points except ∞ are isolated.
- The domain $\mathbf{N}_0^{\infty+1} = \{0 < 1 < 2 < \cdots < \infty < \infty + 1\}$ is algebraic. Again, all points except ∞ are isolated. Point $\infty + 1$ shows that there might be isolated points above non-isolated points. It also shows that calling the isolated points finite is inappropriate in this case.

6.4 Algebraicity and completeness

In this section, we investigate how the containment of an algebraic domain **D** in a completeness class can be decided by looking at the base $\mathbf{D}^0$. We start by considering bounds in algebraic domains.

In an algebraic domain, there is always an isolated point between an iso-finite set and one of its upper bounds.

Proposition 6.4.1 Let E be an iso-finite set in an algebraic domain, and let u be an upper bound of E. Then there is an isolated upper bound of E below u.

Proof: For all $e \in E$, $e \le u = \bigsqcup {\downarrow^0} u$ holds. Since e is isolated, there is a point d_e in ${\downarrow^0} u$ with $e \le d_e$. The set $\{d_e \mid e \in E\}$ is finite since E is finite, and thus has an upper bound d in ${\downarrow^0} u$ because ${\downarrow^0} u$ is directed. Then for all e in E, $e \le d_e \le d$ holds, i.e. d is an upper bound of E. Since d is in ${\downarrow^0} u$, it is below u. □

By Lemma 6.1.5, the least upper bound of an iso-finite set in a domain is isolated if it exists. In an algebraic domain, this statement may be generalized to minimal upper bounds.

Lemma 6.4.2
In an algebraic domain, the minimal upper bounds of an iso-finite set are isolated.

Proof: Let E be an iso-finite set, and let m be minimal in the set of upper bounds of E. By Prop. 6.4.1, there is an isolated upper bound a of E below m. Since m is a minimal upper bound, $m = a$ is isolated. □

Proposition 6.4.3 Let $\mathbf{D}$ be an algebraic domain. Then $\mathbf{D}$ and $\mathbf{D}^0$ have the same minimal elements, and $\mathbf{D} = \uparrow min\, \mathbf{D}$ holds in $\mathbf{D}$ iff $\mathbf{D}^0 = \uparrow min\, \mathbf{D}^0$ holds in $\mathbf{D}^0$.

The addition 'holds in ...' is supplied since the $\uparrow$-operator depends on the underlying poset.

Proof: Every point x of $\mathbf{D}$ is the lub of a directed set of isolated points. Hence, there is an isolated point below every point of x. From this fact, all the statements may be proved.□

The connections between completeness of an algebraic domain and completeness of its base are given by the following theorem:

Theorem 6.4.4 Let $\mathbf{D}$ be an algebraic domain.

CC: $\mathbf{D}$ is complete iff the poset $\mathbf{D}^0$ is finitely complete.

BC: $\mathbf{D}$ is bounded complete iff the poset $\mathbf{D}^0$ is bounded complete.

SC: $\mathbf{D}$ is in SC, iff it is in KC, iff the poset $\mathbf{D}^0$ is in FC.

Proof:

BC: Assume $\mathbf{D}$ is in BC. Let E be a finite bounded set in $\mathbf{D}^0$. Then E is bounded in $\mathbf{D}$, too, whence it has a least upper bound x in $\mathbf{D}$. By Prop. 6.1.5, x is in $\mathbf{D}^0$.

Conversely, assume $\mathbf{D}^0$ is in BC. Then $\mathbf{D}^0$ has a least element $\perp$, and every two bounded points of $\mathbf{D}^0$ have a lub in $\mathbf{D}^0$. Every point x of $\mathbf{D}$ is the lub of a directed set of isolated points. Thus, there is at least one member of $\mathbf{D}^0$ below x, whence $\perp$ is below x. Thus, $\perp$ is also the least element of $\mathbf{D}$.

Let x and y be two points of $\mathbf{D}$ bounded in $\mathbf{D}$. Let $D = \downarrow^0 x \times \downarrow^0 y$. D is directed as product of two directed sets. Every pair (a, b) in D is bounded in $\mathbf{D}$ whence it is also bounded in $\mathbf{D}^0$ by Prop. 6.4.1. Thus, $a \sqcup b$ exists. The set $D' = \{a \sqcup b \mid (a, b) \in D\}$ is directed by Prop. 2.4.7. Let z be its lub in $\mathbf{D}$. $z = x \sqcup y$ may then easily be verified.

CC: The proof is analogous to a simplified version of the proof for BC.

SC: If $\mathbf{D}$ is in SC, then it is also in KC. Now assume $\mathbf{D}$ is in KC, and we want to show $\mathbf{D}^0$ in FC. First we show $\mathbf{D}^0$ is finitary. By Prop. 5.5.2, $\mathbf{D}$ is finitary. By Prop. 6.4.3, $\mathbf{D}^0$ is finitary, too. Let a and b be two points of $\mathbf{D}^0$. In $\mathbf{D}$, the set $\uparrow a \cap \uparrow b$ is open by Prop. 6.1.2 and compact because of KC. By Prop. 4.6.8, it is finitary in $\mathbf{D}$. Since $\{a, b\}$ forms an iso-finite set, the minimal points of $\uparrow a \cap \uparrow b$ are in $\mathbf{D}^0$. Thus, the set of upper bounds of a and b in $\mathbf{D}^0$ is finitary.

Now assume $\mathbf{D}^0$ is in FC, and we want to show $\mathbf{D}$ is in SC. $\mathbf{D}$ is finitary since $\mathbf{D}^0$ is. Let x and y be two points of $\mathbf{D}$. Let $D = \downarrow^0 x \times \downarrow^0 y$. D is directed as product of two directed sets. For every pair (a, b) in D, the set of common upper bounds of a and b is finitary in $\mathbf{D}^0$, whence $\uparrow a \cap \uparrow b$ is finitary in $\mathbf{D}$. Thus, using Prop. 2.4.6, one obtains $\uparrow x \cap \uparrow y = \bigcap_{a \in \downarrow^0 x} \uparrow a \ \cap \ \bigcap_{b \in \downarrow^0 y} \uparrow b = \bigcap_{(a,b) \in D} \uparrow a \cap \uparrow b$ is a $\supseteq$-directed intersection of finitary sets, whence strongly compact by Prop. 4.7.8. □

By the last statement of the theorem, the domains in KC & ALG are just the domains whose bases have property M. These domains enjoy many useful properties, whence they were used in early papers of mine such as [Hec90c, Hec90a]. The nice properties of the domains in KC & ALG now turned out to depend on KC, not on algebraicity. Examples for such properties are the Lemmas 8.10.1 and 8.10.2, and in connection with power domains, the theorems 21.5.1 and 22.6.1.

6.5 Continuous extension and ideal completion

In this section, we consider the relation between an algebraic domain and its base a bit closer. A monotonic function defined on the base may be uniquely extended to a continuous function on the whole domain. This also holds for embeddings. Two domains with isomorphic bases are completely isomorphic. From a given non-empty poset $\mathbf{P}$, an algebraic domain may be constructed whose base is isomorphic to $\mathbf{P}$.

Theorem 6.5.1 (Continuous extension)
Let $\mathbf{X}$ be an algebraic domain and $\mathbf{Y}$ a domain. For every continuous function $f : [\mathbf{X} \to \mathbf{Y}]$, the restriction of f to the base $\mathbf{X}^0$ of $\mathbf{X}$ is monotonic. Conversely, for every monotonic function $f^0 : \mathbf{X}^0 \to \mathbf{Y}$, there is exactly one continuous extension $f : [\mathbf{X} \to \mathbf{Y}]$, i.e. f is continuous and equals f^0 when restricted to $\mathbf{X}^0$.

Proof: The first claim concerning the restriction of f to $\mathbf{X}^0$ is trivial.

First, we show the uniqueness of the continuous extension. If f_1 and f_2 are two continuous functions from $\mathbf{X}$ to $\mathbf{Y}$ which both equal f^0 on $\mathbf{X}^0$, then with Prop. 6.2.2, $f_1 x = f_1(\bigsqcup \downarrow^0 x) = \bigsqcup f_1[\downarrow^0 x] = \bigsqcup f^0[\downarrow^0 x] = \bigsqcup f_2[\downarrow^0 x] = f_2(\bigsqcup \downarrow^0 x) = f_2 x$ holds. This also shows how the continuous extension has to be constructed.

For x in $\mathbf{X}$, we define $fx = \bigsqcup f^0[\downarrow^0 x] = \bigsqcup\{f^0 a \mid a \in \downarrow^0 x\}$. This is well defined, since $\downarrow^0 x$ is directed by Prop. 6.2.2, and the image of a directed set by a monotonic function is directed by Prop. 3.1.4.

To show the monotonicity of f, let $x \le y$. Then $\downarrow^0 x \subseteq \downarrow^0 y$, whence $f^0[\downarrow^0 x] \subseteq f^0[\downarrow^0 y]$, whence $fx \le fy$ by Prop. 2.4.7.

To show the continuity of f, let D be a directed set in $\mathbf{X}$. Then

$$\begin{aligned}
\bigsqcup f[D] &= \bigsqcup\{fd \mid d \in D\} = \bigsqcup\{\bigsqcup f^0[\downarrow^0 d] \mid d \in D\} \\
&= \bigsqcup f^0[\textstyle\bigcup_{d \in D} \downarrow^0 d] && \text{by Prop. 2.4.8} \\
&= \bigsqcup f^0[\downarrow^0(\bigsqcup D)] && \text{by isolatedness} \\
&= f(\bigsqcup D)
\end{aligned}$$

Finally, we have to show $fa = f^0 a$ for points a in $\mathbf{X}^0$. By definition of f, $fa = \bigsqcup f^0[\downarrow^0 a]$ holds. a is an upper bound of $\downarrow^0 a$, whence $f^0 a$ is an upper bound of $f^0[\downarrow^0 a]$ by monotonicity of f^0. $f^0 a$ is in $f^0[\downarrow^0 a]$, since a is in $\downarrow^0 a$. By Prop. 2.4.3, $f^0 a$ is the lub of $f^0[\downarrow^0 a]$. □

Next, we consider embeddings. If a poset embedding $f^0 : \mathbf{X}^0 \to \mathbf{Y}$ is given, its continuous extension f may not be an embedding. The poset $\mathbf{X} = \mathbf{N}_0^{\infty+1}$ is algebraic with base $\mathbf{X}^0 = \mathbf{N}_0^{\infty+1} - \{\infty\} = \{0, 1, \ldots, \infty + 1\}$. Let $\mathbf{Y} = \mathbf{N}_0^{\infty}$ and $f^0 : \mathbf{X}^0 \to \mathbf{Y}$ be defined by $f^0 n = n$ for $n \in \mathbf{N}_0$ and $f^0(\infty + 1) = \infty$. This function is an embedding, but its continuous extension $f : \mathbf{X} \to \mathbf{Y}$ is not since $f(\infty) = f(\infty + 1) = \infty$ holds.

To ensure that the continuous extension is an embedding, the function f^0 has to map base points into isolated points.

Proposition 6.5.2 Let $\mathbf{X}$ be an algebraic domain and $\mathbf{Y}$ a domain. A continuous function $f : [\mathbf{X} \to \mathbf{Y}]$ is a domain embedding if its restriction $f^0 : \mathbf{X}^0 \to \mathbf{Y}$ is a poset embedding and maps into $\mathbf{Y}^0$.

Proof: Let $f^0 : \mathbf{X}^0 \to \mathbf{Y}^0$ be a poset embedding, and let x and x' be points of $\mathbf{X}$ such that $fx \leq fx'$ holds, i.e. $\bigsqcup f^0[\downarrow^0 x] \leq \bigsqcup f^0[\downarrow^0 x']$. Let a be a point of $\downarrow^0 x$. Then $f^0 a$ is in $f^0[\downarrow^0 x]$, whence $f^0 a \leq \bigsqcup f^0[\downarrow^0 x] \leq \bigsqcup f^0[\downarrow^0 x']$ follows. Since $f^0 a$ is in $\mathbf{Y}^0$, there is a point b in $\downarrow^0 x'$ such that $f^0 a \leq f^0 b$. Since f^0 is an embedding, $a \leq b \leq x'$ holds. Because $a \leq x'$ holds for all $a \in \downarrow^0 x$, $x = \bigsqcup \downarrow^0 x \leq x'$ follows. □

Finally, we present how to construct an algebraic domain from a given base.

Theorem 6.5.3 (Ideal completion)
Let $\mathbf{P}$ be a given non-empty poset, and let $\mathcal{I}\mathbf{P}$ be the poset of ideals of $\mathbf{P}$ ordered by set inclusion. Then $\downarrow : \mathbf{P} \to \mathcal{I}\mathbf{P}$ is a poset embedding, and $\mathcal{I}\mathbf{P}$ is an algebraic domain whose base is the image of $\mathbf{P}$ under '$\downarrow$'. Thus, the base of $\mathcal{I}\mathbf{P}$ is isomorphic to $\mathbf{P}$.

Proof: For all x in $\mathbf{P}$, the cone $\downarrow x$ is an ideal since it is a lower set and directed because every subset of $\downarrow x$ has the upper bound $x \in \downarrow x$. The mapping $\downarrow : \mathbf{P} \to \mathcal{I}\mathbf{P}$ is a poset embedding by Prop. 2.2.5. The poset $\mathcal{I}\mathbf{P}$ is directed complete by Lemma 3.1.5; the lub of a directed set of ideals is its union.

Let $B = \{\downarrow x \mid x \in \mathbf{P}\} \subseteq \mathcal{I}\mathbf{P}$. We first show that all members of B are isolated. Let $\mathcal{D}$ be a directed set of ideals, and $\downarrow x \leq \bigsqcup \mathcal{D}$, i.e. $x \in \downarrow x \subseteq \bigsqcup \mathcal{D}$. Then, there is I in $\mathcal{D}$ such that $x \in I$. Since I is a lower set, $\downarrow x \subseteq I$ holds.

Next, we show that all ideals are directed lubs of members of B. Let I be an ideal. Then $I = \bigcup_{i \in I}\{i\} \subseteq \bigcup_{i \in I} \downarrow i \subseteq I$ holds, since I is a lower set. Because it is directed, $\bigcup_{i \in I} \downarrow i$ is a directed lub of members of B.

By Prop. 6.2.6, $\mathcal{I}\mathbf{P}$ is algebraic with base B. □

6.6 The way-below relation

The class of algebraic domains is generalized to the class of continuous domains by generalizing the concept of isolated points to the way-below relation. We say that a point x is *way-below* or *essentially below* a point y iff every computation converging to a point above y must eventually produce partial results above x.

Definition 6.6.1 A point x in a domain $\mathbf{X}$ is *way-below* a point y, written $x \ll y$, iff for all directed sets D of $\mathbf{X}$, $y \leq \bigsqcup D$ implies $x \in \downarrow D$.

For points x in $\mathbf{X}$, we write $\Downarrow x = \{y \mid y \ll x\}$ and $\Uparrow x = \{y \mid x \ll y\}$.

By comparing the definitions, it is obvious that a point x is isolated iff $x \ll x$ holds. We start by proving some simple properties of '$\ll$'.

Proposition 6.6.2 $x \ll y$ implies $x \leq y$. Hence, $\Downarrow x \subseteq \downarrow x$ and $\Uparrow x \subseteq \uparrow x$.

Proof: Let $D = \{y\}$. D is a directed set with lub y. $x \ll y$ implies there is a point d in D such that $x \leq d$. $d = y$ implies $x \leq y$. □

Proposition 6.6.3 If $x \leq y$ and $y \ll z$, then $x \ll z$ follows. If $x \ll y$ and $y \leq z$, then $x \ll z$ also follows. Hence, '$\ll$' is transitive, $\Downarrow x$ is lower, $\Uparrow x$ is upper, $x \leq y$ implies $\Downarrow x \subseteq \Downarrow y$ and $\Uparrow x \supseteq \Uparrow y$.

Proof:

(1) $z \leq \bigsqcup D$ implies $y \in \downarrow D$, whence $x \in \downarrow D$.

(2) $z \leq \bigsqcup D$ implies $y \leq \bigsqcup D$, whence $x \in \downarrow D$.

The statements about '$\Downarrow$' and '$\Uparrow$' are direct conclusions. □

Proposition 6.6.4 If $\mathbf{X}$ has a least element $\perp$, then $\perp \ll x$ holds for all x in $\mathbf{X}$.

Proof: For every directed set D, $\perp \in \downarrow D$ holds since D is not empty. □

6.7 Continuous domains

In a continuous domain, every point is a directed limit of points that are way-below it.

Definition 6.7.1 A domain $\mathbf{X}$ is *continuous* iff for all points x of $\mathbf{X}$, there is a directed subset D of $\Downarrow x$ with lub x. The class of continuous domains is denoted by CONT.

Continuous domains could also have been defined by a different condition.

Proposition 6.7.2
A domain $\mathbf{X}$ is continuous iff for all points x of $\mathbf{X}$, the set $\Downarrow x$ is directed with lub x.

Proof: The implication from right to left is trivial. For the opposite implication, let D be a directed subset of $\Downarrow x$ with lub x. We show that D is cofinal in $\Downarrow x$. Then $\Downarrow x$ is directed with the same lub x by Prop. 3.1.8.

We have to show $\Downarrow x \subseteq \downarrow D$. Let u be a member of $\Downarrow x$. Then $u \ll x = \bigsqcup D$, whence $u \in \downarrow D$ by definition of '$\ll$'. □

Proposition 6.7.3 Every algebraic domain is continuous.

Proof: In an algebraic domain, the set $\downarrow^0 x$ is directed with lub x for every point x. We show $\downarrow^0 x \subseteq \Downarrow x$. If a is in $\downarrow^0 x$, then $a \ll a \leq x$ holds, whence $a \ll x$ by Prop. 6.6.3. □

In continuous domains, the way-below relation satisfies an interpolation property.

Proposition 6.7.4 (Interpolation) Let $\mathbf{X}$ be a continuous domain. For every two points x and z in $\mathbf{X}$ with $x \ll z$, there is a point y in $\mathbf{X}$ such that $x \ll y \ll z$.

Proof: Since $\mathbf{X}$ is continuous, $\Downarrow z$ is directed and $z = \bigsqcup \Downarrow z$ holds. By Prop. 6.6.3 and continuity of $\mathbf{X}$, the set $\{\Downarrow a \mid a \in \Downarrow z\}$ is $\subseteq$-directed. By Prop. 3.1.5, its union D is directed. By Prop. 2.4.8, the lub of D equals $\bigsqcup_{a \in \Downarrow z} \bigsqcup \Downarrow a = \bigsqcup \Downarrow z = z$. Because of $x \ll z = \bigsqcup D$, there is some d in D such that $x \leq d$. d is contained in $\Downarrow y$ for some y in $\Downarrow z$. Hence, $x \leq d \ll y \ll z$ follows. □

The next results will finally lead to a topological characterization of domain continuity.

Proposition 6.7.5 Let $\mathbf{X}$ be a continuous domain, x a point of $\mathbf{X}$ and D a directed subset of $\mathbf{X}$. Then $x \ll \bigsqcup D$ implies $x \ll d$ for some d in D.

Proof: By interpolation, there is a point y in $\mathbf{X}$ such that $x \ll y \ll \bigsqcup D$. By $y \ll \bigsqcup D$, there is some d in D such that $y \leq d$. Then $x \ll y$ implies $x \ll d$. □

The '$\Downarrow$' operator is continuous:

Proposition 6.7.6
For all directed sets D in a continuous domain, $\Downarrow(\bigsqcup D) = \bigcup_{d \in D} \Downarrow d$ holds.

Proof: By Prop. 6.6.3, $d \leq \bigsqcup D$ implies $\Downarrow d \subseteq \Downarrow(\bigsqcup D)$, whence '$\supseteq$' holds. Conversely, if x is in $\Downarrow(\bigsqcup D)$, then $x \ll \bigsqcup D$, whence $x \ll d$ for some d in D by Prop. 6.7.5. □

Proposition 6.7.7 Let $\mathbf{X}$ be a continuous domain. For every x in $\mathbf{X}$, the set $\Uparrow x$ is open.

Proof: $\Uparrow x$ is upper by Prop. 6.6.3. By Prop. 6.7.5, it is d-open. □

The way-below relation may be characterized topologically.

Proposition 6.7.8 In a continuous domain $\mathbf{X}$, $x \ll y$ holds iff there is an open set O such that $\uparrow x \supseteq O \ni y$. The implication from right to left holds in every domain.

Proof: Assume there is an open set O such that $\uparrow x \supseteq O \ni y$. Let D be directed such that $y \leq \bigsqcup D$. $y \in O$ implies $\bigsqcup D$ in O. Since O is open, there is some d in D contained in O. $d \in O \subseteq \uparrow x$ implies $x \leq d$.

Conversely, assume $x \ll y$ holds in a continuous domain. Then $O = \Uparrow x$ is open by Prop. 6.7.7. $x \ll y$ implies $y \in \Uparrow x = O$. $O = \Uparrow x \subseteq \uparrow x$ holds by Prop. 6.6.2. □

Now, we are ready for the topological characterization of domain continuity.

Theorem 6.7.9 A domain $\mathbf{X}$ is continuous iff it has the 'local upper cone' property: for every point x in every open set O, there are an open set O' and a point x' such that $x \in O' \subseteq \uparrow x' \subseteq O$.

Proof: Assume first $\mathbf{X}$ is continuous. Let x be a point in an open set O. $\bigsqcup \Downarrow x = x \in O$ implies there is some x' in $\Downarrow x$ contained in O. $x' \in O$ implies $\uparrow x' \subseteq O$. $x' \in \Downarrow x$ implies $x' \ll x$, whence there is an open set O' by Prop. 6.7.8 such that $x \in O' \subseteq \uparrow x'$.

Conversely, let $\mathbf{X}$ be a domain satisfying the topological property. Let x be a point of $\mathbf{X}$. We have to provide a directed subset D of $\Downarrow x$ with lub x.

Let D be the set of all points u in $\mathbf{X}$ such that there is an open set O_u with $x \in O_u \subseteq \uparrow u$. By the direction of Prop. 6.7.8 that holds for all domains, D is a subset of $\Downarrow x$. Since $\mathbf{X}$ itself is open, $x \in \mathbf{X}$ implies there are a point y and an open set O_y such that $x \in O_y \subseteq \uparrow y$. Thus, D is not empty. Let u and v be two points of D. We look for a common upper bound of u and v in D. u and v in D implies there are open sets O_u and O_v such that $x \in O_u \subseteq \uparrow u$ and $x \in O_v \subseteq \uparrow v$. Then $x \in O_u \cap O_v$ follows, whence by precondition there are a point w and an open set O_w such that $x \in O_w \subseteq \uparrow w \subseteq O_u \cap O_v$. Then w is in D, and $\uparrow w \subseteq O_u \subseteq \uparrow u$ implies $w \geq u$. Similarly, $w \geq v$ follows.

Now, we know D is directed. We have to show $\bigsqcup D = x$. For every d in D, $x \in O_d \subseteq \uparrow d$ holds, whence $d \leq x$. $\bigsqcup D \leq x$ follows. For the opposite relation, we apply Prop. 4.3.1. Let x be a point of an open set O. We have to show $\bigsqcup D \in O$. By the precondition, there are a point y and an open set O_y such that $x \in O_y \subseteq \uparrow y \subseteq O$. y in D and y in O follow. By $y \leq \bigsqcup D$, $\bigsqcup D \in O$ follows. □

6.8 Further properties and examples

In this section, we present some further properties of continuous domains, and provide some examples for continuous and non-continuous domains.

We start by a sample domain that is continuous without being algebraic. Let $\mathbf{X} = [0..1]$ be the unit interval of the real line ordered by the usual order such that 0 is least and 1 is greatest. $\mathbf{X}$ forms a complete domain since every subset has a supremum. In $\mathbf{X}$, $x \ll y$ holds iff $x < y$ or $x = y = 0$. $\mathbf{X}$ is continuous since $x = \bigsqcup\{y \mid y < x\}$ holds for all $x > 0$. It is however not algebraic because it contains only one isolated point, namely 0, and this single point is not capable to generate all domain points by directed lubs.

In section 5.3, we presented a complete domain whose binary infimum operation is not continuous. It was $\mathbf{X} = \{p, 0, 1, 2, \ldots, \infty\}$ where $0 < 1 < 2 < \cdots < \infty$ and $0 < p < \infty$ holds. This domain cannot be continuous because of the following proposition:

Proposition 6.8.1 Let $\mathbf{X}$ be a continuous domain, where every two points x and y have a greatest lower bound $x \sqcap y$. Then $\sqcap : \mathbf{X} \times \mathbf{X} \to \mathbf{X}$ is continuous.

Proof: By Prop. 3.3.6 and commutativity of '$\sqcap$', we only have to show $a \sqcap \bigsqcup D = \bigsqcup_{d \in D}(a \sqcap d)$. Let the point to the left be x and the point to the right be y. For every d in D, $a \sqcap d \leq a \sqcap \bigsqcup D$ holds, whence $x \geq y$ follows.

For the opposite relation $x \leq y$, we apply Prop. 4.3.1. Let $x \in O$ for some open set O. We have to show $y \in O$.

By Th. 6.7.9, there are a point x' and an open set O' such that $x \in O' \subseteq \uparrow x' \subseteq O$. $x = a \sqcap \bigsqcup D$ in O' implies $a \in O'$ and $\bigsqcup D \in O'$ since O' is upper. Because O' is open, some point d of D is in O'. Then $a, d \in O' \subseteq \uparrow x'$ implies $a, d \geq x'$, whence $a \sqcap d \geq x'$. Thus, $a \sqcap d \in \uparrow x' \subseteq O$ follows, whence y is in O because it is above $a \sqcap d$. □

In section 8.4, we shall prove that the classes ALG and CONT are closed w.r.t. finite products. They are however not closed w.r.t. function domain forming. The example for this is prepared by the following proposition.

Proposition 6.8.2 All minimal points in a continuous domain are isolated.

Proof: Let x be a minimal point in a continuous domain $\mathbf{X}$. Then $\Downarrow x$ is directed and in particular not empty. Let y be a point of $\Downarrow x$. Then $y \ll x$ holds whence $y \leq x$ by Prop. 6.6.2. Since x is minimal, $y = x$ follows. Thus, $x \ll x$ holds, i.e. x is isolated. $\square$

Let $\mathbf{X} = \{[a_1, b_1] < [a_2, b_2] < \cdots < \infty\}$:

$$\begin{tikzcd}
a_1 \arrow[r] \arrow[dr] & a_2 \arrow[r] \arrow[dr] & a_3 & & \\
& & & \cdots & \infty \\
b_1 \arrow[r] \arrow[ur] & b_2 \arrow[r] \arrow[ur] & b_3 & &
\end{tikzcd}$$

We already met this domain in section 5.4 as an example of a domain $\mathbf{X}$ in class FC where the function domain $[\mathbf{X} \to \mathbf{X}]$ is not even contained in the larger class KC. Now, we shall show that $\mathbf{X}$ is algebraic whereas $[\mathbf{X} \to \mathbf{X}]$ is not even continuous. Algebraicity of $\mathbf{X}$ is proved easily; all points except ∞ are isolated.

The identity function $id : [\mathbf{X} \to \mathbf{X}]$ is minimal in $[\mathbf{X} \to \mathbf{X}]$. Let f be a function below id. Then $f(a_1) \leq a_1$, whence $f(a_1) = a_1$. Similarly, $f(b_1) = b_1$ holds. Assume as an induction hypothesis $f(a_n) = a_n$ and $f(b_n) = b_n$ hold. Then $f(a_{n+1}) \geq f(a_n) = a_n$ and $f(a_{n+1}) \geq f(b_n) = b_n$ hold by monotonicity, and $f(a_{n+1}) \leq a_{n+1}$ since $f \leq id$. These relations imply $f(a_{n+1}) = a_{n+1}$. Similarly, $f(b_{n+1}) = b_{n+1}$ is shown. By induction, we obtain that f and id coincide for all points a_n and b_n. By monotonicity, $f(\infty) \geq f(a_n) = a_n$ holds for all n, whence $f(\infty) = \infty$.

Let D be the set of all constant functions $\lambda x.\, c$ where $c < \infty$. D is easily shown to be directed. Its lub is the function $\lambda x.\, \infty$. The identity function is below $\lambda x.\, \infty$, but not below any function in D. Thus, id is not isolated.

The identity is a minimal element of $[\mathbf{X} \to \mathbf{X}]$ that is not isolated. By Prop. 6.8.2, $[\mathbf{X} \to \mathbf{X}]$ is not continuous.

Chapter 7

Functions and domain classes

In section 7.1, we introduce two domain classes, the finitely algebraic (also: bifinite, profinite, or strongly algebraic) and the finitely continuous domains, that are defined by properties of the function domain. The former class was studied in [Gun87], whereas I learnt about the latter class from [Law88]. These two classes are inserted in the hierarchy of the previous two chapters in sections 7.1 and 7.2. In section 7.3, their Cartesian closedness is shown.

In section 7.4, we define functors and show that locally continuous functors that map finite domains to finite domains, also preserve finite algebraicity and finite continuity. In section 7.5, we show that arbitrary functors that preserve algebraicity also preserve continuity of domains. The proof is done by using retracts that provide a close relationship between algebraic and continuous domains. I found the notion of retracts in [Law88].

7.1 Finite algebraicity and finite continuity

The classes of finitely algebraic and of finitely continuous domains are defined by approximations of the identity function.

Definition 7.1.1 Let $\mathbf{X}$ be a domain. A function $f : [\mathbf{X} \to \mathbf{X}]$ is a *deflation* iff $f \leq id$ holds and f has a finite image $f[\mathbf{X}]$. A function $f : [\mathbf{X} \to \mathbf{X}]$ is *idempotent* iff $f \circ f = f$.

$\mathbf{X}$ is *finitely continuous* iff the identity function $id : [\mathbf{X} \to \mathbf{X}]$ is the lub of a directed set $\mathcal{D}$ of deflations. $\mathbf{X}$ is *finitely algebraic* (or: bifinite, or: profinite) iff the identity is the lub of a directed set $\mathcal{D}$ of idempotent deflations.

The class of all finitely algebraic domains is called F-ALG, and the class of finitely continuous domains is denoted by F-CONT.

Obviously, F-ALG is a subclass of F-CONT by definition. The names finitely *algebraic* and *continuous* suggest that F-ALG is a subclass of ALG, and F-CONT of CONT. This is indeed true as we shall see soon.

Proposition 7.1.2 Every finite poset is a finitely algebraic domain.

Proof: Every finite poset is a domain by Prop. 6.3.1. If $\mathbf{X}$ is finite, then the identity $id : [\mathbf{X} \to \mathbf{X}]$ has finite image. Thus, $\mathcal{D} = \{id\}$ is a directed set of idempotent deflations, whose lub is id. □

The next Lemma holds for arbitrary domains, not only for finitely continuous ones.

Lemma 7.1.3 Let $\mathbf{X}$ be a domain, and $f : [\mathbf{X} \to \mathbf{X}]$ a deflation. Then $fx \ll x$ holds for all x in $\mathbf{X}$. If f is furthermore idempotent, then fx is isolated for all x in $\mathbf{X}$.

Proof: We first show $fx \ll x$. Let D be a directed set such that $x \leq \bigsqcup D$. Then $fx \leq f(\bigsqcup D) = \bigsqcup f[D]$ follows by continuity of f. $f[D]$ is a *finite* directed set, whence $\bigsqcup f[D] \in f[D]$ by Prop. 3.1.2. Thus, there is d in D such that $\bigsqcup f[D] = fd$. Hence, $fx \leq \bigsqcup f[D] = fd \leq d$ holds for some d in D.

If f is idempotent, then for all x in $\mathbf{X}$, $fx = f(fx) \ll fx$ holds as we just proved. Thus, fx is isolated. □

Now we are ready to show the inclusions of F-ALG in ALG and F-CONT in CONT.

Proposition 7.1.4 Every finitely algebraic domain is algebraic, and every finitely continuous domain is continuous.

Proof: Let $\mathcal{D}$ be the directed set of deflations approximating identity. For all points x, $x = id\, x = (\bigsqcup \mathcal{D})x = \bigsqcup\{fx \mid f \in \mathcal{D}\}$ holds by Prop. 3.4.2. The set $\{fx \mid f \in \mathcal{D}\}$ is directed and has lub x. Lemma 7.1.3 shows that it is a subset of $\Downarrow x$ in case of F-CONT, and of $\downarrow^0 x$ in case of F-ALG. □

7.2 Relations to completeness

By Prop. 7.1.4, class F-ALG is a subclass of ALG. In this section, we investigate its position inside ALG more closely. BC & ALG is a subclass of F-ALG which in turn is a subclass of SC & ALG. Analogous inclusions hold for F-CONT.

Theorem 7.2.1 Every bounded complete algebraic domain is finitely algebraic. Every bounded complete continuous domain is finitely continuous.

Proof: We start by showing the statement about F-CONT. Let $\mathbf{X}$ be a bounded complete continuous domain. Then $\mathbf{X}$ has a least element $\perp$, and $[\mathbf{X} \to \mathbf{X}]$ is also bounded complete. Let $\mathcal{D}$ be the set of all deflations $f : [\mathbf{X} \to \mathbf{X}]$. In particular, $\mathcal{D}$ contains $\lambda x.\, \perp$ whose image is $\{\perp\}$. If f and g are two members of $\mathcal{D}$, then they are bounded by id, whence $h = f \sqcup g \leq id$ exists in $[\mathbf{X} \to \mathbf{X}]$. By Prop. 3.4.2, $hx = fx \sqcup gx$ holds for all x in $\mathbf{X}$, whence $h[\mathbf{X}] = \{u \sqcup v \mid u \in f[\mathbf{X}],\, v \in g[\mathbf{X}]\}$ is finite. Hence, h is in $\mathcal{D}$. Thus, we showed $\mathcal{D}$ is directed.

Since all members of $\mathcal{D}$ are below identity, $\bigsqcup \mathcal{D} \leq id$ obviously holds. For all points a in $\mathbf{X}$, we define an auxiliary function f_a mapping $\Uparrow a$ to a and all other points to $\perp$. f_a is continuous by Lemma 3.7.1 since $\Uparrow a$ is open by Prop. 6.7.7. f_a is below identity since a is below all members of $\Uparrow a$. Finally, $f_a[\mathbf{X}] \subseteq \{\perp, a\}$ holds. Thus, f_a is in $\mathcal{D}$ for all a in $\mathbf{X}$.

By continuity of $\mathbf{X}$, we obtain $id\, x = x = \bigsqcup \Downarrow x = \bigsqcup\{a \mid x \in \Uparrow a\} = \bigsqcup\{f_a x \mid x \in \Uparrow a\} \leq \bigsqcup\{fx \mid f \in \mathcal{D}\} = (\bigsqcup \mathcal{D})x$ whence $id \leq \bigsqcup \mathcal{D}$.

Now, we turn to the algebraic case. Here, we define $\mathcal{D}$ to contain all idempotent deflations in $[\mathbf{X} \to \mathbf{X}]$. $\lambda x.\,\bot$ is idempotent, whence in $\mathcal{D}$. If both f and g are idempotent and below identity, then $f \sqcup g$ is also idempotent: $(f \sqcup g) \circ (f \sqcup g) \leq id \circ (f \sqcup g) = f \sqcup g$ holds, and conversely $(f \sqcup g) \circ (f \sqcup g) \geq f \circ f = f$ and also $(f \sqcup g) \circ (f \sqcup g) \geq g \circ g = g$, whence $(f \sqcup g) \circ (f \sqcup g) \geq f \sqcup g$. Thus, $\mathcal{D}$ is directed.

Again, $\bigsqcup \mathcal{D} \leq id$ holds obviously, and for the opposite direction, we define some auxiliary functions. For every isolated point b, let g_b be the function mapping the open set $\uparrow b$ to b and all other points to $\bot$. In particular, this function maps b to b and $\bot$ to $\bot$, whence it is idempotent. Thus, it is in $\mathcal{D}$.

By algebraicity of $\mathbf{X}$, we obtain $id\,x = x = \bigsqcup \downarrow^0 x = \bigsqcup\{b \mid b \in \mathbf{X}^0,\ x \in \uparrow b\} = \bigsqcup\{g_b\, x \mid b \in \mathbf{X}^0,\ x \in \uparrow b\} \leq \bigsqcup\{fx \mid f \in \mathcal{D}\} = (\bigsqcup \mathcal{D})x$ whence $id \leq \bigsqcup \mathcal{D}$. □

Since there are finite domains which are not bounded complete, the inclusion of BC & ALG in F-ALG is proper.

Theorem 7.2.2 Every finitely continuous domain is in SC. Hence, F-ALG is a subclass of SC & ALG, and F-CONT a subclass of SC & CONT.

Proof: We have to show that the whole domain $\mathbf{X}$ is finitary and that for every two points a and b, the set of upper bounds $\uparrow a \cap \uparrow b$ is strongly compact.

The directed set $\mathcal{D}$ approximating identity is not empty. Let f be a member of $\mathcal{D}$. The set $E = f[\mathbf{X}]$ is finite. For all x in $\mathbf{X}$, $fx \leq x$ holds, whence $\mathbf{X} \subseteq \uparrow E$. Thus, $\mathbf{X}$ is finitary.

Let a and b be two points and $U = \uparrow a \cap \uparrow b$. For all f in $\mathcal{D}$, the image $f[U]$ is finite. For all x in U and f in $\mathcal{D}$, $x \geq fx \in f[U]$ holds, whence $U \subseteq \uparrow f[U]$ for all f in $\mathcal{D}$. Thus, we obtain $U \subseteq \bigcap\{\uparrow f[U] \mid f \in \mathcal{D}\}$.

Next, we show the opposite inclusion. Let y be a point of the intersection. Then for all f in $\mathcal{D}$, there is some u in U such that $y \geq fu$. By $u \geq a$, $y \geq fa$ follows for all f in $\mathcal{D}$, whence $y \geq \bigsqcup_{f \in \mathcal{D}} fa = a$. Similarly, $y \geq b$ holds. Together, this means $y \in U$.

Let $f \leq g$. Then for $y \in g[U]$, $y \geq gu \geq fu \in f[U]$ holds for some u in U. Hence, $f \leq g$ implies $\uparrow f[U] \supseteq \uparrow g[U]$, whence the set $\{\uparrow f[U] \mid f \in \mathcal{D}\}$ is $\supseteq$-directed as $\mathcal{D}$ is directed. Thus, U is a $\supseteq$-directed intersection of finitary upper sets, whence it is strongly compact by Prop. 4.7.8. □

The domain $\mathbf{X} = \{[a_1, b_1] < [a_2, b_2] < \cdots < \infty\}$, which already occurred in sections 5.4 and 6.8, is in SC & ALG — even in FC & ALG. It is however not finitely algebraic since the identity is minimal in it as shown in section 6.8. Hence, the identity cannot be the lub of a directed set of functions with finite image.

7.3 Cartesian closedness

In this section, we show that the classes F-ALG and F-CONT are Cartesian closed — in contrast to ALG and CONT. Let $\mathbf{X}$ and $\mathbf{Y}$ be two domains in F-CONT whose identities are approximated by $\mathcal{D}_1$ and $\mathcal{D}_2$ respectively. Then let $\mathcal{D} = \{f \times g \mid f \in \mathcal{D}_1,\ g \in \mathcal{D}_2\}$ where '$\times$' is the Cartesian combinator of Prop. 3.4.7. It is defined by $(f \times g)(x, y) = (fx, gy)$.

The members of $\mathcal{D}$ have the correct typing $[\mathbf{X} \times \mathbf{Y} \to \mathbf{X} \times \mathbf{Y}]$. By continuity of '$\times$', $\bigsqcup \mathcal{D} = \bigsqcup \mathcal{D}_1 \times \bigsqcup \mathcal{D}_2 = id_{\mathbf{X}} \times id_{\mathbf{Y}} = id_{\mathbf{X}\times\mathbf{Y}}$ holds as required.

All members of $\mathcal{D}$ have finite image since $(f \times g)[\mathbf{X} \times \mathbf{Y}] \subseteq f[\mathbf{X}] \times g[\mathbf{Y}]$. This shows that $\mathbf{X}\times\mathbf{Y}$ is in F-CONT again. If f and g are idempotent functions, then $f \times g$ is also idempotent since $(f \times g) \circ (f \times g) = (f \circ f) \times (g \circ g)$. Thus, F-ALG is also closed w.r.t. product.

Now turning to the function domain, we construct $\mathcal{D} = \{f \to g \mid f \in \mathcal{D}_1,\ g \in \mathcal{D}_2\}$ where $f \to g = \lambda h.\, g \circ h \circ f$. If $f : [\mathbf{X} \to \mathbf{X}]$ and $g : [\mathbf{Y} \to \mathbf{Y}]$, then $f \to g$ maps functions in $[\mathbf{X} \to \mathbf{Y}]$ into functions in $[\mathbf{X} \to \mathbf{Y}]$. Since composition is a continuous operation (Prop. 3.4.6), $\mathcal{D}$ approximates the identity of $[\mathbf{X} \to \mathbf{Y}]$: $(\bigsqcup \mathcal{D})h = (\bigsqcup \mathcal{D}_2) \circ h \circ (\bigsqcup \mathcal{D}_1) = id \circ h \circ id = h$ holds for all h in $[\mathbf{X} \to \mathbf{Y}]$.

Next, we have to show that $f \to g$ has finite image. For all functions h, the function $(f \to g)h = g \circ h \circ f$ maps $\mathbf{X}$ first to the finite set $f[\mathbf{X}]$, then to $h[f[\mathbf{X}]] \subseteq \mathbf{Y}$, and finally into the finite set $g[\mathbf{Y}]$. Hence, only the restriction of $g \circ h$ to the finite function space $f[\mathbf{X}] \to g[\mathbf{Y}]$ matters, and $(f \to g)[\mathbf{X} \to \mathbf{Y}] \subseteq \{\varphi \circ f \mid \varphi : [f[\mathbf{X}] \to g[\mathbf{Y}]]\}$ is finite. This shows F-CONT is closed w.r.t. function domain forming.

Finally, we show that $f \to g$ is idempotent whenever f and g are idempotent:
$(f \to g)((f \to g)\, h) = (f \to g)(g \circ h \circ f) = g \circ (g \circ h \circ f) \circ f = g \circ h \circ f = (f \to g)h$
This proves that F-ALG is also closed w.r.t. function domains.

Theorem 7.3.1 The classes CC & ALG, BC & ALG, F-ALG, CC & CONT, BC & CONT, and F-CONT are all Cartesian closed.

Proof: For F-ALG and F-CONT, this was shown above. Since BC & ALG is a subclass of F-ALG by Th. 7.2.1, it coincides with the class BC & F-ALG. This class is the intersection of the two Cartesian closed classes BC (Prop. 5.2.4 and 5.2.5) and F-ALG, and hence Cartesian closed itself. The proof for the remaining classes is analogous. □

7.4 Functors

In this section, we define functors for domains as an instance of a more general categorical concept. We show that functors which preserve finiteness also preserve finite algebraicity and finite continuity. In the next section, we shall show that functors preserving ALG also preserve CONT.

Definition 7.4.1 A *functor* $\mathcal{F}$ defined on a class C of domains maps every domain $\mathbf{X}$ in C to a domain $\mathcal{F}\mathbf{X}$, and also maps every function $f : [\mathbf{X} \to \mathbf{Y}]$ where $\mathbf{X}$ and $\mathbf{Y}$ are in C into a function $\mathcal{F}f : [\mathcal{F}\mathbf{X} \to \mathcal{F}\mathbf{Y}]$ such that the two functorial properties hold:

(1) For every $\mathbf{X}$ in C, $\mathcal{F}\, id_{\mathbf{X}} = id_{\mathcal{F}\mathbf{X}}$ holds.

(2) For every domains $\mathbf{X}$, $\mathbf{Y}$, and $\mathbf{Z}$ in C and functions $f : [\mathbf{X} \to \mathbf{Y}]$ and $g : [\mathbf{Y} \to \mathbf{Z}]$, the equality $\mathcal{F}(g \circ f) = \mathcal{F}g \circ \mathcal{F}f$ holds in $[\mathcal{F}\mathbf{X} \to \mathcal{F}\mathbf{Z}]$.

The functor $\mathcal{F}$ is *locally continuous* iff the mapping $\mathcal{F} : [\mathbf{X} \to \mathbf{Y}] \to [\mathcal{F}\mathbf{X} \to \mathcal{F}\mathbf{Y}]$ is continuous for all domains $\mathbf{X}$ and $\mathbf{Y}$ in C.

We already met a functor, namely ideal completion $\mathcal{I}$ where $\mathcal{I}f(A) = \downarrow f[A]$ for all ideals A. $\mathcal{I}$ is not locally continuous. The main field of application of the theory developed below is however the theory of power constructions: every power construction is a locally continuous functor.

Definition 7.4.2 A functor $\mathcal{F}$ is said to map a class C_1 into a class C_2 iff it is defined on a superclass of C_1, and $\mathcal{F}\mathbf{X}$ is in C_2 for every domain $\mathbf{X}$ in C_1.

A functor $\mathcal{F}$ is said to *preserve* a class C iff it maps C into C.

Our goal is to provide a criterion for functors to preserve the classes F-ALG and F-CONT. The first step is to consider the behavior of a functor operating on functions with finite image.

Proposition 7.4.3 Let $\mathcal{F}$ be a functor which preserves the class FIN of finite domains and is defined on a superclass C of FIN. Then for all domains $\mathbf{X}$ and $\mathbf{Y}$ in C holds: If $f : [\mathbf{X} \to \mathbf{Y}]$ is a morphism with finite image $f[\mathbf{X}]$, then $\mathcal{F}f : [\mathcal{F}\mathbf{X} \to \mathcal{F}\mathbf{Y}]$ has finite image again.

Proof: Let $\mathbf{Z} = f[\mathbf{X}]$ be the image of f. It is a domain in the order inherited from $\mathbf{Y}$ since it is finite, and the embedding $e : [\mathbf{Z} \to \mathbf{Y}]$ is continuous. The original morphism f may be corestricted to $f' : [\mathbf{X} \to \mathbf{Z}]$ such that $f = e \circ f'$. Then $\mathcal{F}f = \mathcal{F}(e \circ f') = \mathcal{F}e \circ \mathcal{F}f'$ follows. $\mathcal{F}f'$ maps from $\mathcal{F}\mathbf{X}$ to $\mathcal{F}\mathbf{Z}$, and the latter is finite as $\mathcal{F}$ preserves FIN. Thus, the image of $\mathcal{F}f$ is finite because $(\mathcal{F}f)[\mathcal{F}\mathbf{X}] \subseteq (\mathcal{F}e)[\mathcal{F}\mathbf{Z}]$. $\square$

The claim of the Proposition is needed to prove the following theorem:

Theorem 7.4.4 Let $\mathcal{F}$ be a locally continuous functor defined on a superclass of F-CONT. If $\mathcal{F}$ preserves FIN, then it also preserves F-ALG and F-CONT.

Proof: Let $\mathbf{X}$ be a finitely continuous domain. Then, there is a directed set $\mathcal{D}$ of functions from $\mathbf{X}$ to $\mathbf{X}$ with finite image such that $\bigsqcup \mathcal{D} = id$. By local continuity of $\mathcal{F}$ and Prop. 7.4.3, $\mathcal{F}[\mathcal{D}]$ is a directed set of functions from $\mathcal{F}\mathbf{X}$ to $\mathcal{F}\mathbf{X}$ with finite image, whose lub is $\bigsqcup \mathcal{F}[\mathcal{D}] = \mathcal{F}(\bigsqcup \mathcal{D}) = \mathcal{F}\, id = id$. Thus, $\mathcal{F}\mathbf{X}$ is finitely continuous again.

If $\mathbf{X}$ is finitely algebraic, there is the additional condition $f \circ f = f$ for all f in $\mathcal{D}$. It implies $\mathcal{F}f \circ \mathcal{F}f = \mathcal{F}(f \circ f) = \mathcal{F}f$, whence $\mathcal{F}\mathbf{X}$ is finitely algebraic again. $\square$

7.5 Retracts

In this section, we want to prove that a functor preserving ALG also preserves CONT. The technical means for performing the proof are retracts.

Definition 7.5.1 A domain $\mathbf{Y}$ is a *retracts* of a domain $\mathbf{X}$ if there are continuous functions $r : [\mathbf{X} \to \mathbf{Y}]$ and $e : [\mathbf{Y} \to \mathbf{X}]$ such that $r \circ e = id_{\mathbf{Y}}$.

A class C is *retract closed* if all retracts of all domains in C are in C again.

A factor domain $\mathbf{X}$ is a retract of a product $\mathbf{X} \times \mathbf{Y}$ by means of $e = \lambda x.(x, p)$ and $r = \pi_1$ where p is an arbitrary point of $\mathbf{Y}$. Similarly, $\mathbf{Y}$ is a retract of $\mathbf{X} \times \mathbf{Y}$. A domain $\mathbf{Y}$ is also retract of $[\mathbf{X} \to \mathbf{Y}]$ by means of $e = \lambda y.\, \lambda x.\, y$ and $r = \lambda f.\, fp$ where p is an arbitrary point of $\mathbf{X}$.

We already know several retract closed domain classes.

Proposition 7.5.2 The class $\mathrm{DOM}_\perp$ of domains with least element, the class of finitary domains, the completeness classes CC, BC, FC, SC, and KC, the class DIS of discrete domains, the class FIN of finite domains, and the classes F-CONT and CONT are retract closed.

In connection with the remarks above, we get a host of corollaries, e.g. if $\mathbf{X} \times \mathbf{Y}$ is continuous, then $\mathbf{X}$ and $\mathbf{Y}$ are continuous, if $[\mathbf{X} \to \mathbf{Y}]$ is continuous, then so is $\mathbf{Y}$, etc.

Proof: Let always $\mathbf{Y}$ be a retract of $\mathbf{X}$ by means of $r : [\mathbf{X} \to \mathbf{Y}]$ and $e : [\mathbf{Y} \to \mathbf{X}]$. Assuming that $\mathbf{X}$ is in the class under consideration, we have to show that $\mathbf{Y}$ is also in it.

Least element: Let $\perp$ be the least element of $\mathbf{X}$. For all y in $\mathbf{Y}$, $\perp \leq ey$ holds, whence $r(\perp) \leq r(ey) = y$ holds in $\mathbf{Y}$.

Finitary: Let M be a finite subset of $\mathbf{X}$ such that $\mathbf{X} = \uparrow M$. For all y in $\mathbf{Y}$, there is some m in M such that $m \leq ey$ holds, whence $rm \leq r(ey) = y$ holds in $\mathbf{Y}$. Thus, we obtain $\mathbf{Y} = \uparrow r[M]$ where $r[M]$ is finite.

BC: Let a and b be two points of $\mathbf{Y}$ bounded by u. Then ea and eb are bounded by eu in $\mathbf{X}$, whence $ea \sqcup eb$ exists. Let $c = r(ea \sqcup eb)$. c is an upper bound of a and b since $ea \sqcup eb \geq ea$ implies $c \geq r(ea) = a$. c is the least upper bound of a and b since $a, b \leq u$ implies $ea \sqcup eb \leq eu$, whence $c \leq r(eu) = u$.

CC: Analogous.

FC / SC / KC: $\mathbf{Y}$ is finitary as shown above. For two points a and b of $\mathbf{Y}$, we claim $\uparrow a \cap \uparrow b = r[\uparrow ea \cap \uparrow eb]$. If u is a point of the left hand set, then $u \geq a, b$ holds, whence $eu \geq ea, eb$. By $u = r(eu)$, u is then in the right hand set. Conversely, if w is in $r[\uparrow ea \cap \uparrow eb]$, then $w = rv$ holds for some v in $\uparrow ea \cap \uparrow eb$. $v \geq ea$ implies $w = rv \geq r(ea) = a$.

The set $\uparrow a \cap \uparrow b$ is finitary / strongly compact / compact as continuous image by r of the finitary / strongly compact / compact set $\uparrow ea \cap \uparrow eb$.

DIS: $y \leq y'$ in $\mathbf{Y}$ implies $ey \leq ey'$ in $\mathbf{X}$. Since $\mathbf{X}$ is discrete, $ey = ey'$ follows, whence $y = r(ey) = r(ey') = y'$.

FIN: $e[\mathbf{Y}]$ is a subdomain of $\mathbf{X}$ that is isomorphic to $\mathbf{Y}$. $\mathbf{X}$ being finite implies $e[\mathbf{Y}]$ is finite.

F-CONT: Let $\mathcal{D}$ be a directed set of deflations approximating the identity of $\mathbf{X}$. We define $\mathcal{D}' = \{r \circ f \circ e \mid f \in \mathcal{D}\}$. By continuity of composition, $\mathcal{D}'$ is a directed set with lub $\bigsqcup \mathcal{D}' = r \circ (\bigsqcup \mathcal{D}) \circ e = r \circ id_{\mathbf{X}} \circ e = id_{\mathbf{Y}}$. All members of $\mathcal{D}'$ have finite image since $r[f[e[\mathbf{Y}]]] \subseteq r[f[\mathbf{X}]]$ is finite.

CONT: We apply the topological criterion of Th. 6.7.9. Let y be a point of an open set O in $\mathbf{Y}$. By $r(ey) = y \in O$, point ey is in the open set $r^{-1}[O]$. By continuity of $\mathbf{X}$, there are an open set O' and a point x such that $ey \in O' \subseteq \uparrow x \subseteq r^{-1}[O]$. First, y is in $e^{-1}[O']$. Second, every point u of $e^{-1}[O']$ satisfies $eu \in O' \subseteq \uparrow x$, whence $eu \geq x$, whence in turn $u = r(eu) \geq rx$. Third, every point v above rx is in O since x is in $r^{-1}[O]$. Summarizing, we obtain $y \in e^{-1}[O'] \subseteq \uparrow rx \subseteq O$. $\square$

By the proposition above, any retract of any continuous domain is continuous. Hence, every retract of any algebraic domain is continuous. The converse of this statement also holds.

Proposition 7.5.3 Every continuous domain $\mathbf{X}$ is a retract of $\mathcal{I}\mathbf{X}$ by $r : [\mathcal{I}\mathbf{X} \to \mathbf{X}]$, $r(A) = \bigsqcup A$, and $e : [\mathbf{X} \to \mathcal{I}\mathbf{X}]$, $ex = \Downarrow x$.

Thus, every continuous domain is a retract of some algebraic domain.

Proof: $\mathcal{I}\mathbf{X}$ is algebraic by Th. 6.5.3.

For every x in $\mathbf{X}$, the set $\Downarrow x$ is lower by Prop. 6.6.3 and directed by Prop. 6.7.2. Hence, it is an ideal. Conversely, every ideal of $\mathbf{X}$ has a lub in $\mathbf{X}$ since $\mathbf{X}$ is a domain. This defines mapping r. By continuity of $\mathbf{X}$, $r(ex) = \bigsqcup \Downarrow x = x$ holds for all x in $\mathbf{X}$.

For continuity of e, we have to show $e(\bigsqcup D) = \bigsqcup e[D]$ for all directed sets D. The left hand side equals $\Downarrow(\bigsqcup D)$, and the right hand side is $\bigcup_{d \in D} \Downarrow d$. Both sides are equal by Prop. 6.7.6.

For continuity of r, we have to show $r(\bigsqcup \mathcal{D}) = \bigsqcup r[\mathcal{D}]$ for all directed sets $\mathcal{D}$ of ideals. The left hand side equals $\bigsqcup \bigcup_{A \in \mathcal{D}} A$, and the right hand side is $\bigsqcup_{A \in \mathcal{D}}(\bigsqcup A)$. Both sides are equal by Prop. 2.4.8. □

Now, we return to functors.

Proposition 7.5.4 Let $\mathcal{F}$ be a functor defined for two domains $\mathbf{X}$ and $\mathbf{Y}$. If $\mathbf{Y}$ is a retract of $\mathbf{X}$, then $\mathcal{F}\mathbf{Y}$ is a retract of $\mathcal{F}\mathbf{X}$.

Proof: $\mathbf{Y}$ is a retract of $\mathbf{X}$ if there are continuous $r : [\mathbf{X} \to \mathbf{Y}]$ and $e : [\mathbf{Y} \to \mathbf{X}]$ with $r \circ e = id_{\mathbf{Y}}$. Then $\mathcal{F}r \circ \mathcal{F}e = \mathcal{F}(r \circ e) = \mathcal{F}id_{\mathbf{Y}} = id_{\mathcal{F}\mathbf{Y}}$, whence $\mathcal{F}\mathbf{Y}$ is a retract of $\mathcal{F}\mathbf{X}$. □

By now, we are able to prove the announced preservation property for ALG and CONT.

Proposition 7.5.5 Let $\mathcal{F}$ be a functor defined on a superclass of CONT. If $\mathcal{F}$ maps ALG to CONT, then it preserves CONT.

Proof: If $\mathbf{X}$ is continuous, then it is a retract of some algebraic domain $\mathbf{Y}$ by Prop. 7.5.3. Then, $\mathcal{F}\mathbf{X}$ is a retract of the continuous domain $\mathcal{F}\mathbf{Y}$ by Prop. 7.5.4, whence it is continuous by Prop. 7.5.2. □

By strengthening the pre-condition, we obtain:

Proposition 7.5.6 Let $\mathcal{F}$ be a functor defined on a superclass of CONT. If $\mathcal{F}$ preserves ALG, then it also preserves CONT.

We shall meet retracts again in the theory of power constructions in section 14.4.

Chapter 8

Large domain classes

In this chapter, we introduce some domain classes that are much larger than the class CONT of continuous domains. We shall see that for these classes still powerful structural properties may be shown. All these classes are defined topologically. Hence, they would primarily provide a classification of (T0)-spaces. By employing the Scott topology as the standard topology for domains, the space classes induce however domain classes.

We mostly consider domains in this chapter, not topological spaces. The reason is that we sometimes apply Rudin's Lemma in the versions Lemma 3.9.3 and 3.9.4 that do not hold in arbitrary (T0)-spaces.

In sections 8.1 through 8.4, we introduce generalizations of algebraicity and continuity, namely multi-algebraicity and multi-continuity. A long time, I believed these classifications to be new. Multi-continuity as defined topologically in this thesis is however equivalent to the notion of quasicontinuity defined order-theoretically in [GLS83].

The topological notion of sobriety is closely related with upper power domain constructions as pointed out in [Smy83]. Some authors e.g. [Law88] define sobriety via irreducible closed sets (see section 8.5), whereas others e.g. [Smy78] define it via prime or open filters (see section 8.6). The equivalence of all three definitions is shown in [HM81] by means of lattice theory. We present the definitions together with new, more direct proofs of their equivalence in section 8.7. In section 8.8, we show that all multi-continuous domains are sober. This is probably a new result since multi-continuity was never considered topologically before.

In section 8.9, we introduce the novel classes K-RD and S-RD characterized by certain generalizations of Rudin's Lemma. All sober domains are in K-RD, and K-RD is a subclass of S-RD. These classes provide the maximal classes where the upper power domains in terms of (strongly) compact sets make sense as power domains.

In section 8.10, we prove a couple of interesting Lemmas that hold for domains in KC & K-RD as well as in SC & S-RD. Section 8.11 then presents Johnstone's non-sober domain published in [Joh81, Joh82]. This domain is also not in S-RD. Thus, we do not know any domain that would separate the classes SOB of sober domains, K-RD, and S-RD.

The final section 8.12 summarizes all domain classes introduced in the last four chapters.

8.1 Locality classes

The topological criteria for algebraicity (Th. 6.2.5) and continuity (Th. 6.7.9) deal with upper cones. They may easily be generalized by replacing the upper cones by finitary, strongly compact, or compact sets.

Definition 8.1.1 Let $\mathbf{X}$ be a domain. $\mathbf{X}$ is *multi-algebraic* iff for all points x in an open set O, there is a finitary open set F such that $x \in F \subseteq O$.

$\mathbf{X}$ is *locally finitary* or *multi-continuous* iff for all points x in an open set O, there are an open set O' and a finitary set F such that $x \in O' \subseteq F \subseteq O$.

$\mathbf{X}$ is *locally compact* iff for all points x in an open set O, there are an open set O' and a compact set K such that $x \in O' \subseteq K \subseteq O$.

The class of all multi-algebraic domains is called M-ALG, and the class of multi-continuous domains is denoted by M-CONT. L-COMP is the class of locally compact domains.

In the definition of multi-algebraicity, it makes no difference to replace 'finitary open' by 'compact open' because of Th. 4.6.8. We did not define locally strongly compact domains since in the situation $x \in O' \subseteq S \subseteq O$ where S is strongly compact and O is open, there is a finitary set F such that $S \subseteq F \subseteq O$. Thus, local strong compactness coincides with local finitarity.

For multi-algebraic and multi-continuous domains, a *degree of multiplicity* may be defined. If F is a finitary set, then $min\,F$ is finite. We define the size of F to be the cardinality of $min\,F$. For every pair (x, O) of a point x in an open set O, we choose a finitary set $F(x, O)$ with minimal size that satisfies the conditions in Def. 8.1.1. The degree of multiplicity of a domain $\mathbf{X}$ is then the supremum of the sizes of the sets $F(x, O)$. It may be ∞. Since finitary sets of size 1 are upper cones, a multi-algebraic domain has degree 1 iff it is algebraic.

By means of the degree, an infinite hierarchy is obtained between ALG and M-ALG, and between CONT and M-CONT. If we refer to the class of all domains in M-ALG with degree at most d, then we write M^d-ALG. The class of all multi-algebraic domains with finite degree is denoted by M*-ALG. Thus, we obtain ALG = M^1-ALG $\subset$ M^2-ALG $\subset \cdots \subset$ M*-ALG $\subset$ M-ALG. An analogous hierarchy is contained in M-CONT.

Proposition 8.1.2 M-CONT and all classes M^k-CONT as well as M*-CONT are retract closed. L-COMP is also retract closed.

Proof: We first show the statement for M-CONT. Let $\mathbf{Y}$ be a retract of $\mathbf{X}$ by means of $r : [\mathbf{X} \to \mathbf{Y}]$ and $e : [\mathbf{Y} \to \mathbf{X}]$ where $r \circ e = id_{\mathbf{Y}}$. Let y be a point in an open set O. By $r(ey) = y \in O$, the point ey is in $r^{-1}[O]$. As $\mathbf{X}$ is multi-continuous, there are an open set O' and a finitary upper set F such that $ey \in O' \subseteq F \subseteq r^{-1}[O]$. First, y is in the open set $e^{-1}[O']$. Second, if u is in $e^{-1}[O']$, then eu is in $O' \subseteq F$, whence $u = r(eu)$ is in $r[F]$. Set $r[F]$ is finitary by Prop. 2.7.2 (5). Third, if v is in $r[F]$, then $v = rw$ where $w \in F \subseteq r^{-1}[O]$, whence $v = rw \in O$. Thus, we obtain $y \in e^{-1}[O'] \subseteq r[F] \subseteq O$.

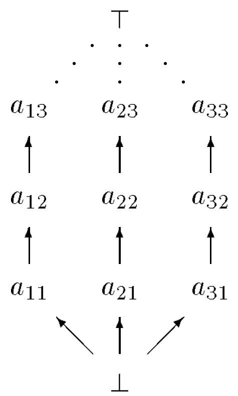
If F is finitary, then $F \subseteq {\uparrow}(min\,F)$ holds. By Prop. 2.6.2 (7), $min\,r[F] \subseteq r[min\,F]$ follows. Hence, the size of $r[F]$ is bounded by the size of F. Thus, all the intermediate classes are retract closed.

The proof for local compactness is analogous to the proof for multi-continuity. $r[K]$ is compact by Prop. 4.6.5. □

8.2 Examples for multi-algebraic domains

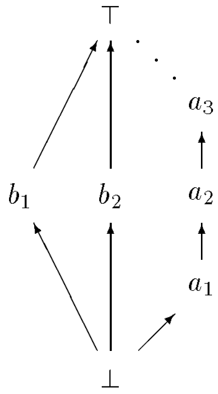
In this section, we shall provide examples for multi-algebraic domains of any degree. We shall also meet domains that are not multi-continuous. All the domains presented in this section will be complete.

For all k in $\{1, 2, \ldots, \infty\}$, we define a domain $\mathbf{X}_k$ that consists of a least point $\perp$, a greatest point $\top$, and k incomparable ascending chains $A_1, \ldots, A_k$ where A_i is given by $\{a_{i1} < a_{i2} < a_{i3} < \cdots\}$. All the ascending chains share the same lub $\top$. The domain $\mathbf{X}_3$ is depicted below.



For every $k < \infty$, the domain $\mathbf{X}_k$ is multi-algebraic of degree k. This is because every non-empty open set contains $\top$ and thus also the ends of all k chains. The domain $\mathbf{X}_\infty$ is not multi-continuous however since all open sets containing $\top$ must contain the ends of all the infinitely many chains and thus cannot be subset of a finitary set.

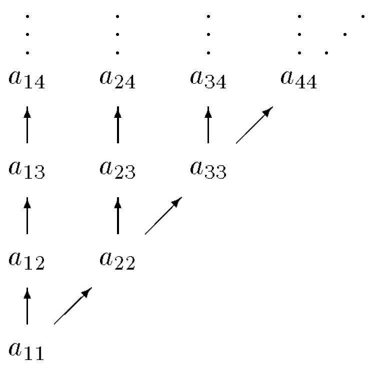
As a second series of examples, for all n in $\{0, 1, 2, \ldots, \infty\}$, we define a domain $\mathbf{Y}_n = \{\perp, a_1, a_2, a_3, \ldots, b_1, b_2, \ldots, b_n, \top\}$ ordered such that $\perp$ is least, $\top$ is greatest, the a-points form an ascending chain $a_1 < a_2 < \cdots$ with lub $\top$, and the b-points are incomparable to each other and to the a-points. The domains $\mathbf{Y}_n$ differ in the number of b-points they contain. The domain $\mathbf{Y}_2$ is depicted below.



The domain $\mathbf{Y}_0$, which consists of a chain only, is algebraic. All other domains $\mathbf{Y}_n$, even $\mathbf{Y}_\infty$, are multi-algebraic of degree 2. Let a point x be in an open set O. Then O contains at least x, $\top$, and some end of the chain of a-points starting at some point a_k. The set $F = \uparrow\{x, a_k\}$ is open no matter what x and k are. $x \in F \subseteq O$ holds. Thus, the degree is at most 2. It is not 1 since b_1 is neither isolated nor a lub of isolated points. These examples show that the domains in M^2-ALG may be arbitrarily complex.

In all the domains $\mathbf{X}_k$ where $k > 1$, and $\mathbf{Y}_n$ where $n > 0$, the binary infimum '$\sqcap$' is not continuous. Prop. 6.8.1 states that '$\sqcap$' is continuous in continuous domains. The examples above show that this fact cannot be generalized to M-CONT, not even to M^2-ALG. One might believe that the converse of Prop. 6.8.1 holds, i.e. that a complete domain with continuous '$\sqcap$' operation be continuous. This belief is however wrong; we shall meet examples for this in the theory of upper power constructions.

We have seen examples for multi-algebraic domains of any *finite* degree and for a domain that is not multi-algebraic. An example for a multi-algebraic domain of degree ∞ is provided now. The domain $\mathbf{Z}$ consists of points a_{ij} where $1 \le i \le j < \infty$ and a special point $\top$. The points a_{ij} form an infinite number of ascending chains $a_{ii} < a_{i,i+1} < \cdots$ with the common lub $\top$. All the chains are incomparable except that they rise at a diagonal chain $a_{11} < a_{22} < \cdots$ whose lub is also $\top$. The domain is depicted below; the point $\top$ is omitted for simplicity.



Every non-empty open set of $\mathbf{Z}$ contains $\top$, and hence some end of the diagonal chain from say a_{kk}. Furthermore, it then contains some ends of the first $k-1$ vertical chains. Thus, a finitary open set of size k is in it. Conversely, let $x = a_{kk}$ be a point of the finitary open set $O = \uparrow\{a_{1k}, \ldots, a_{kk}\}$. There is no finitary open set of size smaller than k between x and O. Thus, $\mathbf{Z}$ is multi-algebraic with unbounded degree of multiplicity.

8.3 The main properties of multi-continuous domains

From the three domain classes defined in the beginning of this chapter, we mainly consider the class of multi-continuous domains. Local compactness is too general to allow to prove analogous properties, whereas multi-algebraicity does not allow to prove significantly stronger statements.

The main theorem about multi-continuous domains provides several criteria equivalent to multi-continuity. Before we state and prove it, we define some auxiliary notions.

Definition 8.3.1 Let $\mathbf{X}$ be a domain. For a set A of $\mathbf{X}$, let $\mathcal{F}(A)$ be the set of all finitary upper environments of A, i.e. the set of all finitary upper sets F such that there is an open set O with $A \subseteq O \subseteq F$. For points x, $\mathcal{F}(\{x\})$ is abbreviated by $\mathcal{F}(x)$.

Theorem 8.3.2 For a domain $\mathbf{X}$ are equivalent:

(1) $\mathbf{X}$ is multi-continuous.

(2) For all points x and open sets O of $\mathbf{X}$, $x \in O$ implies there is F in $\mathcal{F}(x)$ such that $F \subseteq O$.

(3) For all points x, the set $\mathcal{F}(x)$ is $\supseteq$-directed and $\bigcap \mathcal{F}(x) = \uparrow x$ holds.

(4) For all compact sets K and open sets O of $\mathbf{X}$, $K \subseteq O$ implies there is F in $\mathcal{F}(K)$ such that $F \subseteq O$.

(5) For all compact sets K, the set $\mathcal{F}(K)$ is $\supseteq$-directed and $\bigcap \mathcal{F}(K) = \uparrow K$ holds.

Statement (3) allows to prove that our notion of multi-continuity is equivalent to the quasi-continuity of [GLS83].

Proof: Statements (1) and (2) are equivalent by the definitions of multi-continuity and $\mathcal{F}(.)$. The remaining equivalences are shown in the order (2) $\Rightarrow$ (4) $\Rightarrow$ (5) $\Rightarrow$ (3) $\Rightarrow$ (2).

(2) $\Rightarrow$ (4) : Let $K \subseteq O$ where K is compact and O is open. By (2), for every x in K, there are a finitary upper set F_x and an open set O_x such that $x \in O_x \subseteq F_x \subseteq O$. Thus, $K \subseteq \bigcup_{x\in K} O_x$ whence by compactness there is a finite set $E \subseteq K$ such that $K \subseteq \bigcup_{e\in E} O_e \subseteq F \subseteq O$ where $F = \bigcup_{e\in E} F_e$ is finitary upper and in $\mathcal{F}(K)$.

(4) $\Rightarrow$ (5) : K is a subset of the open set $\mathbf{X}$, whence $\mathcal{F}(K)$ is not empty by (4). Let F_1 and F_2 be two members of $\mathcal{F}(K)$. Then there are open sets O_1 and O_2 such that $K \subseteq O_i \subseteq F_i$. $K \subseteq O_1 \cap O_2$ follows, whence there is a set F in $\mathcal{F}(K)$ by (4) such that $K \subseteq F \subseteq O_1 \cap O_2$. $K \subseteq F \subseteq F_i$ follows, whence $\mathcal{F}(K)$ is $\supseteq$-directed.

All members of $\mathcal{F}(K)$ are upper supersets of K. Hence, $\uparrow K \subseteq F$ holds for all F in $\mathcal{F}(K)$, whence $\uparrow K \subseteq \bigcap \mathcal{F}(K)$. For the opposite inclusion $\bigcap \mathcal{F}(K) \subseteq \uparrow K$, we apply Lemma 4.4.4. Let $\uparrow K \subseteq O$ for some open set O. We have to show $\bigcap \mathcal{F}(K) \subseteq O$.

$\uparrow K \subseteq O$ implies $K \subseteq O$ whence by (4) there is F in $\mathcal{F}(K)$ such that $F \subseteq O$. Thus, $\bigcap \mathcal{F}(K) \subseteq F \subseteq O$ holds.

(5) $\Rightarrow$ (3) is immediate since $\{x\}$ is compact.

(3) $\Rightarrow$ (2) : Let x be in an open set O. By (3), $\bigcap \mathcal{F}(x) = \uparrow x \subseteq O$ holds. Since $\mathcal{F}(x)$ is a $\supseteq$-directed set of finitary upper sets, Rudin's Lemma 3.9.4 applies and states $F \subseteq O$ for some F in $\mathcal{F}(x)$. □

As an immediate conclusion of statement (4) of this theorem, one obtains:

Proposition 8.3.3 In a multi-continuous domain every compact set is strongly compact.

For multi-algebraicity, an analogous theorem holds. Instead of $\mathcal{F}(.)$, we have to consider a slightly different set:

Definition 8.3.4 Let $\mathbf{X}$ be a domain. For a set A of $\mathbf{X}$, let $\mathcal{FO}(A)$ be the set of all finitary open supersets of A. For points x, $\mathcal{FO}(\{x\})$ is abbreviated by $\mathcal{FO}(x)$.

Using this notion, we may state:

Theorem 8.3.5 For a domain $\mathbf{X}$ are equivalent:

(1) $\mathbf{X}$ is multi-algebraic.

(2) For all points x and open sets O of $\mathbf{X}$, $x \in O$ implies there is F in $\mathcal{FO}(x)$ such that $F \subseteq O$.

(3) For all points x, the set $\mathcal{FO}(x)$ is $\supseteq$-directed and $\bigcap \mathcal{FO}(x) = \uparrow x$ holds.

(4) For all compact sets K and open sets O of $\mathbf{X}$, $K \subseteq O$ implies there is F in $\mathcal{FO}(K)$ such that $F \subseteq O$.

(5) For all compact sets K, the set $\mathcal{FO}(K)$ is $\supseteq$-directed and $\bigcap \mathcal{FO}(K) = \uparrow K$ holds.

The proof of this theorem is very similar to that of Th. 8.3.2. One mainly has to replace 'continuous' by 'algebraic' and '$\mathcal{F}(.)$' by '$\mathcal{FO}(.)$', and to simplify some arguments involving a finitary set F and an open set O to arguments involving finitary open sets.

8.4 Multi-continuity and product

Generally, the Scott topology of the product of two domains cannot be directly obtained from the Scott topologies of the factors. For multi-continuous domains, this is however possible.

Theorem 8.4.1 Multi-continuous domains are $\times$-conform to each other.

Proof: Let $\mathbf{X}_1$ and $\mathbf{X}_2$ be multi-continuous domains, and let (x_1, x_2) be a point in an open set O of $\mathbf{X}_1 \times \mathbf{X}_2$. We have to show that there are open sets O_i in $\mathbf{X}_i$ such that $(x_1, x_2) \subseteq O_1 \times O_2 \subseteq O$.

By Th. 8.3.2, the sets $\mathcal{F}(x_i)$ are $\supseteq$-directed sets of finitary upper sets of $\mathbf{X}_i$, and $\bigcap \mathcal{F}(x_i) = \uparrow x_i$ holds. Let $\mathcal{F} = \{F_1 \times F_2 \mid F_i \in \mathcal{F}(x_i)\}$. $\mathcal{F}$ is also $\supseteq$-directed and consists of finitary upper sets by Prop. 4.8.1. The intersection of $\mathcal{F}$ is $\uparrow(x_1, x_2)$.

$\bigcap \mathcal{F} = \uparrow(x_1, x_2) \subseteq O$ implies $F \subseteq O$ for some F in $\mathcal{F}$ by Rudin's Lemma 3.9.4. $F = F_1 \times F_2$ holds for some F_i in $\mathcal{F}(x_i)$. Hence, there are open sets O_i in $\mathbf{X}_i$ such that $x_i \in O_i \subseteq F_i$. Thus, we get $(x_1, x_2) \in O_1 \times O_2 \subseteq F_1 \times F_2 = F \subseteq O$. □

Using this theorem, we may show that some subclasses of M-CONT are closed w.r.t. products.

Theorem 8.4.2
The classes ALG, CONT, M-ALG, and M-CONT are closed w.r.t. finite products.

Proof: Let $\mathbf{X}_1$ and $\mathbf{X}_1$ be the factors, and let $x = (x_1, x_2)$ be a point of some open set O in $\mathbf{X}_1 \times \mathbf{X}_2$. By Th. 8.4.1, there are open sets O_i in $\mathbf{X}_i$ such that $x \in O_1 \times O_2 \subseteq O$, whence x_i in O_i.

M-CONT: x_i in O_i open implies there are open sets O_i' and finitary sets F_i such that $x_i \in O_i' \subseteq F_i \subseteq O_i$. Hence, $x \in O_1' \times O_2' \subseteq F_1 \times F_2 \subseteq O_1 \times O_2 \subseteq O$ holds as required. $O_1' \times O_2'$ is open and $F_1 \times F_2$ is finitary by Prop. 4.8.1.

M-ALG: x_i in O_i open implies there are open finitary sets F_i such that $x_i \in F_i \subseteq O_i$. Hence, $x \in F_1 \times F_2 \subseteq O_1 \times O_2 \subseteq O$ holds as required. $F_1 \times F_2$ is finitary open by Prop. 4.8.1.

The proofs for ALG and CONT are analogous; one only has to replace the finitary sets by upper cones. □

8.5 Sobriety in terms of irreducible closed sets

The notion of sobriety is closely related with upper power domain constructions as pointed out in [Smy83]. Some authors e.g. [Law88] define sobriety via irreducible closed sets, whereas others e.g. [Smy78] define it via prime filters. The equivalence of both definitions is shown in [HM81] by means of lattice theory. We present both definitions together with new, more direct proofs of their equivalence.

We first present the definition by means of irreducible closed sets. As often, there are several equivalent definitions of this concept.

Proposition 8.5.1 Let $\mathbf{X}$ be a topological space. For a closed set C of $\mathbf{X}$, the following statements are equivalent.

(1) $C = A \cup B$ for closed sets A and B implies $C = A$ or $C = B$.

(2) $C \subseteq A \cup B$ for closed sets A and B implies $C \subseteq A$ or $C \subseteq B$.

(3) Whenever two open sets meet C, their intersection meets C.

Closed sets with these properties are called *irreducible closed.*

Proof: Let C be a closed set satisfying (1) and let O_1 and O_2 be two open sets with $O_i \cap C \neq \emptyset$. Let $C_i = C \setminus O_i$. The sets C_i are closed since they are intersections of C and the closed complements of O_i. $C_i \neq C$ holds since at least one point of C is in O_i. Since C is irreducible, $C_1 \cup C_2$ cannot be C, i.e. there is a point x in C with $x \notin C_1 \cup C_2$, whence $x \in O_1 \cap O_2 \cap C$.

Let C be a closed set satisfying (3). Let $C \subseteq C_1 \cup C_2$ for two closed sets C_1 and C_2. If O_i is the complement of C_i, then $O_1 \cap O_2$ is the complement of $C_1 \cup C_2 \supseteq C$, whence $C \cap O_1 \cap O_2 = \emptyset$. By the criterion, $C \cap O_i = \emptyset$ holds for at least one i in $\{1, 2\}$, whence $C \subseteq C_i$.

Let $C = C_1 \cup C_2$. Then $C \subseteq C_1 \cup C_2$ holds, whence by (2), $C \subseteq C_i$ for some i. $C_i \subseteq C$ also holds. $\square$

Proposition 8.5.2 Let $f : [\mathbf{X} \to \mathbf{Y}]$ be continuous. If C is irreducible closed in $\mathbf{X}$, then $\mathsf{cl}\, f[C]$ is irreducible closed in $\mathbf{Y}$.

Proof: Let $\mathsf{cl}\, f[C] \subseteq C_1 \cup C_2$. Then $C \subseteq f^{-1}[C_1] \cup f^{-1}[C_2]$ follows, whence $C \subseteq f^{-1}[C_k]$ for some k in $\{1, 2\}$. This inclusion implies $f[C] \subseteq C_k$, whence $\mathsf{cl}\, f[C] \subseteq C_k$. $\square$

Closures of singleton sets are irreducible.

Proposition 8.5.3 For all x in $\mathbf{X}$, $\mathsf{cl}\, x$ is irreducible closed.

Proof: Let $\mathsf{cl}\, x \subseteq A \cup B$. Then x is in one of A and B, say A. A being closed implies $\mathsf{cl}\, x \subseteq A$. $\square$

The class of sober topological spaces is characterized by the converse of the proposition above.

Definition 8.5.4 A topological space is *sober* iff all non-empty irreducible closed sets are the closure of a unique point.

Sobriety may be considered a separation property. As such, it is between (T0) and (T2), but incomparable with (T1).

Proposition 8.5.5 Every sober space satisfies axiom (T0).

Proof: Let $\mathsf{cl}\, x = \mathsf{cl}\, y$. Set $\mathsf{cl}\, x$ is irreducible closed by Prop. 8.5.3. By sobriety, there is exactly one point p with $\mathsf{cl}\, x = \mathsf{cl}\, p$. Hence, $x = y$ follows. $\square$

Proposition 8.5.6 Every (T2)-space is sober.

Proof: Let C be an irreducible closed set. Assume there were two different points x_1 and x_2 in C. By (T2), there would be two disjoint open sets O_1 and O_2 with $x_i \in O_i$. Hence, both open sets meet C, but their intersection does not because it is empty. This would be a contradiction to Prop. 8.5.1. Thus, the assumption is wrong, i.e. all non-empty irreducible closed sets are singletons. Hence, they are closures of a unique point. $\square$

Alternatively, sobriety may be considered a completeness property since it states that certain points exist.

8.6 Filters of open sets

The notion of filters is fundamental in topology. Usually, topological filters are sets of sets satisfying the properties of the definition below. The filters we are going to define here however contain only open sets.

Definition 8.6.1 (Filters) Let $\mathbf{X}$ be a topological space. A *filter* in $\mathbf{X}$ is a set $\mathcal{O}$ of open sets satisfying the following properties:

(1) $\mathbf{X} \in \mathcal{O}$

(2) If O_1 and O_2 are in $\mathcal{O}$, then so is their intersection $O_1 \cap O_2$.

(3) If O is in $\mathcal{O}$ and $O \subseteq O'$ holds for some open set O', then O' is in $\mathcal{O}$.

In domain-theoretic view, a filter is an ideal in the poset $(\Omega\mathbf{X}, \supseteq)$. We distinguish different classes of filters:

Definition 8.6.2

- A filter $\mathcal{O}$ is *comprising* iff for all open sets O, $O \supseteq \bigcap \mathcal{O}$ implies $O \in \mathcal{O}$.
- A filter $\mathcal{O}$ is *open* iff for all $\subseteq$-directed sets $\mathcal{D}$ of open sets, $\bigcup \mathcal{D} \in \mathcal{O}$ implies $\mathcal{D} \cap \mathcal{O} \neq \emptyset$.
- A filter $\mathcal{O}$ is *prime* iff for all sets $\mathcal{S}$ of open sets, $\bigcup \mathcal{S} \in \mathcal{O}$ implies $\mathcal{S} \cap \mathcal{O} \neq \emptyset$.

The notion 'open' coincides with the notion of Scott open sets in the domain $(\Omega\mathbf{X}, \subseteq)$.

Filters naturally occur as sets of open environments of sets.

Proposition 8.6.3 For every set $A \subseteq \mathbf{X}$, the set $\mathcal{O}(A) = \{O \text{ open} \mid O \supseteq A\}$ of open environments of A is a comprising filter with intersection $\bigcap \mathcal{O}(A) = \uparrow A$. Conversely, if $\mathcal{O}$ is a comprising filter, then $\bigcap \mathcal{O}$ is an upper set, and $\mathcal{O} = \mathcal{O}(\bigcap \mathcal{O})$ holds.

Proof: $A \subseteq \mathbf{X}$ holds. $A \subseteq O_1$ and $A \subseteq O_2$ imply $A \subseteq O_1 \cap O_2$. $A \subseteq O$ and $O \subseteq O'$ imply $A \subseteq O'$.

$\bigcap \mathcal{O}(A) = \uparrow A$ holds by Lemma 4.4.4. If $O \supseteq \uparrow A$, then obviously $O \supseteq A$, i.e. $O \in \mathcal{O}(A)$. Thus, $\mathcal{O}(A)$ is comprising.

For the converse direction, $\bigcap \mathcal{O}$ is an upper set as intersection of upper sets. $O \in \mathcal{O}$ implies $O \supseteq \bigcap \mathcal{O}$. The opposite inclusion is the definition of 'comprising'. □

The next proposition is as trivial as the previous one.

Proposition 8.6.4 A set K is compact iff $\mathcal{O}(K)$ is a comprising open filter. Hence, if $\mathcal{O}$ is a comprising open filter, then $\bigcap \mathcal{O}$ is a compact upper set.

Proof: The equivalence holds since $\mathcal{O}(K)$ being open is just statement (2) of Prop. 4.6.1, the definition of compactness. The 'hence' part is due to Prop. 8.6.3. □

An analogue statement for prime filters is a bit more difficult (and also more useful).

Lemma 8.6.5 If a is a point, then $\mathcal{O}(a)$ is a comprising prime filter. Conversely, if $\mathcal{O}$ is a comprising prime filter, then there is a point a such that $\bigcap \mathcal{O} = \uparrow a$ and thus $\mathcal{O} = \mathcal{O}(a)$.

Proof: $\mathcal{O}(a)$ is prime since $a \in \bigcup \mathcal{S}$ implies $a \in O$ for some $O \in \mathcal{S}$.

For the converse statement, let $\mathcal{S} = \{\mathsf{co}\downarrow x \mid x \in \bigcap \mathcal{O}\}$. $\mathcal{S}$ is a set of open sets. Assume $\bigcap \mathcal{O} \subseteq \bigcup \mathcal{S}$. Then $\bigcup \mathcal{S}$ is in $\mathcal{O}$ since $\mathcal{O}$ is comprising, whence there is M in $\mathcal{S}$ with $M \in \mathcal{O}$ since $\mathcal{O}$ is prime. Hence, $M \supseteq \bigcap \mathcal{O}$. Remember $M = \mathsf{co}\downarrow x$ for some $x \in \bigcap \mathcal{O}$. For this x, $x \notin \downarrow x$ would hold.

Thus, $\bigcap \mathcal{O}$ is not a subset of $\bigcup \mathcal{S}$. This means, there is a in $\bigcap \mathcal{O}$ with $a \notin \bigcup \mathcal{S}$, i.e. for all M in $\mathcal{S}$, $a \notin M$. Hence, for all $x \in \bigcap \mathcal{O}$, $a \in \downarrow x$ holds, or equivalently $x \in \uparrow a$. Thus, $\bigcap \mathcal{O} \subseteq \uparrow a$. Conversely, $a \in \bigcap \mathcal{O}$ and this set being upper implies $\uparrow a \subseteq \bigcap \mathcal{O}$. □

8.7 Filters and sobriety

Filters induce equivalent definitions of sobriety:

Theorem 8.7.1 For a (T0)-space $\mathbf{X}$, the following statements are equivalent:

(1) For every non-empty irreducible closed set C, there is a point c such that $C = \mathsf{cl}\, c$. ($\mathbf{X}$ is sober.)

(2) Every open filter is comprising.

(3) Every prime filter is comprising.

Proof:

$(1) \Rightarrow (2)$: Let $\mathcal{O}$ be an open filter. Let $\mathcal{O}' = \{O \text{ open} \mid O \supseteq \bigcap \mathcal{O},\ O \notin \mathcal{O}\}$ be the set of all witnesses for $\mathcal{O}$ not being comprising. We have to show $\mathcal{O}' = \emptyset$.

Assume $\mathcal{O}'$ were not empty. Let $\mathcal{D}$ be a $\subseteq$-directed subset of $\mathcal{O}'$. We claim $\bigcup \mathcal{D} \in \mathcal{O}'$. $\bigcup \mathcal{D}$ is certainly open and a superset of $\bigcap \mathcal{O}$. If $\bigcup \mathcal{D}$ were in $\mathcal{O}$, then some member of $\mathcal{D}$ would be in $\mathcal{O}$ since $\mathcal{O}$ is open. Thus, $\mathcal{O}'$ is a directed complete poset, a domain.

By Zorn's Lemma 3.2.2, there is a maximal element M in $\mathcal{O}'$. Let C be the complement of M. We claim C is irreducible closed. C is closed as complement of the open set M. Let $C = C_1 \cup C_2$ with closed sets C_i. By complementing, one obtains $M = M_1 \cap M_2$ with open sets M_i. $M_i \supseteq M \supseteq \bigcap \mathcal{O}$ holds. If both M_1 and M_2 were in $\mathcal{O}$, then so would be their intersection M since $\mathcal{O}$ is a filter (condition (2)). Thus, one of M_1 and M_2 — say M_1 — is in $\mathcal{O}'$. By maximality of M, $M_1 = M$ holds, whence $C_1 = C$.

Hence, we got an irreducible closed set C. C is not empty since otherwise M would be $\mathbf{X}$ contradicting $\mathbf{X} \in \mathcal{O}$ (condition (1) of filters). By (1), there is a point c with $C = \mathsf{cl}\, c$. c is not in M since M is the complement of C. $\bigcap \mathcal{O} \subseteq M$ implies $c \notin \bigcap \mathcal{O}$, i.e. there is a set $O \in \mathcal{O}$ with $c \notin O$. Hence, c is in $\mathsf{co}\, O$. This set is closed, whence $\mathsf{cl}\, c = C \subseteq \mathsf{co}\, O$. By complementing, $O \subseteq M$ follows. By condition (3) of filters, M itself is in $\mathcal{O}$ — a contradiction.

$(2) \Rightarrow (3)$: Every prime filter is open, and every open filter is comprising by (2).

$(3) \Rightarrow (1)$: Let C be a non-empty irreducible closed set, and let $\mathcal{O} = \{O \text{ open} \mid C \cap O \neq \emptyset\}$. We show $\mathcal{O}$ is a prime filter.

(1) $\mathbf{X} \in \mathcal{O}$ since $C \neq \emptyset$.

(2) If O_1 and O_2 are in $\mathcal{O}$, then so is their intersection $O_1 \cap O_2$ by Prop. 8.5.1.

(3) If O is in $\mathcal{O}$, i.e. $C \cap O \neq \emptyset$, and $O \subseteq O'$ holds for some open set O', then $C \cap O' \neq \emptyset$, whence O' is in $\mathcal{O}$.

(4) If $\mathcal{S}$ is a set of open sets such that C meets $\bigcup \mathcal{S}$, then C meets some member of $\mathcal{S}$. Hence, $\mathcal{O}$ is prime.

By precondition (3), $\mathcal{O}$ is comprising. By Lemma 8.6.5, there is a point c such that $\bigcap \mathcal{O} = \uparrow c$. Let O be the complement of $\mathsf{cl}\, c$. If O were in $\mathcal{O}$, then $c \in \bigcap \mathcal{O} \subseteq O$ would imply $c \notin \mathsf{cl}\, c$. Hence, $C \cap O = \emptyset$, whence $C \subseteq \mathsf{cl}\, c$.

If U is any open environment of c, then $\bigcap \mathcal{O} = \uparrow c \subseteq U$ implies $U \in \mathcal{O}$ since $\mathcal{O}$ is comprising. Thus, every open environment of c meets C, whence c is in $\mathsf{cl}\, C = C$. $c \in C$ implies $\mathsf{cl}\, c \subseteq C$. Together, $C = \mathsf{cl}\, c$ follows. $\square$

8.8 Sober domains

After having established the main theorem about sobriety, we now turn our attention to sober domains. A domain is sober iff the corresponding Scott space is. The class of sober domains is called SOB. It is quite large as the following theorem shows. Nevertheless, non-sober domains exist. We present one in section 8.11.

Theorem 8.8.1 Every multi-continuous domain is sober.

Proof: Let C be a non-empty irreducible closed set of the multi-continuous domain $\mathbf{X}$. Let $\mathcal{F} = \bigcup_{c \in C} \mathcal{F}(c)$. From multi-continuity, we know every set $\mathcal{F}(c)$ is $\supseteq$-directed with intersection $\uparrow c$. We demonstrate the set $\mathcal{F}$ to be $\supseteq$-directed, too.

$\mathcal{F}$ is not empty since C is not empty and every set $\mathcal{F}(c)$ is not empty. Let F_1, F_2 be in $\mathcal{F}$. By definition, there are open sets O_1, O_2 and points $c_1, c_2 \in C$ with $c_i \in O_i \subseteq F_i$. Hence, both open sets O_1 and O_2 meet C whence their intersection meets C because C is irreducible closed (Prop. 8.5.1). Thus, there is a point c in C with $c \in O_1 \cap O_2$. By multi-continuity, there is F' in $\mathcal{F}(c) \subseteq \mathcal{F}$ with $F' \subseteq O_1 \cap O_2 \subseteq F_1 \cap F_2$. F' is a common lower bound of F_1 and F_2 in $\mathcal{F}$.

$\mathcal{F}$ is a $\supseteq$-directed set of upper finitary sets that all meet C. By Rudin's Lemma 3.9.3, the intersection $\bigcap \mathcal{F}$ meets C. Let x be a point of $C \cap \bigcap \mathcal{F}$. Then $\mathsf{cl}\, x \subseteq C$ holds.

Assume C were not a subset of $\mathsf{cl}\, x$. Then, there would be a point c in C lying in $\mathsf{co}\,(\mathsf{cl}\, x)$. By multi-continuity, there is F in $\mathcal{F}(c) \subseteq \mathcal{F}$ with $F \subseteq \mathsf{co}\,(\mathsf{cl}\, x)$. F in $\mathcal{F}$ implies $x \in \bigcap \mathcal{F} \subseteq F \subseteq \mathsf{co}\,(\mathsf{cl}\, x)$ — a contradiction.

Hence, C is a subset of $\mathsf{cl}\, x$, and together, $C = \mathsf{cl}\, x$ follows. □

An analogous proof for locally compact domains is not correct since Rudin's Lemma is not generally true if finitary sets are replaced by compact sets (compare section 8.11).

In section 8.2, we presented a domain $\mathbf{X}_\infty$ consisting of a least point $\perp$, a greatest point $\top$, and an infinite number of incomparable ascending chains with lub $\top$. This domain is not multi-continuous, but still sober. If a closed set C contains infinitely many points of at least one chain, then it contains $\top$ and is a lower cone. Otherwise, it contains only finitely many points of all chains. If it meets at least two chains, then it is not irreducible, and otherwise, it is a lower cone.

Proposition 8.8.2 The class SOB is retract closed.

Proof: Let $\mathbf{X}$ be sober and $\mathbf{Y}$ a retract of $\mathbf{X}$ by means of $r : [\mathbf{X} \to \mathbf{Y}]$ and $e : [\mathbf{Y} \to \mathbf{X}]$. If C is non-empty irreducible closed in $\mathbf{Y}$, then $\mathsf{cl}\, e[C]$ is non-empty irreducible closed in $\mathbf{X}$ by Prop. 8.5.2. Hence, there is x in $\mathbf{X}$ such that $\mathsf{cl}\, e[C] = \mathsf{cl}\, x = \downarrow x$. We claim $C = \downarrow rx$.

If c is a point of C, then ec is in $e[C] \subseteq \downarrow x$, whence $ec \leq x$. Thus, $c = r(ec) \leq rx$ follows. Now, we proved $C \subseteq \downarrow rx$.

Assume rx were not in C. Then rx is in $\mathsf{co}\, C$, whence x is in $r^{-1}[\mathsf{co}\, C]$. x is also in $\mathsf{cl}\, e[C]$. By Prop. 4.2.2, the open environment $r^{-1}[\mathsf{co}\, C]$ meets $e[C]$. Let u be a point of the intersection. Then ru is both in $\mathsf{co}\, C$ and in $r[e[C]] = C$ — a contradiction. $rx \in C$ implies $\downarrow rx \subseteq C$. □

It is possible to prove that the product of two $\times$-conform sober domains is sober. We did however not succeed in proving or disproving the sobriety of the product of arbitrary sober domains, nor could we generalize Prop. 8.4.1 stating the $\times$-conformity of multi-continuous domains to sober domains.

Problem 4 Is the product of sober domains sober?

Problem 5 Is there any relation between $\times$-conformity and sobriety?

8.9 The Rudin classes S-RD and K-RD

Rudin's Lemma in the version of Lemma 3.9.3 or 3.9.4 deals with finitary upper sets. Analogous statements involving compact or strongly compact sets instead do not generally hold as we shall see in section 8.11. We now define classes of domains characterized by these conditions.

Definition 8.9.1 (Rudin classes) The class S-RD of domains is defined by the condition: for all $\supseteq$-directed sets $\mathcal{D}$ of *strongly compact upper sets* and open sets O, $\bigcap \mathcal{D} \subseteq O$ implies $S \subseteq O$ for some S in $\mathcal{D}$.

The class K-RD of domains is defined analogously by means of compact sets instead of strongly compact sets.

Equivalent conditions are provided by using closed sets instead of open sets: Let $\mathcal{D}$ be a $\supseteq$-directed set of (strongly) compact upper sets. If all members of $\mathcal{D}$ meet a closed set C, then $\bigcap \mathcal{D}$ meets C. The equivalence is proved by letting C be the complement of O and vice versa (cf. the proof of Lemma 3.9.4 from Lemma 3.9.3).

By Lemma 3.9.4, a corresponding class F-RD defined by finitary sets contains all domains. Since strongly compact sets are compact, the class K-RD is a subclass of S-RD. The class S-RD is very large since it contains both SOB and FC (see below); section 8.11 will however provide an example of a domain not contained in S-RD.

Theorem 8.9.2 Every sober domain satisfies property K-RD.

Proof: Let $\mathbf{X}$ be a sober domain and $\mathcal{D}$ a $\supseteq$-directed set of compact upper sets such that $\bigcap \mathcal{D} \subseteq O$ for some open set O. For every K in $\mathcal{D}$, the set $\mathcal{O}(K)$ is an open filter by Prop. 8.6.4, i.e. open in $(\Omega\mathbf{X}, \subseteq)$ and directed in $(\Omega\mathbf{X}, \supseteq)$. Let $\mathcal{O} = \bigcup_{K \in \mathcal{D}} \mathcal{O}(K)$. $\mathcal{O}$ is again open in $(\Omega\mathbf{X}, \subseteq)$, and by Prop. 3.1.5, directed in $(\Omega\mathbf{X}, \supseteq)$. Thus $\mathcal{O}$ is an open filter. Since $\mathbf{X}$ is sober, $\mathcal{O}$ is comprising.

For the open set O, $O \supseteq \bigcap \mathcal{D} = \bigcap\{\bigcap \mathcal{O}(K) \mid K \in \mathcal{D}\} = \bigcap(\bigcup_{K \in \mathcal{D}} \mathcal{O}(K)) = \bigcap \mathcal{O}$ holds. Since $\mathcal{O}$ is comprising, $O \supseteq \bigcap \mathcal{O}$ implies $O \in \mathcal{O}$, i.e. $O \supseteq K$ for some K in $\mathcal{D}$. □

Theorem 8.9.3 Every finitarily complete domain is in S-RD.

Proof: For every strongly compact set S let $\mathcal{F}'(S)$ be the set of finitary upper supersets of S.[1] By Prop. 5.4.4, $\mathcal{F}'(S)$ is $\supseteq$-directed and its intersection is S.

[1] $\mathcal{F}'(S)$ contains *all* finitary upper supersets, whereas $\mathcal{F}(S)$ contains the finitary upper *environments* only.

Let $\mathcal{D}$ be a $\supseteq$-directed set of strongly compact sets. Then $\{\mathcal{F}'(S) \mid S \in \mathcal{D}\}$ forms a $\subseteq$-directed set of $\supseteq$-directed sets of finitary upper sets. By Prop. 3.1.5, its union $\mathcal{F}$ is $\supseteq$-directed. By set-theoretic arguments, $\bigcap \mathcal{F}$ equals $\bigcap \mathcal{D}$. Thus, if $\bigcap \mathcal{D}$ is a subset of some open set O, then by Lemma 3.9.4, there is some F in $\mathcal{F}$ such that $F \subseteq O$. F is contained in $\mathcal{F}'(S)$ for some S in $\mathcal{D}$. For this S, $S = \bigcap \mathcal{F}'(S) \subseteq F \subseteq O$ holds. $\Box$

The compact sets of domains in K-RD form a domain when ordered by '$\supseteq$':

Proposition 8.9.4 Let $\mathbf{X}$ be a domain in K-RD (S-RD). Then the intersection of a $\supseteq$-directed set of (strongly) compact upper sets is (strongly) compact.

Proof: Let $\mathcal{D}$ be a $\supseteq$-directed set of strongly compact upper sets in a domain in S-RD. Let O be an open set with $\bigcap \mathcal{D} \subseteq O$. By S-RD, there is some S in $\mathcal{D}$ with $S \subseteq O$. Since S is strongly compact, there is a finitary set F with $\bigcap \mathcal{D} \subseteq S \subseteq F \subseteq O$.

Let $\mathcal{D}$ be a $\supseteq$-directed set of compact upper sets in a domain in K-RD. Let $\mathcal{O}$ be an open cover of $\bigcap \mathcal{D}$, i.e. $\bigcap \mathcal{D} \subseteq \bigcup \mathcal{O}$. $\bigcup \mathcal{O}$ is an open set. By K-RD, there is some K in $\mathcal{D}$ with $K \subseteq \bigcup \mathcal{O}$. As K is compact, there is finite subset $\mathcal{F}$ of $\mathcal{O}$ with $K \subseteq \bigcup \mathcal{F}$. By $\bigcap \mathcal{D} \subseteq K$, $\mathcal{F}$ also covers $\bigcap \mathcal{D}$. $\Box$

Proposition 8.9.5 The classes K-RD and S-RD are retract closed.

Proof: Let $\mathbf{X}$ be in K-RD and $\mathbf{Y}$ a retract of $\mathbf{X}$ by means of $r : [\mathbf{X} \to \mathbf{Y}]$ and $e : [\mathbf{Y} \to \mathbf{X}]$. Let $\mathcal{K}$ be a $\supseteq$-directed set of compact upper sets of $\mathbf{Y}$ such that $\bigcap \mathcal{K} \subseteq O$ for some open set O. Let $\mathcal{K}' = \{\uparrow e[K] \mid K \in \mathcal{K}\}$. $\mathcal{K}'$ is a $\supseteq$-directed set of compact upper sets by Prop. 4.6.5 and 4.6.2. If x is in $\bigcap \mathcal{K}'$, then for all K in $\mathcal{K}$, there is some y_K in K such that $x \geq e y_K$. $rx \geq r(e y_K) = y_K \in K$ holds, whence $rx \in K$ since K is upper. Thus, rx is in $\bigcap \mathcal{K}$.

Now, we obtained $\bigcap \mathcal{K}' \subseteq r^{-1}[\bigcap \mathcal{K}] \subseteq r^{-1}[O]$. Since $\mathbf{X}$ is in K-RD, there is some K' in $\mathcal{K}'$ such that $K' \subseteq r^{-1}[O]$. Hence, there is K in $\mathcal{K}$ such that $e[K] \subseteq r^{-1}[O]$. This means, $y = r(ey) \in O$ holds for all y in K, i.e. $K \subseteq O$.

The proof for S-RD is analogous. $\Box$

In the region of these large classes, we met a number of open problems:

Problem 6 Are the classes SOB, K-RD, and S-RD different at all?

Problem 7 Are the classes K-RD and S-RD closed w.r.t. products?

Problem 8 Is FC (or any of the smaller classes BC or CC) a subclass of K-RD or of SOB?

8.10 Combinations of completeness and Rudin classes

In this section, we prove two Lemmas that hold for domains that are in SC & S-RD or in KC & K-RD. The first of these two classes contains FC as a subclass by Prop. 8.9.3, and also F-CONT by Prop. 7.2.2 and the inclusions F-CONT $\subset$ CONT $\subset$ M-CONT $\subset$ SOB $\subseteq$ K-RD $\subseteq$ S-RD. The classes KC & K-RD and SC & S-RD are not obviously comparable. SC is a subclass of KC, but conversely K-RD is a subclass of S-RD.

The first Lemma tells that certain lower sets are closed.

Lemma 8.10.1 Let $\mathbf{X}$ be a domain in SC & S-RD or in KC & K-RD. Let F be a finitary upper set and C a closed set of $\mathbf{X}$. Then $\mathsf{cl}\,(C \cap F) = \downarrow(C \cap F)$ holds.

Proof: We present the proof for SC & S-RD. The other proof is analogous.

$\downarrow(C \cap F) \subseteq \mathsf{cl}\,(C \cap F)$ holds by Prop. 4.4.2. For the opposite inclusion, we have to show that $\downarrow(C \cap F)$ is Scott closed.

Let D be a directed subset of $\downarrow(C \cap F)$. Then for every d in D, there is a point x_d in $C \cap F$ above d. Hence, for all d in D, the set $\uparrow d \cap F$ meets C. The set $\{\uparrow d \cap F \mid d \in D\}$ is $\supseteq$-directed since D is directed. By SC, it consists of strongly compact sets only. By S-RD, its intersection meets C since all its members meet C.

Let x be a point in $C \cap \bigcap_{d\in D}(\uparrow d \cap F)$. Then x is in $C \cap F$ and also in $\bigcap_{d \in D} \uparrow d$. Thus, x is an upper bound of D, whence it is above $\bigsqcup D$. $\bigsqcup D \leq x \in C \cap F$ implies $\bigsqcup D \in \downarrow(C \cap F)$. □

The Lemma has some interesting corollaries. A particular special case is $C = \mathbf{X}$. Then the Lemma tells that $\downarrow F$ is closed for all finitary upper sets F. Every upper cone is finitary, whence $\downarrow\uparrow x$ is closed for all points x. In other words, if all members of a directed set D have a common upper bound with x, then $\bigsqcup D$ has a common upper bound with x.

The next Lemma tells that the lower closure operator commutes over certain intersections.

Lemma 8.10.2 Let $\mathbf{X}$ be a domain in SC & S-RD or in KC & K-RD. Let $\mathcal{F}$ be a $\supseteq$-directed set of finitary upper sets and C a closed set of $\mathbf{X}$. Then $\downarrow(\bigcap_{F\in\mathcal{F}}(C \cap F)) = \bigcap_{F\in\mathcal{F}} \downarrow(C \cap F)$ holds.

Proof: Again, we present the proof for SC & S-RD.

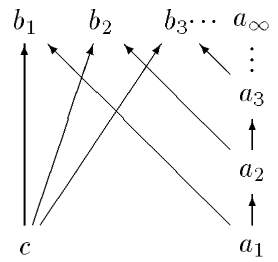
For every F in $\mathcal{F}$, the intersection $\bigcap_{F\in\mathcal{F}} C \cap F$ is a subset of $C \cap F$. Applying '$\downarrow$', the left hand side is a subset of $\downarrow(C \cap F)$ for all F in $\mathcal{F}$, whence the inclusion '$\subseteq$' follows.

For the opposite inclusion, let x be a member of the right hand side. Then for all F in $\mathcal{F}$, x is in $\downarrow(C \cap F)$, whence $\uparrow x$ meets $C \cap F$. Or the other way round, for all F in $\mathcal{F}$, $\uparrow x \cap F$ meets C. Since $\mathcal{F}$ is $\supseteq$-directed, $\{\uparrow x \cap F \mid F \in \mathcal{F}\}$ is $\supseteq$-directed, too. By SC, it consists of strongly compact sets only, and by S-RD, its intersection meets C.

Let y be a point in $C \cap \bigcap_{F\in\mathcal{F}}(\uparrow x \cap F)$. Hence, y is in both $\uparrow x$ and $\bigcap_{F\in\mathcal{F}}(C \cap F)$. Thus, x is in $\downarrow(\bigcap_{F\in\mathcal{F}}(C \cap F))$. □

In the special case $C = \mathbf{X}$, the Lemma tells $\downarrow\bigcap\mathcal{F} = \bigcap\downarrow[\mathcal{F}]$ for all $\supseteq$-directed sets $\mathcal{F}$ of finitary upper sets. This is a kind of continuity statement for the operation '$\downarrow$'. If the finitary upper sets are restricted to upper cones, we obtain $\downarrow\uparrow(\bigsqcup D) = \bigcap_{d\in D} \downarrow\uparrow d$ for all directed sets D.

To show the dependence of the two lemmas on property SC, we present some example domain. Let $\mathbf{X} = \{a_1, a_2, a_3, \ldots, a_\infty, b_1, b_2, b_3, \ldots, c\}$. There is no point b_∞. The a-points form an ascending sequence: $a_1 < a_2 < \cdots < a_\infty$, whereas the b-points are incomparable. Every a-point is below the corresponding b-point: $a_n < b_n$. The remaining point c is below all b-points, but not below any a-point, not even below a_∞. The domain is visualized by the following picture:



This domain is algebraic. The condition SC is not satisfied since $\uparrow c \cap \uparrow a_1 = \{b_1, b_2, \ldots\}$ is an infinite discrete open set, whence it cannot be strongly compact. The set $\uparrow c$ is finitary, but $\downarrow\uparrow c$ is not closed since it contains all points a_n except a_∞. The sets $\uparrow a_n$ where $n < \infty$ provide a $\supseteq$-directed set of finitary sets. c is in the intersection of the sets $\downarrow\uparrow a_n$, but not in $\downarrow \bigcap_{n\in\mathbf{N}} \uparrow a_n = \downarrow a_\infty$.

The defect of this domain seems to be cured by establishing the additional inequality $c \leq a_\infty$. The resulting domain is indeed in SC, but no longer algebraic. It is multi-algebraic of degree 2. Hence, it is in SC & S-RD, and the two Lemmas may be applied.

8.11 A non-sober domain

An example for a non-sober domain was found by Johnstone and published in [Joh81] and [Joh82]. It is also not in S-RD.

The domain $\mathbf{X}$ consists of a least element $\bot$, points a_{ij} where $1 \leq i, j < \infty$, and points b_k where $1 \leq k < \infty$. For every i, the points a_{ij} form an ascending chain $A_i = \{a_{i1} < a_{i2} < \cdots\}$ with lub b_i. The chains are incomparable to each other. In addition, all points of level number j, i.e. $L_j = \{a_{1j}, a_{2j}, \ldots\}$, are below the points b_j, b_{j+1}, etc. Hence, the set of upper bounds of a point a_{ij} is $\uparrow a_{ij} = \{a_{ij}, a_{i,j+1}, a_{i,j+2}, \ldots, b_i, b_j, b_{j+1}, b_{j+2}, \ldots\}$. The points b_k are maximal. Their lower cone is $\downarrow b_k = \{b_k\} \cup \{a_{kj} \mid 1 \leq j < \infty\} \cup \{a_{ij} \mid 1 \leq i < \infty, 1 \leq j \leq k\} \cup \{\bot\}$.

The whole domain is closed. We show it is irreducible closed. Let $\mathbf{X} = A \cup B$ where A and B are closed. One of A and B contains an infinite number of b-points. Because level L_k is below point b_k, it thus contains an infinite number of levels, whence all levels. Thus, it also contains all chains A_i and so all limit points b_i. Summarizing, it contains all of $\mathbf{X}$. Thus, there is an irreducible closed set that obviously is not a lower cone. The corresponding prime filter is the set of all non-empty open sets.

Next, we show that $\mathbf{X}$ is not in S-RD. The sets $B_k = \{b_k, b_{k+1}, b_{k+2}, \ldots\}$ are all strongly compact. For, if $B_k \subseteq O$ for some open set O, then O contains some member of the chain A_k, say a_{kl} where $l > k$. The points b_l, b_{l+1}, etc. are all above a_{kl}. Hence, $B_k \subseteq F_{kl} = \uparrow\{a_{kl}, b_{k+1}, b_{k+2}, \ldots, b_{l-1}\} \subseteq O$. The set of all sets B_k is $\supseteq$-directed and its intersection is $\emptyset$. $\emptyset$ is open, whence some B_k would have to be empty if $\mathbf{X}$ were in S-RD. The sets F_{kl} prove that B_k may also be reached as $\supseteq$-directed intersection of finitary upper sets, namely the sets F_{kl} where $l > k$.

Finally, we consider the completeness properties of $\mathbf{X}$. Since we let $\mathbf{X}$ contain a least element, the whole domain is an upper cone. Let x and y be two points of $\mathbf{X}$, and $U = \uparrow x \cap \uparrow y$

be the set of their common upper bounds. If $x \leq y$, then $U = \uparrow y$ is an upper cone. If $y = b_k$ for some k, then $U \subseteq \{b_k\}$ is empty or an upper cone since b_k is maximal. The interesting case is $x = a_{ij}$ and $y = a_{i'j'}$ where $i \neq i'$. Let k be the maximum of j and j'. Then $B_k \subseteq U \subseteq \{b_i, b_{i'}\} \cup B_k$ holds where the exact value of U depends on the relative sizes of i, i', j, and j'. In any case, U is strongly compact. Hence, domain $\mathbf{X}$ is in SC, but not in FC. Thus, $\mathbf{X}$ shows that class S-RD is not a superclass of SC.

8.12 The domain hierarchy

In this chapter and the previous ones, many domain classes were introduced. In this section, we shortly repeat their definitions and provide pictures showing their relationships. The definitions in the following list are not meant to be mathematically exact since quantifiers are missing. The domain under consideration is $\mathbf{X}$, x and y are points, O, O_1 etc. are open sets, C closed sets, F finitary upper sets, S strongly compact sets, and K compact sets.

DOM	(domains)	all domains
DIS	discrete	$x \leq y$ implies $x = y$
FIN	finite	$\mathbf{X}$ is finite
CC	(cone) complete	$\mathbf{X}$ and $\uparrow x \cap \uparrow y$ are upper cones
BC	bounded complete	$\mathbf{X}$ and $\uparrow x \cap \uparrow y$ are empty or upper cones
FC	finitarily complete	$\mathbf{X}$ and $\uparrow x \cap \uparrow y$ are finitary
SC	strongly compactly complete	$\mathbf{X}$ and $\uparrow x \cap \uparrow y$ are strongly compact
KC	compactly complete	$\mathbf{X}$ and $\uparrow x \cap \uparrow y$ are compact
F-ALG	finitely algebraic (bifinite)	$id_{\mathbf{X}}$ is limit of idempotent deflations
F-CONT	finitely continuous	$id_{\mathbf{X}}$ is limit of deflations
ALG	algebraic	$x \in O$ implies $x \in O' = \uparrow y \subseteq O$
CONT	continuous	$x \in O$ implies $x \in O' \subseteq \uparrow y \subseteq O$
M-ALG	multi-algebraic	$x \in O$ implies $x \in O' = F \subseteq O$
M-CONT	multi-continuous	$x \in O$ implies $x \in O' \subseteq F \subseteq O$
L-COMP	locally compact	$x \in O$ implies $x \in O' \subseteq K \subseteq O$
SOB	sober	C non-empty irreducible closed $\Rightarrow C = \mathsf{cl}\, x$
K-RD	K-Rudin	K_i $\supseteq$-directed, $\bigcap K_i \subseteq O$ implies $K_k \subseteq O$
S-RD	S-Rudin	S_i $\supseteq$-directed, $\bigcap S_i \subseteq O$ implies $S_k \subseteq O$
	($\mathbf{X}$, $\mathbf{Y}$ $\times$-conform)	$(x, y) \in O$ implies $(x, y) \in O_X \times O_Y \subseteq O$

The following picture shows the relationships among the various classes and the one-point domain $\mathbf{1}$. The arrows indicate inclusions. The arrow from KC to DOM is omitted for the sake of clarity.

$$
\begin{tikzcd}
 & & (\times\text{-conform}) & \text{L-COMP} \arrow[rr] & & \text{DOM} \\
 & \text{M-ALG} \arrow[r] & \text{M-CONT} \arrow[u] \arrow[ur] \arrow[r] & \text{SOB} \arrow[r] & \text{K-RD} \arrow[r] & \text{S-RD} \arrow[u] \\
\text{DIS} \arrow[r] & \text{ALG} \arrow[u] \arrow[r] & \text{CONT} \arrow[u] & \text{KC} & & \\
 & \text{F-ALG} \arrow[u] \arrow[r] & \text{F-CONT} \arrow[u] \arrow[r] & \text{SC} \arrow[u] & & \\
 & \text{FIN} \arrow[u] \arrow[rr] & & \text{FC} \arrow[u] \arrow[uuurr] & & \\
 & & & \text{BC} \arrow[u] & & \\
\mathbf{1} \arrow[uuuu] \arrow[uur] \arrow[rrr] & & & \text{CC} \arrow[u] & &
\end{tikzcd}
$$

As the picture shows, the classes form two almost independent hierarchies: the completeness hierarchy from CC to KC, and another hierarchy from FIN via CONT and SOB to S-RD. The next picture shows inclusions among combinations of these classes. The completeness hierarchy is spread along the x-axis, whereas the other hierarchy forms the y-axis. For simplicity, M-ALG, K-RD, BC, and KC are left off.

$$
\begin{tikzcd}
 & & & \text{SC} \arrow[r] & \text{DOM} \\
\text{CC} \arrow[r] & \text{FC} \arrow[rr] & & \text{SC \& S-RD} \arrow[u] \arrow[r] & \text{S-RD} \arrow[u] \\
\text{CC \& SOB} \arrow[u] \arrow[r] & \text{FC \& SOB} \arrow[u] \arrow[rr] & & \text{SC \& SOB} \arrow[u] \arrow[r] & \text{SOB} \arrow[u] \\
\text{CC \& M-CONT} \arrow[u] \arrow[r] & \text{FC \& M-CONT} \arrow[u] \arrow[rr] & & \text{SC \& M-CONT} \arrow[u] \arrow[r] & \text{M-CONT} \arrow[u] \\
\text{CC \& CONT} \arrow[u] \arrow[r] & \text{FC \& F-CONT} \arrow[u] \arrow[r] & \text{F-CONT} \arrow[r] & \text{SC \& CONT} \arrow[u] \arrow[r] & \text{CONT} \arrow[u] \\
\text{CC \& ALG} \arrow[u] \arrow[r] & \text{FC \& F-ALG} \arrow[u] \arrow[r] & \text{F-ALG} \arrow[u] \arrow[r] & \text{SC \& ALG} \arrow[u] \arrow[r] & \text{ALG} \arrow[u] \\
\text{CC \& FIN} \arrow[u] \arrow[r] & \text{FIN} \arrow[u] & & & \text{DIS} \arrow[u]
\end{tikzcd}
$$

Part II

General power constructions

This part contains the *algebraic theory of power domain constructions.* Power constructions are defined by axioms concerning existence and properties of some basic operations. One might worry about the actual choice of these axioms, but we think that our choice is quite natural. This opinion is strengthened by the fact that our definition covers the known power constructions, and allows to characterize them algebraically.

The basic operations and their axioms are presented in chapter 9. The subsequent chapters contain consequences of these axioms i.e. facts that hold for *all* power domain constructions. In chapter 10, a host of derived operations is defined by means of the basic operations, and their algebraic properties are studied. Among those, there is a multiplication that makes the power domain $\mathcal{P}\mathbf{1}$ over the one-point domain into a semiring, and the remaining power domains into $\mathcal{P}\mathbf{1}$-modules. The relations between some classifying properties of power constructions and of the semiring $\mathcal{P}\mathbf{1}$ are studied in chapter 11.

As power constructions are algebraic structures on a higher level, it is possible and useful to define homomorphisms between power constructions. These *power homomorphisms* are introduced in chapter 12. They allow to define *initial* and *final* constructions for a given semiring $\mathcal{P}\mathbf{1}$. The discussion of the initial construction is prepared by considering R-$\mathbf{X}$-modules in chapter 13. In chapters 14 and 15, we then prove that initial and final constructions exist for every semiring, and we derive their basic properties.

Since the concept of a semiring is very general, we thus obtain a host of power domain constructions. Their number is further increased by various methods to obtain new constructions from given ones.

The *core* of a power domain is the smallest sub-domain that still admits the power operations including multiplication. Taking the cores of all power domains of a power construction is shown to be a power construction again with the same characteristic semiring. Power constructions whose power domains equal their core, i.e. are minimal, are called *reduced*. Reduced power constructions enjoy many pleasant properties. The core and the notion of being reduced are studied in chapter 14.

A family of given power constructions may be combined to form a product power construction. In chapter 16, we present the definition of the product and its main properties. This chapter also deals with the question of factorization of power constructions.

Chapter 9

Specification of power constructions

In this chapter, we present the algebraic specification of power domain constructions. Power constructions are algebraic structures on a higher level: they have not just one carrier set with some operations, but instead map every domain $\mathbf{X}$ to a power domain $\mathcal{P}\mathbf{X}$ (see section 9.1). The power domains are equipped with special elements playing the role of an empty set and a binary operation representing set union (section 9.2). These operations together make the power domains into monoid domains (section 9.3). In addition, there has to be a singleton map for every ground domain that maps ground domain members into power domain elements representing formal singletons (section 9.4).

The most complex family of operations connected with a power construction is function extension. For every two domains $\mathbf{X}$ and $\mathbf{Y}$, there has to be a second order function *ext* that maps functions from $\mathbf{X}$ to $\mathcal{P}\mathbf{Y}$ into functions from $\mathcal{P}\mathbf{X}$ to $\mathcal{P}\mathbf{Y}$. The extension functional *ext* has to satisfy 7 axioms enumerated in section 9.5.

Section 9.6 shows that the well known powerset construction is a partial power domain construction that is defined for discretely ordered domains, i.e. sets, only. The final section 9.7 summarizes the specification of power domain constructions.

9.1 Constructions

A power construction is something like a function which applied to a domain $\mathbf{X}$ yields a new domain, the power domain over $\mathbf{X}$. It is not really a function since there is no *set* of all domains. There may be total constructions that are applicable to all domains, as well as partial constructions applicable to a special class of domains only.

Definition 9.1.1 A *(domain) construction* $\mathcal{F} : \mathbf{X} \mapsto \mathcal{F}\mathbf{X}$ attaches a domain $\mathcal{F}\mathbf{X}$ to every domain $\mathbf{X}$ belonging to a distinguished class *def* $\mathcal{F}$. $\mathcal{F}$ is a *total* construction if *def* $\mathcal{F}$ is the class of all domains, otherwise a *partial* one.

A *power (domain) construction* $\mathcal{P}$ is a domain construction satisfying the axioms presented in the next paragraphs. $\mathcal{P}\mathbf{X}$ is called the *power domain* over the *ground domain* $\mathbf{X}$. The elements of (the carrier of) $\mathcal{P}\mathbf{X}$ are called *formal sets*.

If a power construction $\mathcal{P}$ is defined for a class $C = \mathit{def}\ \mathcal{P}$, then the power domains $\mathcal{P}\mathbf{X}$ are not required to be in C again.

Often, a power domain cannot be realized concretely as a set of subsets of the ground domain. Hence the notion of formal sets in contrast to actual sets, i.e. the ordinary subsets of the ground domain. Formal set operations will be notationally distinguished from actual set operations by means of additional bars, e.g. $\uplus$ vs. $\cup$.

In the following, the symbol $\mathcal{P}$ denotes a generic partial power construction defined for a class $\mathrm{D} = \mathit{def}\ \mathcal{P}$ of domains. We immediately require the class D to contain the one-point-domain $\mathbf{1}$ because the power domain $\mathcal{P}\mathbf{1}$ plays an important algebraic role.

9.2 Empty set and finite union

As a first requirement, we want the power domain $\mathcal{P}\mathbf{X}$ to contain a formal empty set and to provide formal set union. Both the existence of an empty set and the axioms for union may be subject to discussions.

None of the original power domain constructions contained the empty set. However, they were sometimes extended by the empty set in later developments. The power constructions without empty set are briefly considered in section 17.7. For our work, the empty set is important and cannot be dispensed with.

Mathematical set theory suggests that union be commutative, associative, and idempotent. The last requirement turns out to be the least important one. For the sake of generality, we omit it as far as possible. Thus, the following results apply for 'multi-power' domain constructions as well.

For a (generalized) power construction $\mathcal{P}$, all power domains $\mathcal{P}\mathbf{X}$ have to be equipped with a commutative and associative operation $\uplus : [\mathcal{P}\mathbf{X} \times \mathcal{P}\mathbf{X} \to \mathcal{P}\mathbf{X}]$. In addition, there has to be a point θ in $\mathcal{P}\mathbf{X}$ which is the neutral element of union '$\uplus$'. If union is idempotent, it is a real power construction, and otherwise a multi-power construction.

9.3 Monoid domains

To have generally applicable notions, we define the algebra of domains with empty set and union in a more abstract setting.

Definition 9.3.1 (Monoid domains and additive maps)
A *monoid domain* (or simply monoid) $(M, +, 0)$ is a domain M together with an associative operation $+ : [M \times M \to M]$ and an element 0 of (the carrier of) M which is the neutral element of '$+$'.
The monoid is *commutative* iff '$+$' is.

A map $f : [X \to Y]$ between two monoids is *additive* iff it is a *monoid homomorphism*, i.e. $f(0_X) = 0_Y$ and $f(a + b) = fa + fb$ hold.

Many authors, including myself in previous papers, call the additive maps linear. However, the term 'linear' is more appropriate for the module homomorphisms introduced in section 11.1. In many common cases, including the usual power constructions, additivity and linearity coincide as indicated in section 11.12.

9.4 Singleton sets

Returning to the power construction, we next require a morphism which maps elements into singleton sets. We denote it by $\iota = \{\!|.|\!\} : [\mathbf{X} \to \mathcal{P}\mathbf{X}]$, $x \mapsto \{\!|x|\!\}$.

By means of the operations θ and $\uplus$, we may extend $\{\!|.|\!\}$ to finite sequences of ground domain points:

$$\{\!|x_1, \ldots, x_n|\!\} = \begin{cases} \{\!|x_1|\!\} \uplus \cdots \uplus \{\!|x_n|\!\} & \text{if } n > 0 \\ \theta & \text{if } n = 0 \end{cases}$$

Because of commutativity and associativity, one is free to permute the n arguments of $\{\!|x_1, \ldots, x_n|\!\}$. If union is idempotent, one additionally might delete and add multiple occurrences of elements. Thus $\{\!|.|\!\}$ becomes a mapping from finite actual sets to formal sets in this case.

9.5 Function extension

So far, we required the existence of singletons, empty set, and binary union. Singleton and union are not yet interrelated by axioms, and there are no axioms yet relating power domains over different ground domains. Both relationships are established by the extension functional. It takes a set-valued function defined on points of a ground domain and extends it to formal sets.

Definition 9.5.1 Let $\mathbf{X}$ be a domain in D and $\mathbf{Z}$ an arbitrary domain. A function $F : [\mathcal{P}\mathbf{X} \to \mathbf{Z}]$ is an *extension* of a function $f : [\mathbf{X} \to \mathbf{Z}]$ iff $F\{\!|x|\!\} = fx$ holds for all x in X, or equivalently iff $F \circ \iota = f$.

For every two domains $\mathbf{X}$ and $\mathbf{Y}$ in D, *ext* is a morphism mapping morphisms from $\mathbf{X}$ to $\mathcal{P}\mathbf{Y}$ into morphisms from $\mathcal{P}\mathbf{X}$ to $\mathcal{P}\mathbf{Y}$. For every $f : [\mathbf{X} \to \mathcal{P}\mathbf{Y}]$, the extended function $\overline{f} = \mathit{ext}\, f$ should be an additive extension of f. These axioms imply $\overline{f}\{\!|x_1, \ldots, x_n|\!\} = fx_1 \uplus \cdots \uplus fx_n$ for $n > 0$.

$$\begin{tikzcd}
\mathcal{P}\mathbf{X} \arrow[r, "\overline{f}"] & \mathcal{P}\mathbf{Y} \\
\mathbf{X} \arrow[u, "\iota"] \arrow[ur, "f"'] &
\end{tikzcd}$$

We call the *ext* axioms indicated above primary axioms because their relevance is immediate. In addition, we require some 'secondary axioms' which will be stated below as (Si). (S1) and (S2) specify additivity in the functional argument. In the next section, power constructions are shown to be functors by means of (S3) and (S4).

- For all domains $\mathbf{X}$, $\mathbf{Y}$ in D, there is a morphism $ext = \overline{} : [[\mathbf{X} \to \mathcal{P}\mathbf{Y}] \to [\mathcal{P}\mathbf{X} \to \mathcal{P}\mathbf{Y}]]$ with

(P1) $\overline{f}\,\theta = \theta$

(P2) $\overline{f}\,(A \uplus B) = (\overline{f}\,A) \uplus (\overline{f}\,B)$

(P3) $\overline{f}\,\{\!|x|\!\} = f x$ or: $\overline{f} \circ \iota = f$

Together, (P1) through (P3) mean $\overline{f}$ is an additive extension of f.

(S1) $ext\,(\lambda x.\,\theta)\,A = \theta$ or shortly $ext\,\underline{\theta} = \underline{\theta}$ where $\underline{\theta}$ denotes the constant function $\lambda x.\,\theta$.

(S2) $ext\,(\lambda x.\,f x \uplus g x)\,A = (ext\,f\,A) \uplus (ext\,g\,A)$.
Raising '$\uplus$' to functions, one may shortly write $\overline{f \uplus g} = \overline{f} \uplus \overline{g}$.

(S3) $ext\,(\lambda x.\,\{\!|x|\!\})\,A = A$ or: $\overline{\iota} = id$

(S4) For every two morphisms $f : [\mathbf{X} \to \mathcal{P}\mathbf{Y}]$ and $g : [\mathbf{Y} \to \mathcal{P}\mathbf{Z}]$,

$$ext\,g\,(ext\,f\,A) = ext\,(\lambda a.\,ext\,g\,(f a))\,A$$

holds for all A in $\mathcal{P}\mathbf{X}$, or: $\overline{g} \circ \overline{f} = \overline{\overline{g} \circ f}$

$$\begin{tikzcd}
\mathcal{P}\mathbf{X} \arrow[r, "\overline{f}"] & \mathcal{P}\mathbf{Y} \arrow[r, "\overline{g}"] & \mathcal{P}\mathbf{Z} \\
\mathbf{X} \arrow[u, "\iota"] \arrow[ur, "f"'] & \mathbf{Y} \arrow[u, "\iota"] \arrow[ur, "g"'] &
\end{tikzcd}$$

Note that we do not require $\overline{f}$ to be the only morphism satisfying (P1) through (P3) for given f. However, an important class of power constructions will have this property. For these constructions, (S1) through (S4) become provable (see section 14.5). That is why we call them secondary axioms.

9.6 Examples

Sets may be conceived as discrete domains, and all functions between discrete domains are continuous. Hence, ordinary is a partial power domain construction defined for discrete domains.

- $\mathcal{P}_{set}\,\mathbf{X} = \mathcal{P}\mathbf{X} = \{A \mid A \subseteq \mathbf{X}\}$ ordered discretely for discrete domains $\mathbf{X}$,
- $\theta = \emptyset$,
- $A \uplus B = A \cup B$,
- $\{\!|x|\!\} = \{x\}$,
- $ext\, f\, A = \bigcup_{a \in A} fa$.

Union is obviously commutative, associative, and the empty set is its neutral element. The axioms for extension read as follows:

(P1) $\bigcup_{a\in\emptyset} fa = \emptyset$ (P2) $\bigcup_{c\in A\cup B} fc \;=\; \bigcup_{a\in A} fa \,\cup\, \bigcup_{b\in B} fb$

(P3) $\bigcup_{x\in\{a\}} fx = fa$

(S1) $\bigcup_{a\in A} \emptyset = \emptyset$ (S2) $\bigcup_{a\in A}(fa \cup ga) \;=\; \bigcup_{a\in A} fa \,\cup\, \bigcup_{a\in A} ga$

(S3) $\bigcup_{a\in A}\{a\} = A$ (S4) $\bigcup\{gb \mid b \in \bigcup_{a\in A} fa\} = \bigcup_{a\in A}\bigcup_{b\in fa} gb$

All these equations hold, i.e. $\mathcal{P}_{set}$ is a power construction.

ext f is not the only additive extension of f if $\mathbf{X}$ is infinite. Another additive extension of $f : \mathbf{X} \to \mathcal{P}_{set}\mathbf{Y}$ is $FA = \begin{cases} \bigcup_{a\in A} fa & \text{if } A \text{ is finite} \\ \mathbf{Y} & \text{otherwise} \end{cases}$
An extension functional defined in this manner would however violate axiom (S3).

The empty set and all singletons are finite, and finite unions of finite sets are finite. Hence, there is another power construction for sets:

$$\mathcal{P}_{fin}\,\mathbf{X} = \{A \subseteq \mathbf{X} \mid A \text{ is finite}\}$$

whose operations are the restrictions of the operations above. In this construction, every function $f : \mathbf{X} \to \mathcal{P}_{fin}\,\mathbf{Y}$ has a unique additive extension.

The two partial constructions introduced in this section are considered to some more extent in section 23.5.

9.7 Summary

A power construction is a tuple (D, $\mathcal{P}$, θ, $\uplus$, ι, $^{-}$) where

- D is a class of domains;
- $\mathcal{P}$ maps domains belonging to class D into domains;
- $\theta \;=\; (\theta_X)_{X \in \mathrm{D}}$ with $\theta_X \;:\; \mathcal{P}X$
- $\uplus \;=\; (\uplus_X)_{X \in \mathrm{D}}$ with $\uplus_X \;:\; [\mathcal{P}X \times \mathcal{P}X \to \mathcal{P}X]$
- $\iota \;=\; (\iota_X)_{X \in \mathrm{D}}$ with $\iota_X \;:\; [X \to \mathcal{P}X]$
- $^{-} \;=\; (ext_{XY})_{X,Y \in \mathrm{D}}$ with $ext_{XY} :\; [[X \to \mathcal{P}Y] \to [\mathcal{P}X \to \mathcal{P}Y]]$

satisfying the axioms (domain indices are dropped!)

(C) $A \uplus B = B \uplus A$

(A) $A \uplus (B \uplus C) = (A \uplus B) \uplus C$

(N) $\theta \uplus A = A \uplus \theta = A$

(P1) $\overline{f}\,\theta = \theta$

(P2) $\overline{f}\,(A \uplus B) = (\overline{f}\,A) \uplus (\overline{f}\,B)$

(P3) $\overline{f} \circ \iota = f$

(S1) $\overline{\lambda x.\,\theta} = \lambda X.\,\theta$

(S2) $\overline{f \uplus g} = \overline{f} \uplus \overline{g}$ with '$\uplus$' raised to functions

(S3) $\overline{\iota} = id$

(S4) $\overline{g} \circ \overline{f} = \overline{\overline{g} \circ f}$

Chapter 10

Derived operations in a power construction

The operations as specified in the previous chapter allow to derive many other operations with useful algebraic properties. We first consider some set operations including function mapping (10.1), big union (10.2), and Cartesian product (10.3). Function mapping turns the power construction into a locally continuous functor.

In section 10.4, we concentrate on the power domain $\mathcal{P}\mathbf{1}$ over the one-point-domain $\mathbf{1}$ and show that it incorporates the inherent logic of the power construction in its operations. In section 10.5, existential quantification $\mathcal{E}$ is introduced. Given a formal set and a predicate, $\mathcal{E}$ intuitively tells whether some member of the set satisfies the predicate. In chapter 15, $\mathcal{E}$ will be used to define power domain constructions in terms of second order predicates. In section 10.6, other set predicates are derived from $\mathcal{E}$.

Elements of a power domain $\mathcal{P}\mathbf{X}$ may be multiplied by logical values, i.e. members of $\mathcal{P}\mathbf{1}$ (see section 10.7). Intuitively, multiplication of A by the logical value b results in the conditional *if* b *then* A *else* $\emptyset$. In case $\mathbf{X} = \mathbf{1}$, this operation induces a binary operation within $\mathcal{P}\mathbf{1}$. This operation may be interpreted as conjunction (section 10.8). The algebraic properties of these products make $\mathcal{P}\mathbf{1}$ into a semiring with union as addition, whereas the power domains become left semiring modules (cf. section 11.1).

Further derived operations are filtering of formal sets through predicates (section 10.9) and multiplication by logical values from the right (section 10.10). This alternative product has dual properties to the left product of section 10.7. It makes the power domains into right semiring modules.

10.1 Mapping of functions over sets

Given a morphism $f : [\mathbf{X} \to \mathbf{Y}]$, it can be composed with the singleton operation to obtain $\iota \circ f : [\mathbf{X} \to \mathcal{P}\mathbf{Y}]$. The resulting set-valued function can be extended to set arguments. Thus, we obtain

$$map = \widehat{\ }: \ [[\mathbf{X} \to \mathbf{Y}] \to [\mathcal{P}\mathbf{X} \to \mathcal{P}\mathbf{Y}]] \qquad \widehat{f} = \overline{\iota \circ f}.$$

The primary and some secondary axioms of extension may be translated into corresponding properties of *map*.

(P1)' $\widehat{f}\,\theta = \theta$
(P2)' $\widehat{f}\,(A \uplus B) = (\widehat{f}\,A) \uplus (\widehat{f}\,B)$
(P3)' $\widehat{f} \circ \iota = \iota \circ f$ or: $\widehat{f}\,\{\!|x|\!\} = \{\!|f\,x|\!\}$
(S3)' $\widehat{id} = id$
(S4)' $\widehat{g} \circ \widehat{f} = \widehat{g \circ f}$

$$\begin{tikzcd} \mathcal{P}\mathbf{X} \arrow[r, "\widehat{f}"] & \mathcal{P}\mathbf{Y} \\ \mathbf{X} \arrow[u, "\iota"] \arrow[r, "f"] & \mathbf{Y} \arrow[u, "\iota"'] \end{tikzcd}$$

Proof:

(P1)' $\widehat{f}\,\theta = \overline{\iota \circ f}\,(\theta) = \theta$ by (P1)

(P2)' immediately by (P2)

(P3)' $\widehat{f} \circ \iota = \overline{\iota \circ f} \circ \iota = \iota \circ f$ by (P3)

(S3)' $\widehat{id} = \overline{\iota \circ id} = \overline{\iota} = id$ by (S3)

(S4)' $\widehat{g} \circ \widehat{f} = \overline{\iota \circ g} \circ \overline{\iota \circ f} \stackrel{(S4)}{=} \overline{\overline{\iota \circ g} \circ \iota \circ f} \stackrel{(P3)}{=} \overline{\iota \circ g \circ f} = \widehat{g \circ f}$ □

The properties (P1)' through (P3)' imply $\widehat{f}\,\{\!|x_1, \ldots, x_n|\!\} = \{\!|f\,x_1, \ldots, f\,x_n|\!\}$. The last two properties show that $\mathcal{P}$ becomes a functor by means of *map*. Since *map* is continuous when considered a second order function, this functor is locally continuous. Thus, theorems 7.4.4 and 7.5.6 apply, i.e.

- If $\mathcal{P}$ preserves FIN, then it also preserves F-ALG and F-CONT.
- If $\mathcal{P}$ preserves ALG, then it also preserves CONT.

10.2 Big union

If $\mathbf{X}$ is in D such that $\mathcal{P}\mathbf{X}$ is back in D again, the identity $id : [\mathcal{P}\mathbf{X} \to \mathcal{P}\mathbf{X}]$ may be extended to a morphism $U = \overline{id} : [\mathcal{P}(\mathcal{P}\mathbf{X}) \to \mathcal{P}\mathbf{X}]$. The axioms (P1) through (P3) of extension imply
(1) $U\,\theta = \theta$
(2) $U\,(A \uplus B) = U\,A \uplus U\,B$
(3) $U\,\{\!|S|\!\} = S$
whence $U\,\{\!|S_1, \ldots, S_n|\!\} = S_1 \uplus \cdots \uplus S_n$. Thus, U is a formal big union of formal sets of formal sets.

$$\begin{tikzcd} \mathcal{P}(\mathcal{P}\mathbf{X}) \arrow[dr, "U"] & \\ \mathcal{P}\mathbf{X} \arrow[u, "\iota"] \arrow[r, "id"] & \mathcal{P}\mathbf{X} \end{tikzcd}$$

10.3 Double extension

Let $\mathbf{X}$, $\mathbf{Y}$, and $\mathbf{Z}$ be three domains in D, and let $\star : [\mathbf{X} \times \mathbf{Y} \to \mathcal{P}\mathbf{Z}]$ be a binary operation written in infix notation. By double extension, one obtains

$$A \overrightarrow{\star} B = ext\ (\lambda a.\ ext\ (\lambda b.\ a \star b)\ B)\ A \quad \text{and} \quad A \overleftarrow{\star} B = ext\ (\lambda b.\ ext\ (\lambda a.\ a \star b)\ A)\ B$$

The results are two morphisms $\overrightarrow{\star}$, $\overleftarrow{\star} : [\mathcal{P}\mathbf{X} \times \mathcal{P}\mathbf{Y} \to \mathcal{P}\mathbf{Z}]$.

A power construction is *symmetric* iff $A \overrightarrow{\star} B = A \overleftarrow{\star} B$ holds for all $\mathbf{X}$, $\mathbf{Y}$, and $\mathbf{Z}$ in D, A in $\mathcal{P}\mathbf{X}$, B in $\mathcal{P}\mathbf{Y}$, and $\star : [\mathbf{X} \times \mathbf{Y} \to \mathcal{P}\mathbf{Z}]$. Power constructions are not automatically symmetric. Later, we shall meet examples for this.

Our two sample power constructions for discrete domains — set of arbitrary subsets and set of finite subsets — are both symmetric because of

$$\bigcup_{a \in A} \bigcup_{b \in B} a \star b = \bigcup_{b \in B} \bigcup_{a \in A} a \star b$$

For two singletons, $\{\!|a|\!\} \overrightarrow{\star} \{\!|b|\!\} = \{\!|a|\!\} \overleftarrow{\star} \{\!|b|\!\} = a \star b$ may be shown using (P3) twice. Because of (P1) and (P2), '$\overrightarrow{\star}$' is obviously additive in its first argument:

$$\theta \overrightarrow{\star} B = \theta \qquad (A_1 \uplus A_2) \overrightarrow{\star} B = (A_1 \overrightarrow{\star} B) \uplus (A_2 \overrightarrow{\star} B)$$

For additivity in the second argument, (S1) and (S2) have to be employed in addition because B appears in the functional argument of the outer occurrence of *ext*. Thus, we get

$$A \overrightarrow{\star} \theta = \theta \qquad A \overrightarrow{\star} (B_1 \uplus B_2) = (A \overrightarrow{\star} B_1) \uplus (A \overrightarrow{\star} B_2)$$

'$\overleftarrow{\star}$' has the same properties; the proofs are however exchanged.
For formal finite sets, one then obtains

$$\begin{aligned} &\{\!|x_1, \ldots, x_n|\!\} \overrightarrow{\star} \{\!|y_1, \ldots, y_m|\!\} = \{\!|x_1, \ldots, x_n|\!\} \overleftarrow{\star} \{\!|y_1, \ldots, y_m|\!\} = \\ &\quad (x_1 \star y_1) \uplus (x_1 \star y_2) \uplus \cdots \uplus (x_1 \star y_m) \uplus (x_2 \star y_1) \uplus \cdots \uplus (x_n \star y_m) \end{aligned}$$

Cartesian product of formal sets is a special instance of double extension. If $\mathbf{X}$ and $\mathbf{Y}$ are in D such that $\mathbf{X} \times \mathbf{Y}$ is also in D, then

$$\begin{aligned} &A \overrightarrow{\times} B = ext\ (\lambda a.\ ext\ (\lambda b.\ \{\!|(a,b)|\!\})\ B)\ A \quad \text{and} \\ &A \overleftarrow{\times} B = ext\ (\lambda b.\ ext\ (\lambda a.\ \{\!|(a,b)|\!\})\ A)\ B \end{aligned}$$

are formal Cartesian products.

If the class D where the power construction is defined is closed w.r.t. Cartesian product, then symmetry may be defined in terms of formal Cartesian products because of the following proposition:

Proposition 10.3.1 Let $\mathbf{X}$ and $\mathbf{Y}$ be in D such that $\mathbf{X} \times \mathbf{Y}$ also is in D. Then for all $\mathbf{Z}$ in D and $\star : [\mathbf{X} \times \mathbf{Y} \to \mathcal{P}\mathbf{Z}]$, $A \overrightarrow{\star} B = ext\,(\star)\,(A \overrightarrow{\times} B)$ and $A \overleftarrow{\star} B = ext\,(\star)\,(A \overleftarrow{\times} B)$ hold.

Proof:

$$\begin{aligned} ext\,(\star)\,(A \overrightarrow{\times} B) &= ext\ (\star)\ (ext\ (\lambda a.\ ext\ (\lambda b.\ \{\!|(a,b)|\!\})\ B)\ A) \\ &\overset{(S4)}{=} ext\ (\lambda a.\ ext\ (\star)\ (ext\ (\lambda b.\ \{\!|(a,b)|\!\})\ B))\ A \\ &\overset{(S4)}{=} ext\ (\lambda a.\ ext\ (\lambda b.\ ext\ (\star)\ \{\!|(a,b)|\!\})\ B)\ A \\ &\overset{(P3)}{=} ext\ (\lambda a.\ ext\ (\lambda b.\ a \star b)\ B)\ A \end{aligned}$$

The statement about '$\overleftarrow{\star}$' and '$\overleftarrow{\times}$' is proved analogously. □

Corollary 10.3.2 Let $\mathcal{P}$ be a power construction such that D = $def\,\mathcal{P}$ is closed w.r.t. product, i.e. $\mathbf{X}$, $\mathbf{Y}$ in D implies $\mathbf{X} \times \mathbf{Y}$ in D. Then $\mathcal{P}$ is symmetric iff for all $\mathbf{X}$, $\mathbf{Y}$ in D, A in $\mathcal{P}\mathbf{X}$, and B in $\mathcal{P}\mathbf{Y}$, $A \overrightarrow{\times} B = A \overleftarrow{\times} B$ holds.

10.4 The logic of power constructions

Each power construction is equipped with an inherent logic. In this section, we present the domain of logical values together with disjunction and existential quantification. The corresponding conjunction is defined in section 10.8.

The domain of logical values is obtained by interpreting the power domain $\mathcal{P}\mathbf{1}$ where $\mathbf{1} = \{\diamond\}$. It has at least two elements: θ and $\{\!|\diamond|\!\}$, and is equipped with the binary operation '$\uplus$'. We interpret θ as F, $\{\!|\diamond|\!\}$ as T, and '$\uplus$' as formal disjunction '$\curlyvee$'. From the power axioms, one gets the following properties:

- '$\curlyvee$' is commutative and associative.
- $\mathsf{F} \curlyvee a = a \curlyvee \mathsf{F} = a$ for all a in $\mathcal{P}\mathbf{1}$.
- In case of a real power construction, one additionally has $a \curlyvee a = a$ for all a in $\mathcal{P}\mathbf{1}$.

Table of values for a generalized power construction:

$\curlyvee$	F	T
F	F	T
T	T	?

for a real power construction:

$\curlyvee$	F	T
F	F	T
T	T	T

Further statements about $\mathcal{P}\mathbf{1}$ beyond the ones above are not possible for generic power constructions. In particular, one does not know whether there are further logical values besides T and F, and $a \curlyvee \mathsf{T} = \mathsf{T}$ does not generally hold, even for real power constructions. There is no information about the relative order of F and T; F might be below T, above T, or incomparable to T.

The two power set constructions — set of arbitrary subsets and set of finite subsets — both have the same logic: $\mathcal{P}\mathbf{1}$ is $\{\emptyset, \{\diamond\}\}$ or $\{\mathsf{F}, \mathsf{T}\}$ with ordinary disjunction.

10.5 Existential quantification

Extension ext: $[[\mathbf{X} \to \mathcal{P}\mathbf{Y}] \to [\mathcal{P}\mathbf{X} \to \mathcal{P}\mathbf{Y}]]$ is polymorphic over the domains $\mathbf{X}$ and $\mathbf{Y}$. In this section, we consider the special case $\mathbf{Y} = \mathbf{1}$; section 10.7 is concerned with $\mathbf{X} = \mathbf{1}$.

Extension to the one-point domain $ex : [[\mathbf{X} \to \mathcal{P}\mathbf{1}] \to [\mathcal{P}\mathbf{X} \to \mathcal{P}\mathbf{1}]]$ [1] may be logically interpreted along the lines of the previous section. It has the following properties:

[1]This morphism is called ex to distinguish it from the fully polymorphic ext.

(P1) $ex\ p\ \theta = \mathsf{F}$
(P2) $ex\ p\ (A \uplus B) = (ex\ p\ A) \curlyvee (ex\ p\ B)$
(P3) $ex\ p\ \{\!|x|\!\} = p\ x$
(S1) $ex\ (\lambda x.\, \mathsf{F})\ A = \mathsf{F}$
(S2) $ex\ (\lambda x.\, p\,x \curlyvee q\,x)\ A = (ex\ p\ A) \curlyvee (ex\ q\ A)$
(S4) $ex\ p\ (ext\ f\ A) = ex\ (\lambda a.\, ex\ p\ (f a))\ A$

$$\begin{tikzcd}
\mathcal{P}\mathbf{X} \arrow[r, "ex\ p"] & \mathcal{P}\mathbf{1} \\
\mathbf{X} \arrow[u, "\iota"] \arrow[ur, "p"'] &
\end{tikzcd}$$

whence $ex\ f\ \{\!|x_1, \ldots, x_n|\!\} = f x_1 \curlyvee \cdots \curlyvee f x_n$. Thus, ex means existential quantification. It takes a predicate $p : [\mathbf{X} \to \mathcal{P}\mathbf{1}]$ and a formal set A and tells whether some member of A satisfies p. (S4) then informally reads: There is x in $\bigcup_{a \in A} f a$ satisfying p iff there is a in A such that there is x in $f a$ satisfying p.

Existential quantification may also be used to translate formal sets into second order predicates. For this end, we exchange the order of arguments of ex by uncurrying, twisting, and then currying again. The outcome is a morphism $\mathcal{E} : [\mathcal{P}\mathbf{X} \to [[\mathbf{X} \to \mathcal{P}\mathbf{1}] \to \mathcal{P}\mathbf{1}]]$ mapping formal sets into second order predicates. The properties of ex presented above translate easily into properties of $\mathcal{E}$:

(P1) $\mathcal{E}\ \theta = \lambda p.\, \mathsf{F}$

(P2) $\mathcal{E}\ (A \uplus B) = \lambda p.\, (\mathcal{E}\ A\ p) \curlyvee (\mathcal{E}\ B\ p)$

(P3) $\mathcal{E}\ \{\!|x|\!\} = \lambda p.\, p\ x$

(S4) $\mathcal{E}\ (ext\ f\ A) = \lambda p.\, \mathcal{E}\ A\ (\lambda a.\, \mathcal{E}\ (f a)\ p)$

These results suggest to define a power construction for given domain $\mathcal{P}\mathbf{1}$ by (a slight variant of) $\mathcal{P}\mathbf{X} = [[\mathbf{X} \to \mathcal{P}\mathbf{1}] \to \mathcal{P}\mathbf{1}]$. This method to obtain power constructions will be presented and explored in chapter 15.

By (S1), $ex\ (\lambda x.\, \mathsf{F})\ A = \mathsf{F}$ holds for all formal sets A. And what about $ex\ (\lambda x.\, \mathsf{T})$? It maps the empty set to F by (P1), and singletons to T by (P3). Hence, it is a predicate checking for non-emptiness — at least in the case of real power constructions with idempotent union. In case of multi-power constructions, it is better interpreted as cardinality.

- $ne = ex\ (\lambda x.\, \mathsf{T}) :\ [\mathcal{P}\mathbf{X} \to \mathcal{P}\mathbf{1}]$

(P1) $ne\ \theta = \mathsf{F}$

(P2) $ne\ (A \uplus B) = (ne\ A) \curlyvee (ne\ B)$

(P3) $ne\ \{\!|x|\!\} = \mathsf{T}$

(S4) $ne\ (ext\ f\ A) = ext\ (ne \circ f)\ A$

- For all B in $\mathcal{P}\mathbf{1}$, $ne\ B = B$.

Proof: Properties (P1) through (P3) are immediate. For (S4), note $ne \circ ext\ f = ext\ \underline{\mathsf{T}} \circ ext\ f = ext\ (ext\ \underline{\mathsf{T}} \circ f) = ext\ (ne \circ f)$ holds by (S4) where $\underline{\mathsf{T}}$ abbreviates $\lambda x.\, \mathsf{T}$.

The function $\lambda x.\, \mathsf{T} = \lambda x.\, \{\diamond\}$ on domain $\mathbf{1}$ coincides with the singleton map ι since the only possible argument is $\diamond$. Thus, $ne = ext\ \iota = id$ on $\mathcal{P}\mathbf{1}$ because of (S3). □

For a real power construction, $\mathsf{T} \curlyvee \mathsf{T} = \mathsf{T}$ holds. Then by (P1) through (P3), $ne\ \{\!|x_1, \ldots, x_n|\!\}$ is T if $n > 0$, and F if $n = 0$.

10.6 Other set predicates

Assume $\mathcal{P}$ is a real power construction i.e. $A \uplus A = A$, and assume $\mathcal{P}\mathbf{1}$ provides a negation $\neg : [\mathcal{P}\mathbf{1} \to \mathcal{P}\mathbf{1}]$ such that $\neg\mathsf{F} = \mathsf{T}$ and $\neg\mathsf{T} = \mathsf{F}$. (There are real power constructions that do not admit such a negation.) Then one may define a conjunction by $a \mathbin{\bar\wedge} b = \neg(\neg a \mathbin{\bar\vee} \neg b)$. (In section 10.8, we define a conjunction for every power construction without using negation.)

With these assumptions, one may derive further predicates on formal sets.

- $\mathit{forall} : [[\mathbf{X} \to \mathcal{P}\mathbf{1}] \to [\mathcal{P}\mathbf{X} \to \mathcal{P}\mathbf{1}]]$ defined by $\mathit{forall}\ p\ S = \neg(\mathit{ex}\ (\lambda x.\, \neg(p\, x))\ S)$
  This implies $\mathit{forall}\ p\ \{\!|x_1, \ldots, x_n|\!\} = p\, x_1 \mathbin{\bar\wedge} \cdots \mathbin{\bar\wedge} p\, x_n$ for $n > 0$.
- $\mathit{empty} : [\mathcal{P}\mathbf{X} \to \mathcal{P}\mathbf{1}]$ defined by $\mathit{empty}\ S = \neg(\mathit{ex}\ (\lambda x.\, \mathsf{T})\ S) = \mathit{forall}\ (\lambda x.\, \mathsf{F})\ S$
  Hence, we obtain $\mathit{empty}\ \{\!|x_1, \ldots, x_n|\!\} = \begin{cases} \mathsf{T} & \text{if } n = 0 \\ \mathsf{F} & \text{if } n > 0 \end{cases}$

For the tests of containment, inclusion, and set equality, we need an equality morphism $==: [\mathbf{X} \times \mathbf{X} \to \mathcal{P}\mathbf{1}]$ given on the ground domain.

- $\_\ \mathit{in}\ \_ : \quad [\mathbf{X} \times \mathcal{P}\mathbf{X} \to \mathcal{P}\mathbf{1}] \quad x\ \mathit{in}\ B \ = \ \mathit{ex}\ (\lambda b.\, x == b)\ B$
- $\_ \mathrel{\overline{\subseteq}} \_ : \quad [\mathcal{P}\mathbf{X} \times \mathcal{P}\mathbf{X} \to \mathcal{P}\mathbf{1}] \quad A \mathrel{\overline{\subseteq}} B \ = \ \mathit{forall}\ (\lambda a.\, a\ \mathit{in}\ B)\ A$
- $\_ == \_ : \quad [\mathcal{P}\mathbf{X} \times \mathcal{P}\mathbf{X} \to \mathcal{P}\mathbf{1}] \quad (A == B) \ = \ (A \mathrel{\overline{\subseteq}} B) \mathbin{\bar\wedge} (B \mathrel{\overline{\subseteq}} A)$

10.7 Multiplication by a logical value

In this section, we consider extension of a morphism with domain $\mathbf{1}$, i.e. the instance $\mathit{ext} : [[\mathbf{1} \to \mathcal{P}\mathbf{X}] \to [\mathcal{P}\mathbf{1} \to \mathcal{P}\mathbf{X}]]$. The function space $[\mathbf{1} \to \mathcal{P}\mathbf{X}]$ is isomorphic to $\mathcal{P}\mathbf{X}$. Thus, we get a morphism $[\mathcal{P}\mathbf{X} \to [\mathcal{P}\mathbf{1} \to \mathcal{P}\mathbf{X}]]$. Uncurrying and exchanging arguments leads to the 'product' $\cdot : [\mathcal{P}\mathbf{1} \times \mathcal{P}\mathbf{X} \to \mathcal{P}\mathbf{X}]$. The definition is $b \cdot S = \mathit{ext}\,(\lambda\diamond.\, S)\, b$. We call this product external since its left operand is not a member of $\mathcal{P}\mathbf{X}$. The axioms of ext imply the characteristic properties of the product.

Proposition 10.7.1

(P1·) $\mathsf{F} \cdot S = \theta$

(P2·) $(a \mathbin{\bar\vee} b) \cdot S = (a \cdot S) \uplus (b \cdot S)$

(P3·) $\mathsf{T} \cdot S = S$

(S1·) $b \cdot \theta = \theta$

(S2·) $b \cdot (S_1 \uplus S_2) = (b \cdot S_1) \uplus (b \cdot S_2)$

(S4·) $\mathit{ext}\, f\, (b \cdot S) = b \cdot (\mathit{ext}\, f\, S)$

(S4a·) $(a \cdot b) \cdot S = a \cdot (b \cdot S)$

(SY·) If $\mathcal{P}$ is symmetric, then $\mathit{ext}\,(\lambda x.\, b \cdot f x)\, S = b \cdot (\mathit{ext}\, f\, S)$

Algebraists will notice that these properties essentially are the axioms of left modules. This topic will be further explored in section 11.1.

Proof:

(P1$\cdot$) $\mathsf{F} \cdot S \;=\; ext\,(\lambda\diamond.\,S)\,\theta \;=\; \theta$

(P2$\cdot$) $(a \curlyvee b) \cdot S \;=\; ext\,(\lambda\diamond.\,S)\,(a \uplus b)$ etc.

(P3$\cdot$) $\mathsf{T} \cdot S \;=\; ext\,(\lambda\diamond.\,S)\,\{\!|\diamond|\!\} \;=\; (\lambda\diamond.\,S)\,\diamond \;=\; S$

(S1$\cdot$) $b \cdot \theta \;=\; ext\,(\lambda\diamond.\,\theta)\,b \;=\; \theta$

(S2$\cdot$) $b \cdot (S_1 \uplus S_2) \;=\; ext\,(\lambda\diamond.\,S_1 \uplus S_2)\,b$ etc.

(S4$\cdot$)
$$\begin{aligned} ext\,f\,(b \cdot S) &= ext\,f\,(ext\,(\lambda\diamond.\,S)\,b) \\ \text{(S4):} \quad &= ext\,(\lambda\diamond.\,ext\,f\,((\lambda\diamond.\,S)\,\diamond))\,b \\ &= ext\,(\lambda\diamond.\,ext\,f\,S)\,b \\ &= b \cdot (ext\,f\,S) \end{aligned}$$

(S4a$\cdot$)
$$\begin{aligned} (a \cdot b) \cdot S &= ext\,(\lambda\diamond.\,S)\,(a \cdot b) \\ \text{(S4}\cdot\text{):} \quad &= a \cdot (ext\,(\lambda\diamond.\,S)\,b) \\ &= a \cdot (b \cdot S) \end{aligned}$$

(SY$\cdot$)
$$\begin{aligned} ext\,(\lambda x.\,b \cdot f x)\,S &= ext\,(\lambda x.\,ext\,(\lambda\diamond.\,f x)\,b)\,S \\ \text{by symmetry:} \quad &= ext\,(\lambda\diamond.\,ext\,(\lambda x.\,f x)\,S)\,b \\ &= b \cdot (ext\,f\,S) \qquad \Box \end{aligned}$$

Interpreted logically, the product $b \cdot S$ resembles the conditional 'if b then S else θ'. At least for the cases $b = \mathsf{T}$ and $b = \mathsf{F}$, product and conditional coincide because of $\mathsf{T} \cdot S = S$ and $\mathsf{F} \cdot S = \theta$.

10.8 Conjunction

Up to now, the logical domain $\mathcal{P}\mathbf{1}$ was only equipped with constants F and T and a disjunction '$\curlyvee$'. We now interpret the external product on $\mathcal{P}\mathbf{1}$ as conjunction since $a \cdot b$ resembles 'if a then b else θ'.

$$\curlywedge : [\mathcal{P}\mathbf{1} \times \mathcal{P}\mathbf{1} \to \mathcal{P}\mathbf{1}],\; a \curlywedge b = a \cdot b$$

The algebraic properties of conjunction are given by the next proposition:

Proposition 10.8.1

- $\mathsf{F} \curlywedge b = b \curlywedge \mathsf{F} = \mathsf{F}$
- Distributivities: $(a_1 \curlyvee a_2) \curlywedge b = (a_1 \curlywedge b) \curlyvee (a_2 \curlywedge b)$, $a \curlywedge (b_1 \curlyvee b_2) = (a \curlywedge b_1) \curlyvee (a \curlywedge b_2)$
- Neutral element: $\mathsf{T} \curlywedge b = b \curlywedge \mathsf{T} = b$
- Associativity: $(a \curlywedge b) \curlywedge c = a \curlywedge (b \curlywedge c)$
- If the construction $\mathcal{P}$ is symmetric, then '$\curlywedge$' is commutative.

Proof:

- F: immediate by (P1$\cdot$) and (S1$\cdot$)
- Distributivities: (P2$\cdot$) and (S2$\cdot$)

- T: $\mathsf{T} \mathbin{\bar\wedge} b = b$ holds by (P3·). $b \mathbin{\bar\wedge} \mathsf{T} = \mathit{ext}\,(\lambda\diamond.\ \{\!|\diamond|\!\})\, b = b$ holds by (S3).
- Associativity is just (S4a·).
- Commutativity: $a \mathbin{\bar\wedge} b = \mathit{ext}\,(\lambda\diamond.\ b)\, a = \mathit{ext}\,(\lambda\diamond.\ b \cdot \mathsf{T})\, a =$ using (SY·) $\quad\square$
  $b \cdot \mathit{ext}\,(\lambda\diamond.\ \mathsf{T})\, a = b \mathbin{\bar\wedge} (a \mathbin{\bar\wedge} \mathsf{T}) = b \mathbin{\bar\wedge} a$

The axioms of generic power constructions do not allow to derive more algebraic properties for conjunction. In particular, idempotence of conjunction, the opposite distributivities, and the laws of absorption do not generally hold. On the other hand, the existing laws are powerful enough to obtain the following table of values:

$\bar\wedge$	F	T
F	F	F
T	F	T

10.9 Filtering a set through a predicate

We also want to provide an operation that filters a set through a given predicate w.r.t. the $\mathcal{P}\mathbf{1}$-logic. The operation $\mathit{filter} : [[\mathbf{X} \to \mathcal{P}\mathbf{1}] \to [\mathcal{P}\mathbf{X} \to \mathcal{P}\mathbf{X}]]$ should be additive in its set argument as well as in its predicate argument and operate on singletons as one expects i.e.

$$\mathit{filter}\, p\, \{\!|x|\!\} \;=\; \text{if } p\,x \text{ then } \{\!|x|\!\} \text{ else } \theta \;=\; p\,x \cdot \{\!|x|\!\}$$

This suggests to define $\quad \mathit{filter}\, p\, S = \mathit{ext}\,(\lambda x.\, p\,x \cdot \{\!|x|\!\})\, S$
Employing the axioms of extension, one immediately gets

(P1f) $\mathit{filter}\ p\ \theta = \theta$

(P2f) $\mathit{filter}\ p\ (A \uplus B) = (\mathit{filter}\ p\ A) \uplus (\mathit{filter}\ p\ B)$

(P3f) $\mathit{filter}\ p\ \{\!|x|\!\} = p\,x \cdot \{\!|x|\!\}$

(S1f) $\mathit{filter}\ (\lambda x.\, \mathsf{F})\ S = \theta$

(S2f) $\mathit{filter}\ (p \mathbin{\bar\vee} q)\ A = (\mathit{filter}\ p\ A) \uplus (\mathit{filter}\ q\ A)$

(S3f) $\mathit{filter}\ (\lambda x.\, \mathsf{T})\ S = S$

(S4f) $\mathit{filter}\ p\ (b \cdot S) = b \cdot (\mathit{filter}\ p\ S)$ by (S4·).

Property (P3f) implies $\mathit{filter}\ (\lambda x.\, b)\ \{\!|x|\!\} = b \cdot \{\!|x|\!\}$. One might believe that this equation may be generalized from singletons to arbitrary sets. However, this is only valid for symmetric constructions.

(SYf) If $\mathcal{P}$ is symmetric then (SYf) $\mathit{filter}\,(\lambda x.\, b)\, S = b \cdot S$.

Proof: $\mathit{filter}\,(\lambda x.\, b)\, S = \mathit{ext}\,(\lambda x.\, b \cdot \{\!|x|\!\})\, S = b \cdot (\mathit{ext}\,(\lambda x.\, \{\!|x|\!\})\, S) = b \cdot S$ using (SY·) and then (S3). (SY·) was shown for symmetric constructions only. $\square$

Filtering by a predicate p first and then by a predicate q is intuitively equivalent to filtering by $p \mathbin{\bar\wedge} q$.

Proposition 10.9.1 (Composition of filter operations)
$\mathit{filter}\, q \circ \mathit{filter}\, p \;=\; \mathit{filter}\,(p \mathbin{\bar\wedge} q)$

Proof:

$$\begin{array}{lcl}
\mathit{filter}\, q \circ \mathit{filter}\, p & = & \mathit{ext}\,(\lambda x.\, q\, x \cdot \{\!|x|\!\}) \circ \mathit{ext}\,(\lambda y.\, p\, y \cdot \{\!|y|\!\}) \\
(\mathrm{S4}): & = & \mathit{ext}\,(\lambda y.\, \mathit{ext}\,(\lambda x.\, q\, x \cdot \{\!|x|\!\})\,(p\, y \cdot \{\!|y|\!\})) \\
(\mathrm{S4f}): & = & \mathit{ext}\,(\lambda y.\, p\, y \cdot (\mathit{ext}\,(\lambda x.\, q\, x \cdot \{\!|x|\!\})\, \{\!|y|\!\})) \\
(\mathrm{P3}): & = & \mathit{ext}\,(\lambda y.\, p\, y \cdot (q\, y \cdot \{\!|y|\!\})) \\
(\mathrm{Associativity}): & = & \mathit{ext}\,(\lambda y.\, (p\, y \curlywedge q\, y) \cdot \{\!|y|\!\}) \\
 & = & \mathit{filter}\,(p \curlywedge q) \qquad \square
\end{array}$$

10.10 Multiplication by a logical value from the right

In section 10.7, we defined a left external product, i.e. multiplication by a logical value from the left:

$$\cdot_L = \cdot : [\mathcal{P}\mathbf{1} \times \mathcal{P}\mathbf{X} \to \mathcal{P}\mathbf{X}],\ b \cdot S = \mathit{ext}\,(\lambda\diamond.\, S)\, b$$

In the previous section, we saw that $\mathit{filter}\,(\lambda x.\, b)\, S$ does not generally coincide with $b \cdot S$ except in the symmetric case. The idea is now to write filtering by a constant predicate as a right product.

$$\cdot_R = \cdot : [\mathcal{P}\mathbf{X} \times \mathcal{P}\mathbf{1} \to \mathcal{P}\mathbf{X}],\ S \cdot b = \mathit{filter}\,(\lambda x.\, b)\, S$$

We want to use the same operation symbol '$\cdot$' for left and right product. The following proposition assures that this may be safely done even in the case $\mathbf{X} = \mathbf{1}$.

Proposition 10.10.1 For all a, b in $\mathcal{P}\mathbf{1}$, $a \cdot_L b = a \cdot_R b$ holds.

Proof:

$$\begin{array}{lcl}
a \cdot_R b & = & \mathit{filter}\,(\lambda x.\, b)\, a \ = \ \mathit{filter}\,(\lambda x.\, b)\,(a \cdot_L \mathsf{T}) \\
(\mathrm{S4f}): & = & a \cdot_L \mathit{filter}\,(\lambda x.\, b)\, \mathsf{T} \\
\mathsf{T} = \{\diamond\},\ (\mathrm{P3f}): & = & a \cdot_L (b \cdot_L \mathsf{T}) \ = \ a \cdot_L b
\end{array}$$

$\square$

The algebraic properties of the right product are analogous to that of the left product. In the following, capital letters range over $\mathcal{P}\mathbf{X}$ and small letters over $\mathcal{P}\mathbf{1}$.

Proposition 10.10.2

(1) $S \cdot \mathsf{F} = \theta$

(2) $S \cdot (a \curlyvee b) = (S \cdot a) \uplus (S \cdot b)$

(3) $S \cdot \mathsf{T} = S$

(4) $\theta \cdot b = \theta$

(5) $(S_1 \uplus S_2) \cdot b = (S_1 \cdot b) \uplus (S_2 \cdot b)$

(6) $\mathit{ext}\,(\lambda x.\, f\, x \cdot b)\, S = (\mathit{ext}\, f\, S) \cdot b$

(7) $S \cdot (a \cdot b) = (S \cdot a) \cdot b$

(8) $(a \cdot S) \cdot b = a \cdot (S \cdot b)$

(9) $a \cdot \{\!|x|\!\} = \{\!|x|\!\} \cdot a$

(10) If $\mathcal{P}$ is symmetric, then $a \cdot S = S \cdot a$

Proof: For a constant b in $\mathcal{P}\mathbf{1}$, we abbreviate $\lambda x.\, b$ by $\underline{b}$.

(1) $S \cdot \mathsf{F} \;=\; \mathit{filter}\, \underline{\mathsf{F}}\, S \;=\; \theta$ by (S1f)

(2) $S \cdot (a \curlyvee b) \;=\; \mathit{filter}\, (\lambda x.\, a \curlyvee b)\, S$ Use (S2f)

(3) $S \cdot \mathsf{T} \;=\; \mathit{filter}\, \underline{\mathsf{T}}\, S \;=\; S$ by (S3f)

(4) $\theta \cdot b \;=\; \mathit{filter}\, \underline{b}\, \theta \;=\; \theta$ by (P1f)

(5) $(S_1 \uplus S_2) \cdot b \;=\; \mathit{filter}\, \underline{b}\, (S_1 \uplus S_2)$ Use (P2f)

(6)
$$\begin{aligned} & (\mathit{ext}\, f\, S) \cdot b &=\;& \mathit{ext}\, (\lambda x.\, b \cdot \{\!|x|\!\})\, (\mathit{ext}\, f\, S) \\ & \text{(S4)}: &=\;& \mathit{ext}\, (\lambda y.\, \mathit{ext}\, (\lambda x.\, b \cdot \{\!|x|\!\})\, (f y))\, S \\ & &=\;& \mathit{ext}\, (\lambda y.\, \mathit{filter}\, \underline{b}\, (f y))\, S \;=\; \mathit{ext}\, (\lambda y.\, (f y) \cdot b)\, S \end{aligned}$$

(7)
$$\begin{aligned} S \cdot (a \cdot b) &=\; \mathit{filter}\, (\underline{a} \curlywedge \underline{b})\, S \qquad \text{using Prop. 10.9.1} \\ &=\; \mathit{filter}\, \underline{b}\, (\mathit{filter}\, \underline{a}\, S) \;=\; (S \cdot a) \cdot b \end{aligned}$$

(8)
$$\begin{aligned} & (a \cdot S) \cdot b &=\;& (\mathit{ext}\, (\lambda \diamond.\, S)\, a) \cdot b \\ & (6): &=\;& \mathit{ext}\, (\lambda \diamond.\, S \cdot b)\, a \;=\; a \cdot (S \cdot b) \end{aligned}$$

(9) $\{\!|x|\!\} \cdot a \;=\; \mathit{filter}\, \underline{a}\, \{\!|x|\!\} \;=\; a \cdot \{\!|x|\!\}$ by (P3f)

(10) $a \cdot S = S \cdot a$ is just a re-formulation of (SYf) which holds for symmetric constructions.□

In the general non-symmetric case, left factors may be drawn off the set argument of *ext* by (S4·), and right factors may be drawn off the functional argument by part (6) of the proposition above. Unless $a \cdot S = S \cdot a$ holds, it is impossible to draw right factors off the set argument and left factors off the functional argument.

Chapter 11

The characteristic semiring

In the previous chapter, we saw that the power domain $\mathcal{P}\mathbf{1}$ becomes a semiring with addition being union, 0 the empty set, 1 a singleton, and multiplication derived from extension. We call this semiring the *characteristic semiring* of the power domain construction $\mathcal{P}$. In this chapter, we study the relation between a power construction and its semiring. We introduce certain classifications of power constructions and investigate to what extent these classifications are determined by properties of the characteristic semiring.

Section 11.1 contains the definitions of the algebraic concepts of semirings, left and right modules, and linearity of functions. In section 11.2, we present examples of power constructions and their semirings. In particular, the semirings of the known power constructions are presented. The notion of characteristic semiring is generalized in section 11.3. The characteristic semiring of $\mathcal{P}$ need no longer be identical to $\mathcal{P}\mathbf{1}$, but may also be isomorphic to $\mathcal{P}\mathbf{1}$ if the isomorphism is assumed to be fixed. In section 11.4, we then consider the effect of the power operations to finite linear combinations in the $\mathcal{P}\mathbf{1}$-modules $\mathcal{P}\mathbf{X}$.

In section 11.5, power domain constructions are called real iff union is idempotent in all power domains. This property is equivalent to $1 + 1 = 1$ in the characteristic semiring. The next section 11.6 compares symmetric constructions where two extensions commute, commutative constructions where the product by a logical value commutes, and constructions with commutative characteristic semiring. In section 11.7, constructions are studied where linear maps on the power domains are uniquely determined by their values on singletons.

In section 11.8, all power constructions are classified into four principal classes according to whether the singletons are incomparable to the empty set, strictly below the empty set, strictly above the empty set, or equal to the empty set. These classes correspond to semiring classes where 0 and 1 are incomparable, $1 < 0$, $1 > 0$, or $1 = 0$ respectively holds.

In section 11.9, criteria are found for the singleton map being an embedding of the ground domain into the power domain. Section 11.10 shows that even if the ground domain and its image under the singleton map are not isomorphic, this image may be taken as an alternative ground domain.

All modules belonging to proper rings are shown to be discrete in section 11.11. Thus power constructions $\mathcal{P}$ with $\mathcal{P}\mathbf{1}$ being a ring are degenerated from the point of view of domain theory. The final section 11.12 investigates *additive semirings* characterized by the fact that all additive functions between their modules are linear.

11.1 Semirings and modules

The host of algebraic properties of power constructions may be described in terms of well-known algebraic structures.

Definition 11.1.1 (Semiring)
A *semiring domain* or shortly *semiring* $(R, +, 0, \cdot, 1)$ is a domain R with elements 0 and 1, and continuous operations $+, \cdot : [R \times R \to R]$ such that $(R, +, 0)$ is a commutative monoid, $(R, \cdot, 1)$ is a monoid, and multiplication '$\cdot$' is additive in both arguments, i.e.

$$a \cdot 0 = 0 \cdot a = 0 \qquad a \cdot (b_1 + b_2) = (a \cdot b_1) + (a \cdot b_2) \qquad (a_1 + a_2) \cdot b = (a_1 \cdot b) + (a_2 \cdot b)$$

The semiring is *commutative* iff its multiplication is, and it is *idempotent* iff its addition is, i.e. $a + a = a$ holds.

A *semiring homomorphism* $h : [R \to R']$ between two semirings is a continuous mapping that preserves the semiring operations:

$$h\,(a + b) = h\,a + h\,b \qquad h\,(0) = 0' \qquad h\,(a \cdot b) = h\,a \cdot h\,b \qquad h\,(1) = 1'$$

The power domain $\mathcal{P}\mathbf{1}$ is such a semiring with $0 = \mathsf{F} = \theta$, $a + b = a \curlyvee b = a \uplus b$, $1 = \mathsf{T} = \{\!|\diamond|\!\}$, and $a \cdot b = a \curlywedge b = ext\,(\lambda\diamond.\, b)\, a$ as shown in the previous sections.

Semirings are generalizations of both rings and distributive lattices. These in turn are generalizations of fields and Boolean algebras. Hence, both the notations $(R, +, 0, \cdot, 1)$ of the definition above and $(R, \curlyvee, \mathsf{F}, \curlywedge, \mathsf{T})$ as used in the previous sections seem to be adequate.

When semiring domains are considered which are lattices, there is a high risk to confuse the order '$\le$' of the domain and the lattice order '$\sqsubseteq$' defined by $a + b = b$. Generally, there is no relation between these two orders. In special cases only, they are equal or just opposite.

Definition 11.1.2 (Modules)
Let $R = (R, +, 0, \cdot, 1)$ be a semiring domain and $M = (M, +, 0)$ be a commutative monoid domain. $(R, M, \cdot)$ is a *(left) module* iff

$$\begin{array}{ll} \cdot : [R \times M \to M] & \\ a \cdot 0_M = 0_M & a \cdot (B_1 + B_2) = (a \cdot B_1) + (a \cdot B_2) \\ 0_R \cdot A = 0_M & (a_1 + a_2) \cdot B = (a_1 \cdot B) + (a_2 \cdot B) \\ 1_R \cdot A = A & a \cdot (b \cdot C) = (a \cdot b) \cdot C \end{array}$$

We also say 'M is a (left) R-module'.

Let M_1 and M_2 be two (left) R-modules. A morphism $f : [M_1 \to M_2]$ is *(left) linear* iff

$$f\,(A + B) = f A + f B \quad \text{and} \quad f\,(r \cdot A) = r \cdot f A$$

$(R, M, \cdot)$ is a *right module* iff

$$\begin{array}{ll} \cdot : [M \times R \to M] & \\ 0_M \cdot b = 0_M & (A_1 + A_2) \cdot b = (A_1 \cdot b) + (A_2 \cdot b) \\ A \cdot 0_R = 0_M & A \cdot (b_1 + b_2) = (A \cdot b_1) + (A \cdot b_2) \\ A \cdot 1_R = A & (A \cdot b) \cdot c = A \cdot (b \cdot c) \end{array}$$

We also say 'M is a right R-module'.

Let M_1 and M_2 be two right R-modules. A morphism $f : [M_1 \to M_2]$ is *right linear* iff

$$f\,(A + B) = f A + f B \quad \text{and} \quad f\,(A \cdot r) = f A \cdot r$$

We assume left modules as the standard; thus, the word 'left' may be omitted. Right modules do not become left modules by simply twisting the arguments. The very last equation of right modules becomes $c \cdot (b \cdot A) = (b \cdot c) \cdot A$ by twisting. Thus, a right R-module is a left R^t-module where R^t is R with multiplication twisted. For commutative semirings however, the notions of left and right module coincide.

Particularly prominent modules are those over a field; they are called vector spaces. The notion of linearity is drawn from there.

The most important results of the previous sections may be summarized to

Theorem 11.1.3 Let $\mathcal{P}$ be a power construction and let

$$\begin{array}{ll} + = \uplus : & [\mathcal{P}\mathbf{X} \times \mathcal{P}\mathbf{X} \to \mathcal{P}\mathbf{X}] \\ 0 = \theta : & \mathcal{P}\mathbf{X} \\ \cdot = \lambda(a,\, S).\, ext\,(\lambda\diamond.\, S)\, a : & [\mathcal{P}\mathbf{1} \times \mathcal{P}\mathbf{X} \to \mathcal{P}\mathbf{X}] \\ 1 = \{\!|\diamond|\!\} : & \mathcal{P}\mathbf{1} \end{array}$$

Then $\mathcal{P}\mathbf{1}$ with these operations is a semiring domain, and $\mathcal{P}\mathbf{X}$ is a $\mathcal{P}\mathbf{1}$-module for all domains $\mathbf{X}$. For $f : [\mathbf{X} \to \mathcal{P}\mathbf{Y}]$, the extension $\overline{f} : [\mathcal{P}\mathbf{X} \to \mathcal{P}\mathbf{Y}]$ is linear, and $\overline{f} \circ \iota = f$ holds.

The semiring $\mathcal{P}\mathbf{1}$ is called the *characteristic semiring* of the power construction $\mathcal{P}$. Generalizing a bit, a power construction $\mathcal{P}$ is said to have characteristic semiring R iff $\mathcal{P}\mathbf{1}$ and R are isomorphic semirings where the isomorphism is assumed to be fixed (cf. section 11.3).

Different power constructions may have the same characteristic semiring. For instance, the construction of the set of all subsets and the construction of the set of finite subsets for the class of discrete domains both have characteristic semiring $\{0, 1\}$ with $1 + 1 = 1$.

Conversely, one may wonder whether there is a power construction for every given semiring. The answer is yes; in chapters 14 and 15, two distinguished constructions with given semiring are presented.

11.2 Examples for characteristic semirings

In this section, we informally present some examples for power constructions and their characteristic semirings.

- The *lower power construction* has characteristic semiring $\{0 < 1\}$, or logically $\{\mathsf{F} < \mathsf{T}\}$ where $\mathsf{T} \vee \mathsf{T} = \mathsf{T}$. In this logic, F is unstable because it may become T while the computation proceeds. Thus, F actually means 'don't know' since only positive answers are reliable.
- The *upper power construction* has the dual semiring $\{1 < 0\}$, or logically $\{\mathsf{T} < \mathsf{F}\}$. Here, T is unstable and may change to F in the course of a computation. Only negative answers are reliable.

- The convex or *Plotkin power construction* has semiring $\{0, 1\}$ with $1 + 1 = 1$. The elements are not comparable. Logically, T and F are both stable here. The semiring is discrete, whence computations with logical result cannot proceed. They have immediately to decide whether the result is T or F, and cannot change their 'opinion' afterwards.

  The constructions of the set of all subsets and of the set of finite subsets have the same characteristic semiring as Plotkin's construction. Indeed, the construction of finite subsets is just a special instance of Plotkin's.

The three examples above show the importance of the empty set in our algebraic theory. Without empty set resp. 0, all three semirings would collapse to $\{1\}$ and could not be distinguished.

- A power construction with a more reasonable logic should have the Boolean domain $\mathbf{B} = \{\perp, \mathsf{F}, \mathsf{T}\}$ as semiring. Such constructions are called set domain constructions in [Hec90c]. The interpretation of $\perp$ is 'I do not (yet) know'. Computations with logical results start in this state which may change to F or T if the computation proceeds.

  The *sandwich power construction* [BDW88] or *big set domain construction* [Hec90c] and the *mixed power construction* [Gun89b, Gun90] or *small set domain construction* [Hec90c] both have characteristic semiring $\mathbf{B}$ with parallel conjunction and disjunction.

- *Multi-power domain constructions* containing formal multi-sets should have the natural numbers as their semiring. There are several ways how to arrange the naturals to form a semiring domain:
  - $\mathbf{N}_0^\infty = \{0 < 1 < 2 < \cdots < \infty\}$ with addition extended by $n + \infty = \infty$ and multiplication extended by $0 \cdot \infty = 0$ and $n \cdot \infty = \infty$ for $n > 0$.
  - The order dual $\mathbf{N}_\infty^0 = \{0 > 1 > 2 > \cdots > \infty\}$ of the semiring above.
  - '∞' is not needed here. Thus, $\mathbf{N}^0 = \{0 > 1 > 2 > \cdots\}$ with usual arithmetic is also a semiring domain.
  - $\mathbf{N}_0 = \{0, 1, 2, \ldots\}$ with usual arithmetic and discrete order.
  - The flat domain $\mathbf{N}_\perp = \{\perp, 0, 1, 2, \ldots\}$ with usual arithmetic extended by $n + \perp = \perp$, $0 \cdot \perp = 0$, and $n \cdot \perp = \perp$ for $n > 0$.
  - The domain of 'lazy naturals'.
  - The only multi-power construction that was ever proposed occurs in [Bro82] and is defined for algebraic ground domains. It excludes the empty set as many other proposals do. If the empty set is added properly, then the characteristic semiring becomes $\{0, 1 < 2 < 3 < \cdots < \infty\}$, i.e. 0 is totally uncomparable as in Plotkin's construction, and the remaining numbers form an infinite ascending chain.
- In [Mai85], *discrete probabilistic non-determinism* is modeled by a power construction with characteristic semiring $\mathbf{R}_0^\infty$ — the non-negative reals including infinity ordered as usual with ordinary addition and multiplication.
- In [Mai85] again, *oracle non-determinism* is modeled by a construction whose semiring is the power set of a fixed set. The power set is ordered by inclusion '$\subseteq$', addition is union, and multiplication is intersection.

- A third construction in [Mai85] models *ephemeral non-determinism.* Its semiring is the so-called *tropical semiring* $\mathbf{T} = \{t_0 < t_1 < t_2 < \cdots < t_\infty\}$ where $t_n + t_k = t_{min\,(n,\,k)}$, $0 = t_\infty$, $t_n \cdot t_k = t_{n+k}$, and $1 = t_0$, i.e. addition in $\mathbf{T}$ is minimum, and multiplication in $\mathbf{T}$ is arithmetic addition.

11.3 Power constructions with semirings

It is generally useful not to stick to the fact that the characteristic semiring be exactly $\mathcal{P}\mathbf{1}$. It is better to be more flexible and let the characteristic semiring be some isomorphic copy of $\mathcal{P}\mathbf{1}$. In this case, it is important to fix an isomorphism.

Definition 11.3.1 A tuple $(\mathcal{P},\, R,\, \varphi)$ is a power construction with semiring R, or shortly an R-construction, iff $\mathcal{P}$ is a power construction, R a semiring, and $\varphi : [R \to \mathcal{P}\mathbf{1}]$ is a semiring isomorphism.

If R allows non-trivial automorphisms, then there are several different isomorphisms between $\mathcal{P}\mathbf{1}$ and R. Hence, we fix an isomorphism in the definition. The importance of this fixing will be seen in the subsequent chapters. Nevertheless, we shall mostly use the sloppy notation '$\mathcal{P}$ is an R-construction' without explicitly mentioning the fixed isomorphism $\varphi : [R \to \mathcal{P}\mathbf{1}]$.

Various derived power operations involved the power domain $\mathcal{P}\mathbf{1}$ in their functionality. By means of the isomorphisms φ and φ^{-1}, they may be turned into operations involving R instead. For the sake of clarity, we mark the original operations by an asterisk in the following, and denote the original products by '$*$'.

$$
\begin{array}{lll}
\cdot : & [R \times \mathcal{P}\mathbf{X} \to \mathcal{P}\mathbf{X}] & r \cdot A = \varphi r * A \\
\cdot : & [\mathcal{P}\mathbf{X} \times R \to \mathcal{P}\mathbf{X}] & A \cdot r = A * \varphi r \\
ex : & [[\mathbf{X} \to R] \to [\mathcal{P}\mathbf{X} \to R]] & ex\, p = \varphi^{-1} \circ ex^*\,(\varphi \circ p) \\
ne : & [\mathcal{P}\mathbf{X} \to R] & ne = \varphi^{-1} \circ ne^* \\
\mathcal{E} : & [\mathcal{P}\mathbf{X} \to [[\mathbf{X} \to R] \to R]] & \mathcal{E} A\, p = \varphi^{-1}\,(\mathcal{E}^* A\,(\varphi \circ p)) \\
\mathit{filter} : & [[\mathbf{X} \to R] \to [\mathcal{P}\mathbf{X} \to \mathcal{P}\mathbf{X}]] & \mathit{filter}\, p = \mathit{filter}^*\,(\varphi \circ p)
\end{array}
$$

These new operations enjoy the same algebraic properties as the original operations. The proofs may be performed by simple equational reasoning. In the sequel, we shall mostly use the new operations.

11.4 Finite linear combinations

In section 9.4, we already defined finite formal sets in a power domain $\mathcal{P}\mathbf{X}$ by $\{\!|x_1,\, \ldots,\, x_n|\!\} = \{\!|x_1|\!\} \uplus \cdots \uplus \{\!|x_n|\!\}$. This notion may be generalized by employing multiplication. If $\mathcal{P}$ is an R-construction, we define *finite linear combinations* in a power domain $\mathcal{P}\mathbf{X}$ by

$$\{\!|r_1 \cdot x_1,\, \ldots,\, r_n \cdot x_n|\!\} = r_1 \cdot \{\!|x_1|\!\} \uplus \cdots \uplus r_n \cdot \{\!|x_n|\!\}$$

where the coefficients r_i are in R and the points x_i are in $\mathbf{X}$.

The coefficients indicate the degree of membership of the corresponding point in the formal set. A coefficient $0 = \mathsf{F}$ at some point x means this point is not in the formal set, whereas $1 = \mathsf{T}$ means it is. Other coefficients have other meanings depending on the structure of R. For instance, if $R = \{\perp, \mathsf{F}, \mathsf{T}\}$, then a coefficient $\perp$ denotes an uncertain membership because of $\perp \le \mathsf{T}$ and $\perp \le \mathsf{F}$.

We now show how the power operations act on such finite linear combinations.

- $\theta \;=\; \{\!|\,|\!\}$
- $\{\!|x|\!\} \;=\; \{\!|1 \cdot x|\!\}$
- $\{\!|r_1 \cdot x_1,\, \ldots,\, r_n \cdot x_n|\!\} \uplus \{\!|s_1 \cdot y_1,\, \ldots,\, s_m \cdot y_m|\!\} \;=\; \{\!|r_1 \cdot x_1,\, \ldots,\, r_n \cdot x_n,\, s_1 \cdot y_1,\, \ldots,\, s_m \cdot y_m|\!\}$
- $\mathit{ext}\, f\, \{\!|r_1 \cdot x_1,\, \ldots,\, r_n \cdot x_n|\!\} \;=\; r_1 \cdot f x_1 \uplus \cdots \uplus r_n \cdot f x_n$
- $\mathit{map}\, f\, \{\!|r_1 \cdot x_1,\, \ldots,\, r_n \cdot x_n|\!\} \;=\; \{\!|r_1 \cdot f x_1,\, \ldots,\, r_n \cdot f x_n|\!\}$
- $\mathcal{E}\, \{\!|r_1 \cdot x_1,\, \ldots,\, r_n \cdot x_n|\!\} \;=\; \lambda p.\, (r_1 \mathbin{\curlywedge} p\, x_1) \mathbin{\curlyvee} \cdots \mathbin{\curlyvee} (r_n \mathbin{\curlywedge} p\, x_n)$
- $s \cdot \{\!|r_1 \cdot x_1,\, \ldots,\, r_n \cdot x_n|\!\} \;=\; \{\!|(s \cdot r_1) \cdot x_1,\, \ldots,\, (s \cdot r_n) \cdot x_n|\!\}$
- $\{\!|r_1 \cdot x_1,\, \ldots,\, r_n \cdot x_n|\!\} \cdot s \;=\; \{\!|(r_1 \cdot s) \cdot x_1,\, \ldots,\, (r_n \cdot s) \cdot x_n|\!\}$
- $\mathit{filter}\, p\, \{\!|r_1 \cdot x_1,\, \ldots,\, r_n \cdot x_n|\!\} \;=\; \{\!|(r_1 \cdot p\, x_1) \cdot x_1,\, \ldots,\, (r_n \cdot p\, x_n) \cdot x_n|\!\}$
- $\{\!|r_1 \cdot x_1,\, \ldots,\, r_n \cdot x_n|\!\} \overrightarrow{\times} \{\!|s_1 \cdot y_1,\, \ldots,\, s_m \cdot y_m|\!\} \;=\; \{\!|(r_i \cdot s_j) \cdot (x_i,\, y_j) \mid 1 \le i \le n,\, 1 \le j \le m|\!\}$
- $\{\!|r_1 \cdot x_1,\, \ldots,\, r_n \cdot x_n|\!\} \overleftarrow{\times} \{\!|s_1 \cdot y_1,\, \ldots,\, s_m \cdot y_m|\!\} \;=\; \{\!|(s_j \cdot r_i) \cdot (x_i,\, y_j) \mid 1 \le i \le n,\, 1 \le j \le m|\!\}$

The equation for the right product is due to Prop. 10.10.2 (9): $\{\!|x|\!\} \cdot s = s \cdot \{\!|x|\!\}$. Hence, the left product operates from the outside, whereas the right product operates at the inside.

The formulae for the Cartesian products are derived by

$$
\begin{array}{lcl}
(r \cdot \{\!|x|\!\}) \overrightarrow{\times} (s \cdot \{\!|y|\!\}) & = & \mathit{ext}\ (\lambda a.\, \mathit{ext}\ (\lambda b.\, \{\!|(a,\, b)|\!\})\ (s \cdot \{\!|y|\!\}))\ (r \cdot \{\!|x|\!\}) \\
\text{(S4}\cdot\text{):} & = & r \cdot \mathit{ext}\ (\lambda a.\, \mathit{ext}\ (\lambda b.\, \{\!|(a,\, b)|\!\})\ (s \cdot \{\!|y|\!\}))\ \{\!|x|\!\} \\
\text{(P3):} & = & r \cdot \mathit{ext}\ (\lambda b.\, \{\!|(x,\, b)|\!\})\ (s \cdot \{\!|y|\!\}) \\
\text{(S4}\cdot\text{):} & = & r \cdot s \cdot \mathit{ext}\ (\lambda b.\, \{\!|(x,\, b)|\!\})\ \{\!|y|\!\} \\
\text{(P3):} & = & r \cdot s \cdot \{\!|(x,\, y)|\!\}
\end{array}
$$

and

$$
\begin{array}{lcl}
(r \cdot \{\!|x|\!\}) \overleftarrow{\times} (s \cdot \{\!|y|\!\}) & = & \mathit{ext}\ (\lambda b.\, \mathit{ext}\ (\lambda a.\, \{\!|(a,\, b)|\!\})\ (r \cdot \{\!|x|\!\}))\ (s \cdot \{\!|y|\!\}) \\
\text{(S4}\cdot\text{):} & = & s \cdot \mathit{ext}\ (\lambda b.\, \mathit{ext}\ (\lambda a.\, \{\!|(a,\, b)|\!\})\ (r \cdot \{\!|x|\!\}))\ \{\!|y|\!\} \\
\text{(P3):} & = & s \cdot \mathit{ext}\ (\lambda a.\, \{\!|(a,\, y)|\!\})\ (r \cdot \{\!|x|\!\}) \\
\text{(S4}\cdot\text{):} & = & s \cdot r \cdot \mathit{ext}\ (\lambda a.\, \{\!|(a,\, y)|\!\})\ \{\!|x|\!\} \\
\text{(P3):} & = & s \cdot r \cdot \{\!|(x,\, y)|\!\}
\end{array}
$$

In our opinion, these results show that '$\overrightarrow{\times}$' is more natural than '$\overleftarrow{\times}$'.

According to the algebraic laws of the power operations, syntactically different linear combinations may yield equal values, e.g. $\{\!|r \cdot x,\, s \cdot y|\!\} = \{\!|s \cdot y,\, r \cdot x|\!\}$, $\{\!|0 \cdot x,\, s \cdot y|\!\} = \{\!|s \cdot y|\!\}$, and $\{\!|r \cdot x,\, s \cdot x|\!\} = \{\!|(r + s) \cdot x|\!\}$ hold.

In addition, there are equalities among linear combinations that cannot be explained by algebraic laws only. They result from constraints due to monotonicity. Assume for instance that R be a semiring where $1+1=1$ holds. Then in every R-powerdomain, $x \le y \le z$ implies $\{\!|x, y, z|\!\} = \{\!|x, z|\!\}$ because of $\{\!|x, z|\!\} = \{\!|x, x, z|\!\} \le \{\!|x, y, z|\!\} \le \{\!|x, z, z|\!\} = \{\!|x, z|\!\}$.

Besides the finite linear combinations, a power domain usually contains limits of directed sets of these. These limits cannot always be conceived as infinite linear combinations. Worse, many power domains contain 'junk elements' that are neither finite linear combinations nor limits of these. Power domains that do not contain junk are called *reduced*; this topic is further explored in chapter 14.

11.5 Real power constructions

The subsequent sections are devoted to the definition and investigation of various classes of power constructions characterized by additional properties. We always try to attribute these properties to the characteristic semiring.

A *real power construction* is characterized by the additional property $A \uplus A = A$ for all A in $\mathcal{P}\mathbf{X}$ for all domains $\mathbf{X}$ in D. An equivalent criterion is $1+1=1$ in the characteristic semiring $\mathcal{P}\mathbf{1}$ since this is a special instance of the general property, and on the other hand, it implies $A \uplus A = 1 \cdot A \uplus 1 \cdot A = (1+1) \cdot A = 1 \cdot A = A$. Hence, it is sufficient to consider $\mathcal{P}\mathbf{1}$ to decide whether union in all power domains $\mathcal{P}\mathbf{X}$ is idempotent.

11.6 Symmetry and commutativity

We first repeat the definitions and introduce a new notion in between:

Definition 11.6.1 Let $\mathcal{P}$ be a power construction.

- $\mathcal{P}$ is *symmetric* iff for all domains $\mathbf{X}$, $\mathbf{Y}$, and $\mathbf{Z}$ in D, and all A in $\mathcal{P}\mathbf{X}$, B in $\mathcal{P}\mathbf{Y}$, and $\star : [\mathbf{X} \times \mathbf{Y} \to \mathcal{P}\mathbf{Z}]$:

  $$ext\,(\lambda a.\, ext\,(\lambda b.\, a \star b)\, B)\, A = ext\,(\lambda b.\, ext\,(\lambda a.\, a \star b)\, A)\, B$$

- $\mathcal{P}$ is *commutative* iff for all domains $\mathbf{Y}$ in D and all a in $\mathcal{P}\mathbf{1}$ and B in $\mathcal{P}\mathbf{Y}$:

  $$a \cdot B = B \cdot a$$

- $\mathcal{P}\mathbf{1}$ is *commutative* iff for all a, b in $\mathcal{P}\mathbf{1}$: $a \cdot b = b \cdot a$

Obviously, commutativity of the characteristic semiring is a special instance of commutativity of the construction. The relation between symmetry and commutativity of $\mathcal{P}$ is clarified by expanding left and right multiplication:

$$\begin{aligned} a \cdot B &= ext\,(\lambda\diamond.\, B)\, a = ext\,(\lambda\diamond.\, ext\,(\lambda b.\, \{\!|b|\!\})\, B)\, a \quad \text{by (S3)} \\ B \cdot a &= ext\,(\lambda b.\, a \cdot \{\!|b|\!\})\, B = ext\,(\lambda b.\, ext\,(\lambda\diamond.\, \{\!|b|\!\})\, a)\, B \end{aligned}$$

Thus, commutativity is a special instance of symmetry where $\mathbf{X} = \mathbf{1}$, $\mathbf{Y} = \mathbf{Z}$, and $\diamond \star b = \{\!|b|\!\}$.

Symmetry implies commutativity of $\mathcal{P}$ which in turn implies commutativity of $\mathcal{P}\mathbf{1}$. The opposite implications seem to be false in general; we do not know of examples for this however. The next section provides a special case where they are true.

11.7 Unique extensions

A power construction $\mathcal{P}$ has *unique linear extensions* iff for all linear functions $F_1, F_2 : [\mathcal{P}\mathbf{X} \to \mathcal{P}\mathbf{Y}]$, $F_1 \circ \iota = F_2 \circ \iota$ implies $F_1 = F_2$. In other words, two linear functions that coincide for singletons are equal. $\mathcal{P}$ has *unique additive extensions* iff two additive functions that coincide for singletons are equal. Since all linear functions are additive, the property of having unique additive extensions implies unique linear extensions.

Proposition 11.7.1 A power construction with unique linear extensions is symmetric iff its characteristic semiring is commutative. A power construction with unique additive extensions is symmetric.

Proof: For the first statement, we only need to prove the symmetry of the power construction in the case of commutative semiring R; the opposite implication was already handled in the previous section. The proof is done in two steps:

- In the first step, we claim

  $$ext\,(\lambda x.\, r \cdot f x) = \lambda A.\, r \cdot ext\, f\, A$$

  The left hand function is linear as an extension. The right hand function is linear only because of the commutativity of R. A linear factor at A has to be moved across r:

  $$\begin{aligned}(\lambda A.\, r \cdot ext\, f\, A)\,(s \cdot A) &= r \cdot ext\, f\,(s \cdot A) = r \cdot s \cdot ext\, f\, A \\ &= s \cdot r \cdot ext\, f\, A = s \cdot (\lambda A.\, r \cdot ext\, f\, A)\, A\end{aligned}$$

  Both functions map a singleton $\{\!|a|\!\}$ to $r \cdot f a$, whence they are equal.

- Next, we claim

  $$ext\,(\lambda a.\, ext\,(\lambda b.\, a \star b)\, B) = \lambda A.\, ext\,(\lambda b.\, ext\,(\lambda a.\, a \star b)\, A)\, B$$

  Both functions are linear; the function to the right because of the equation proved in the first step. Both functions map a singleton $\{\!|a|\!\}$ to $ext\,(\lambda b.\, a \star b)\, B$.

For the second statement, we only have to show commutativity of $\mathcal{P}\mathbf{1}$, then the first statement applies. For all a in $\mathcal{P}\mathbf{1}$, we claim $\lambda x.\, a \cdot x = \lambda x.\, x \cdot a$ where x ranges over $\mathcal{P}\mathbf{1}$. Because of distributivity, both functions are additive. Applied to the only singleton $\{\!|\diamond|\!\} = 1$, they both yield a. Hence, they are equal. □

11.8 The four kinds of power constructions

According to the relations between empty set and singletons, one may distinguish among four kinds of power constructions: the *lower kind* where the empty set is strictly below all singletons, the *upper kind* where it is strictly above, the *degenerated kind* where all singletons are empty, and the *separated kind* where the empty set is uncomparable to all singletons. The following propositions show that it is impossible that some singletons are strictly below θ whereas others are not. The reason is that the four kinds are completely determined by the characteristic semiring; they correspond to the cases $0 < 1$, $0 > 1$, $0 = 1$, and incomparability of 0 and 1.

Proposition 11.8.1 For power constructions, the following statements are equivalent:

(1) There is a domain $\mathbf{X}$ in D and a point x in $\mathbf{X}$ with $\{\!|x|\!\} = \theta$.

(2) In $\mathcal{P}\mathbf{1}$, $0 = 1$ holds.

(3) For all domains $\mathbf{X}$ in D and all A in $\mathcal{P}\mathbf{X}$, $A = \theta$ holds.

(4) All $\mathcal{P}\mathbf{1}$-modules are isomorphic to $\{0\}$.

Proof:

$(1) \Rightarrow (2)$: $ne\ \{\!|x|\!\} = 1$ and $ne\ \theta = 0$.

$(2) \Rightarrow (1)$: Let $\mathbf{X} = \mathbf{1}$ and $x = \diamond$. $0 = \theta$ and $1 = \{\!|\diamond|\!\}$ hold.

$(2) \Rightarrow (4)$: $A = 1 \cdot A = 0 \cdot A = 0$

$(4) \Rightarrow (3) \Rightarrow (2)$ is obvious. □

Let the *degenerated power construction* be defined by $\mathcal{P}\mathbf{X} = \{\theta\}$ for all domains $\mathbf{X}$. Trivially, this is a symmetric real power construction. Prop. 11.8.1 shows that it is the only construction where $\{\!|x|\!\} = \theta$ is possible for some x in some $\mathbf{X}$.

Proposition 11.8.2 For power constructions, the following statements are equivalent:

(1) There is a domain $\mathbf{X}$ in D and a point x in $\mathbf{X}$ with $\theta \le \{\!|x|\!\}$.

(2) In $\mathcal{P}\mathbf{1}$, $0 \le 1$ holds.

(3) For all domains $\mathbf{X}$ in D and all A in $\mathcal{P}\mathbf{X}$, $\theta \le A$ holds.

(4) In all $\mathcal{P}\mathbf{1}$-modules, 0 is the least element.

The dual equivalence obtained by replacing '$\le$' by '$\ge$' and 'least' by 'greatest' also holds.

Proof:

$(1) \Rightarrow (2)$: $\theta \le \{\!|x|\!\}$ implies $0 = ne\ \theta \le ne\ \{\!|x|\!\} = 1$.

$(2) \Rightarrow (1)$: Let $\mathbf{X} = \mathbf{1}$ and $x = \diamond$. $\theta = 0 \le 1 = \{\!|\diamond|\!\}$ holds.

$(2) \Rightarrow (4)$: $0 = 0 \cdot A \le 1 \cdot A = A$

$(4) \Rightarrow (3) \Rightarrow (2)$ is obvious. □

11.9 Faithful power constructions

Definition 11.9.1 A power domain construction $\mathcal{P}$ is *faithful* iff for all domains $\mathbf{X}$ in D the map $\iota : \mathbf{X} \to \mathcal{P}\mathbf{X}$ is an embedding, i.e. $\{\!|x|\!\} \le \{\!|x'|\!\}$ implies $x \le x'$. $\mathcal{P}$ is *injective* iff ι is injective for all domains $\mathbf{X}$ in D.

If $\mathcal{P}$ is a faithful construction, then $\mathbf{X}$ may be considered a subdomain of $\mathcal{P}\mathbf{X}$. There is a useful criterion for faithfulness:

Lemma 11.9.2 Let $\mathcal{P}$ be a power construction defined for a class D. If there is a domain $\mathbf{Y}$ in D such that $\mathcal{P}\mathbf{Y}$ is not discrete, then $\mathcal{P}$ is faithful.

Here, being discrete means $a \le b$ implies $a = b$. Hence, a domain is not discrete iff there are points a and b with $a < b$ in it.

Proof: Let $\mathbf{Y}$ be a (fixed) domain such that $\mathcal{P}\mathbf{Y}$ is not discrete. Let B and B' be two fixed members of $\mathcal{P}\mathbf{Y}$ with $B < B'$.

Let $\mathbf{X}$ be an arbitrary domain in D, and let x and x' be two points of $\mathbf{X}$ such that $\{\!|x|\!\} \leq \{\!|x'|\!\}$ holds. We have to show $x \leq x'$. Applying Prop. 4.3.1, let O be an open set with $x \in O$. Then we must show $x' \in O$.

Let $f : \mathbf{X} \to \mathcal{P}\mathbf{Y}$ be defined by $fu = \begin{cases} B' & \text{if } u \in O \\ B & \text{otherwise} \end{cases}$

f is continuous by Lemma 3.7.1, and may be extended to the continuous function $\overline{f} : [\mathcal{P}\mathbf{X} \to \mathcal{P}\mathbf{Y}]$. $x \in O$ implies $\overline{f}\{\!|x|\!\} = fx = B'$. By monotonicity of $\overline{f}$, $fx' = \overline{f}\{\!|x'|\!\} \geq \overline{f}\{\!|x|\!\} = B'$ holds. f is able to produce only the two values B and B', whence $fx' = B'$ follows, i.e. x' is in O. □

By this theorem, a total power construction is not faithful iff all power domains $\mathcal{P}\mathbf{X}$ are discrete.

The non-discrete power domain $\mathcal{P}\mathbf{Y}$ may well be $\mathcal{P}\mathbf{1}$. Thus, we obtain

Corollary 11.9.3
A power domain construction with a non-discrete characteristic semiring is faithful.

The final theorem tells that being injective and being faithful is almost the same.

Theorem 11.9.4 Let $\mathcal{P}$ be a power construction that is defined for at least one non-discrete domain $\mathbf{X}$. Then, the following statements are equivalent:

(1) $\mathcal{P}$ is faithful.

(2) $\mathcal{P}$ is injective.

(3) $\mathcal{P}\mathbf{X}$ is not discrete.

(4) There is a domain $\mathbf{Y}$ in D such that $\mathcal{P}\mathbf{Y}$ is not discrete.

Proof:

(1) $\Rightarrow$ (2): Every embedding is injective.

(2) $\Rightarrow$ (3): Let a and b be two points of $\mathbf{X}$ with $a < b$. By monotonicity, $\{\!|a|\!\} \leq \{\!|b|\!\}$ holds. The two singletons are different because of injectivity. Thus, $\{\!|a|\!\} < \{\!|b|\!\}$ holds, and $\mathcal{P}\mathbf{X}$ is not discrete.

(3) $\Rightarrow$ (4): is obvious; let $\mathbf{Y} = \mathbf{X}$.

(4) $\Rightarrow$ (1): Immediately by Lemma 11.9.2. □

11.10 Ground domains as sub-domains of power domains

If $\mathcal{P}$ is a faithful power construction, then power domains $\mathcal{P}\mathbf{X}$ contain an isomorphic image of the ground domain $\mathbf{X}$ as a sub-domain. If $\mathcal{P}$ is not faithful, then this statement is wrong. Even in this case however, $\mathcal{P}\mathbf{X}$ contains a sub-domain $\mathbf{Y}$ such that $\mathcal{P}\mathbf{X}$ is isomorphic to $\mathcal{P}\mathbf{Y}$. The alternative ground domain $\mathbf{Y}$ is the image of $\mathbf{X}$ in $\mathcal{P}\mathbf{X}$.

Theorem 11.10.1 Let $\mathbf{X}$ be a ground domain and $\mathcal{P}$ a power domain construction defined for $\mathbf{X}$. Then the image $\mathbf{Y} = \iota[\mathbf{X}]$ of $\mathbf{X}$ by the singleton mapping is a sub-domain of $\mathcal{P}\mathbf{X}$ and the two power domains $\mathcal{P}\mathbf{X}$ and $\mathcal{P}\mathbf{Y}$ are isomorphic by a $\mathcal{P}\mathbf{1}$-linear mapping.

Proof: If $\mathcal{P}$ is faithful, then $\mathbf{Y}$ is isomorphic to $\mathbf{X}$ by ι, whence it is a sub-domain of $\mathcal{P}\mathbf{X}$. Otherwise, the power domain $\mathcal{P}\mathbf{X}$ is discrete by Lemma 11.9.2, whence $\mathbf{Y}$ is also discrete and a sub-domain of $\mathcal{P}\mathbf{X}$.

Let $\iota_{\mathbf{X}} : [\mathbf{X} \to \mathcal{P}\mathbf{X}]$ and $\iota_{\mathbf{Y}} : [\mathbf{Y} \to \mathcal{P}\mathbf{Y}]$ be the two singleton mappings. $\iota_{\mathbf{X}}$ may be co-restricted to a surjective mapping $e : [\mathbf{X} \to \mathbf{Y}]$. By composition with $\iota_{\mathbf{Y}}$, one obtains $f = \iota_{\mathbf{Y}} \circ e : [\mathbf{X} \to \mathcal{P}\mathbf{Y}]$. The natural inclusion map of $\mathbf{Y}$ into $\mathcal{P}\mathbf{X}$ is denoted by $g : [\mathbf{Y} \to \mathcal{P}\mathbf{X}]$. Extension yields a pair of functions $\overline{f} : [\mathcal{P}\mathbf{X} \to \mathcal{P}\mathbf{Y}]$ and $\overline{g} : [\mathcal{P}\mathbf{Y} \to \mathcal{P}\mathbf{X}]$. We show that $\overline{f}$ and $\overline{g}$ are inverse to each other.

One part of the proof is fairly easy:

$$\overline{g} \circ \overline{f} \stackrel{(S4)}{=} \overline{\overline{g} \circ f} = \overline{\overline{g} \circ \iota_{\mathbf{Y}} \circ e} \stackrel{(P3)}{=} \overline{g \circ e} = \overline{\iota_{\mathbf{X}}} \stackrel{(S3)}{=} id_{\mathcal{P}\mathbf{X}}$$

using the equation $g \circ e = \iota_{\mathbf{X}}$.

Similarly, $\overline{f} \circ \overline{g} = \overline{\overline{f} \circ g}$ holds by (S4). If we are able to prove $\overline{f} \circ g = \iota_{\mathbf{Y}}$, then $\overline{f} \circ \overline{g} = id_{\mathcal{P}\mathbf{Y}}$ follows by (S3).

Let y be a point of $\mathbf{Y}$. Then $y = ex$ holds for some x in $\mathbf{X}$, and $\overline{f}(g\,y) = \overline{f}(g\,(e\,x)) = \overline{f}(\iota_{\mathbf{X}} x) = f x = \iota_{\mathbf{Y}}\,(e\,x) = \iota_{\mathbf{Y}}\,y$ follows. □

11.11 Modules over rings

A *ring* is a semiring that has a member -1 such that $1 + (-1) = 0$ holds. One might believe that power constructions whose characteristic semiring is a ring have particularly nice properties. Because we consider domains and continuous operations, the contrary is true however.

Proposition 11.11.1 If R is a ring, then every R-module is discrete.

Hence, power constructions with a characteristic ring cannot be faithful if they are defined for some non-discrete ground domain.

Proof: Let M be an R-module where R is a ring. Then for all c in M, $c + (-1) \cdot c = (1 + (-1)) \cdot c = 0 \cdot c = 0$ holds.

Now assume $a \leq b$ holds for some points a and b of M. By monotonicity of the external product, $(-1) \cdot a \leq (-1) \cdot b$ follows. By monotonicity of addition, we obtain $a + b + (-1) \cdot a \leq a + b + (-1) \cdot b$. By the equation above, the left hand side equals b and the right hand side equals a. Thus, $a \leq b$ implies $b \leq a$, whence $a = b$. □

11.12 Additive semirings

Modules M over a semiring R were defined as commutative monoids with an external product. A morphism $f : [M_1 \to M_2]$ between two R-modules is *additive* iff $f(0) = 0$ and

$f(a+b) = fa + fb$, and *linear* iff $f(a+b) = fa + fb$ and $f(r \cdot a) = r \cdot fa$. Every linear map is additive by $f(0) = f(0 \cdot 0) = 0 \cdot f0 = 0$. The opposite implication not being generally true, it however holds for many common cases. We attribute this property to the semiring and define:

Definition 11.12.1 A semiring domain R is *additive* iff all additive morphisms between any two R-modules are linear.

The quantification over all additive *morphisms* makes this notion depend on the underlying domains. There are semirings where continuity is necessary to derive linearity from additivity.

Some examples will clarify the situation:

- A semiring R with carrier $\{0, 1\}$ with any order is additive since $f(0 \cdot x) = f(0) = 0 = 0 \cdot fx$ holds by additivity of f, and also $f(1 \cdot x) = fx = 1 \cdot fx$ holds anyway.
- $\mathbf{N}_0^\infty = \{0 < 1 < 2 < \cdots < \infty\}$ with usual addition and multiplication extended by $n + \infty = \infty$ for all n, and $0 \cdot \infty = 0$, and $n \cdot \infty = \infty$ for all $n > 0$ becomes a semiring domain. It is additive since $f(n \cdot x) = n \cdot fx$ may be shown by induction for $n < \infty$:

  $$f((n+1) \cdot x) = f(n \cdot x + x) = f(n \cdot x) + fx = n \cdot fx + fx = (n+1) \cdot fx$$

  and follows from continuity of f and '$\cdot$' in case $n = \infty$.

  The monotonic, but non-continuous map

  $$g : \mathbf{N}_0^\infty \to \mathbf{N}_0^\infty, \quad g\, n = \begin{cases} 0 & \text{if } n < \infty \\ \infty & \text{if } n = \infty \end{cases}$$

  is additive, but not linear because of $g(\infty \cdot 1) = g(\infty) = \infty$, but $\infty \cdot g(1) = \infty \cdot 0 = 0$.
- The 'tropical semiring' is $\mathbf{T} = \{t_0 < t_1 < \cdots < t_\infty\}$ with $t_n + t_k = t_{min\,(n,k)}$, $0 = t_\infty$, $t_n \cdot t_k = t_{n+k}$, and $1 = t_0$. The operation of doubling $d(t_n) = t_{2 \cdot n}$ is continuous and additive, but not linear since $d(t_1 \cdot t_1) = d(t_2) = t_4$, whereas $t_1 \cdot d(t_1) = t_1 \cdot t_2 = t_3$. Hence, the tropical semiring is not additive.

Proposition 11.12.2 (Consequences of additivity)

(1) Every additive semiring is commutative.

(2) A power construction with additive characteristic semiring is commutative.

Proof: We show only (2); the proof of (1) is similar.

Let S be a member of $\mathcal{P}\mathbf{X}$ for some $\mathbf{X}$ in D. The map $f = \lambda r.\, S \cdot r : [\mathcal{P}\mathbf{1} \to \mathcal{P}\mathbf{X}]$ is additive, whence linear because of additivity of $\mathcal{P}\mathbf{1}$. Thus,

$$S \cdot r = S \cdot (r \cdot 1) = f(r \cdot 1) = r \cdot f(1) = r \cdot (S \cdot 1) = r \cdot S \qquad \square$$

Chapter 12

Power homomorphisms

Homomorphisms between algebraic structures are mappings preserving all operations of these structures. Power constructions may be considered algebraic structures on a higher level. Thus, it is also possible and useful to define corresponding homomorphisms.

Power homomorphisms are defined in section 12.1. In section 12.2, power isomorphisms are considered. In section 12.3, the behavior of the derived power operations under power homomorphisms is studied. It turns out that power homomorphisms are usually not linear. The linear ones are investigated in section 12.4. They can be used to define initial and final power constructions (section 12.5). A power construction with semiring R is *initial* iff there is exactly one linear power homomorphism from it to every other construction with semiring R. *Finality* is the dual notion. The existence of initial and final power constructions for every semiring is shown in chapters 14 and 15 respectively.

12.1 Definition

A *power homomorphism* $H : \mathcal{P} \dot{\rightarrow} \mathcal{Q}$ between two power constructions $\mathcal{P}$ and $\mathcal{Q}$ with $def\ \mathcal{P} \subseteq def\ \mathcal{Q}$ is a 'family' of morphisms $H = (H_{\mathbf{X}})_{\mathbf{X} \in def\ \mathcal{P}} : [\mathcal{P}\mathbf{X} \to \mathcal{Q}\mathbf{X}]$ commuting over all power operations, i.e.

- The empty set in $\mathcal{P}\mathbf{X}$ is mapped to the empty set in $\mathcal{Q}\mathbf{X}$: $H\theta = \theta$.
- The image of a union is the union of the images: $H(A \uplus B) = (HA) \uplus (HB)$. This axiom and the previous one together mean H is additive.
- Singletons in $\mathcal{P}\mathbf{X}$ are mapped to singletons in $\mathcal{Q}\mathbf{X}$: $H\{\!|x|\!\}_{\mathcal{P}} = \{\!|x|\!\}_{\mathcal{Q}}$, or: $H \circ \iota_{\mathcal{P}} = \iota_{\mathcal{Q}}$.
- Let $f : [\mathbf{X} \to \mathcal{P}\mathbf{Y}]$. Then $H \circ f : [\mathbf{X} \to \mathcal{Q}\mathbf{Y}]$, and $H(ext_{\mathcal{P}}\, f\, A) = ext_{\mathcal{Q}}(H \circ f)(HA)$ has to hold for all A in $\mathcal{P}\mathbf{X}$. This axiom may also be written $H \circ (ext_{\mathcal{P}}\, f) = ext_{\mathcal{Q}}(H \circ f) \circ H$ (see the figure to the right).

$$\begin{tikzcd}
\mathbf{X} \arrow[d, "\iota"'] \arrow[dr, "H \circ f"] & \\
\mathcal{Q}\mathbf{X} \arrow[r, "\overline{H \circ f}"] & \mathcal{Q}\mathbf{Y} \\
\mathcal{P}\mathbf{X} \arrow[u, "H"] \arrow[r, "\overline{f}"] & \mathcal{P}\mathbf{Y} \arrow[u, "H"'] \\
\mathbf{X} \arrow[u, "\iota"] \arrow[ur, "f"'] &
\end{tikzcd}$$

Obviously, there is an identity power homomorphism $I : \mathcal{P} \dot{\to} \mathcal{P}$ where all morphisms $I_{\mathbf{X}}$ are identities. Furthermore, two power homomorphisms $G : \mathcal{P} \dot{\to} \mathcal{Q}$ and $H : \mathcal{Q} \dot{\to} \mathcal{R}$ may be composed 'pointwise', i.e. $(H \circ G)_{\mathbf{X}} = H_{\mathbf{X}} \circ G_{\mathbf{X}}$. It is easy to show that the outcome is again a power homomorphism $H \circ G : \mathcal{P} \dot{\to} \mathcal{R}$.

12.2 Power isomorphisms

A *power isomorphism* between two constructions $\mathcal{P}$ and $\mathcal{Q}$ is a family of isomorphisms $H = H_{\mathbf{X}} : [\mathcal{P}\mathbf{X} \to \mathcal{Q}\mathbf{X}]$ such that both $(H_{\mathbf{X}})_{\mathbf{X} \in\, def\, \mathcal{P}}$ and $(H_{\mathbf{X}}^{-1})_{\mathbf{X} \in\, def\, \mathcal{Q}}$ are power homomorphisms. Hence, two isomorphic constructions are defined for the same class of domains.

In contrast to continuity, the inverse of a bijective power homomorphism is again a power homomorphism.

Proposition 12.2.1 Let $H : \mathcal{P} \dot{\to} \mathcal{Q}$ be a power homomorphism such that the individual maps $H_{\mathbf{X}} : [\mathcal{P}\mathbf{X} \to \mathcal{Q}\mathbf{X}]$ are all bijective, and their inverses are monotonic. Then H is a power isomorphism.

Proof: The inverses are continuous by Prop. 2.5.3. They satisfy the axioms of a power homomorphism:

- $H^{-1}(\theta_{\mathcal{Q}}) = H^{-1}(H(\theta_{\mathcal{P}})) = \theta_{\mathcal{P}}$.
- $H^{-1}(A \uplus B) = H^{-1}((H(H^{-1}A)) \uplus (H(H^{-1}B))) = H^{-1}(H(H^{-1}A \uplus H^{-1}B)) = H^{-1}A \uplus H^{-1}B$.
- $H^{-1}\{\!|x|\!\}_{\mathcal{Q}} = H^{-1}(H\{\!|x|\!\}_{\mathcal{P}}) = \{\!|x|\!\}_{\mathcal{P}}$.
- $H^{-1} \circ ext_{\mathcal{Q}} f = H^{-1} \circ ext_{\mathcal{Q}}(H \circ H^{-1} \circ f) \circ H \circ H^{-1} = H^{-1} \circ H \circ ext_{\mathcal{Q}}(H^{-1} \circ f) \circ H^{-1} = ext_{\mathcal{Q}}(H^{-1} \circ f) \circ H^{-1}$ □

12.3 Some properties of power homomorphisms

Since power homomorphisms preserve all basic power operations, it is not surprising that they also preserve the derived operations.

Proposition 12.3.1 Let $H : \mathcal{P} \dot{\to} \mathcal{Q}$ be a power homomorphism.

(1) Let $f : [\mathbf{X} \to \mathbf{Y}]$. Then $H \circ (map_{\mathcal{P}} f) = (map_{\mathcal{Q}} f) \circ H : [\mathcal{P}\mathbf{X} \to \mathcal{Q}\mathbf{Y}]$ (see the figure).

(2) Let b be in $\mathcal{P}\mathbf{1}$ and S in $\mathcal{P}\mathbf{X}$. Then $H(b \cdot S) = Hb \cdot HS$ and $H(S \cdot b) = HS \cdot Hb$.

(3) $H_{\mathbf{1}} : [\mathcal{P}\mathbf{1} \to \mathcal{Q}\mathbf{1}]$ is a semiring homomorphism.

(4) Let $p : [\mathbf{X} \to \mathcal{P}\mathbf{1}]$ and A in $\mathcal{P}\mathbf{X}$. Then $H\,(\mathit{filter}_{\mathcal{P}}\; p\; A) = \mathit{filter}_{\mathcal{Q}}\,(H \circ p)\,(HA)$. Using composition, one may also write $H \circ (\mathit{filter}_{\mathcal{P}}\; p) = (\mathit{filter}_{\mathcal{Q}}\,(H \circ p)) \circ H$.

$$\begin{tikzcd}
\mathcal{Q}\mathbf{X} \arrow[rrr, "map_{\mathcal{Q}} f"] & & & \mathcal{Q}\mathbf{Y} \\
& \mathbf{X} \arrow[r, "f"] \arrow[ul, "\iota"] \arrow[dl, "\iota"'] & \mathbf{Y} \arrow[ur, "\iota"'] \arrow[dr, "\iota"] & \\
\mathcal{P}\mathbf{X} \arrow[rrr, "map_{\mathcal{P}} f"] \arrow[uu, "H"] & & & \mathcal{P}\mathbf{Y} \arrow[uu, "H"']
\end{tikzcd}$$

In categorical terms, (1) means H is a *natural transformation* between the functors $\mathcal{P}$ and $\mathcal{Q}$.

Proof:

(1)
$$\begin{aligned}
H \circ (map_{\mathcal{P}}\, f) &= H \circ (ext_{\mathcal{P}}\,(\iota_{\mathcal{P}} \circ f)) \\
&= (ext_{\mathcal{Q}}\,(H \circ \iota_{\mathcal{P}} \circ f)) \circ H \\
&= (ext_{\mathcal{Q}}\,(\iota_{\mathcal{Q}} \circ f)) \circ H \\
&= (map_{\mathcal{Q}}\, f) \circ H
\end{aligned}$$

(2) $H(b \cdot S) = H(\mathit{ext}\,(\lambda\diamond.\, S)\, b) = \mathit{ext}\,(\lambda\diamond.\, HS)\,(Hb) = (Hb) \cdot (HS)$

(4)
$$\begin{aligned}
H(\mathit{filter}\; p\; A) &= H(\mathit{ext}\,(\lambda x.\, p\, x \cdot \{\!|x|\!\}_{\mathcal{P}})\; A) \\
&= \mathit{ext}\,(\lambda x.\, H(p\, x \cdot \{\!|x|\!\}_{\mathcal{P}}))\,(HA) \\
(2): \quad &= \mathit{ext}\,(\lambda x.\, H(p\, x) \cdot \{\!|x|\!\}_{\mathcal{Q}})\,(HA) \\
&= \mathit{filter}\,(H \circ p)\,(HA)
\end{aligned}$$

(2) $H(S \cdot b) = H(\mathit{filter}\,(\lambda x.\, b)\, S) = \mathit{filter}\,(\lambda x.\, Hb)\,(HS) = (HS) \cdot (Hb)$

(3) $H_{\mathbf{1}}$ respects $+ = \uplus$, $0 = \theta$, and $1 = \{\!|\diamond|\!\}$ by the definition of power homomorphisms. It respects '$\cdot$' by (2). □

12.4 Linear power homomorphisms

In the following, we want to compare power constructions with the same characteristic semiring by means of power homomorphisms. For fixed semiring, the notion of linearity of a power homomorphism makes sense.

Definition 12.4.1 Let $(\mathcal{P}, R, \varphi)$ and $(\mathcal{P}', R, \varphi')$ be two power constructions with the same characteristic semiring. A power homomorphism $H : \mathcal{P} \dot{\to} \mathcal{Q}$ is called *linear* iff the morphisms $H_{\mathbf{X}} : [\mathcal{P}\mathbf{X} \to \mathcal{P}'\mathbf{X}]$ are R-linear.

Remember that we always fix a semiring isomorphism $\varphi : [R \to \mathcal{P}\mathbf{1}]$ when considering a power construction $\mathcal{P}$ with semiring R.

Linearity of the morphisms is not a matter of course. Prop. 12.3.1 (2) tells $H(b \cdot S) = Hb \cdot HS$ instead for b in $\mathcal{P}\mathbf{1}$. From this, it becomes evident that a power homomorphism is linear iff it acts on R as an identity.

Proposition 12.4.2 Let $(\mathcal{P}, R, \varphi)$ and $(\mathcal{P}', R, \varphi')$ be two power constructions with the same characteristic semiring. A power homomorphism $H : \mathcal{P} \dot{\to} \mathcal{Q}$ is linear iff the composition ${\varphi'}^{-1} \circ H_{\mathbf{1}} \circ \varphi : [R \to R]$ is the identity.

Proof: To be sufficiently distinctive, we denote the product with members of $\mathcal{P}\mathbf{1}$ and $\mathcal{P}'\mathbf{1}$ by '$*$' in this proof.

Let H be a linear power homomorphism. Then for all r in R,

$$\begin{array}{rcll} {\varphi'}^{-1}(H_{\mathbf{1}}(\varphi r)) & = & {\varphi'}^{-1}(H_{\mathbf{1}}(\varphi r * \{\!|\diamond|\!\})) & \{\!|\diamond|\!\} \text{ is neutral in } \mathcal{P}\mathbf{1} \\ & = & {\varphi'}^{-1}(H_{\mathbf{1}}(r \cdot \{\!|\diamond|\!\})) & R\text{-product '}\cdot\text{' defined by } \varphi \\ & = & {\varphi'}^{-1}(r \cdot H_{\mathbf{1}}\{\!|\diamond|\!\}) & H \text{ is linear} \\ & = & {\varphi'}^{-1}(\varphi' r * \{\!|\diamond|\!\}') & H \text{ is power homomorphism} \\ & = & {\varphi'}^{-1}(\varphi' r) \;=\; r & \end{array}$$

Conversely,

$$H(r \cdot S) = H(\varphi r * S) = H(\varphi r) * HS = {\varphi'}^{-1}(H(\varphi r)) \cdot HS = r \cdot HS$$

holds applying the definition of '$\cdot$' in terms of '$*$'. □

Hence, if R allows non-trivial automorphisms there are non-linear power homomorphisms besides the linear ones.

12.5 Initial and final constructions

Initial and final power constructions are defined relative to the characteristic semiring by means of *linear* power homomorphisms. Without the assumption of linearity, their existence could not be guaranteed.

A construction $\mathcal{P}$ is initial if for all constructions $\mathcal{Q}$ with the same characteristic semiring there is exactly one linear power homomorphism $\mathcal{P} \dot{\to} \mathcal{Q}$. Finality is dual. The exact definitions however are more complex. To prevent a construction from being initial simply because it is almost undefined, we concentrate on total constructions defined for all domains.

Definition 12.5.1

A total power construction $(\mathcal{P},\ R,\ \varphi)$ is *initial* for semiring R if for all total R-constructions $(\mathcal{Q},\ R,\ \varphi')$ there is exactly one linear power homomorphism $H : \mathcal{P} \dot{\rightarrow} \mathcal{Q}$.

A total power construction $(\mathcal{P},\ R,\ \varphi)$ is *final* for semiring R if for all R-constructions $(\mathcal{Q},\ R,\ \varphi')$ there is exactly one linear power homomorphism $H : \mathcal{Q} \dot{\rightarrow} \mathcal{P}$.

These definitions imply the existence and uniqueness of an initial and a final construction for every given semiring domain R, as pointed out in chapters 14.5 and 15. If the definitions did not refer to *linear* power homomorphisms, there would be no initial and final constructions for semirings with non-trivial automorphisms.

Next, we state some simple properties of initial and final constructions.

Proposition 12.5.2 If $\mathcal{P}$ is isomorphic to an initial (a final) power construction $\mathcal{P}'$ for R, then $\mathcal{P}$ is also an initial (a final) power construction for R.

Proposition 12.5.3 For given semiring R, initial and final power constructions are unique up to isomorphism.

Note that this proposition does not claim the existence of initial and final constructions.

The proofs of the propositions are done by standard algebraic arguments — provided that 'isomorphic' is understood as isomorphic by a linear power isomorphism.

The main result is the following theorem:

Theorem 12.5.4 For every semiring R, initial and final power constructions exist.

In chapter 14, we demonstrate the initial construction. Chapter 15 is then devoted to the final construction. Before introducing the initial construction, we first investigate the theory of R-$\mathbf{X}$-*modules* because the results of this theory are used when considering the initial construction.

Chapter 13

R-X-Modules

Before introducing the initial and final power constructions for a semiring R, we consider R-$\mathbf{X}$-*modules* in this chapter. R-$\mathbf{X}$-modules are R-modules together with a map from $\mathbf{X}$. Power domains are R-$\mathbf{X}$-modules by the singleton map. The theory of R-$\mathbf{X}$-modules allows to prove a host of theorems that are applied to the theory of power domain constructions in the next chapter.

The chapter starts by the definition of R-$\mathbf{X}$-modules and R-$\mathbf{X}$-linear maps in section 13.1. In section 13.2, we introduce the *core* of an R-$\mathbf{X}$-module as its least subset that still admits all operations. R-$\mathbf{X}$-modules that coincide with their core are called *reduced*. They have particularly nice properties, which are presented in section 13.3. An R-$\mathbf{X}$-module is *free* iff there is a unique R-$\mathbf{X}$-linear map to every other R-$\mathbf{X}$-module. The existence of free R-$\mathbf{X}$-modules is generally shown in section 13.5. Section 13.6 then provides a more explicit construction of the free R-$\mathbf{X}$-module in case of algebraic R and $\mathbf{X}$.

13.1 Definition

An R-$\mathbf{X}$-module is an R-module together with a mapping from $\mathbf{X}$ to it.

Definition 13.1.1 An R-$\mathbf{X}$-*module* $\mathbf{M}$ is a pair $\mathbf{M} = (M, \eta)$ of an R-module domain M and a morphism $\eta : [\mathbf{X} \to M]$.

A morphism $f : (M, \eta) \to (M', \eta')$ is R-$\mathbf{X}$-*linear* iff $f : [M \to M']$ is R-linear and $f \circ \eta = \eta'$, i.e. $f(\eta x) = \eta' x$ for all x in $\mathbf{X}$.

We already met examples for such R-$\mathbf{X}$-modules and R-$\mathbf{X}$-linear mappings. If $H : \mathcal{P} \dot{\to} \mathcal{Q}$ is a linear power homomorphism between two power constructions with the same semiring R, then for every ground domain $\mathbf{X}$, the instance $H_{\mathbf{X}}$ is an R-$\mathbf{X}$-linear mapping between the two R-$\mathbf{X}$-modules $(\mathcal{P}\mathbf{X}, \iota_{\mathcal{P}})$ and $(\mathcal{Q}\mathbf{X}, \iota_{\mathcal{Q}})$. If $f : [\mathbf{X} \to \mathcal{P}\mathbf{Y}]$, then the extension $ext\, f$ is R-$\mathbf{X}$-linear between the R-$\mathbf{X}$-modules $(\mathcal{P}\mathbf{X}, \iota)$ and $(\mathcal{P}\mathbf{Y}, f)$ since $ext\, f \circ \iota = f$. Thus, the R-$\mathbf{X}$-modules with R-$\mathbf{X}$-linear mappings provide a common abstraction of extension and power homomorphisms.

In the sequel, we need some more definitions.

Definition 13.1.2 Let $\mathbf{M} = (M, \eta)$ where $M = (M, +, 0, \cdot)$ is an R-module. A subset S of (the carrier of) M is called an R-$\mathbf{X}$-*submodule* of $\mathbf{M}$ iff

(1) ηx is in S for all $x \in \mathbf{X}$, i.e. $\eta[\mathbf{X}] \subseteq S$.

(2) 0 is in S.

(3) If a and b are in S, then so is $a + b$.

(4) If a is in S, then $r \cdot a$ is in S for all $r \in R$.

(5) S is a subdomain of M, i.e. if D is a directed subset of S, then the limit of D w.r.t. M is in S.

By definition, S may be assumed to be an R-$\mathbf{X}$-module again, and the natural inclusion map $e : S \to M$ is an R-$\mathbf{X}$-linear morphism.

13.2 The core of an R-X-module

It is easily verified that the intersection of a family of R-$\mathbf{X}$-submodules of a fixed R-$\mathbf{X}$-module is again an R-$\mathbf{X}$-submodule. Hence, the R-$\mathbf{X}$-submodules form a complete lattice, and there is a least R-$\mathbf{X}$-submodule for every given R-$\mathbf{X}$-module $\mathbf{M}$. We call it the *core* $\mathbf{M}^c$ of $\mathbf{M}$. In the following, we develop a more explicit description of the core.

Theorem 13.2.1
If $\mathbf{M} = (M, \eta)$ is an R-$\mathbf{X}$-module, then its core is given by $\mathbf{M}^c = \overline{\mathbf{M}^\#}$ where

$$\mathbf{M}^\# = \{r_1 \cdot \eta x_1 + \cdots + r_n \cdot \eta x_n \mid n \in \mathbf{N}_0,\ r_i \in R,\ x_i \in \mathbf{X}\}$$

and $\overline{B}$ is the least d-closed superset of B.

The size of $\mathbf{M}^c$ is bounded by $|\mathbf{M}^c| \leq 2^{(|R|^{|\mathbf{X}|})}$.

The proof of the theorem is split into a sequence of propositions and their proofs.

Proposition 13.2.2 $\mathbf{M}^\#$ satisfies the properties (1) through (4) of Def. 13.1.2.

Proof:

(1) $\eta x = 1 \cdot \eta x$ implies $\eta[\mathbf{X}] \subseteq \mathbf{M}^\#$.

(2) $r_1 \cdot \eta x_1 + \cdots + r_n \cdot \eta x_n = 0$ in case $n = 0$.

(3) Obvious.

(4) $r \cdot (r_1 \cdot \eta x_1 + \cdots + r_n \cdot \eta x_n) = (r \cdot r_1) \cdot \eta x_1 + \cdots + (r \cdot r_n) \cdot \eta x_n$. □

Proposition 13.2.3 $|\mathbf{M}^\#| \leq |R|^{|\mathbf{X}|}$

Proof: Because of $r \cdot \eta x + r' \cdot \eta x = (r + r') \cdot \eta x$, one can arrange $r_1 \cdot \eta x_1 + \cdots + r_n \cdot \eta x_n$ such that every x in $\mathbf{X}$ occurs at most once. Those x that do not occur may be added as $0 \cdot x$. Thus, $|\mathbf{M}^\#| \leq |\mathbf{X} \to R| = |R|^{|\mathbf{X}|}$. □

Proposition 13.2.4 $\overline{\mathbf{M}^\#}$ satisfies properties (1) through (5) of Def. 13.1.2, i.e. it is an R-$\mathbf{X}$-submodule of $\mathbf{M}$.

Proof: (1) and (2) hold because of Prop. 13.2.2 and $\mathbf{M}^{\#} \subseteq \overline{\mathbf{M}^{\#}}$. Property (3) for $\mathbf{M}^{\#}$ means this set is closed w.r.t. the binary continuous operation '+', whence $\overline{\mathbf{M}^{\#}}$ is also closed w.r.t. '+' by Prop. 4.8.4 (2).

For $r \in R$, let $p_r : [M \to M]$ be given by $p_r\, m = r \cdot m$. Property (4) for $\mathbf{M}^{\#}$ means this set is closed w.r.t. the unary continuous operation p_r, whence $\overline{\mathbf{M}^{\#}}$ is also closed w.r.t. p_r by Prop. 4.8.4 (1) for all r. □

Proposition 13.2.5 $\mathbf{M}^c = \overline{\mathbf{M}^{\#}}$

Proof: By Prop. 13.2.4, $\overline{\mathbf{M}^{\#}}$ is an R-$\mathbf{X}$-submodule of $\mathbf{M}$. Since $\mathbf{M}^c$ is the least such set, $\mathbf{M}^c \subseteq \overline{\mathbf{M}^{\#}}$ holds. Conversely, $\mathbf{M}^c$ being d-closed and $\mathbf{M}^{\#} \subseteq \mathbf{M}^c$ implies $\overline{\mathbf{M}^{\#}} \subseteq \mathbf{M}^c$. □

Proposition 13.2.6 $|\mathbf{M}^c| \leq 2^{(|R|^{|\mathbf{X}|})}$

Proof: By Prop. 13.2.3, $|\mathbf{M}^{\#}| \leq |R|^{|\mathbf{X}|}$ holds, and Prop. 3.5.9 yields $|\overline{B}| \leq 2^{|B|}$. □

13.3 Reduced R-X-modules

Definition 13.3.1 An R-$\mathbf{X}$-module is *reduced* iff it coincides with its core.

Equivalently, an R-$\mathbf{X}$-module is reduced iff it does not allow proper R-$\mathbf{X}$-submodules.

For every R-$\mathbf{X}$-module $\mathbf{M}$, the core $\mathbf{M}^c$ is reduced. Hence, every R-$\mathbf{X}$-module contains a reduced R-$\mathbf{X}$-submodule.

Reduced R-$\mathbf{X}$-modules enjoy many interesting properties listed in the sequel.

Lemma 13.3.2 Let $\mathbf{M} = (M, \eta)$ be a reduced R-$\mathbf{X}$-module, and M' an R-module. If $F, G : [M \to M']$ are two R-linear morphisms with $F(\eta x) \leq G(\eta x)$ for all $x \in \mathbf{X}$, then $F \leq G$ holds.

Proof: Let $S = \{a \in M \mid Fa \leq Ga\}$. We show that S satisfies the properties of Def. 13.1.2 whence $S = M$ follows because $\mathbf{M}$ admits no proper R-$\mathbf{X}$-submodules.

(1) For $x \in \mathbf{X}$, ηx is in S since $F(\eta x) \leq G(\eta x)$.

(2) $F(0) = 0_{M'} = G(0)$ whence $0 \in S$.

(3) Let $a, b \in S$. Then $F(a+b) = Fa + Fb \leq Ga + Gb = G(a+b)$.

(4) Let $a \in S$ and $r \in R$. Then $F(r \cdot a) = r \cdot Fa \leq r \cdot Ga = G(r \cdot a)$.

(5) Let D be a directed subset of S with limit a w.r.t. M. Then for all $d \in D$, $Fd \leq Gd \leq Ga$ holds, whence $Fa \leq Ga$ by continuity of F. □

The Lemma allows several conclusions:

Proposition 13.3.3 In Lemma 13.3.2, '$\leq$' may be replaced by '=':
Let $\mathbf{M} = (M, \eta)$ be a reduced R-$\mathbf{X}$-module, and M' an R-module. If $F, G : [M \to M']$ are two R-linear morphisms with $F(\eta x) = G(\eta x)$ for all $x \in \mathbf{X}$, then $F = G$ holds.

Proof: By anti-symmetry: $a = b$ iff $a \leq b$ and $b \leq a$. □

As a special instance of this proposition, one obtains:

Proposition 13.3.4 If $\mathbf{M}$ is a reduced R-$\mathbf{X}$-module, then there is at most one R-$\mathbf{X}$-linear mapping from $\mathbf{M}$ to any other R-$\mathbf{X}$-module $\mathbf{M}'$.

Proof: If F and G are two R-$\mathbf{X}$-linear morphisms, then they are in particular linear, and $F(\eta_{\mathbf{M}}x) = \eta_{\mathbf{M}'}x = G(\eta_{\mathbf{M}}x)$ holds for all x in $\mathbf{X}$. □

Proposition 13.3.5 If the semiring R has a least element $\perp_R$ and $\mathbf{X}$ has a least element $\perp_{\mathbf{X}}$, then every reduced R-$\mathbf{X}$-module $\mathbf{M} = (M,\, \eta)$ has a least element, namely $\perp_R \cdot \eta(\perp_{\mathbf{X}})$.

Proof: Let $S = \{a \in M \mid a \geq \perp_R \cdot \eta(\perp_{\mathbf{X}})\}$.

(1) Let $x \in \mathbf{X}$. Then $\eta x = 1 \cdot \eta x \geq \perp_R \cdot \eta(\perp_{\mathbf{X}})$.

(2) $0 = 0 \cdot \eta(\perp_{\mathbf{X}}) \geq \perp_R \cdot \eta(\perp_{\mathbf{X}})$.

(3) Let $a, b \in S$. Then $a + b \geq \perp_R \cdot \eta(\perp_{\mathbf{X}}) + \perp_R \cdot \eta(\perp_{\mathbf{X}}) = (\perp_R + \perp_R) \cdot \eta(\perp_{\mathbf{X}}) \geq \perp_R \cdot \eta(\perp_{\mathbf{X}})$.

(4) For $r \in R$ and $a \in S$, $r \cdot a \geq r \cdot \perp_R \cdot \eta(\perp_{\mathbf{X}}) \geq \perp_R \cdot \eta(\perp_{\mathbf{X}})$.

(5) S is obviously closed w.r.t. limits of directed sets in M.

Hence, S satisfies the conditions of Def. 13.1.2. Thus, $S = M$ holds. □

The following Lemma will be needed in the next chapter.

Lemma 13.3.6 Let $\mathbf{M} = (M,\, \eta)$ and $\mathbf{M}' = (M',\, \eta')$ be two R-$\mathbf{X}$-modules, and $F : [M \to M']$ an R-linear morphism that maps all points ηx into the core of $\mathbf{M}'$. If $\mathbf{M}$ is reduced, then F maps all points of $\mathbf{M}$ into the core of $\mathbf{M}'$.

Proof: Let $S = \{a \in \mathbf{M} \mid FA \in \mathbf{M}'^c\}$. We show that S satisfies conditions (1) through (5) of Def. 13.1.2. This implies $S = \mathbf{M}$ whence the claim of the proposition follows.

(1) $F(\eta_{\mathbf{M}}x) \in \mathbf{M}'^c$ by precondition.

(2) $F(0) = 0 \in \mathbf{M}'^c$.

(3) If a and b are in S, then $F(a + b) = Fa + Fb$ is in $\mathbf{M}'^c$.

(4) If a is in S, then $F(r \cdot a) = r \cdot Fa \in \mathbf{M}'^c$.

(5) If D is a directed set in S, then $F(\bigsqcup D) = \bigsqcup F[D] \in \mathbf{M}'^c$. □

13.4 Free R-X-modules

By Prop. 13.3.4, there is at most one R-$\mathbf{X}$-linear mapping from every reduced R-$\mathbf{X}$-module. In this section, we consider an even more special class of R-$\mathbf{X}$-modules.

Definition 13.4.1 An R-$\mathbf{X}$-module $\mathbf{F}$ is *free* iff for every R-$\mathbf{X}$-module $\mathbf{M}$, there is exactly one R-$\mathbf{X}$-linear morphism from $\mathbf{F}$ to $\mathbf{M}$.

The existence of free R-$\mathbf{X}$-modules is shown in section 13.5. For algebraic R and $\mathbf{X}$, a more explicit construction is provided in section 13.6. In this section, we study the properties of free R-$\mathbf{X}$-modules.

Proposition 13.4.2 All free R-$\mathbf{X}$-modules are isomorphic to each other.

Proof: Let $\mathbf{F}_1$ and $\mathbf{F}_2$ be two free R-$\mathbf{X}$-modules. Then there are R-$\mathbf{X}$-linear morphisms $f_1 : \mathbf{F}_1 \to \mathbf{F}_2$ and $f_2 : \mathbf{F}_2 \to \mathbf{F}_1$. Their composition $f_2 \circ f_1$ is an R-$\mathbf{X}$-linear morphism from $\mathbf{F}_1$ to itself. Identity is another such. By uniqueness, $f_2 \circ f_1 = id$ follows. The dual equation $f_1 \circ f_2 = id$ is shown analogously. □

Proposition 13.4.3 Every free R-$\mathbf{X}$-module is reduced.

Proof:
Let $\mathbf{F}$ be a free R-$\mathbf{X}$-module and S an R-$\mathbf{X}$-submodule of $\mathbf{F}$. We have to show $S = \mathbf{F}$.

The embedding $\epsilon : S \to \mathbf{F}$ is R-$\mathbf{X}$-linear since S is an R-$\mathbf{X}$-submodule. Since $\mathbf{F}$ is free, there is an R-$\mathbf{X}$-linear morphism $\zeta : [\mathbf{F} \to S]$. The composition $\epsilon \circ \zeta$ is R-$\mathbf{X}$-linear and maps $\mathbf{F}$ to itself as the identity does. By freedom, $\epsilon \circ \zeta = id$ holds. Hence, for every y in $\mathbf{F}$, $y = \epsilon(\zeta y) \in S$ holds. □

If $\mathbf{F}$ is a free R-$\mathbf{X}$-module, then for every morphism $f : [\mathbf{X} \to M]$ from $\mathbf{X}$ to some R-module M, there is a unique R-$\mathbf{X}$-linear extension $\overline{f} : [\mathbf{F} \to (M, f)]$ to the R-$\mathbf{X}$-module (M, f). Thus, '$^-$' itself is a function from $[\mathbf{X} \to M]$ to $[\mathbf{F} \to M]$.

Theorem 13.4.4 If $\mathbf{F}$ is a free R-$\mathbf{X}$-module, then for every R-module M, the mapping $^- : [\mathbf{X} \to M] \to [\mathbf{F} \to M]$ as introduced above is continuous.

Proof: '$^-$' is monotonic by Lemma 13.3.2 telling that $f \le g$ implies $\overline{f} \le \overline{g}$.

Now, we show the continuity of '$^-$'. Let η be the morphism from $\mathbf{X}$ to $\mathbf{F}$. Let $\mathcal{D}$ be a directed set of morphisms from $\mathbf{X}$ to M, and let f be its limit. We have to show $\overline{f} = \bigsqcup_{d \in \mathcal{D}} \overline{d}$. The function on the right hand side is R-linear by continuity of '$+$' and '$\cdot$'. It maps ηx to $f x$ by continuity of application and $\overline{d}(\eta x) = d\, x$. By uniqueness, it thus equals $\overline{f}$. □

In the special case $\mathbf{X} = \mathbf{1}$, R itself is a free R-$\mathbf{X}$-module:

Proposition 13.4.5 $(R, \lambda x.\, 1)$ is a free R-$\mathbf{1}$-module.

Proof: Let $\mathbf{M} = (M, \eta)$ be an R-$\mathbf{1}$-module. Let $f : [R \to M]$ be given by $f(r) = r \cdot \eta \diamond$. This mapping is R-linear because of the module axioms. For instance, $f(r \cdot r') = (r \cdot r') \cdot \eta \diamond = r \cdot (r' \cdot \eta \diamond) = r \cdot f(r')$ holds. f is R-$\mathbf{1}$-linear since $f((\lambda x.\, 1) \diamond) = f(1) = 1 \cdot \eta \diamond = \eta \diamond$.

Let F be an arbitrary R-$\mathbf{1}$-linear map from $(R, \lambda x.1)$ to $\mathbf{M}$. Then $F(r) = F(r \cdot 1) = r \cdot F(1) = r \cdot F((\lambda x.\, 1) \diamond) = r \cdot \eta \diamond = f(r)$ holds, i.e. f is unique. □

13.5 Existence of free modules

In this section, we show the existence of the free R-$\mathbf{X}$-module for arbitrary semiring domains R and ground domains $\mathbf{X}$. The proof follows the lines of [Hoo87] who proved the existence of the free commutative idempotent monoid over $\mathbf{X}$. Hoofman used the categorical Freyd Adjoint Functor Theorem. We avoid its usage for the sake of a slightly more explicit construction. Our proof looks much simpler than that of Hoofman because we apply the notion of R-$\mathbf{X}$-modules.

We first construct the so-called solution set required by the Adjoint Functor Theorem. Instead of applying this theorem after verifying its remaining preconditions and thus obtaining the mere existence of the free module, we present a simple explicit construction based on the solution set.

The problem with the class of all R-$\mathbf{X}$-modules is that it is not a set. The problem is solved by providing a set of R-$\mathbf{X}$-modules $\{\mathbf{M}_i \mid i \in I\}$ that may be used as representatives for all R-$\mathbf{X}$-modules.

Let c be the cardinal number $2^{(|R|^{|\mathbf{X}|})}$, and let C be a set of cardinality c. From C, we construct the set

$$D = \bigcup_{A \subseteq C} \{A\} \times (A \times A \to 2) \times (A \times A \to A) \times A \times (R \times A \to A) \times (\mathbf{X} \to A)$$

where $2 = \{0, 1\}$. Next, let I be the set of all tuples $(A, \leq, +, 0, \cdot, f)$ in D such that $\mathbf{A} = (A, \leq)$ is a domain, $M = (\mathbf{A}, +, 0, \cdot)$ is an R-module domain, and $f : \mathbf{X} \to \mathbf{A}$ is continuous, i.e. (M, f) is an R-$\mathbf{X}$-module. By construction, I contains isomorphic copies of all R-$\mathbf{X}$-modules up to cardinality c. Indexing I by itself, we obtain a family $(\mathbf{M}_i)_{i\in I}$ of R-$\mathbf{X}$-modules.

Now let $\mathbf{M} = (M, f)$ be an arbitrary R-$\mathbf{X}$-module. Let $\mathbf{M}^c$ be the core of $\mathbf{M}$ and $e : [\mathbf{M}^c \to \mathbf{M}]$ the natural inclusion. Note that e is R-$\mathbf{X}$-linear.

By Th. 13.2.1, $|\mathbf{M}^c| \leq 2^{(|R|^{|\mathbf{X}|})} = c$ holds. Hence, there is an isomorphic copy $\mathbf{M}_i$ of $\mathbf{M}^c$ in I. Let $\varphi : [\mathbf{M}_i \to \mathbf{M}^c]$ be the R-$\mathbf{X}$-linear isomorphism between $\mathbf{M}_i$ and $\mathbf{M}^c$.

Given the 'solution set' $(\mathbf{M}_i)_{i\in I}$, it is now easy to construct the free module. Let $\mathbf{P} = \prod_{i\in I} \mathbf{M}_i$. The operations in $\mathbf{P}$ are defined as follows:

- $a \leq b$ iff $a_i \leq b_i$ for all i in I,
- $a + b = (a_i + b_i)_{i\in I}$,
- $r \cdot a = (r \cdot a_i)_{i\in I}$ for r in R,
- $\eta x = (\eta_i\, x)_{i\in I}$ for $x \in \mathbf{X}$.

It is not difficult to see that all these functions are continuous, and make $\mathbf{P}$ into an R-$\mathbf{X}$-module. The projections $\pi_i : [\mathbf{P} \to \mathbf{M}_i]$ are R-$\mathbf{X}$-linear.

Finally, let $\mathbf{F}$ be the core of $\mathbf{P}$. Then the inclusion $p : \mathbf{F} \to \mathbf{P}$ is R-$\mathbf{X}$-linear. Summarizing, we get for each R-$\mathbf{X}$-module $\mathbf{M}$ the following chain of R-$\mathbf{X}$-linear mappings for some i:

$$\mathbf{F} \xrightarrow{\;p\;} \mathbf{P} \xrightarrow{\;\pi_i\;} \mathbf{M}_i \xrightarrow{\;\varphi\;} \mathbf{M}^c \xrightarrow{\;e\;} \mathbf{M}$$

Thus, we get an R-$\mathbf{X}$-linear map f from $\mathbf{F}$ to every R-$\mathbf{X}$-module $\mathbf{M}$. f is unique since $\mathbf{F}$ is reduced (Prop. 13.3.3).

13.6 Free modules in the algebraic case

A more explicit construction of the free R-$\mathbf{X}$-module is possible at least in the case of *structurally algebraic* semiring R and *algebraic* domain $\mathbf{X}$.

In this section, we construct the free R-$\mathbf{X}$-module from a structurally algebraic semiring domain R and an algebraic ground domain $\mathbf{X}$. We first build up an unordered set of finite linear combinations from the bases R^0 and $\mathbf{X}^0$. This set is then pre-ordered and finally completed to an algebraic domain.

Definition 13.6.1 A semiring domain R is *structurally algebraic* if it is algebraic and its base R^0 contains 0 and 1 and is closed w.r.t. '$+$' and '$\cdot$'.

Examples:

- Every finite and every discrete semiring domain R is structurally algebraic since $R^0 = R$ holds in these cases.
- $\mathbf{N}_0^\infty$ is structurally algebraic since sum and product of finite numbers are finite.
- The tropical semiring $\mathbf{T}$ is algebraic but not structurally algebraic since t_∞ is the neutral element of its addition.
- The powerset of an infinite set X ordered by inclusion with union as addition and intersection as multiplication is algebraic by Prop. 6.3.3, but not structurally algebraic since $1 = X$ is infinite.

Now, let R be a fixed structurally algebraic semiring and $\mathbf{X}$ a fixed algebraic ground domain. As a first step, we consider the set $\widehat{X}$ of finite linear combinations over $\mathbf{X}^0$.

Definition 13.6.2 For functions $\alpha : \mathbf{X}^0 \to R^0$, let $|\alpha| = \{x \in \mathbf{X}^0 \mid \alpha x \neq 0\}$. Then $\widehat{X} = \{\alpha : \mathbf{X}^0 \to R^0 \mid |\alpha| \text{ is finite}\}$.

$\widehat{X}$ consists of arbitrary functions; it is not restricted to continuous or even monotonic functions. For the moment, no order is assumed for $\widehat{X}$. The elements of $\widehat{X}$ stand for finite linear combinations over $\mathbf{X}^0$ where $\alpha\, x$ is the coefficient of x. Consequently, the following operations are defined in $\widehat{X}$:

- $\widehat{0} = \lambda x.\, 0$. It is in $\widehat{X}$ since $|\widehat{0}| = \emptyset$.
- $\alpha \mathbin{\widehat{+}} \beta = \lambda x.\, \alpha\, x + \beta\, x$. $|\alpha \mathbin{\widehat{+}} \beta| \subseteq |\alpha| \cup |\beta|$ holds.
- $r \mathbin{\widehat{\cdot}} \alpha = \lambda x.\, r \cdot \alpha\, x$ for $r \in R^0$. $|r \mathbin{\widehat{\cdot}} \alpha| \subseteq |\alpha|$ holds.
- $\widehat{\iota} a = \lambda x. \begin{cases} 1 & \text{if } x = a \\ 0 & \text{otherwise} \end{cases}$ for $a \in \mathbf{X}^0$. $|\widehat{\iota} a| = \{a\}$ holds.
- Let $(M,\, f)$ be an R-$\mathbf{X}$-module. Then let $\widetilde{f} : \widehat{X} \to M$ be defined by $\widetilde{f}\alpha = \sum_{x \in |\alpha|} \alpha\, x \cdot f x$.

The operations (and $\widehat{X}$ itself) are well defined because R is *structurally* algebraic. The sum in $\widetilde{f}\alpha$ may also be assumed to range over $x \in \mathbf{X}^0$ since αx is 0 outside of $|\alpha|$.

$\widetilde{f}$ is R-$\mathbf{X}$-linear, i.e. is a linear extension of f:

- $\widetilde{f}(\alpha \mathbin{\widehat{+}} \beta) = \sum(\alpha\, x + \beta\, x) \cdot f x = \sum \alpha\, x \cdot f x + \sum \beta\, x \cdot f x = \widetilde{f}\alpha + \widetilde{f}\beta$ using commutativity and associativity of '$+$' in M.
- $\widetilde{f}(r \mathbin{\widehat{\cdot}} \alpha) = \sum(r \cdot \alpha\, x) \cdot f x = r \cdot \sum \alpha\, x \cdot f x = r \cdot \widetilde{f}\alpha$
- $\widetilde{f}(\widehat{\iota} a) = \sum(\widehat{\iota} a)\, x \cdot f x = 1 \cdot f a = f a$

Function $\widetilde{f}$ is unique with these properties since $|\alpha| = \{x_1, \ldots, x_n\}$ implies $\alpha = \alpha\, x_1 \mathbin{\widehat{\cdot}} \widehat{\iota} x_1 \mathbin{\widehat{+}} \cdots \mathbin{\widehat{+}} \alpha\, x_n \mathbin{\widehat{\cdot}} \widehat{\iota} x_n$.

The next step is pre-ordering $\widehat{X}$. Let '$\preceq$' the least binary relation in $\widehat{X}$ with the following properties:

(1) $\alpha \preceq \alpha$.

(2) If $\alpha \preceq \beta$ and $\beta \preceq \gamma$, then $\alpha \preceq \gamma$.

(3) If $x \leq x'$ in $\mathbf{X}^0$, then $\hat{\iota}x \preceq \hat{\iota}x'$.

(4) If $\alpha \preceq \alpha'$ and $\beta \preceq \beta'$, then $\alpha \mathbin{\hat{+}} \beta \preceq \alpha' \mathbin{\hat{+}} \beta'$.

(5) If $r \leq r'$ in R^0 and $\alpha \preceq \alpha'$, then $r \mathbin{\hat{\cdot}} \alpha \preceq r' \mathbin{\hat{\cdot}} \alpha'$.

More formally, '$\preceq$' is the intersection of all subsets of $\widehat{X} \times \widehat{X}$ with the indicated properties. One easily verifies that the intersection then also has these properties.

'$\preceq$' is a pre-order by (1) and (2), and the operations $\hat{\iota}$, '$\hat{+}$', and '$\hat{\cdot}$' are monotonic by (3) through (5). The extended functions $\tilde{f}$ are also monotonic.

Proof:
For fixed $f : [\mathbf{X} \to M]$, define a relation '$\preceq_f$' in $\widehat{X}$ by $\alpha \preceq_f \beta$ iff $\tilde{f}\alpha \leq \tilde{f}\beta$. Then '$\preceq_f$' obviously is reflexive and transitive, and it also satisfies the properties (3) through (5) above:

(3) $x \leq x'$ implies $\tilde{f}(\hat{\iota}x) = fx \leq fx' = \tilde{f}(\hat{\iota}x')$, whence $\hat{\iota}x \preceq_f \hat{\iota}x'$.

(4) $\tilde{f}(\alpha \mathbin{\hat{+}} \beta) = \tilde{f}\alpha + \tilde{f}\beta \leq \tilde{f}\alpha' + \tilde{f}\beta' = \tilde{f}(\alpha' \mathbin{\hat{+}} \beta')$

(5) $\tilde{f}(r \mathbin{\hat{\cdot}} \alpha) = r \cdot \tilde{f}\alpha \leq r' \cdot \tilde{f}\alpha' = \tilde{f}(r' \mathbin{\hat{\cdot}} \alpha')$

Since '$\preceq$' is the least relation with these properties, $\alpha \preceq \beta$ implies $\alpha \preceq_f \beta$, i.e. $\tilde{f}\alpha \leq \tilde{f}\beta$. □

As a next step, we turn the pre-ordered set $\widehat{X}$ into the poset $\widehat{X}^*$ by forming classes w.r.t. the equivalence relation '$\alpha \preceq \beta$ and $\beta \preceq \alpha$'. All operations of $\widehat{X}$ may also be defined for $\widehat{X}^*$ because they are monotonic. The resulting operations on $\widehat{X}^*$ are monotonic again.

As final step, let $\overline{X}$ be the ideal completion of the poset $\widehat{X}^*$. Hence, $\overline{X}$ is an algebraic domain whose base is isomorphic to $\widehat{X}^*$. All the monotonic functions $\hat{+} : \widehat{X}^* \times \widehat{X}^* \to \widehat{X}^*$, $\hat{\cdot} : R^0 \times \widehat{X}^* \to \widehat{X}^*$, and $\tilde{f} : \widehat{X}^* \to M$ for every $f : [\mathbf{X} \to M]$ have unique continuous extensions $+ : [\overline{X} \times \overline{X} \to \overline{X}]$, $\cdot : [R \times \overline{X} \to \overline{X}]$, and $\overline{f} : [\overline{X} \to M]$ for every $f : [\mathbf{X} \to M]$. These operations possess the necessary algebraic properties such as associativity and linearity because they are inherited from the base. The linear extension $\overline{f}$ of f on $\overline{X}$ is unique because $\overline{X}$ trivially is a reduced R-$\mathbf{X}$-module. Thus, $\overline{X}$ is the free R-$\mathbf{X}$-module.

The results above allow to state the following theorem:

Theorem 13.6.3 If R is a structurally algebraic semiring domain and $\mathbf{X}$ is algebraic, then the free R-$\mathbf{X}$-module is algebraic.

Problem 9 What happens if R is algebraic without being structurally algebraic?

Proposition 13.6.4 If R and $\mathbf{X}$ are discrete, then so is the free R-$\mathbf{X}$-module. It equals $\widehat{X}$, the set of finite R-linear combinations over $\mathbf{X}$.

Proof: Discrete semirings are structurally algebraic. Consider the set $\widehat{X}$. We claim $\alpha \preceq \beta$ iff $\alpha = \beta$.

$\alpha = \beta$ implies $\alpha \preceq \beta$ by reflexivity. For the opposite direction, note that the relation '$=$' satisfies the properties (1) through (5) of '$\preceq$'. In proving this, discreteness of $\mathbf{X}$ is needed for (3), and discreteness of R for (5). Hence, $\widehat{X}$ is a discrete poset and coincides with its ideal completion. □

Proposition 13.6.5 If R and $\mathbf{X}$ are finite, then so is the free R-$\mathbf{X}$-module. It equals $\widehat{X}^*$, the poset of equivalence classes of R-linear combinations over $\mathbf{X}$.

Proof: Finite semirings are structurally algebraic. Because the restriction that $|\alpha|$ be finite is satisfied always, $\widehat{X}$ is the set of all functions from $\mathbf{X}$ to R. This set is finite. Thus, $\widehat{X}^*$ is also finite, whence it is isomorphic to its ideal completion. $\square$

Chapter 14

Sub-constructions, reduced and initial constructions

In this chapter, we consider some notions that were already introduced and investigated for R-$\mathbf{X}$-modules in the previous chapter. Here, we abstract from the ground domain $\mathbf{X}$ and apply the notions to power constructions.

A power construction $\mathcal{Q}$ is a sub-construction of $\mathcal{P}$ if the power domains $\mathcal{Q}\mathbf{X}$ are subsets of $\mathcal{P}\mathbf{X}$ such that $\mathcal{Q}$ inherits all operations of $\mathcal{P}$ (see section 14.1). Sub-constructions of $\mathcal{P}$ may or may not have the same characteristic semiring as $\mathcal{P}$. The *core* is the least sub-construction of $\mathcal{P}$ with the same semiring (section 14.2). *Reduced power constructions* coincide with their core, whence they do not possess proper sub-constructions with the same characteristic semiring. Their properties are investigated in sections 14.3 and 14.4. The initial power construction for semiring R is introduced in section 14.5. It results from mapping every ground domain $\mathbf{X}$ to the free R-$\mathbf{X}$-module. If R' is a sub-semiring of $\mathcal{P}\mathbf{1}$, then there is a greatest sub-construction of $\mathcal{P}$ that has characteristic semiring R'. In section 14.6, this sub-construction is explicitly characterized in terms of formal existential quantification on $\mathcal{P}$.

If $\mathcal{P}$ is an R-construction, then the power domains $\mathcal{P}\mathbf{X}$ induce R-$\mathbf{X}$-modules ($\mathcal{P}\mathbf{X}$, $\iota_{\mathbf{X}}$). In the sequel, we always refer to this R-$\mathbf{X}$-module when saying $\mathcal{P}\mathbf{X}$ is reduced or $\mathcal{P}\mathbf{X}$ and $\mathcal{Q}\mathbf{X}$ are isomorphic. Consequently, isomorphisms between two power domains $\mathcal{P}\mathbf{X}$ and $\mathcal{Q}\mathbf{X}$ are R-$\mathbf{X}$-linear maps.

14.1 Sub-constructions

Let $\mathcal{P}$ be a given power construction. $\mathcal{Q}$ is called a *sub-construction* of $\mathcal{P}$ iff $\mathcal{Q}$ maps ground domains $\mathbf{X}$ into subsets of $\mathcal{P}\mathbf{X}$ such that

- $\mathcal{Q}\mathbf{X}$ is closed w.r.t. lubs of directed sets,
- $\theta \in \mathcal{Q}\mathbf{X}$,
- If A and B are in $\mathcal{Q}\mathbf{X}$, then $A \uplus B$ is in $\mathcal{Q}\mathbf{X}$,
- $\{\!|x|\!\}$ is in $\mathcal{Q}\mathbf{X}$ for all x in $\mathbf{X}$,
- If $f : [\mathbf{X} \to \mathcal{Q}\mathbf{Y}]$ and A in $\mathcal{Q}\mathbf{X}$, then $\mathit{ext}\, f\, A$ is in $\mathcal{Q}\mathbf{Y}$.

In shorter terms, $\mathcal{Q}\mathbf{X}$ is closed w.r.t. all power operations of $\mathcal{P}$.

$\mathcal{Q}$ is obviously a power construction since the validity of the power axioms for $\mathcal{Q}$ is inherited from $\mathcal{P}$. By the same argument, one concludes

Proposition 14.1.1
If $\mathcal{P}$ is commutative / symmetric / faithful, then every sub-construction of $\mathcal{P}$ is so.

One easily verifies that the intersection of a family of sub-constructions of a power construction $\mathcal{P}$ is again a sub-construction of $\mathcal{P}$, if we define $(\bigcap_{i\in I} \mathcal{Q}_i)\mathbf{X} = \bigcap_{i\in I}(\mathcal{Q}_i\mathbf{X})$. Hence, the sub-constructions of $\mathcal{P}$ form a complete lattice.

Sub-semirings are defined analogously to sub-constructions.

Definition 14.1.2
Let $(R, +, 0, \cdot, 1)$ be a semiring domain. A subset R' of R is a *sub-semiring* of R iff

(1) 0 is in R';

(2) if a and b are in R', then $a + b$ is in R';

(3) 1 is in R';

(4) if a and b are in R', then $a \cdot b$ is in R';

(5) if D is a directed set in R' with limit x w.r.t. R, then x is in R'.

Because the operations in the characteristic semiring are derived from the power operations, the semiring of a sub-construction $\mathcal{Q}$ of $\mathcal{P}$ is a sub-semiring of the semiring of $\mathcal{P}$. Similarly, the semiring of an intersection of sub-constructions is the intersection of the semirings of the sub-constructions. In section 14.6, a method is presented to obtain a sub-construction for every given semiring.

The sub-constructions with the same characteristic semiring attract special interest.

Proposition 14.1.3 Let $\mathcal{Q}$ be a sub-construction of a power construction $\mathcal{P}$. Then $\mathcal{Q}\mathbf{1} = \mathcal{P}\mathbf{1}$ holds iff all power domains $\mathcal{Q}\mathbf{X}$ are closed w.r.t. $\mathcal{P}\mathbf{1}$-multiplication.

Proof: Assume $\mathcal{Q}\mathbf{1} = \mathcal{P}\mathbf{1}$ holds. Let A be in $\mathcal{Q}\mathbf{X}$ and r in $\mathcal{P}\mathbf{1}$, whence it is also in $\mathcal{Q}\mathbf{1}$. Extension yields $r \cdot A = \mathit{ext}\,(\lambda\diamond.\, A)\, r$ in $\mathcal{Q}\mathbf{X}$ by the definition of sub-constructions.

If all power domains $\mathcal{Q}\mathbf{X}$ are closed w.r.t. $\mathcal{P}\mathbf{1}$-multiplication, then in particular $\mathcal{Q}\mathbf{1}$ is so. Hence, for all r in $\mathcal{P}\mathbf{1}$, $r = r \cdot 1$ is in $\mathcal{Q}\mathbf{1}$ since 1 is in $\mathcal{Q}\mathbf{1}$. □

If $\mathcal{Q}$ is a sub-construction of $\mathcal{P}$, then the natural inclusion maps $E_{\mathbf{X}} : \mathcal{Q}\mathbf{X} \to \mathcal{P}\mathbf{X}$ yield a power homomorphism $E : \mathcal{Q} \dot{\to} \mathcal{P}$. If $\mathcal{Q}$ moreover has the same characteristic semiring as $\mathcal{P}$, then E is a linear power homomorphism.

14.2 The core of a power construction

Let $\mathcal{P}$ be a power construction with characteristic semiring R. Then the power domains $\mathcal{P}\mathbf{X}$ form R-$\mathbf{X}$-modules together with the singleton mappings $\iota : [\mathbf{X} \to \mathcal{P}\mathbf{X}]$. We denote the core of $(\mathcal{P}\mathbf{X}, \iota)$ by $(\mathcal{P}^c\mathbf{X}, \iota)$ and show that $\mathcal{P}^c$ induces a power domain construction with the same characteristic semiring as $\mathcal{P}$.

The operations θ, '$\uplus$', and $\{\!|.|\!\}$ obviously cut down to operations for $\mathcal{P}^c$ because $\mathcal{P}^c\mathbf{X}$ was defined to allow these operations together with R-multiplication. We only have to provide an extension satisfying the power axioms.

For every $f : [\mathbf{X} \to \mathcal{P}^c\mathbf{Y}]$, the linear extension $ext_{\mathcal{P}}\, f$ maps singletons $\{\!|x|\!\}$ of $\mathcal{P}^c\mathbf{X}$ into $\mathcal{P}^c\mathbf{Y}$ since $ext_{\mathcal{P}}\{\!|x|\!\} = fx$. Hence by Lemma 13.3.6, $ext_{\mathcal{P}}\, f$ maps all members of $\mathcal{P}^c\mathbf{X}$ to $\mathcal{P}^c\mathbf{Y}$. Thus, the extension of $\mathcal{P}$ cuts down to the extension of $\mathcal{P}^c$. It satisfies the power axioms because $ext_{\mathcal{P}}$ did. Because the power operations for $\mathcal{P}^c$ are derived from those of $\mathcal{P}$ by restriction and corestriction, the embeddings $E : \mathcal{P}^c\mathbf{X} \to \mathcal{P}\mathbf{X}$ form a power homomorphism.

$\mathcal{P}^c\mathbf{1}$ equals $\mathcal{P}\mathbf{1}$ since every member b of $\mathcal{P}\mathbf{1}$ equals $b \cdot \{\!|\diamond|\!\}$, whence $b \in \mathcal{P}^c\mathbf{1}$. Thus, $\mathcal{P}^c$ has the same characteristic semiring as $\mathcal{P}$.

If $\mathcal{Q}$ is another sub-construction of $\mathcal{P}$ with semiring $\mathcal{P}\mathbf{1}$, then the power domains $(\mathcal{Q}\mathbf{X},\, \iota)$ form R-$\mathbf{X}$-submodules of $(\mathcal{P}\mathbf{X},\, \iota)$ by the definition of sub-constructions and by Prop. 14.1.3. Hence, $\mathcal{P}^c\mathbf{X} \subseteq \mathcal{Q}\mathbf{X}$ holds for all ground domains $\mathbf{X}$. Thus, $\mathcal{P}^c$ is the least sub-construction of $\mathcal{P}$ that still has characteristic semiring $\mathcal{P}\mathbf{1}$.

Let $\mathcal{P}$ and $\mathcal{Q}$ be two power constructions with the same characteristic semiring. For every power homomorphism $H : \mathcal{P} \dot\to \mathcal{Q}$, $H\{\!|x|\!\}_{\mathcal{P}} = \{\!|x|\!\}_{\mathcal{Q}}$ is in $\mathcal{Q}^c\mathbf{X}$. If H is linear, then H cuts down to a power homomorphism $H^c : \mathcal{P}^c \dot\to \mathcal{Q}^c$ by Lemma 13.3.6.

All results of this section are collected in the following theorem:

Theorem 14.2.1 For every power construction $\mathcal{P}$, the core $\mathcal{P}^c$ is the least sub-construction of $\mathcal{P}$ with the same characteristic semiring. The embeddings of the cores $\mathcal{P}^c\mathbf{X}$ into the power domains $\mathcal{P}\mathbf{X}$ provide a linear power homomorphism $E : \mathcal{P}^c \dot\to \mathcal{P}$.

For every linear power homomorphism $H : \mathcal{P} \dot\to \mathcal{Q}$, there is a linear homomorphism H^c : $\mathcal{P}^c \dot\to \mathcal{Q}^c$ with $E \circ H^c = H^c \circ E$.

14.3 Reduced power constructions

Definition 14.3.1 A power construction is *reduced* iff it coincides with its core.

For every power construction $\mathcal{P}$, the core $\mathcal{P}^c$ is reduced. Hence, every power construction contains a reduced sub-construction.

Since the core construction $\mathcal{P}^c$ maps ground domains $\mathbf{X}$ to the core modules $(\mathcal{P}\mathbf{X},\, \iota)^c$, a power construction is reduced iff all R-$\mathbf{X}$-modules $(\mathcal{P}\mathbf{X},\, \iota)$ are reduced. Thus, all results of the previous chapter about reduced R-$\mathbf{X}$-modules apply.

Proposition 14.3.2 Let $\mathcal{P}$ be a reduced R-construction, $\mathbf{X}$ a ground domain, and M an R-module. Two linear morphisms $F, G : [\mathcal{P}\mathbf{X} \to M]$ that coincide on singletons are equal.

Proof: If $F, G : [\mathcal{P}\mathbf{X} \to M]$ are two linear morphisms that coincide on singletons, i.e. $F \circ \iota = G \circ \iota$, then F and G induce two R-$\mathbf{X}$-linear morphisms from $(\mathcal{P}\mathbf{X},\, \iota)$ to $(M,\, F \circ \iota)$. By Prop. 13.3.3, $F = G$ follows. □

Furthermore, we obtain the following properties:

Proposition 14.3.3

(1) Reduced power constructions have unique linear extensions.

(2) Every reduced power construction with a commutative semiring is symmetric.

(3) Every reduced power construction with finite semiring preserves FIN, F-ALG, and F-CONT.

(4) Let $\mathcal{P}$ be a reduced power construction with a semiring R that has a least element $\perp_R$. If $\mathbf{X}$ has a least element $\perp_{\mathbf{X}}$, then $\mathcal{P}\mathbf{X}$ has a least element, namely $\perp_R \cdot \{\!|\perp_{\mathbf{X}}|\!\}$.

Proof:

(1) By Prop. 14.3.2 in case $M = \mathcal{P}\mathbf{Y}$.

(2) From (1) by Prop. 11.7.1.

(3) Since $(\mathcal{P}\mathbf{X}, \iota)$ is reduced, its cardinality is bounded by $2^{|R|^{|\mathbf{X}|}}$ by Th. 13.2.1. Hence, $\mathcal{P}\mathbf{X}$ is finite if R and $\mathbf{X}$ are finite, i.e. $\mathcal{P}$ preserves FIN. Preservation of F-ALG and F-CONT is then due to Th. 7.4.4.

(4) directly follows from Prop. 13.3.5. □

Finally, we investigate the behavior of reduced power constructions w.r.t. power homomorphisms. First, linear power homomorphisms are nothing else but families of linear mappings.

Proposition 14.3.4 Let $\mathcal{P}$ be a reduced power construction for semiring R, and $\mathcal{Q}$ a power construction with the same semiring. If $L_{\mathbf{X}} : [\mathcal{P}\mathbf{X} \to \mathcal{Q}\mathbf{X}]$ is a family of linear mappings indexed by domains $\mathbf{X}$ such that $L_{\mathbf{X}}\{\!|x|\!\}_{\mathcal{P}} = \{\!|x|\!\}_{\mathcal{Q}}$ holds for every x in every $\mathbf{X}$, then $L : \mathcal{P} \dot{\to} \mathcal{Q}$ is a linear power homomorphism.

Proof: We only have to show $L \circ (ext_{\mathcal{P}}\, f) = ext_{\mathcal{Q}}\,(L \circ f) \circ L$. Since L and all extensions are linear, both sides are linear morphisms from $\mathcal{P}\mathbf{X}$ to $\mathcal{Q}\mathbf{Y}$. They coincide on singletons: $L\,(ext_{\mathcal{P}}\, f\, \{\!|x|\!\}_{\mathcal{P}}) = L\,(f x)$ and $ext_{\mathcal{Q}}\,(L \circ f)\,(L\{\!|x|\!\}_{\mathcal{P}}) = ext_{\mathcal{Q}}\,(L \circ f)\,\{\!|x|\!\}_{\mathcal{Q}} = L(f x)$ hold. By Prop. 14.3.2, both sides are equal. □

Proposition 14.3.5 Let $\mathcal{P}$ be a reduced power construction, and $\mathcal{Q}$ an arbitrary power construction with the same semiring as $\mathcal{P}$.

(1) There is at most one linear power homomorphism $\mathcal{P} \dot{\to} \mathcal{Q}$.

(2) For every linear power homomorphism $H : \mathcal{P} \dot{\to} \mathcal{Q}$, there is a unique linear power homomorphism $H' : \mathcal{P} \dot{\to} \mathcal{Q}^c$ such that $E \circ H' = H$ where $E : \mathcal{Q}^c \dot{\to} \mathcal{Q}$ is the natural inclusion.

Proof:

(1) Let H_1 and H_2 be two linear power homomorphisms from $\mathcal{P}$ to $\mathcal{Q}$. Then for every ground domain $\mathbf{X}$, the mappings H_1 and H_2 are linear and coincide for singletons: $H_1\{\!|x|\!\}_{\mathcal{P}} = \{\!|x|\!\}_{\mathcal{Q}} = H_2\{\!|x|\!\}_{\mathcal{P}}$, whence they are equal by Prop. 14.3.2.

(2) H' is the power homomorphism H^c of Prop. 14.2.1. It is unique because of (1). □

14.4 Reduced constructions and retracts

In section 7.5, we introduced the notion of *retracts*. A domain $\mathbf{Y}$ is a retract of a domain $\mathbf{X}$ iff there are morphisms $r : [\mathbf{X} \to \mathbf{Y}]$ and $e : [\mathbf{Y} \to \mathbf{X}]$ with $r \circ e = id_{\mathbf{Y}}$. Theorem 7.5.3 stated that every continuous domain is a retract of some algebraic domain.

In this section, we show two theorems about retracts. The first theorem states that reducedness carries over to retracts, whereas the second theorem handles isomorphisms between power domains of different power constructions.

Theorem 14.4.1 Let $\mathcal{P}$ be an R-construction defined for two domains $\mathbf{X}$ and $\mathbf{Y}$ where $\mathbf{Y}$ is a retract of $\mathbf{X}$. If $\mathcal{P}\mathbf{X}$ is reduced, then $\mathcal{P}\mathbf{Y}$ is reduced, too.

Proof: Let S be an R-$\mathbf{X}$-submodule of the R-$\mathbf{X}$-module $(\mathcal{P}\mathbf{Y}, \iota_{\mathbf{Y}})$. We have to show $S = \mathcal{P}\mathbf{Y}$. The two morphisms $r : [\mathbf{X} \to \mathbf{Y}]$ and $e : [\mathbf{Y} \to \mathbf{X}]$ are raised to $R = map\, r : [\mathcal{P}\mathbf{X} \to \mathcal{P}\mathbf{Y}]$ and $E = map\, e : [\mathcal{P}\mathbf{Y} \to \mathcal{P}\mathbf{X}]$. These mappings are linear and satisfy $R \circ E = id_{\mathcal{P}\mathbf{X}}$ because of the properties of '*map*'. Let $T = R^{-1}[S]$ be the inverse image of S in $\mathcal{P}\mathbf{X}$. We show that T is an R-$\mathbf{X}$-submodule of $\mathcal{P}\mathbf{X}$.

(1) For x in $\mathbf{X}$, $R\{\!|x|\!\} = \{\!|r\,x|\!\} \in S$ holds, whence $\{\!|x|\!\}$ is in T.

(2) θ is in T by $R\theta = \theta \in S$.

(3) If A and B are in T, then $A \uplus B$ also is in T by $R(A \uplus B) = RA \uplus RB \in S$.

(4) Similarly, A in T implies $r \cdot A$ in T for all r in R.

(5) T is d-closed by Prop. 3.5.6 since S is d-closed.

Since $\mathcal{P}\mathbf{X}$ is reduced, T equals $\mathcal{P}\mathbf{X}$. If A is an arbitrary member of $\mathcal{P}\mathbf{Y}$, then EA is in $\mathcal{P}\mathbf{X} = T$, whence $A = R(EA)$ is in S. Thus, S equals $\mathcal{P}\mathbf{Y}$. □

The Theorem immediately implies the following corollary:

Corollary 14.4.2
Let $\mathcal{P}$ be a power construction defined for continuous domains. If $\mathcal{P}\mathbf{X}$ is reduced for all algebraic domains $\mathbf{X}$, then $\mathcal{P}\mathbf{Y}$ is also reduced for all continuous domains $\mathbf{Y}$.

Now we turn to isomorphisms between power constructions.

Theorem 14.4.3
Let $\mathcal{P}$ and $\mathcal{Q}$ be two power constructions with the same semiring R defined for two domains $\mathbf{X}$ and $\mathbf{Y}$ where $\mathbf{Y}$ is a retract of $\mathbf{X}$. If the power domains $\mathcal{P}\mathbf{X}$ and $\mathcal{Q}\mathbf{X}$ are reduced and isomorphic, then the power domains $\mathcal{P}\mathbf{Y}$ and $\mathcal{Q}\mathbf{Y}$ are also reduced and isomorphic.

Proof: The retract mappings $r : [\mathbf{X} \to \mathbf{Y}]$ and $e : [\mathbf{Y} \to \mathbf{X}]$, and the pair of isomorphisms $\varphi : [\mathcal{P}\mathbf{X} \to \mathcal{Q}\mathbf{X}]$ and $\psi : [\mathcal{Q}\mathbf{X} \to \mathcal{P}\mathbf{X}]$ induce the following diagram of mappings:

$$
\begin{tikzcd}
\mathcal{P}\mathbf{X} \arrow[rrr, shift left, "map_{\mathcal{P}}\, r"] \arrow[dd, shift left, "\varphi"] & & & \mathcal{P}\mathbf{Y} \arrow[lll, shift left, "map_{\mathcal{P}}\, e"] \arrow[dd, shift left, "\alpha"] \\
& \mathbf{X} \arrow[ul, "\iota_{\mathcal{P},\mathbf{X}}"'] \arrow[dl, "\iota_{\mathcal{Q},\mathbf{X}}"] \arrow[r, shift left, "r"] & \mathbf{Y} \arrow[l, shift left, "e"] \arrow[ur, "\iota_{\mathcal{P},\mathbf{Y}}"] \arrow[dr, "\iota_{\mathcal{Q},\mathbf{Y}}"'] & \\
\mathcal{Q}\mathbf{X} \arrow[uu, shift left, "\psi"] \arrow[rrr, shift left, "map_{\mathcal{Q}} r"] & & & \mathcal{Q}\mathbf{Y} \arrow[lll, shift left, "map_{\mathcal{Q}} e"] \arrow[uu, shift left, "\beta"]
\end{tikzcd}
$$

Here, α and β are given by $\alpha = map_{\mathcal{Q}} r \circ \varphi \circ map_{\mathcal{P}} e$ and $\beta = map_{\mathcal{P}} r \circ \psi \circ map_{\mathcal{Q}} e$. Both α and β are R-linear as compositions of R-linear maps. They are also R-$\mathbf{X}$-linear:

$$
\begin{aligned}
\alpha \circ \iota_{\mathcal{P},\mathbf{Y}} &= map_{\mathcal{Q}} r \circ \varphi \circ map_{\mathcal{P}} e \circ \iota_{\mathcal{P},\mathbf{Y}} \\
&= map_{\mathcal{Q}} r \circ \varphi \circ \iota_{\mathcal{P},\mathbf{X}} \circ e \\
&= map_{\mathcal{Q}} r \circ \iota_{\mathcal{Q},\mathbf{X}} \circ e \\
&= \iota_{\mathcal{Q},\mathbf{Y}} \circ r \circ e \\
&= \iota_{\mathcal{Q},\mathbf{Y}}
\end{aligned}
$$

The R-$\mathbf{X}$-linearity of β, i.e. $\beta \circ \iota_{\mathcal{Q},\mathbf{Y}} = \iota_{\mathcal{P},\mathbf{Y}}$, is shown analogously. The composition $\beta \circ \alpha$ forms an R-$\mathbf{X}$-linear map from $\mathcal{P}\mathbf{Y}$ to itself. Identity is another such map. By Th. 14.4.1, $\mathcal{P}\mathbf{Y}$ is reduced. By Prop. 13.3.4, $\beta \circ \alpha = id$ follows. Analogously, $\alpha \circ \beta = id$ is shown. □

Corollary 14.4.4 If $\mathcal{P}$ and $\mathcal{Q}$ are two power constructions defined for continuous domains such that $\mathcal{P}\mathbf{X}$ and $\mathcal{Q}\mathbf{X}$ are reduced and isomorphic for all algebraic ground domains $\mathbf{X}$, then $\mathcal{P}\mathbf{Y}$ and $\mathcal{Q}\mathbf{Y}$ are also reduced and isomorphic for all continuous ground domains $\mathbf{Y}$.

14.5 Initial constructions

In this section, the existence of the initial construction for given semiring R is shown and its properties are studied. The idea to consider initial power constructions dates back to [HP79]. Hoofman [Hoo87] showed the existence of the initial construction for semiring $\{0, 1\}$. Main [Mai85] then proposed initial constructions for some fancy semirings as indicated in section 11.2. In contrast to our work, he requires the singleton mapping to be strict without telling exactly why. Our singleton mappings are generally non-strict as indicated by Prop. 13.3.5 and 15.7.4. The singleton maps of mixed and sandwich power domain are also non-strict.

For every domain $\mathbf{X}$ and every semiring R, there is a free R-$\mathbf{X}$-module $R \odot \mathbf{X}$. The construction $\mathbf{X} \mapsto R \odot \mathbf{X}$ is the initial power construction for semiring R as will be shown in the sequel.

Principally, 'the' free R-$\mathbf{X}$-module is only determined up to isomorphism. Henceforth, we assume that $R \odot \mathbf{X}$ is a fixed member of the class of all free R-$\mathbf{X}$-modules. In the special case $\mathbf{X} = \mathbf{1}$, one may choose $R \odot \mathbf{1} = R$ by Prop. 13.4.5.

Theorem 14.5.1

Let R be a semiring. The power construction $\mathcal{P}$ defined by $\mathcal{P}\mathbf{X} = R \odot \mathbf{X}$ is an initial power construction for semiring R. The a priori given external product of the modules $\mathcal{P}\mathbf{X}$ coincides with the external product derived from the power operations. $\mathcal{P}$ is reduced.

Proof: We first show that $\mathcal{P}$ is a power construction. Empty set and union are given by the module operations: $\theta = 0$ and $A \uplus B = A+B$. Singleton is the morphism $\eta : [\mathbf{X} \to R\odot\mathbf{X}]$, i.e. $\{|x|\} = \eta x$. For every $f : [\mathbf{X} \to \mathcal{P}\mathbf{Y}]$, the extension *ext* is given by the unique R-$\mathbf{X}$-linear map from $R \odot \mathbf{X}$ to $(\mathcal{P}\mathbf{Y}, f)$. Function *ext* is continuous by Theorem 13.4.4. We have to demonstrate that it satisfies the power axioms.

The primary axioms of extension are satisfied by definition of *ext*. Linearity of $\mathit{ext}\, f$ implies additivity, i.e. (P1) and (P2), and $\mathit{ext}\, f \circ \eta = f$ is (P3). The secondary axioms are consequences of the uniqueness of the extended map (cf. Prop. 14.3.2).

(S1) $\mathit{ext}\,(\lambda a.\, \theta) = \lambda A.\, \theta$

The function to the right is linear and maps singletons to θ. The function to the left behaves equally, whence they are equal.

(S2) $\mathit{ext}\,(\lambda x.\, f x \uplus g\, x) = \lambda A.\, \mathit{ext}\, f\, A \uplus \mathit{ext}\, g\, A$

Both functions are linear, and both map a singleton $\{|a|\}$ to $fa \uplus ga$. Note that commutativity of the addition in a module is required to prove the additivity of the function ρ to the right because $\rho(A \uplus B) = \overline{f}A+\overline{f}B+\overline{g}A+\overline{g}B$, whereas $\rho A \uplus \rho B = \overline{f}A+\overline{g}A+\overline{f}B+\overline{g}B$.

(S3) $\mathit{ext}\, \iota = \mathit{id}$

Again, both sides are linear and coincide on singletons since both map $\{|a|\}$ to $\{|a|\}$.

(S4) $\mathit{ext}\, g \circ \mathit{ext}\, f = \mathit{ext}\,(\mathit{ext}\, g \circ f)$

Once more, both sides are linear — the left hand side as composition of linear maps. They both map a singleton $\{|a|\}$ to $\mathit{ext}\, g\,(fa)$.

Next, we show that the primarily given external product of the R-module $\mathcal{P}\mathbf{X}$ coincides with the derived external product of the power construction. The latter is denoted by '$*$' for the moment.

$$\begin{aligned} r * A &= \mathit{ext}\,(\lambda\diamond.\, A)\, r = \mathit{ext}\,(\lambda\diamond.\, A)\,(r \cdot 1) \\ &= r \cdot \mathit{ext}\,(\lambda\diamond.\, A)\,(\eta\diamond) = r \cdot A \end{aligned}$$

At the line break, the linearity of extended maps is used.

The power construction $\mathcal{P}$ is reduced since all power domains $\mathcal{P}\mathbf{X}$ are reduced by Prop. 13.4.3.

To show initiality of $\mathcal{P}$, let $\mathcal{Q}$ be another power construction with characteristic semiring R. We have to demonstrate the existence of a unique linear power homomorphism $H : \mathcal{P} \dot{\to} \mathcal{Q}$.

For every domain $\mathbf{X}$, $(\mathcal{Q}\mathbf{X}, \iota_{\mathcal{Q}})$ is an R-$\mathbf{X}$-module. Since $(\mathcal{P}\mathbf{X}, \iota_{\mathcal{P}})$ is a free R-$\mathbf{X}$-module, there is a (unique) R-linear morphism $H : [\mathcal{P}\mathbf{X} \to \mathcal{Q}\mathbf{X}]$ with $H \circ \iota_{\mathcal{P}} = \iota_{\mathcal{Q}}$. By linearity, H is additive, i.e. $H\theta = \theta$ and $H(A \uplus B) = HA \uplus HB$ hold. Since $\mathcal{P}$ is reduced, these properties allow to apply Prop. 14.3.4, whence $H : \mathcal{P} \dot{\to} \mathcal{Q}$ is a linear power homomorphism. It is unique by Prop. 14.3.5 since $\mathcal{P}$ is reduced. □

Since the initial power construction is reduced, it enjoys all properties of reduced power constructions as enumerated in the previous section. In addition, we are able to prove some more properties:

Theorem 14.5.2 If R is structurally algebraic, then the initial power construction for R preserves ALG and CONT.

Proof: Preservation of ALG is due to Th. 13.6.3. Preservation of CONT follows from Prop. 7.5.6. □

Lemma 14.5.3
If there is a non-discrete R-module, then the initial construction for R is faithful.

Proof: The proof is analogous to that of Lemma 11.9.2. One only has to replace the non-discrete power domain $\mathcal{P}\mathbf{Y}$ by the non-discrete R-module. □

14.6 Existential restriction of power constructions

So far, we mainly considered sub-constructions of $\mathcal{P}$ with characteristic semiring $\mathcal{P}\mathbf{1}$. The least such construction is given by the core $\mathcal{P}^c$. In this section, we consider a different problem. Given a specific sub-semiring R' of $\mathcal{P}\mathbf{1}$, we derive sub-constructions of $\mathcal{P}$ with just this characteristic semiring. The principal method to do this is *existential restriction*.

Let $\mathcal{P}$ be an R-construction. According to section 11.3, existential quantification may be defined w.r.t. R and then has type $\mathcal{E} : [\mathcal{P}\mathbf{X} \to [[\mathbf{X} \to R] \to R]]$. The existential restriction of $\mathcal{P}\mathbf{X}$ now is defined as the set of all those formal sets where quantification may be restricted to result in a second order predicate in $[[\mathbf{X} \to R'] \to R']$.

Definition 14.6.1
Let R be a semiring with a sub-semiring R', and let $\mathcal{P}$ be an R-construction. Then we define for all ground domains $\mathbf{X}$ the *existential restriction* of $\mathcal{P}\mathbf{X}$ to R':

$$\mathcal{P}'\mathbf{X} = \{A \in \mathcal{P}\mathbf{X} \mid \forall p : [\mathbf{X} \to R'] : \mathcal{E}A\,p \in R'\}$$

In the following, we verify that $\mathcal{P}'$ is a power construction with characteristic semiring R'.

- $\mathcal{E}\theta\,p = 0 \in R'$ implies $\theta \in \mathcal{P}'\mathbf{X}$.
- If A and B are in $\mathcal{P}'\mathbf{X}$, then for all $p : [\mathbf{X} \to R']$, $\mathcal{E}A\,p$ and $\mathcal{E}B\,p$ are in R', whence $\mathcal{E}\,(A \uplus B)\,p = \mathcal{E}A\,p + \mathcal{E}B\,p$ is in R'.
- For x in $\mathbf{X}$ and $p : [\mathbf{X} \to R']$, $\mathcal{E}\{\!|x|\!\}\,p = p\,x$ is in R'. Hence, $\{\!|x|\!\}$ is in $\mathcal{P}'\mathbf{X}$ for all x in $\mathbf{X}$.
- Let $f : [\mathbf{X} \to \mathcal{P}'\mathbf{Y}]$ and $A \in \mathcal{P}'\mathbf{X}$. We have to show $ext\,f\,A$ in $\mathcal{P}'\mathbf{Y}$.

  For all $p : [\mathbf{Y} \to R']$, $\mathcal{E}\,(ext\,f\,A)\,p = \mathcal{E}A\,(\lambda a.\mathcal{E}(f\,a)\,p)$ holds by the results of section 10.5. $f\,a \in \mathcal{P}'\mathbf{Y}$ implies $\mathcal{E}(f\,a)\,p \in R'$. Thus, $(\lambda a.\,\mathcal{E}(f\,a)\,p) : [\mathbf{X} \to R']$, whence the value of the whole term is in R'.
- Finally, we have to verify that $\mathcal{P}'\mathbf{X}$ is a sub-domain of $\mathcal{P}\mathbf{X}$. Let $(A_i)_{i \in I}$ be a directed family of members of $\mathcal{P}'\mathbf{X}$ with limit A. Then for all $p : [\mathbf{X} \to R']$, $\mathcal{E}(\bigsqcup_{i \in I} A_i)\,p = \bigsqcup_{i \in I}(\mathcal{E}A_i\,p) \in R'$ by continuity of $\mathcal{E}$.

Thus, the power operations of $\mathcal{P}$ imply power operations for $\mathcal{P}'$ by restriction and co-restriction. The power axioms thus are directly inherited from $\mathcal{P}$. Note that we only used parts (1), (2), and (5) of the definition of sub-semirings so far.

Next, we show $\mathcal{P}'\mathbf{1} = R'$. For $p : [\mathbf{1} \to R]$ and $a \in R = \mathcal{P}\mathbf{1}$, we may simplify $\mathcal{E}\,a\,p = ext\,(\lambda\diamond.\,p\,\diamond)\,a = a \cdot p\,\diamond$. Hence, $\mathcal{P}'\mathbf{1} = \{a \in \mathcal{P}\mathbf{1} \mid \forall p : [\mathbf{1} \to R'] : \mathcal{E} a\, p \in R'\} = \{a \in R \mid \forall r \in R' : a \cdot r \in R'\}$. This set is a subset of R', since $a \in \mathcal{P}'\mathbf{1}$ and $1 \in R'$ implies $a = a\cdot 1 \in R'$. Conversely, if r' is in R', then for all r in R', $r' \cdot r$ is in R', whence r' is in $\mathcal{P}'\mathbf{1}$.

Thus, $\mathcal{P}'\mathbf{1} = R'$ is proven. Here, we needed the remaining parts (3) and (4) of the definition of sub-semirings.

If $\mathcal{Q}$ is an arbitrary sub-construction of $\mathcal{P}$ with $\mathcal{Q}\mathbf{1} = R'$, then $\mathcal{Q}\mathbf{X} \subseteq \mathcal{P}'\mathbf{X}$ holds for all ground domains $\mathbf{X}$ since existential quantification in $\mathcal{Q}$ maps $\mathcal{Q}\mathbf{1}$-predicates to $\mathcal{Q}\mathbf{1}$.

Summarizing, we obtain

Theorem 14.6.2 If $\mathcal{P}$ is an R-construction, and R' is a sub-semiring of R, then the existential restriction $\mathcal{P}'$ of $\mathcal{P}$ to semiring R' is the greatest sub-construction of $\mathcal{P}$ with semiring R'. The core of $\mathcal{P}'$ is the least sub-construction of $\mathcal{P}$ with semiring R'.

In the special case $R' = R$, the existential restriction of $\mathcal{P}$ is $\mathcal{P}$ itself. Since the existential restriction $\mathcal{P}'$ is a sub-construction of $\mathcal{P}$, it is commutative, symmetric, or faithful whenever $\mathcal{P}$ is so. It however need not be reduced even if $\mathcal{P}$ is; section 22.1 provides an example.

Let $H : \mathcal{P} \dot{\to} \mathcal{Q}$ be a linear power homomorphism between two R-constructions, and let R' be a sub-semiring of R. Does H cut down to a power homomorphism between the existential restrictions $\mathcal{P}'$ and $\mathcal{Q}'$ of $\mathcal{P}$ and $\mathcal{Q}$?

The answer is yes. We only need to show $H A \in \mathcal{Q}'\mathbf{X}$ for $A \in \mathcal{P}'\mathbf{X}$. Linear power homomorphisms act as identity on the characteristic semiring by Prop. 12.4.2. Hence, for all $p : [\mathbf{X} \to R']$, $\mathcal{E}(H A)\,p = ext\,(H \circ p)\,(H A) = H\,(ext\,p\,A) = \mathcal{E} A\,p \in R'$ holds. Thus, we obtain

Proposition 14.6.3 Let R be a semiring with sub-semiring R', and let ^{e} be the method of existential restriction to R'. Then every R-linear power homomorphism $H : \mathcal{P} \dot{\to} \mathcal{Q}$ between two R-constructions $\mathcal{P}$ and $\mathcal{Q}$ cuts down to an R'-linear power homomorphism $H^e : \mathcal{P}^e \dot{\to} \mathcal{Q}^e$.

Chapter 15

Final power constructions

In contrast to the initial power construction, the final one may be explicitly constructed. As indicated in section 10.5, existential quantification leads to a mapping $\mathcal{E}$ from $\mathcal{P}\mathbf{X}$ to $[[\mathbf{X} \to \mathcal{P}\mathbf{1}] \to \mathcal{P}\mathbf{1}]$ for every power construction $\mathcal{P}$. This suggests to define $\mathcal{P}\mathbf{X}$ as $[[\mathbf{X} \to R] \to R]$ if $R = \mathcal{P}\mathbf{1}$ is given. The equations in section 10.5 also indicate how to define the power operations.

One has to prove that these operations satisfy the axioms of chapter 9, and that the derived semiring $\mathcal{P}\mathbf{1}$ is isomorphic to the original semiring R. For proving the axioms, the outer, second order mappings have to be additive, and for proving the isomorphism between $\mathcal{P}\mathbf{1}$ and R, they even have to be right linear. Thus, the actual definition is $\mathcal{P}\mathbf{X} = [[\mathbf{X} \to R] \stackrel{rlin}{\to} R]$.

The main theorem about the final power construction for semiring R and its power operations is presented in section 15.1. The proof of the main theorem is spread over the subsequent 4 sections. In section 15.6, the derived operations of the final power construction are computed. Properties of the final construction such as preservation of domain classes are studied in section 15.7. Linear functions on final power domains are not generally uniquely determined by their values on singletons. In section 15.8, we present a necessary and sufficient criterion for this. The core of the final construction is briefly looked at in section 15.9.

15.1 The main theorem

Let R be a given semiring, and let $\mathbf{X}$ and $\mathbf{Y}$ be two right R-modules. Remember a mapping $F : [\mathbf{X} \to \mathbf{Y}]$ is right linear iff $F(x + x') = Fx + Fx'$ and $F(x \cdot r) = Fx \cdot r$ for all $x, x' \in \mathbf{X}$ and $r \in R$. The set $\{F : [\mathbf{X} \to \mathbf{Y}] \mid F \text{ is right linear }\}$ is denoted by $[\mathbf{X} \stackrel{rlin}{\to} \mathbf{Y}]$. Ordered as subset of $[\mathbf{X} \to \mathbf{Y}]$, it becomes a domain because the lub of a directed set of linear functions is linear again by continuity of application, sum, and external product.

Every semiring R becomes a right R-module in a canonical way. For given domain $\mathbf{X}$, the function space $[\mathbf{X} \to R]$ also becomes a right R-module by defining $f \cdot r = \lambda x.\,(fx) \cdot r$. Thus, the notation $[[\mathbf{X} \to R] \stackrel{rlin}{\to} R]$ makes sense and denotes a domain.

Theorem 15.1.1 The *final power construction* belonging to a given semiring R is defined by $(\mathcal{P}, R, \varphi)$ where $\mathcal{P}\mathbf{X} = \mathcal{P}_f^R\mathbf{X} = [[\mathbf{X} \to R] \stackrel{rlin}{\to} R]$. Its operations are defined by

- $\theta = \lambda g.\, 0$
- $A \uplus B = \lambda g.\, Ag + Bg$
- $\{\!|x|\!\} = \lambda g.\, gx$ for $x \in \mathbf{X}$.
- $ext\, f\, A = \lambda g.\, A\,(\lambda a.\, fag)$ for $f : [\mathbf{X} \to \mathcal{P}\mathbf{Y}]$ and $A \in \mathcal{P}\mathbf{X}$.
- The isomorphism $\varphi : [R \to \mathcal{P}\mathbf{1}]$ is given by $\varphi(r) = \lambda g.\, r \cdot g\diamond$. Its inverse is $\psi(A) = A(\lambda\diamond.\, 1)$.

To understand the definition of ext, note that a ranges over $\mathbf{X}$. Then a in $\mathbf{X}$ and $f : [\mathbf{X} \to \mathcal{P}\mathbf{Y}]$ imply $fa \in \mathcal{P}\mathbf{Y} = [[\mathbf{Y} \to R] \overset{rlin}{\to} R]$. g ranges over $[\mathbf{Y} \to R]$, whence $fag \in R$ and $\lambda a.\, fag : [\mathbf{X} \to R]$. Thus, $A \in \mathcal{P}\mathbf{X} = [[\mathbf{X} \to R] \overset{rlin}{\to} R]$ implies $A\,(\lambda \ldots) \in R$.

The proof of the theorem proceeds in four steps: First, it is shown that the power operations defined above always create right linear maps when applied to such maps. Second, the validity of the power axioms is shown by λ-conversions. Third, an isomorphism between $\mathcal{P}\mathbf{1}$ and R is established. Fourth, the power construction $\mathcal{P}_f^R$ is demonstrated to be final. These steps are done in the next four sections.

15.2 Proof step 1: Right linearity

In this section, we show that the power operations defined above always preserve right linearity.

- $\theta = \lambda g.\, 0$

  Additive: $\theta\,(g + g') = 0 = 0 + 0 = \theta\, g + \theta\, g'$

  Linear: $\theta\,(g \cdot r) = 0 = 0 \cdot r = (\theta\, g) \cdot r$

- $A \uplus B = \lambda g.\, Ag + Bg$

  $$\begin{aligned} \text{Additive :}\quad (A \uplus B)\,(g + g') &= A\,(g+g') + B\,(g+g') \;=\; Ag + Ag' + Bg + Bg' \\ &= Ag + Bg + Ag' + Bg' \;=\; (A \uplus B)\,g + (A \uplus B)\,g' \end{aligned}$$

  Note that for the equality at the line break, commutativity of '+' is needed, which is ensured by the definition of semirings.

  $$\begin{aligned} \text{Linear :}\quad (A \uplus B)\,(g \cdot r) &= A\,(g \cdot r) + B\,(g \cdot r) \;=\; (Ag) \cdot r + (Bg) \cdot r \\ &= (Ag + Bg) \cdot r \;=\; (A \uplus B)\,g \cdot r \end{aligned}$$

- $\{\!|x|\!\} = \lambda g.\, g\,x$ for $x \in \mathbf{X}$.

  Additive: $\{\!|x|\!\}\,(g + g') = (g + g')\,x = g\,x + g'\,x = \{\!|x|\!\}\,g + \{\!|x|\!\}\,g'$

  Linear: $\{\!|x|\!\}\,(g \cdot r) = (g \cdot r)\,x = (gx) \cdot r = (\{\!|x|\!\}\,g) \cdot r$

- $ext\, f\, A = \lambda g.\, A\,(\lambda a.\, fag)$ for $f : [\mathbf{X} \to \mathcal{P}\mathbf{Y}]$ and $A \in \mathcal{P}\mathbf{X}$.

  Here, the linearity of fa and A is used.

  $$\begin{aligned} \text{Linear :}\quad (ext\, f\, A)\,(g \cdot r) &= A\,(\lambda a.\, f\,a\,(g \cdot r)) \\ &= A\,(\lambda a.\, (fag \cdot r)) = A\,((\lambda a.\, fag) \cdot r) \\ &= (A\,(\lambda a.\, fag)) \cdot r = ((ext\, f\, A)\,g) \cdot r \end{aligned}$$

  Additivity is shown analogously.

15.3 Proof step 2: The power axioms

In this section, we prove the validity of the power axioms for the new construction.

By the definition $A \uplus B = \lambda g.\, Ag + Bg$, the operation '$\uplus$' trivially is commutative, associative, and has neutral element $\theta = \lambda g.\, 0$. The axioms of extension are less easy to prove.

Def.: $\quad ext\, f\, A = \lambda p.\, A\,(\lambda a.\, fap)$

(P1) $\quad ext\, f\, \theta = \lambda p.\, \theta\,(\lambda a.\, fap) = \lambda p.\, 0 = \theta$

(P2)
$$\begin{aligned} ext\, f\,(A \uplus B) &= \lambda p.\,(A \uplus B)\,(\lambda a.\, fap) \\ &= \lambda p.\, A\,(\lambda a.\, fap) + B\,(\lambda a.\, fap) \\ &= ext\, f\, A \uplus ext\, f\, B \end{aligned}$$

(P3)
$$\begin{aligned} ext\, f\, \{\!|x|\!\} &= \lambda p.\, \{\!|x|\!\}\,(\lambda a.\, fap) \\ &= \lambda p.\,(\lambda a.\, fap)\, x \\ &= \lambda p.\, fxp \;=\; fx \end{aligned}$$

(S1)
$$\begin{aligned} ext\,(\lambda x.\, \theta)\, A &= \lambda p.\, A\,(\lambda a.\,(\lambda x.\, \theta)\, a\, p) \\ &= \lambda p.\, A\,(\lambda a.\, \theta\, p) \\ &= \lambda p.\, A\,(\lambda a.\, 0) \\ &= \lambda p.\, 0 \;=\; \theta \end{aligned}$$

At the last line break, the additivity of A is used.

(S2)
$$\begin{aligned} ext\,(f \uplus g)\, A &= \lambda p.\, A\,(\lambda a.\,(f \uplus g)\, a\, p) \\ &= \lambda p.\, A\,(\lambda a.\,(fa \uplus ga)\, p) \\ &= \lambda p.\, A\,(\lambda a.\, fap + gap) && \text{using additivity of } A \text{ here} \\ &= \lambda p.\, A\,(\lambda a.\, fap) + A\,(\lambda a.\, gap) \\ &= ext\, f\, A \uplus ext\, g\, A \end{aligned}$$

(S3)
$$\begin{aligned} ext\, \iota\, A &= \lambda p.\, A\,(\lambda a.\, \iota ap) \;=\; \lambda p.\, A\,(\lambda a.\, \{\!|a|\!\}\, p) \\ &= \lambda p.\, A\,(\lambda a.\, pa) \;=\; \lambda p.\, Ap \;=\; A \end{aligned}$$

(S4) The claim is $ext\, g \circ ext\, f = ext\,(ext\, g \circ f)$, or $\; ext\, g\,(ext\, f\, A) \;=\; ext\,(\lambda x.\, ext\, g\,(fx))\, A$

$$\begin{aligned} ext\, g\,(ext\, f\, A) &= \lambda p.\,(ext\, f\, A)\,(\lambda b.\, gbp) \\ &= \lambda p.\,(\lambda q.\, A\,(\lambda a.\, faq))\,(\lambda b.\, gbp) \\ &= \lambda p.\, A\,(\lambda a.\, f\, a\,(\lambda b.\, gbp)) \\ ext\,(\lambda x.\, ext\, g\,(fx))\, A &= \lambda p.\, A\,(\lambda a.\,(\lambda x.\, ext\, g\,(fx))\, a\, p) \\ &= \lambda p.\, A\,(\lambda a.\,(ext\, g\,(fa))\, p) \\ &= \lambda p.\, A\,(\lambda a.\,(\lambda q.\,(fa)\,(\lambda b.\, gbq))\, p) \\ &= \lambda p.\, A\,(\lambda a.\, f\, a\,(\lambda b.\, gbp)) \end{aligned}$$

15.4 Proof step 3: The characteristic semiring

In this section, we show the power domain $\mathcal{P}\mathbf{1}$ and the original semiring R to be isomorphic. To this end, we first consider how the semiring operations in $\mathcal{P}\mathbf{1}$ are defined.

- $\mathcal{P}\mathbf{1} \;=\; [[\mathbf{1} \to R] \stackrel{rlin}{\to} R]$
- $0 \;=\; \theta \;=\; \lambda p.\, 0$
- $A + B \;=\; A \uplus B \;=\; \lambda p.\, Ap + Bp$
- $1 \;=\; \{\!|\diamond|\!\} \;=\; \lambda p.\, p\diamond$
- $$\begin{aligned} A \cdot B \;=\; ext\,(\lambda\diamond.\, B)\, A \;&=\; \lambda p.\, A\,(\lambda a.\, (\lambda\diamond.\, B)\, a\, p) \\ &=\; \lambda p.\, A\,(\lambda a.\, B\, p) \\ &=\; \lambda p.\, A\,(\lambda\diamond.\, B\, p) \end{aligned}$$

  For the last equality, note that a ranges over $\mathbf{1}$.

There is one obvious choice for a mapping $\psi : [\mathcal{P}\mathbf{1} \to R]$, namely $\psi A \;=\; A\,(\lambda\diamond.\, 1)$. This mapping is a semiring homomorphism:

- $\psi\,(0) \;=\; (\lambda p.\, 0)\,(\lambda\diamond.\, 1) \;=\; 0$
- $\psi\,(A + B) \;=\; (\lambda p.\, Ap + Bp)\,(\lambda\diamond.\, 1) \;=\; \psi A + \psi B$
- $\psi\,(1) \;=\; (\lambda p.\, p\diamond)\,(\lambda\diamond.\, 1) \;=\; (\lambda\diamond.\, 1)\diamond \;=\; 1$
- $$\begin{aligned} \psi\,(A \cdot B) \;&=\; (\lambda p.\, A\,(\lambda\diamond.\, Bp))\,(\lambda\diamond.\, 1) \\ &=\; A\,(\lambda\diamond.\, B\,(\lambda\diamond.\, 1)) \\ &=\; A\,(\lambda\diamond.\, \psi B) \;=\; A\,(\lambda\diamond.\, 1 \cdot \psi B) \qquad \text{use right linearity of } A \text{ now} \\ &=\; A\,(\lambda\diamond.\, 1) \cdot \psi B \;=\; \psi A \cdot \psi B \end{aligned}$$

As announced in the introduction of this chapter, right linearity of the second order functions in $\mathcal{P}\mathbf{X}$ is needed here. With left linearity, the result would be $\psi\,(A \cdot B) = \psi B \cdot \psi A$ instead.

The mapping ψ is shown to be an isomorphism by specifying its inverse. Let $\varphi : [R \to \mathcal{P}\mathbf{1}]$ be defined by $\varphi r = \lambda p.\, r \cdot p\diamond$. The second order mapping φr is right linear in p because

$$\begin{aligned} \varphi r\,(p + p') \;&=\; r \cdot (p + p')\diamond \;=\; r \cdot (p\diamond + p'\diamond) \;=\; r \cdot p\diamond + r \cdot p'\diamond \;=\; \varphi r\,(p) + \varphi r\,(p') \\ \varphi r\,(p \cdot a) \;&=\; r \cdot (p \cdot a)\diamond \;=\; r \cdot (p\diamond \cdot a) \;=\; (r \cdot p\diamond) \cdot a \;=\; \varphi r\,(p) \cdot a \end{aligned}$$

φ is the inverse of ψ since

$$\begin{aligned} \psi\,(\varphi r) \;&=\; (\lambda p.\, r \cdot p\diamond)\,(\lambda\diamond.\, 1) \;=\; r \cdot (\lambda\diamond.\, 1)\diamond \;=\; r \cdot 1 \;=\; r \\ \varphi\,(\psi A) \;&=\; \lambda p.\, \psi A \cdot p\diamond \\ &=\; \lambda p.\, A\,(\lambda\diamond.\, 1) \cdot p\diamond \qquad \text{and by right linearity of } A \\ &=\; \lambda p.\, A\,(\lambda\diamond.\, 1 \cdot p\diamond) \\ &=\; \lambda p.\, A\,(\lambda\diamond.\, p\diamond) \;=\; \lambda p.\, Ap \;=\; A \end{aligned}$$

15.5 Proof step 4: Finality

Let $(\mathcal{Q}, R, \rho)$ be an arbitrary power construction with semiring R and let $(\mathcal{P}, R, \varphi)$ be the existential construction for R. We have to construct a linear power homomorphism $H : \mathcal{Q} \dot{\to} \mathcal{P}$ and then show it is unique. H is given by existential quantification $\mathcal{E} : [\mathcal{Q}\mathbf{X} \to [[\mathbf{X} \to R] \stackrel{rlin}{\to} R]]$ as defined in section 11.3.

Existential quantification in $\mathcal{Q}$ would map functions in $[\mathbf{X} \to \mathcal{Q}\mathbf{1}]$ into elements of $\mathcal{Q}\mathbf{1}$. It can be used to define H if semiring elements can be translated into elements of $\mathcal{Q}\mathbf{1}$ and vice versa by means of ρ and ρ^{-1}. Hence we define for A in $\mathcal{Q}\mathbf{X}$

$$H\,A = \lambda p.\,\rho^{-1}\,(ext_{\mathcal{Q}}\,(\rho \circ p)\,A)$$

Here, p ranges over $[\mathbf{X} \to R]$, whence $\rho \circ p : [\mathbf{X} \to \mathcal{Q}\mathbf{1}]$. Thus, $(ext_{\mathcal{Q}}\,(\rho \circ p)\,A)$ is in $\mathcal{Q}\mathbf{1}$, whence its value by ρ^{-1} is in R. Hence, $H : [\mathcal{Q}\mathbf{X} \to [[\mathbf{X} \to R] \to R]]$.

Adopting this definition of H, we have to show that $H\,A$ is right linear, that H is a linear power homomorphism, and finally that H is unique. We tackle these statements one by one.

$H\,A\,(p_1 + p_2)$ equals $\rho^{-1}\,(ext\,(\rho \circ (p_1 + p_2))\,A)$. Using additivity of ρ^{-1} and ρ, and axiom (S2), the operator '+' may be moved to the outermost level.

$$\begin{aligned}
H\,A\,(p \cdot r) &= \rho^{-1}\,(ext\,(\rho \circ (\lambda x.\,p\,x \cdot r))\,A) \\
&= \rho^{-1}\,(ext\,((\rho \circ p) \cdot \rho r)\,A) \qquad \text{use Prop. 10.10.2, (6) now} \\
&= \rho^{-1}\,((ext\,(\rho \circ p)\,A) \cdot \rho r) \\
&= \rho^{-1}\,(ext\,(\rho \circ p)\,A) \cdot \rho^{-1}\,(\rho r) \;=\; (H\,A\,p) \cdot r
\end{aligned}$$

Now the right linearity of $H\,A$ has been proved. Next, we show that H is a power homomorphism:

- $H\,\theta_{\mathcal{Q}} = \lambda p.\,\rho^{-1}\,(ext\,(\rho \circ p)\,\theta_{\mathcal{Q}}) \overset{(P1)}{=} \lambda p.\,\rho^{-1}\,(0) = \lambda p.\,0 = \theta_{\mathcal{P}}$
- $$\begin{aligned} H\,(A \uplus B) &= \lambda p.\,\rho^{-1}\,(ext\,(\rho \circ p)\,(A \uplus B)) \\ &= \lambda p.\,\rho^{-1}\,(ext\,(\rho \circ p)\,A) \uplus \rho^{-1}\,(ext\,(\rho \circ p)\,B) \;=\; H\,A \uplus H\,B \end{aligned}$$
  using axiom (P2) and additivity of ρ^{-1} at the equality sign in the middle.
- $H\,\{\!|x|\!\}_{\mathcal{Q}} = \lambda p.\,\rho^{-1}\,(ext\,(\rho \circ p)\,\{\!|x|\!\}_{\mathcal{Q}}) \overset{(P3)}{=} \lambda p.\,\rho^{-1}\,(\rho\,(p\,x)) = \lambda p.\,p\,x = \{\!|x|\!\}_{\mathcal{P}}$
- $$\begin{aligned} H\,(ext\,f\,A) &= \lambda p.\,\rho^{-1}\,(ext\,(\rho \circ p)\,(ext\,f\,A)) \\ &\overset{(S4)}{=} \lambda p.\,\rho^{-1}\,(ext\,(\lambda x.\,ext\,(\rho \circ p)\,(f x))\,A) \\ &= \lambda p.\,\rho^{-1}\,(ext\,(\lambda x.\,\rho\,(\rho^{-1}\,(ext\,(\rho \circ p)\,(f x))))\,A) \\ &= \lambda p.\,\rho^{-1}\,(ext\,(\lambda x.\,\rho\,(H\,(f x)\,p))\,A) \\ &= \lambda p.\,H\,A\,(\lambda x.\,H\,(f x)\,p) \\ &= \lambda p.\,H\,A\,(\lambda x.\,(H \circ f)\,x\,p) \\ &= ext_{\mathcal{P}}\,(H \circ f)\,(H\,A) \end{aligned}$$

Now we know H is a power homomorphism. To show its linearity, we have to prove $\psi\,(H_{\mathbf{1}}\,(\rho r)) = r$ for all $r \in R$ by Prop. 12.4.2 where $\psi = \lambda S.\,S\,(\lambda\diamond.\,1)$ is the isomorphism from $\mathcal{P}\mathbf{1}$ to R.

$$\begin{aligned}
\psi\,(H_{\mathbf{1}}\,(\rho r)) &= (\lambda p.\,\rho^{-1}\,(ext_{\mathcal{Q}}\,(\rho \circ p)\,(\rho r)))\,(\lambda\diamond.\,1) \\
&= \rho^{-1}\,(ext_{\mathcal{Q}}\,(\rho \circ (\lambda\diamond.\,1))\,(\rho r)) \\
&= \rho^{-1}\,(ext_{\mathcal{Q}}\,(\lambda\diamond.\,\{\!|\diamond|\!\}_{\mathcal{Q}})\,(\rho r)) \qquad \text{since } \rho\,(1) = \{\!|\diamond|\!\}_{\mathcal{Q}} \\
&\overset{(S3)}{=} \rho^{-1}\,(\rho r) \;=\; r
\end{aligned}$$

The last property to be shown is that H is the only linear power homomorphism from $\mathcal{Q}$ to $\mathcal{P}$. Let G be another linear power homomorphism. Then $\psi \circ G_{\mathbf{1}} \circ \rho = id_R$ holds.

$$
\begin{array}{rcll}
HA & = & \lambda p.\, \rho^{-1}\, (ext_{\mathcal{Q}}\, (\rho \circ p)\, A) & \\
 & = & \lambda p.\, \psi\, (G_{\mathbf{1}}\, (ext_{\mathcal{Q}}\, (\rho \circ p)\, A)) & \text{since } \rho^{-1} = \psi \circ G_{\mathbf{1}} \\
 & = & \lambda p.\, \psi\, (ext_{\mathcal{P}}\, (G_{\mathbf{1}} \circ \rho \circ p)\, (GA)) & \text{because } G \text{ is a power homomorphism} \\
 & = & \lambda p.\, (ext_{\mathcal{P}}\, (\psi^{-1} \circ p)\, (GA))\, (\lambda\diamond.\, 1) & \text{since } G_{\mathbf{1}} \circ \rho = \psi^{-1}, \text{ and } \psi S = S\, (\lambda\diamond.\, 1) \\
 & = & \lambda p.\, (GA)\, (\lambda x.\, (\psi^{-1} \circ p)\, x\, (\lambda\diamond.\, 1)) & \text{by definition of } ext_{\mathcal{P}} \\
 & = & \lambda p.\, (GA)\, (\lambda x.\, \psi\, (\psi^{-1}\, (p\, x))) & \text{since } S(\lambda\diamond.\, 1) = \psi S \\
 & = & \lambda p.\, GA\, (\lambda x.\, p\, x) \;=\; \lambda p.\, GA\, p \;=\; GA &
\end{array}
$$

Now, the theorem is completely proved.

15.6 Derived operations

The definition of the existential construction provides realizations for the principal power operations in terms of higher order functions. The derived operations may also be expressed in functional form.

- $U A = ext\; id\; A = \lambda p.\, A\, (\lambda a.\, id\; a\, p) = \lambda p.\, A\, (\lambda a.\, a\, p)$

- $$
\begin{array}{rcl}
map\, f\, A & = & ext\, (\iota \circ f)\, A \\
 & = & \lambda p.\, A\, (\lambda a.\, (\iota \circ f)\, a\, p) \\
 & = & \lambda p.\, A\, (\lambda a.\, \{\!|f a|\!\}\, p) \\
 & = & \lambda p.\, A\, (\lambda a.\, p\, (f a)) \\
 & = & \lambda p.\, A\, (p \circ f)
\end{array}
$$

- Existential quantification $\mathcal{E} : \mathcal{P}\mathbf{X} \to [[\mathbf{X} \to R] \to R]$ in terms of the semiring R instead of $\mathcal{P}\mathbf{1}$ is simply given by $\mathcal{E} A\, p = A\, p$. The reason is that $\mathcal{E}$ is a linear power homomorphism from $\mathcal{P}$ to $\mathcal{P}$ as we saw in the previous section. Identity also is a linear power homomorphism. Because of the uniqueness result of the previous section, they coincide.

- $ne\; A = \mathcal{E} A\, (\lambda x.\, 1) = A\, (\lambda x.\, 1)$

- As indicated in section 11.3, the external product is defined for elements of R by means of φ.

  $$
  \begin{array}{rcl}
  r \cdot A & = & ext\, (\lambda\diamond.\, A)\, (\varphi r) \\
   & = & \lambda p.\, (\varphi r)\, (\lambda a.\, (\lambda\diamond.\, A)\, a\, p) \\
   & = & \lambda p.\, (\lambda q.\, r \cdot q\, \diamond)\, (\lambda a.\, Ap) \\
   & = & \lambda p.\, r \cdot (\lambda a.\, Ap)\, \diamond \\
   & = & \lambda p.\, r \cdot Ap
  \end{array}
  $$

- $r \cdot \{\!|x|\!\} \;=\; \lambda p.\, r \cdot \{\!|x|\!\}\, p \;=\; \lambda p.\, r \cdot p\, x$

- Similar to the external product, we define *filter* for mappings $f : [\mathbf{X} \to R]$ instead of $f : [\mathbf{X} \to \mathcal{P}\mathbf{1}]$.

$$\begin{aligned} \mathit{filter}\, f\, A &= \mathit{ext}\,(\lambda x.\, fx \cdot \{\!|x|\!\})\, A \\ &= \lambda p.\, A\,(\lambda a.\,(\lambda x.\, fx \cdot \{\!|x|\!\})\, a\, p) \\ &= \lambda p.\, A\,(\lambda a.\,(fa \cdot \{\!|a|\!\})\, p) \\ &= \lambda p.\, A\,(\lambda a.\,(\lambda q.\, fa \cdot q\, a)\, p) \\ &= \lambda p.\, A\,(\lambda a.\, fa \cdot p\, a) \end{aligned}$$

- $$\begin{aligned} A \cdot r &= \mathit{filter}\,(\lambda x.\, r)\, A \\ &= \lambda p.\, A\,(\lambda a.\,(\lambda x.\, r)\, a \cdot p\, a) \\ &= \lambda p.\, A\,(\lambda a.\, r \cdot p\, a) \end{aligned}$$

Strangely, both $r \cdot A$ and $A \cdot r$ denote an expression where r is a left factor. The difference is that with $\mathcal{P}\mathbf{X} = [[\mathbf{X} \to R] \overset{rlin}{\to} R]$, the left product $r \cdot A$ operates at the outer occurrence of R, whereas the right product $A \cdot r$ operates at the inner occurrence.

Proposition 15.6.1
If the semiring R is commutative, then the construction $\mathcal{P}^R_f$ is commutative.

Proof:

$$\begin{aligned} && A \cdot r &= \lambda p.\, A\,(\lambda a.\, r \cdot pa) \\ &\text{by commutativity of } R: & &= \lambda p.\, A\,(\lambda a.\, pa \cdot r) \\ &\text{by right linearity of } A: & &= \lambda p.\, A\,(\lambda a.\, pa) \cdot r \\ &\text{by commutativity of } R: & &= \lambda p.\, r \cdot Ap = r \cdot A \qquad \square \end{aligned}$$

Concerning symmetry, one has to show $A \overset{\rightarrow}{\times} B = A \overset{\leftarrow}{\times} B$ for all A in $\mathcal{P}\mathbf{X}$ and B in $\mathcal{P}\mathbf{Y}$ because of Cor. 10.3.2.

$$\begin{aligned} A \overset{\rightarrow}{\times} B &= \mathit{ext}\ (\lambda a.\, \mathit{ext}\ (\lambda b.\, \{\!|(a,\, b)|\!\})\ B)\ A \\ &= \lambda p.\, A\ (\lambda a.\, \mathit{ext}\ (\lambda b.\, \{\!|(a,\, b)|\!\})\ B\ p) \\ &= \lambda p.\, A\ (\lambda a.\, B\ (\lambda b.\, \{\!|(a,\, b)|\!\}\ p)) \\ &= \lambda p.\, A\ (\lambda a.\, B\ (\lambda b.\, p\,(a,\, b))) \end{aligned}$$

and analogously

$$A \overset{\leftarrow}{\times} B = \lambda p.\, B\ (\lambda b.\, A\ (\lambda a.\, p\,(a,\, b)))$$

The usual equational reasoning does not help in proving $A \overset{\rightarrow}{\times} B = A \overset{\leftarrow}{\times} B$.

Problem 10 Is $\mathcal{P}^R_f$ symmetric whenever R is commutative?

15.7 Further properties

This section is a collection of some simple properties of the final construction.

Proposition 15.7.1 If R is discrete, then $\mathcal{P}^R_f\mathbf{X}$ is discrete for all domains $\mathbf{X}$.

Proof: $\mathcal{P}_f^R\mathbf{X}$ is $[[\mathbf{X} \to R] \overset{rlin}{\to} R]$ ordered pointwise, i.e. $A \leq B$ iff $Ap \leq Bp$ in R for all $p : [\mathbf{X} \to R]$. □

Proposition 15.7.2 The final construction $\mathcal{P}_f^R$ is faithful iff R is non-discrete.

Proof: If R is non-discrete, then $\mathcal{P}_f^R$ is faithful by Cor. 11.9.3. If R is discrete, then $\mathcal{P}_f^R\mathbf{X}$ is discrete even for non-discrete domains $\mathbf{X}$. Hence, embeddings from non-discrete domains $\mathbf{X}$ into $\mathcal{P}_f^R\mathbf{X}$ are impossible. □

Proposition 15.7.3 If R is finite, then $\mathcal{P}_f^R$ preserves FIN, F-ALG, and F-CONT.

Proof: If R and $\mathbf{X}$ are finite, then so is $[[\mathbf{X} \to R] \overset{rlin}{\to} R]$. According to Th. 7.4.4, $\mathcal{P}_f^R$ then maps finitely algebraic (continuous) domains into finitely algebraic (continuous) domains. □

Problem 11 If R and $\mathbf{X}$ are finitely algebraic (R not necessarily being finite), is $\mathcal{P}_f^R\mathbf{X}$ finitely algebraic?

Problem 12 If R and $\mathbf{X}$ are algebraic, is $\mathcal{P}_f^R\mathbf{X}$ algebraic?

Proposition 15.7.4 If R and $\mathbf{X}$ have least elements $\bot_R$ and $\bot_\mathbf{X}$, then $\mathcal{P}_f^R\mathbf{X}$ has a least element, namely $\bot_R \cdot \{\!|\bot_\mathbf{X}|\!\}$.

Proof: We have to show $Ap \geq (\bot_R \cdot \{\!|\bot_\mathbf{X}|\!\})\,p$ for all $A : [[\mathbf{X} \to R] \overset{rlin}{\to} R]$ and all $p : [\mathbf{X} \to R]$.

$$\begin{aligned} Ap &= A\,(\lambda x.\, p\,x) \;\geq\; A\,(\lambda x.\, p\,(\bot_\mathbf{X})) \\ &= A\,(\lambda x.\, 1 \cdot p\,(\bot_\mathbf{X})) \overset{\text{rlin}}{=} A\,(\lambda x.\, 1) \cdot p\,(\bot_\mathbf{X}) \\ &\geq \bot_R \cdot p\,(\bot_\mathbf{X}) \;=\; \bot_R \cdot \{\!|\bot_\mathbf{X}|\!\}\,p \;=\; (\bot_R \cdot \{\!|\bot_\mathbf{X}|\!\})\,p \qquad \Box \end{aligned}$$

Because of its definition in terms of existential quantification, one might believe that the existential restriction of a final construction for R is a final construction for R'. However this is not true as we shall see in section 23.4. There are two reasons for this. First, two distinct second order predicates in $[[\mathbf{X} \to R] \overset{rlin}{\to} R]$ may produce equal results for predicates in $[\mathbf{X} \to R']$. They are then still different in the restriction of the final construction for R, but equal in $[[\mathbf{X} \to R'] \overset{rlin}{\to} R']$. Second, there may be additional members in $[[\mathbf{X} \to R'] \overset{rlin}{\to} R']$ that cannot be obtained by restricting predicates in $[[\mathbf{X} \to R] \overset{rlin}{\to} R]$.

Despite of this general result, we also meet examples for semirings R and R' where the existential restriction of the final construction for R is final for R' — see Th. 23.3.1.

15.8 Unique extensions in final power constructions

Given a power construction $\mathcal{P}$ and two domains $\mathbf{X}$ and $\mathbf{Y}$, one is interested whether mappings $f : [\mathbf{X} \to \mathcal{P}\mathbf{Y}]$ have a unique linear extension $\overline{f} : [\mathcal{P}\mathbf{X} \to \mathcal{P}\mathbf{Y}]$. This means, whenever F_1 and F_2 are linear morphisms from $\mathcal{P}\mathbf{X}$ to $\mathcal{P}\mathbf{Y}$, does $F_1 \circ \iota = F_2 \circ \iota$ imply $F_1 = F_2$?

This useful property is not generally true, not even for final constructions. An example for this will be given in section 23.5. However, there is a nice criterion telling that the target domain $\mathbf{Y}$ does not matter; one may substitute $\mathbf{1}$ for it.

Proposition 15.8.1 Let R be an arbitrary semiring domain. For a fixed domain $\mathbf{X}$, the following statements are equivalent:

(1) For every domain $\mathbf{Y}$, $F_1 \circ \iota = F_2 \circ \iota$ implies $F_1 = F_2$ for all linear morphisms $F_1, F_2 : [\mathcal{P}_f^R\mathbf{X} \to \mathcal{P}_f^R\mathbf{Y}]$.

(2) $G_1 \circ \iota = G_2 \circ \iota$ implies $G_1 = G_2$ for all linear morphisms $G_1, G_2 : [\mathcal{P}_f^R\mathbf{X} \to \mathcal{P}_f^R\mathbf{1}]$.

In (2), $\mathcal{P}_f^R\mathbf{1}$ may be replaced by R using the isomorphisms φ and ψ.

Proof: Abbreviating $\mathcal{P}_f^R$ by $\mathcal{P}$, let F_1 and F_2 be two linear maps from $\mathcal{P}\mathbf{X}$ to $\mathcal{P}\mathbf{Y}$ with $F_1 \circ \iota = F_2 \circ \iota$. We have to show $F_1 = F_2$, i.e. $F_1 A\, p = F_2 A\, p$ for all A in $\mathcal{P}\mathbf{X}$ and $p : [\mathbf{Y} \to R]$. For every mapping $p : [\mathbf{Y} \to R]$, the composite $\varphi \circ p$ maps from $\mathbf{Y}$ to $\mathcal{P}\mathbf{1}$, whence its extension maps from $\mathcal{P}\mathbf{Y}$ to $\mathcal{P}\mathbf{1}$. Let $G_i = \mathit{ext}\,(\varphi \circ p) \circ F_i$. Then G_1 and G_2 are linear mappings from $\mathcal{P}\mathbf{X}$ to $\mathcal{P}\mathbf{1}$ with $G_1 \circ \iota = G_2 \circ \iota$. By (2), $G_1 = G_2$ holds, whence $G_1 A\, q = G_2 A\, q$ for all A in $\mathcal{P}\mathbf{X}$ and $q : [\mathbf{1} \to R]$. We use this equation for the special case $q = \lambda\diamond.\, 1$.

$$\begin{aligned}
G_i A\, q &= \mathit{ext}\,(\varphi \circ p)\,(F_i A)\, q \\
&= (F_i A)\,(\lambda x.\,(\varphi \circ p)\, x\, q) && \text{by the definition of } \mathit{ext} \\
&= F_i A\,(\lambda x.\,(\varphi\,(p\, x))\,(\lambda\diamond.\, 1)) \\
&= F_i A\,(\lambda x.\,\psi\,(\varphi\,(p\, x))) = F_i A p
\end{aligned}$$

Hence, $G_1 A\, q = G_2 A\, q$ implies $F_1 A\, p = F_2 A\, p$. □

15.9 The core of the final construction

Since the final construction need not have unique extensions, it moreover need not be reduced. In this section, we want to consider its core. Being reduced, this core has properties superior to those of the final construction itself. It is symmetric whenever the characteristic semiring is commutative, and it always has unique extensions. Algebraically, it may be characterized as follows:

Proposition 15.9.1
The core of the final construction is final among the reduced constructions.

Proof: Let $\mathcal{P}_f$ be the final construction for semiring R and $\mathcal{P}^c$ its core, and let $\mathcal{Q}$ be a reduced construction with semiring R. Since $\mathcal{P}_f$ is final, there is a linear power homomorphism $H : \mathcal{Q} \dot{\to} \mathcal{P}_f$. By Prop. 14.2.1, it cuts down to a linear power homomorphism $H^c : \mathcal{Q} = \mathcal{Q}^c \dot{\to} \mathcal{P}^c$. This linear homomorphism is unique by Prop. 14.3.5. □

In contrast to this, the initial construction is always reduced by Prop. 13.4.3. Thus, it is not only initial among all constructions, but also initial among the reduced ones.

In the final construction, all power operations and hence all derived operations are representable in terms of λ-expressions involving the semiring operations as constants. Hence, the operations of the final construction may be implemented in a functional language that only has to provide the semiring operations as primitives (usually, these are not sequential). If this implementation is done as an abstract data type that only allows the power operations and nothing else, then the core of the final construction is obtained. This is because the core cannot be left by the power operations including recursion.

Chapter 16

Products of power constructions

The previous chapters showed the existence of three distinguished power constructions for every semiring: the initial construction, the final construction, and the core of the final construction.[1] Now we are going to present a method that allows to produce a power construction $\prod \mathcal{P}_i$ with semiring $\prod R_i$ from given R_i-constructions $\mathcal{P}_i$.

The investigation of the product construction is prepared by investigating products of simpler but related algebraic structures such as modules and semirings in section 16.1. The product construction is then introduced in section 16.2. It is shown that it forms a categorical product in the category of power domain constructions and power homomorphisms. In section 16.3, we prove that the product of final constructions for semirings R_i is final for $\prod_{i \in I} R_i$.

In the remainder of the chapter, we restrict ourselves to binary products. Section 16.4 provides some theory for modules: an $R_1 \times R_2$-module is always isomorphic to a product of an R_1-module and an R_2-module, or more concisely: an $R_1 \times R_2$-module may always be factorized. This result is used to prove that core formation, reducedness, and initiality of power constructions commute over binary products (section 16.5). In section 16.6, we state and prove a necessary and sufficient criterion for the ability to factorize $R_1 \times R_2$-constructions. This criterion holds for large classes of power constructions, e.g. for reduced ones. The final section 16.7 then studies the factorization of linear power homomorphisms.

16.1 Products of algebraic structures

In this section, we consider the products of various algebraic structures we are concerned with: monoids, modules, and semirings. Products of power constructions are then considered in the next section.

Let $(M_i)_{i \in I}$ be a family of monoids, and let $M = \prod_{i \in I} M_i = \{(m_i)_{i \in I} \mid m_i \in M_i \ \forall i \in I\}$ be its Cartesian product. M may be given a monoid structure by defining $0 = (0_i)_{i \in I}$ and $(a_i)_{i \in I} + (b_i)_{i \in I} = (a_i + b_i)_{i \in I}$. This operation is continuous by Prop. 3.3.5. The algebraic

[1] For particular semirings, two or three of these constructions may well coincide.

laws of associativity and neutrality are obviously satisfied. M is commutative / idempotent iff all M_i are so.

For each $k \in I$, there is an out-going map $\pi_k : M \to M_k$ given by projection onto dimension k, i.e. $\pi_k((m_i)_{i\in I}) = m_k$. All these mappings are continuous and additive. For each $k \in I$, there is also an in-going map $\eta_k : M_k \to M$ defined by $(\eta_k m)_i = \begin{cases} m & \text{if } i = k \\ 0 & \text{otherwise} \end{cases}$ These mappings are also continuous and additive.

Turning to semirings, let R be the Cartesian product of the family $(R_i)_{i\in I}$. By the definitions $1 = (1_i)_{i\in I}$ and $(a_i)_{i\in I} \cdot (b_i)_{i\in I} = (a_i \cdot b_i)_{i\in I}$, R becomes a semiring domain. The projections π_k are semiring homomorphisms whereas the injections η_k are not. This is because $\eta_k(1_k) \neq 1$; the remaining requirements hold: $\eta_k(0_k) = 0$, $\eta_k(a+b) = \eta_k(a) + \eta_k(b)$, and $\eta_k(a \cdot b) = \eta_k(a) \cdot \eta_k(b)$.

Finally, let $(R_i)_{i\in I}$ be a family of semirings and $(M_i)_{i\in I}$ a family of commutative monoids such that for all i in I, M_i is an R_i-module. Let R be the product semiring and M the product monoid. M becomes an R-module by defining $(r_i)_{i\in I} \cdot (m_i)_{i\in I} = (r_i \cdot m_i)_{i\in I}$. The 'small' modules M_k also become R-modules by the definition $r \cdot m = \pi_k(r) \cdot m$ for $r \in R$ and $m \in M_k$. With this definition, both the projection $\pi_k : M \to M_k$ and the injection $\eta_k : M_k \to M$ become R-linear. For the injection, this is proved by

$$\eta_k(r \cdot m) \;=\; \begin{pmatrix} r_k \cdot m \\ 0_i \end{pmatrix} \;=\; \begin{pmatrix} r_k \cdot m \\ r_i \cdot 0_i \end{pmatrix} \;=\; r \cdot \begin{pmatrix} m \\ 0_i \end{pmatrix} \;=\; r \cdot \eta_k m$$

where the upper line is the component in dimension k whereas the lower line stands for all other components $i \neq k$.

All these products are also products in a categorical sense. For morphisms $f_i : [\mathbf{X} \to M_i]$, there is a unique $f : [\mathbf{X} \to M]$ such that $\pi_i \circ f = f_i$ for all i in I. This function f is additive / linear / a semiring homomorphism iff the functions f_i are so for all i in I.

16.2 The product power construction

After the preliminaries about semirings and modules, we now turn to the power constructions themselves. Let $(\mathcal{P}_i)_{i\in I}$ be a family of power constructions. Then define $\mathcal{P} = \prod_{i\in I} \mathcal{P}_i$ by $\mathcal{P}\mathbf{X} = \prod_{i\in I} \mathcal{P}_i\mathbf{X}$ for all ground domains $\mathbf{X}$. The power operations in $\mathcal{P}$ are defined 'pointwise':

- $\theta = (\theta_i)_{i\in I}$
- $(A_i)_{i\in I} \uplus (B_i)_{i\in I} = (A_i \uplus B_i)_{i\in I}$
- $\{\!|x|\!\} = (\{\!|x|\!\}_i)_{i\in I}$ for all x in $\mathbf{X}$
- For $f : [\mathbf{X} \to \mathcal{P}\mathbf{Y}]$ let $f_i = \pi_i \circ f$. Then $ext\, f\, (A_i)_{i\in I} = (ext_i\, f_i\, A_i)_{i\in I}$ where ext_i denotes the extension functional of $\mathcal{P}_i$.

The verification of the power axioms for $\mathcal{P}$ is straightforward. For instance, (S4) is shown as follows:

$$\begin{aligned} ext\, f\,(ext\, g\, A) &= (ext_i\, f_i\, \pi_i\,(ext\, g\, A))_{i\in I} \\ &= (ext_i\, f_i\,(ext_i\, g_i\, A_i))_{i\in I} \\ &= (ext_i\,(\lambda x.\, ext_i\, f_i\,(g_i x))A_i)_{i\in I} \\ &= (ext_i\,(\lambda x.\, \pi_i(ext\, f\,(gx)))A_i)_{i\in I} \\ &= ext\,(\lambda x.\, ext\, f\,(gx))\, A \end{aligned}$$

The definition of extension directly implies $\pi_i \circ ext\, f = ext_i\,(\pi_i \circ f) \circ \pi_i$. Together with simple conclusions from the other definitions, one obtains a power homomorphism $\pi_k : \mathcal{P} \dot{\rightarrow} \mathcal{P}_k$ for every k in I. In contrast to this, the injections η_k do not create a power homomorphism since $\eta_k \{\!|x|\!\}_k$ has component θ in all dimensions except k, whereas $\{\!|x|\!\}$ has component $\{\!|x|\!\}_i$ in every dimension i.

Let $H_i : \mathcal{Q} \dot{\rightarrow} \mathcal{P}_i$ be a family of power homomorphisms. Then define H by $H(A_i)_{i\in I} = (H_i A_i)_{i\in I}$. Then $H_i = \pi_i \circ H$ holds. H is a power homomorphism:

- $H\,\theta_{\mathcal{Q}} = (H_i \theta_{\mathcal{Q}})_{i\in I} = (\theta_i)_{i\in I} = \theta_{\mathcal{P}}$
- $H(A \uplus B) = (H_i(A_i \uplus B_i))_{i\in I} = (H_i A \uplus H_i B)_{i\in I}$ etc.
- $H\{\!|x|\!\}_{\mathcal{Q}} = (H_i\{\!|x|\!\}_{\mathcal{Q}})_{i\in I} = (\{\!|x|\!\}_i)_{i\in I} = \{\!|x|\!\}_{\mathcal{P}}$
- $H(ext_{\mathcal{Q}}\, f\, A) = (H_i(ext_{\mathcal{Q}}\, f\, A))_{i\in I} = (ext_i(H_i \circ f)(H_i A))_{i\in I} = ext_{\mathcal{P}}(H \circ f)(HA)$ since $H_i \circ f = \pi_i \circ H \circ f$.

H is unique with $\pi_i \circ H = H_i$. If H' is another one, then $\pi_i(H'A) = H_i A = \pi_i(HA)$ holds for all i in I and A in $\mathcal{Q}\mathbf{X}$, whence $H' = H$.

Now, we determine the characteristic semiring of $\mathcal{P}$. Let $R_i = \mathcal{P}_i\mathbf{1}$ be the characteristic semirings of the factors. Then the characteristic semiring of $\mathcal{P}$ has carrier $R = \prod_{i\in I} R_i$. The operations in R are defined by

- $0 = \theta = (\theta_i)_{i\in I} = (0_i)_{i\in I}$
- $a + b = a \uplus b = (a_i \uplus b_i)_{i\in I} = (a_i + b_i)_{i\in I}$
- $1 = \{\!|\diamond|\!\} = (\{\!|\diamond|\!\}_i)_{i\in I} = (1_i)_{i\in I}$
- $a \cdot b = ext\,(\lambda\diamond.\, b)\, a = (ext_i\,(\lambda\diamond.\, b_i)\, a_i)_{i\in I} = (a_i \cdot b_i)_{i\in I}$

Hence, the characteristic semiring of $\mathcal{P}$ is the product of the semirings of $\mathcal{P}_i$ in the sense of the previous section. Similarly, one shows that the R-module $\mathcal{P}\mathbf{X}$ is the product of the R_i-modules $\mathcal{P}_i\mathbf{X}$.

Summarizing, we obtain the following theorem:

Theorem 16.2.1 For every family $(\mathcal{P}_i)_{i\in I}$ of power constructions with semirings R_i, there is a product construction $\mathcal{P} = \prod_{i\in I} \mathcal{P}_i$ with semiring $R = \prod_{i\in I} R_i$. For every k in I, there is a power homomorphism $\pi_k : \mathcal{P} \dot{\rightarrow} \mathcal{P}_k$. For every family of power homomorphisms $H_i : \mathcal{Q} \rightarrow \mathcal{P}_i$, there is a unique $H : \mathcal{Q} \dot{\rightarrow} \mathcal{P}$ with $\pi_k \circ H = H_k$ for all k in I.

Since all power operations and hence all derived operations are defined componentwise, the following facts become obvious:

Proposition 16.2.2

(1) The product $\prod_{i \in I} \mathcal{P}_i$ is commutative iff all factors are commutative.

(2) The product is symmetric iff all factors are.

(3) If at least one factor is faithful, then the whole product is faithful.

The remainder of the chapter is concerned with more advanced questions, e.g. is the product of reduced / initial / final constructions reduced / initial / final again? Is the core of the product the product of the cores? Is it possible to factorize any power construction with a product semiring? The first question to attack is that of finality.

16.3 Product and finality

This section is concerned with final power constructions. The goal is to prove that the product of final constructions $\mathcal{P}_i$ for semirings R_i is final for the product semiring $R = \prod_{i \in I} R_i$. For this, let $\mathcal{P} = \prod_{i \in I} \mathcal{P}_i$ and let $\mathcal{Q}$ be the final construction for R, i.e. $\mathcal{Q}\mathbf{X} = [[\mathbf{X} \to R] \stackrel{rlin}{\to} R]$. Since $\mathcal{Q}$ is final, there is a (unique) linear power homomorphism $\mathcal{E} : \mathcal{P} \dot{\to} \mathcal{Q}$ defined by $\mathcal{E}A = \lambda p.\, ext\, p\, A$ for A in $\mathcal{P}\mathbf{X}$. Hence, $\mathcal{E}A = \lambda p.\, (ext_i\, p_i\, A_i)_{i \in I} = \lambda p.\, (\mathcal{E}_i\, A_i\, p_i)_{i \in I}$ where $\mathcal{E}_i$ is the unique linear power homomorphism from $\mathcal{P}_i$ to itself, i.e. is the identity. Thus, $\mathcal{E}A = \lambda p.\, (A_i\, p_i)_{i \in I}$. We have to show that $\mathcal{E}$ is a power isomorphism. By Prop. 12.2.1, it suffices to show that $\mathcal{E}$ is a surjective embedding.

Assume $\mathcal{E}A \le \mathcal{E}B$ holds for $A, B \in \mathcal{P}\mathbf{X}$. For all $k \in I$ and all $q : [\mathbf{X} \to R_k]$, let $p = \eta_k \circ q : [\mathbf{X} \to R]$. $\mathcal{E}A \le \mathcal{E}B$ implies $(A_i\, p_i)_{i \in I} = \mathcal{E}Ap \le \mathcal{E}B\, p = (B_i\, p_i)_{i \in I}$. This in particular holds for dimension k. Thus, $A_k\, q \le B_k\, q$ holds for all k in I and $q : [\mathbf{X} \to R_k]$, whence $A_k \le B_k$ for all k, whence $A \le B$.

For surjectivity, let $P : [[\mathbf{X} \to R] \stackrel{rlin}{\to} R]$. Then let $P_i : [[\mathbf{X} \to R_i] \stackrel{rlin}{\to} R_i]$ be defined by $P_i = \lambda q.\, \pi_i\, (P\, (\eta_i \circ q))$. This mapping obviously is additive. Right linearity is shown by the following derivation where $r \in R_i$:

$$\pi_i\, (P\, (\eta_i \circ (q \cdot r))) = \pi_i\, (P\, ((\eta_i \circ q) \cdot (\eta_i\, r))) = \pi_i\, (P\, (\eta_i \circ q)) \cdot \pi_i\, (\eta_i\, r) = \pi_i\, (P\, (\eta_i \circ q)) \cdot r$$

Thus, $A = (P_i)_{i \in I}$ is a member of $\mathcal{P}\mathbf{X}$. We claim $\mathcal{E}A = P$.
$\mathcal{E}A\, p = (P_i\, p_i)_{i \in I} = (\pi_i\, (P\, (\eta_i \circ p_i)))_{i \in I}$ holds where $p_i = \pi_i \circ p$. Note that $\eta_i(\pi_i r) = r \cdot \eta_i\, 1_i$ holds for all r in R. Thus,

$$\begin{aligned} \pi_i\, (P\, (\eta_i \circ \pi_i \circ p)) &= \pi_i\, (P\, (p \cdot \eta_i\, 1_i)) = \pi_i\, (Pp \cdot \eta_i\, 1_i) \\ &= \pi_i\, (Pp) \cdot \pi_i\, (\eta_i\, 1_i) = \pi_i\, (Pp) \cdot 1_i = \pi_i\, (Pp) \end{aligned}$$

whence $(\pi_i\, (P\, (\eta_i \circ p_i)))_{i \in I} = Pp$.

Up to now, we proved the following theorem:

Theorem 16.3.1 If $\mathcal{P}_i$ are final power constructions for the semirings R_i for all $i \in I$, then the product $\prod_{i \in I} \mathcal{P}_i$ is a final construction for the product $\prod_{i \in I} R_i$.

16.4 Factorization of modules

Before attacking the remaining questions, we have to consider the product of module domains a bit closer. Thereby, we concentrate on finite products; most of the following results do not hold or are open for infinite ones. Since binary products may be iterated to obtain any desired finite product, we moreover restrict ourselves to products of just two objects.

Let R_1 and R_2 be two semirings with product $R = R_1 \times R_2$. The projections are given in this special case by $\pi_i(r_1, r_2) = r_i$, and the injections by $\eta_1 r = (r, 0)$ and $\eta_2 r = (0, r)$. If it seems appropriate, we also write $(r_i)_{i \in I}$ instead of (r_1, r_2) where I is understood as $\{1, 2\}$.

Given an R_1-module M_1 and an R_2-module M_2, the product $M_1 \times M_2$ is an R-module. We now investigate the opposite direction: given an R-module M, is it possible to find R_i-modules M_i such that M is isomorphic to $M_1 \times M_2$?

The product $R = R_1 \times R_2$ contains two particularly interesting elements: $e_1 = (1, 0)$ and $e_2 = (0, 1)$. They commute with every other semiring element and enjoy the properties $e_1 \cdot e_1 = e_1$, $e_2 \cdot e_2 = e_2$, $e_1 \cdot e_2 = 0$, and $e_1 + e_2 = 1$.

Proposition 16.4.1
Let M be an R-module. For m in M, the following statements are equivalent:

(1) There is an x in M such that $e_1 \cdot x = m$.

(2) $e_1 \cdot m = m$

(3) $e_2 \cdot m = 0$

Proof:

$(1) \Rightarrow (3):\ e_2 \cdot m = e_2 \cdot e_1 \cdot x = 0 \cdot x = 0$

$(3) \Rightarrow (2):\ m = 1 \cdot m = (e_1 + e_2) \cdot m = e_1 \cdot m + 0$

$(2) \Rightarrow (1):$ Let $x = m$. □

After these preliminaries, we define

- $M_1 = \{e_1 \cdot x \mid x \in M\} = \{m \in M \mid e_1 \cdot m = m\} = \{m \in M \mid e_2 \cdot m = 0\}$

and analogously M_2.

Proposition 16.4.2 If M_1 is equipped with the order and the operations of M, then it is an R-module domain. By the definition $r * m = \eta_1 r \cdot m$ for $r \in R_1$ and $m \in M_1$, it also becomes an R_1-module.

Proof:
For the first part, we have to show that M_1 is closed w.r.t. 0, '+', '·', and directed limits.

- $e_2 \cdot 0 = 0$, whence $0 \in M_1$
- If $a, b \in M_1$, then $e_2 \cdot (a + b) = e_2 \cdot a + e_2 \cdot b = 0 + 0 = 0$
- If $r \in R$ and $m \in M_1$, then $e_2 \cdot (r \cdot m) = r \cdot (e_2 \cdot m) = r \cdot 0 = 0$
- If D is a directed set in M_1, then $e_2 \cdot (\bigsqcup D) = \bigsqcup_{d \in D}(e_2 \cdot d) = \bigsqcup_{d \in D} 0 = 0$

The necessary algebraic properties are inherited from M.

To demonstrate that M_1 is an R_1-module, we only have to verify the module axioms. Since η_1 is additive and multiplicative, the only problem is with the axiom $1_1 * m = m$. It holds because $1_1 * m = \eta_1 1_1 \cdot m = e_1 \cdot m = m$ holds for all $m \in M_1$. □

Since both M_i are R_i-modules, their product $M_1 \times M_2$ becomes an R-module with multiplication defined as in section 16.1. This multiplication looks like

$$\begin{aligned}
(r_1, r_2) * (m_1, m_2) &= (r_1 * m_1, r_2 * m_2) \\
&= (\eta_1 r_1 \cdot m_1, \eta_2 r_2 \cdot m_2) \\
&= ((r_1, 0) \cdot m_1, (0, r_2) \cdot m_2) \qquad \text{and by } (0, r_2) = (0, r_2) \cdot e_2 \\
&= ((r_1, 0) \cdot m_1 + (0, r_2) \cdot m_1, (r_1, 0) \cdot m_2 + (0, r_2) \cdot m_2) \\
&= ((r_1, r_2) \cdot m_1, (r_1, r_2) \cdot m_2)
\end{aligned}$$

whence $r * (m_1, m_2) = (r \cdot m_1, r \cdot m_2)$.

Next, we claim that M and $M_1 \times M_2$ are isomorphic R-modules. Let $\alpha : [M \to M_1 \times M_2]$ be defined by $\alpha x = (e_1 \cdot x, e_2 \cdot x)$. By definition of M_i, the result really is in $M_1 \times M_2$, and the mapping obviously is continuous. Additivity is simple. For linearity note that

$$\alpha(r \cdot x) = (e_1 \cdot r \cdot x, e_2 \cdot r \cdot x) = (r \cdot e_1 \cdot x, r \cdot e_2 \cdot x) = r * (e_1 \cdot x, e_2 \cdot x) = r * \alpha x$$

using that both e_i commute over all semiring elements.

In the opposite direction, let $\beta : [M_1 \times M_2 \to M]$ be defined by $\beta\,(m_1, m_2) = m_1 + m_2$. It is continuous because of the continuity of '+'. For linearity, the commutativity of addition in modules is needed:

- $\beta\,(x_1 + y_1, x_2 + y_2) = x_1 + y_1 + x_2 + y_2 = x_1 + x_2 + y_1 + y_2 = \beta\,(x_1, x_2) + \beta\,(y_1, y_2)$
- $\beta\,(r * (m_1, m_2)) = \beta\,(r \cdot m_1, r \cdot m_2) = r \cdot m_1 + r \cdot m_2 = r \cdot \beta\,(m_1, m_2)$

α and β are inverses of each other:

- $\beta\,(\alpha x) = \beta\,(e_1 \cdot x, e_2 \cdot x) = e_1 \cdot x + e_2 \cdot x = (e_1 + e_2) \cdot x = 1 \cdot x = x$
- $\alpha\,(\beta(m_1, m_2)) = \alpha\,(m_1 + m_2) = (e_1 \cdot (m_1 + m_2), e_2 \cdot (m_1 + m_2))$
  $= (e_1 \cdot m_1 + e_1 \cdot m_2, e_2 \cdot m_1 + e_2 \cdot m_2) = (m_1, m_2)$

Summarizing all results, we proved the following theorem:

Theorem 16.4.3

Every $(R_1 \times R_2)$-module may be factorized into an R_1-module and an R_2-module.

Or more exactly: For every $(R_1 \times R_2)$-module M, there is an R_1-module M_1 and an R_2-module M_2 such that M and $M_1 \times M_2$ are isomorphic R-modules.

This factorization is unique up to isomorphism.

The last statement is proved below.

From modules, we now turn to the corresponding arrows, i.e. to linear morphisms. Let M_1 and M_1' be two R_1-modules, and let M_2 and M_2' be two R_2-modules. Then the products $M = M_1 \times M_2$ and $M' = M_1' \times M_2'$ are R-modules. A pair of R_i-linear mappings $f_i : [M_i \to M_i']$

may be combined to a R-linear $f : [M \to M']$ defined by $f\,(x,\, y) = (f_1\, x,\, f_2\, x)$. We denote f by $f_1 \times f_2$.

Conversely, let $f : [M \to M']$ be a linear morphism. Then $f\,(x,\,0) = f\,((1,\,0)\cdot(x,\,0)) = (1,\,0)\cdot f\,(x,\,0)$ holds, i.e. the M'_2-component of $f\,(x,\,0)$ is 0, i.e. there is a function f_1 such that $f\,(x,\,0) = (f_1 x,\,0)$. f_1 may explicitly built by composition: $f_1 = \pi_1 \circ f \circ \eta_1$. Hence, it is continuous. Additivity is also immediate. f_1 is R_1-linear because of

$$f_1\,(r_1\cdot x) = \pi_1\,(f\,(r_1\cdot x,\,0)) = \pi_1\,(f\,((r_1,\,0)\cdot(x,\,0))) = \pi_1\,((r_1,\,0)\cdot(f_1 x,\,0)) = r_1\cdot f_1 x$$

Obviously, an R_2-linear function f_2 with $f\,(0,\,y) = (0,\,f_2 y)$ also exists.

Finally, $f_1 \times f_2$ is f because of

$$f\,(x,\,y) = f\,((x,\,0)+(0,\,y)) = f\,(x,\,0) + f\,(0,\,y) = (f_1 x,\,0) + (0,\,f_2 y) = (f_1 x,\,f_2 y)$$

It is not difficult to see that $f_1 \times f_2$ is an isomorphism iff f_1 and f_2 are. This proves the uniqueness in the theorem above: if $M_1 \times M_2$ is isomorphic to $M'_1 \times M'_2$ by $f_1 \times f_2$, then M_i and M'_i are isomorphic by f_i.

Summarizing, we obtain:

Theorem 16.4.4 Let M_1 and M'_1 be R_1-modules and M_2 and M'_2 R_2-modules. Then for every pair of R_i-linear functions $f_i : [M_i \to M'_i]$, there is an R-linear function $f_1 \times f_2 : [M_1 \times M_2 \to M'_1 \times M'_2]$ defined by $(f_1 \times f_2)\,(x,\,y) = (f_1 x,\,f_2 y)$. Conversely, for every R-linear $f : [M_1 \times M_2 \to M'_1 \times M'_2]$, there are R_i-linear functions $f_i : [M_i \to M'_i]$ such that $f = f_1 \times f_2$.

16.5 Product, core, and initiality

This section is concerned with the relations between finite products, core formation, and initiality. The first result is

Theorem 16.5.1 The core of the product of two power constructions equals the product of their cores: $(\mathcal{P}_1 \times \mathcal{P}_2)^c = \mathcal{P}_1^c \times \mathcal{P}_2^c$

Proof: For every ground domain $\mathbf{X}$, $(\mathcal{P}_1^c\mathbf{X} \times \mathcal{P}_2^c\mathbf{X},\, \{\!|.|\!\})$ is an R-$\mathbf{X}$-submodule of $\mathcal{P}_1\mathbf{X} \times \mathcal{P}_2\mathbf{X} = (\mathcal{P}_1 \times \mathcal{P}_2)\mathbf{X}$. Since $C = (\mathcal{P}_1 \times \mathcal{P}_2)^c\mathbf{X}$ is the least such submodule, $C \subseteq \mathcal{P}_1^c\mathbf{X} \times \mathcal{P}_2^c\mathbf{X}$ immediately follows.

Now let $S_1 = \{A \subseteq \mathcal{P}_1\mathbf{X} \mid (A,\,\theta_2) \in C\}$. We show that S_1 is an R-$\mathbf{X}$-submodule of $(\mathcal{P}_1\mathbf{X},\, \{\!|.|\!\}_1)$.

(1) $\{\!|x|\!\}_1 \in S_1$, since $(\{\!|x|\!\}_1,\,\theta_2) = (1_1,\,0_2)\cdot(\{\!|x|\!\}_1,\,\{\!|x|\!\}_2) = e_1\cdot\{\!|x|\!\} \in C$.

(2) $\theta_1 \in S_1$, since $(\theta_1,\,\theta_2) = \theta \in C$.

(3) If $A, B \in S_1$, then $(A \uplus B,\,\theta_2) = (A,\,\theta_2) \uplus (B,\,\theta_2) \in C$.

(4) If $r_1 \in R_1$ and $A \in S_1$, then $(r_1\cdot A,\,\theta_2) = (r_1,\,0_2)\cdot(A,\,\theta_2) \in C$.

(5) If D is a directed set in S_1, then $(\bigsqcup D,\,\theta_2) = \bigsqcup_{d\in D}(d,\,\theta_2) \in C$.

Since $\mathcal{P}_1^c\mathbf{X}$ is the least such sub-module, $\mathcal{P}_1^c\mathbf{X} \subseteq S_1$ follows. Similarly, $\mathcal{P}_2^c\mathbf{X} \subseteq S_2$ holds for an analogous set S_2.

Finally, let (A_1, A_2) be in $\mathcal{P}_1^c\mathbf{X} \times \mathcal{P}_2^c\mathbf{X}$. Then, $(A_1, A_2) \in S_1 \times S_2$, whence both (A_1, θ_2) and (θ_1, A_2) are in C. This implies $(A_1, A_2) = (A_1, \theta_2) \uplus (\theta_1, A_2)$ in C. Thus, $\mathcal{P}_1^c\mathbf{X} \times \mathcal{P}_2^c\mathbf{X} \subseteq C$.□

From this theorem, we obtain the following corollary:

Theorem 16.5.2 The product $\mathcal{P}_1 \times \mathcal{P}_2$ is reduced iff both factors $\mathcal{P}_1$ and $\mathcal{P}_2$ are reduced.

Proof: If both $\mathcal{P}_i$ are reduced, then $(\mathcal{P}_1 \times \mathcal{P}_2)^c = \mathcal{P}_1^c \times \mathcal{P}_2^c = \mathcal{P}_1 \times \mathcal{P}_2$, whence the product is reduced.

Conversely, assume $\mathcal{P}_1 \times \mathcal{P}_2$ is reduced. For all $\mathbf{X}$, $\mathcal{P}_1^c\mathbf{X} \times \mathcal{P}_2\mathbf{X}$ is an R-$\mathbf{X}$-submodule of $\mathcal{P}_1\mathbf{X} \times \mathcal{P}_2\mathbf{X}$. Since the product is reduced, $\mathcal{P}_1^c\mathbf{X} \times \mathcal{P}_2\mathbf{X} = \mathcal{P}_1\mathbf{X} \times \mathcal{P}_2\mathbf{X}$ follows, whence $\mathcal{P}_1^c = \mathcal{P}_1$. □

This section is concluded by investigating initiality.

Theorem 16.5.3
If $\mathcal{P}_1$ is initial for R_1 and $\mathcal{P}_2$ initial for R_2, then $\mathcal{P}_1 \times \mathcal{P}_2$ is initial for $R_1 \times R_2$.

Proof: By the results of chapter 14.5, we have to show that $\mathcal{P}_1\mathbf{X} \times \mathcal{P}_2\mathbf{X}$ is a free R-$\mathbf{X}$-module where $R = R_1 \times R_2$. Let $\mathcal{P} = \mathcal{P}_1 \times \mathcal{P}_2$.

Let M be an R-module, and $f : [\mathbf{X} \to M]$ a morphism. By Th. 16.4.3, there are R_i-modules M_i such that M is isomorphic to $M_1 \times M_2$. (In the following, we do not explicitly write down these isomorphisms.) By projection, f splits into morphisms $f_i : [\mathbf{X} \to M_i]$. Since $\mathcal{P}_i\mathbf{X}$ is a free R_i-$\mathbf{X}$-module, there are R_i-linear mappings $F_i : [\mathcal{P}_i\mathbf{X} \to M_i]$ with $F_i \circ \iota_i = f_i$. These mappings may be combined to an R-linear map $F : [\mathcal{P}\mathbf{X} \to M]$. F is an extension of f because of $F\{\!|x|\!\} = (F_i\{\!|x|\!\}_i)_{i\in I} = (f_i x)_{i\in I} = fx$.

Initial constructions are reduced. By Th. 16.5.2, $\mathcal{P}$ is also reduced, whence F is unique by Prop. 14.3.2. □

16.6 Factorization of power constructions

In section 16.4, we saw that every $(R_1 \times R_2)$-module and every $(R_1 \times R_2)$-linear map uniquely factorize into R_i-modules resp. R_i-linear maps. We now investigate to what extent these results hold for power constructions and linear power homomorphisms.

Let $R = R_1 \times R_2$ be a product semiring and let $\mathcal{P}$ be a power construction for R. For every $\mathbf{X}$, $\mathcal{P}\mathbf{X}$ is an R-module. By Th. 16.4.3, it may uniquely be factorized into an R_1-module $\mathcal{P}_1\mathbf{X}$ and an R_2-module $\mathcal{P}_2\mathbf{X}$. These modules are given by $\mathcal{P}_i\mathbf{X} = \{e_i \cdot A \mid A \in \mathcal{P}\mathbf{X}\}$. Members of $\mathcal{P}_1\mathbf{X}$ are equivalently characterized by $e_1 \cdot A = A$ or $e_2 \cdot A = \theta$ by Prop. 16.4.1. $\mathcal{P}\mathbf{X}$ and $\mathcal{P}_1\mathbf{X} \times \mathcal{P}_2\mathbf{X}$ are isomorphic by means of $\alpha A = (e_1 \cdot A, e_2 \cdot A)$ and $\beta(A_1, A_2) = A_1 \uplus A_2$. The question now is whether $\mathcal{P}_i$ are power constructions and whether α and β are inverse power isomorphisms.

Let us check whether $\mathcal{P}_1$ is a power construction. For all $\mathbf{X}$, $\mathcal{P}_1\mathbf{X}$ is a domain equipped with θ and '$\uplus$'. We define $\{\!|x|\!\}_1 = e_1 \cdot \{\!|x|\!\}$. For $f : [\mathbf{X} \to \mathcal{P}_1\mathbf{Y}]$ and A in $\mathcal{P}_1\mathbf{X}$, we simply define $ext_1\, f\, A = ext\, f\, A$. The result is indeed in $\mathcal{P}_1\mathbf{Y}$, since

$$e_1 \cdot ext\, f\, A = ext\, f\,(e_1 \cdot A) = ext\, f\, A$$

Thus, $\mathcal{P}_1$ inherits all power operations of $\mathcal{P}$ except singleton that has a new definition. Hence, all power axioms that do not involve singleton are valid. The only critical axioms are (P3) and (S3).

- $ext\, f\, \{\!|x|\!\}_1 = ext\, f\,(e_1 \cdot \{\!|x|\!\}) = e_1 \cdot ext\, f\, \{\!|x|\!\} = e_1 \cdot f x = f x$
  since fx is in $\mathcal{P}_1\mathbf{X}$.
- $ext\,(\lambda x.\, \{\!|x|\!\}_1)\, A = ext\,(\lambda x.\, e_1 \cdot \{\!|x|\!\})\, A = A \cdot e_1$
  where the last equation is given by the definition of the external product from the right in section 10.10.

  Unfortunately, the factor appears on the wrong side such that one cannot simplify to the desired result A. Remember that $A = e_1 \cdot B$ holds for some $B \in \mathcal{P}\mathbf{X}$ since A is in $\mathcal{P}_1\mathbf{X}$. Thus, one would need the equation $e_1 \cdot B \cdot e_1 = e_1 \cdot B$ for all B in $\mathcal{P}\mathbf{X}$.

Nevertheless, we now check whether α implies a power homomorphism. If so, then α and β are inverse power isomorphisms by Prop. 12.2.1. We know α is linear, whence it respects θ and '$\uplus$'. Singleton and extension remain to be checked.

- $\alpha\{\!|x|\!\} = (e_1 \cdot \{\!|x|\!\},\, e_2 \cdot \{\!|x|\!\}) = (\{\!|x|\!\}_1,\, \{\!|x|\!\}_2)$
  This is the desired outcome.
- $\alpha\,(ext\, f\, A) = (e_i \cdot ext\, f\, A)_{i \in I}$
  On the other hand,
  $ext_{\mathcal{P}_1 \times \mathcal{P}_2}\,(\alpha \circ f)\,(\alpha A) = (ext_i\,(e_i \cdot f)\,(e_i \cdot A))_{i \in I} = (e_i \cdot ext\,(e_i \cdot f)\, A)_{i \in I}$
  As with (S3), there is a problem here: the factor e_i at the function f appears on the wrong side. Left factors at the functional argument of ext cannot generally drawn out, only right factors. A slight generalization of the property needed to prove (S3) helps here, namely $e_i \cdot B = B \cdot e_i$ for all $B \in \mathcal{P}\mathbf{X}$. This equation implies $e_i \cdot B \cdot e_i = e_i \cdot e_i \cdot B = e_i \cdot B$, and also
  $(e_i \cdot ext\,(e_i \cdot f)\, A)_{i \in I} = (e_i \cdot ext\,(f \cdot e_i)\, A)_{i \in I} = (e_i \cdot ext\, f\, A \cdot e_i)_{i \in I} = (e_i \cdot ext\, f\, A)_{i \in I}$
  This is the desired result.

Before formulating the concluding theorem, we consider the additional conditions we need a bit closer. It seems as if we needed both $e_1 \cdot B = B \cdot e_1$ and $e_2 \cdot B = B \cdot e_2$. However, it suffices to explicitly demand only one of these because the other one then holds automatically:

For B in $\mathcal{P}\mathbf{X}$, $e_1 \cdot B = B \cdot e_1$ and $e_2 \cdot B = B \cdot e_2$ are equivalent.

Proof: $e_2 \cdot B = e_2 \cdot B \cdot 1 = e_2 \cdot B \cdot (e_1 + e_2) = e_2 \cdot e_1 \cdot B + e_2 \cdot B \cdot e_2 = e_2 \cdot B \cdot e_2 = B \cdot e_1 \cdot e_2 + e_2 \cdot B \cdot e_2 = (e_1 + e_2) \cdot B \cdot e_2 = B \cdot e_2$ □

Hence, only $e_1 \cdot B = B \cdot e_1$ has to hold to allow factorization. Conversely, if $\mathcal{P}$ may be factorized, then in $\mathcal{P}_1\mathbf{X} \times \mathcal{P}_2\mathbf{X}$, both left and right multiplication operate componentwise, whence

$$e_1 \cdot B = (1,\, 0) \cdot (B_1,\, B_2) = (B_1,\, \theta_2) = (B_1,\, B_2) \cdot (1,\, 0) = B \cdot e_1$$

A linear power homomorphism is also right linear, whence this equation carries over to $\mathcal{P}$. Thus, we obtain the following

Theorem 16.6.1 Let $\mathcal{P}$ be a power domain construction for semiring $R_1 \times R_2$. There are power constructions $\mathcal{P}_1$ for R_1 and $\mathcal{P}_2$ for R_2 such that $\mathcal{P}$ is isomorphic to $\mathcal{P}_1 \times \mathcal{P}_2$ by a linear power isomorphism, if and only if $(1, 0) \cdot B = B \cdot (1, 0)$ holds for all B in all power domains $\mathcal{P}\mathbf{X}$.

Or in short terms: a power construction for $R_1 \times R_2$ can be factorized if and only if $(1, 0)$ commutes over all its formal sets.

It might well be possible that all power constructions for $R_1 \times R_2$ satisfy the criterion and hence may be factorized. However we cannot prove this such that it is an open **problem**. At least, we are able to conclude some interesting corollaries:

Corollary 16.6.2 Every commutative power construction $\mathcal{P}$ for $R_1 \times R_2$ can be factorized. The resulting factors are commutative again. If $\mathcal{P}$ is even symmetric then the factors are symmetric too.

Proof: The first statement is immediate. The remaining ones hold by Prop. 16.2.2. □

Corollary 16.6.3
If $R = R_1 \times R_2$ is additive, then every power construction for R can be factorized.

Proof: By Prop. 11.12.2, every power construction for an additive semiring is commutative.□

Corollary 16.6.4 Every reduced power construction for $R_1 \times R_2$ can be factorized. The resulting factors are reduced again.

Proof: Consider the two functions $F_L, F_R : [\mathcal{P}\mathbf{X} \to \mathcal{P}\mathbf{X}]$ defined by $F_L A = e_1 \cdot A$ and $F_R A = A \cdot e_1$. Both functions obviously are additive. F_R is linear:

$$F_R(r \cdot A) = (r \cdot A) \cdot e_1 = r \cdot (A \cdot e_1) = r \cdot F_R A$$

F_L is linear since e_1 commutes over all semiring elements:

$$F_L(r \cdot A) = e_1 \cdot r \cdot A = r \cdot e_1 \cdot A = r \cdot F_L A$$

F_L and F_R coincide for singletons by Prop. 10.10.2 (9). Thus, they are equal by Prop. 14.3.2. The factors are reduced by Prop. 16.5.2. □

16.7 Factorization of linear power homomorphisms

The goal of this section is to derive a theorem analogous to Th. 16.4.4. This theorem claimed that all linear functions between products of modules can be factorized into linear functions between the factors.

Let $\mathcal{P}_1$ and $\mathcal{P}'_1$ be power constructions with semiring R_1, and let $\mathcal{P}_2$ and $\mathcal{P}'_2$ be power constructions with semiring R_2. Then $\mathcal{P} = \mathcal{P}_1 \times \mathcal{P}_2$ and $\mathcal{P}' = \mathcal{P}'_1 \times \mathcal{P}'_2$ are power constructions with semiring $R = R_1 \times R_2$.

Let $H_i : \mathcal{P}_i \dot{\rightarrow} \mathcal{P}'_i$ be two linear power homomorphisms. Then define $H = H_1 \times H_2 : \mathcal{P} \dot{\rightarrow} \mathcal{Q}$ by $H\,(A_1,\,A_2) = (H_1 A_1,\, H_2 A_2)$. Since the power operations are defined componentwise, H is easily proved to be a linear power homomorphism.

Conversely, if a linear power homomorphism $H : \mathcal{P} \dot{\rightarrow} \mathcal{P}'$ is given, it can be split into two families of R_i-linear mappings $H_i : \mathcal{P}_i \dot{\rightarrow} \mathcal{P}'_i$ such that $H = H_1 \times H_2$ by Th. 16.4.4. We have to show that both H_i are power homomorphisms.

- $H_1 \{\!|x|\!\}_1 = \pi_1\,(H\,(\{\!|x|\!\}_1,\,\theta_2)) = \pi_1\,(H\,((1,\,0) \cdot (\{\!|x|\!\}_1,\,\{\!|x|\!\}_2))) = \pi_1\,((1,\,0) \cdot H\,\{\!|x|\!\}) = \pi_1\,((1,\,0) \cdot \{\!|x|\!\}') = \{\!|x|\!\}'_1$
  Here, linearity of H is used.
- $H_1\,(ext_1\,f_1\,A_1)) = \pi_1\,(H\,(ext_1\,f_1\,A_1,\,\theta_2)) = \pi_1\,(H\,(ext_i\,f_i\,A_i)_{i \in I})$
  where $f_2 = \lambda x.\,\theta_2$ and $A_2 = \theta_2$. Applying the corresponding property of H, one obtains the desired result.

Summarizing, one gets

Theorem 16.7.1 Let $\mathcal{P}_1$ and $\mathcal{P}'_1$ be power constructions with semiring R_1 and $\mathcal{P}_2$ and $\mathcal{P}'_2$ power constructions with semiring R_2. Then for every pair of R_i-linear power homomorphisms $H_i : \mathcal{P}_i \dot{\rightarrow} \mathcal{P}'_i$, there is an R-linear power homomorphism $H_1 \times H_2 : \mathcal{P}_1 \times \mathcal{P}_2 \dot{\rightarrow} \mathcal{P}'_1 \times \mathcal{P}'_2$ defined by $(H_1 \times H_2)\,(A_1,\,A_2) = (H_1 A_1,\,H_2 A_2)$. Conversely, for every R-linear power homomorphism $H : \mathcal{P}_1 \times \mathcal{P}_2 \dot{\rightarrow} \mathcal{P}'_1 \times \mathcal{P}'_2$, there are R_i-linear power homomorphisms $H_i : \mathcal{P}_i \dot{\rightarrow} \mathcal{P}'_i$ such that $H = H_1 \times H_2$.

Part III

Special power constructions

In this part, we investigate how the five known power constructions — lower, upper, convex, mixed, and sandwich power construction — fit into the general algebraic framework. All these constructions have small semirings with at most 3 elements that share many common properties: their addition and multiplication is idempotent, they are all additive, and their modules may be described without referring to multiplication.

The three classical power constructions have two-valued semirings: $\mathbf{L} = \{0 < 1\}$ belongs to the lower construction, $\mathbf{U} = \{1 < 0\}$ belongs to the upper one, and $\mathbf{C} = \{0, 1\}$ belongs to the convex one. The semiring of the mixed and sandwich construction is $\mathbf{B} = \{\perp < [0, 1]\}$ with parallel disjunction as addition and parallel conjunction as multiplication.

To obtain a system of semirings that is better suited for our purposes, we also consider the semiring $\overline{\mathbf{B}} = \{[0, 1] < \top\}$ that is dual to $\mathbf{B}$, and the 'double' semiring $\mathbf{D} = \{\perp < [0, 1] < \top\}$. $\mathbf{C}$ is a sub-semiring of both $\mathbf{B}$ and its dual $\overline{\mathbf{B}}$, and $\mathbf{B}$ and $\overline{\mathbf{B}}$ are sub-semirings of $\mathbf{D}$. Thus, we are able to apply our results about sub-constructions of chapter 14. The 'double' semiring $\mathbf{D}$ is the product of $\mathbf{L}$ and $\mathbf{U}$ such that our results about product and factorization of power constructions (Ch. 16) can be applied.

In chapter 17, we investigate the structure of the modules of these semirings. We show that all these modules may be characterized without referring to multiplication. In particular, $\mathbf{C}$-modules are commutative idempotent monoids, and $\mathbf{B}$-modules are Gunter's mix algebras defined in [Gun89b, Gun90].

Chapter 18 is devoted to the $\mathbf{L}$-constructions. We show that there is a standard $\mathbf{L}$-construction $\mathcal{L}$ that is both initial and final. It coincides with the well known lower power construction in terms of Scott closed sets.

Chapter 19 deals with $\mathbf{U}$-constructions. For sober ground domains, the final $\mathbf{U}$-construction $\mathcal{U}_f$ is shown to be equivalent to the topological construction $\mathcal{U}_K$ in terms of compact upper sets. The initiality of this construction could however be shown for continuous ground domains only. Thus, one has to carefully distinguish between $\mathcal{U}_f$ and the initial $\mathbf{U}$-construction $\mathcal{U}_i$.

In chapter 20, a sub-construction $\mathcal{U}_s$ of $\mathcal{U}_f$ is identified. For sober ground domains, it is equivalent to a topological construction $\mathcal{U}_S$ in terms of strongly compact upper sets. All these constructions $\mathcal{U}_f$, $\mathcal{U}_s$, and $\mathcal{U}_i$ could not be proved to be isomorphic, although we do not know any ground domain where they differ.[2] For continuous ground domains only, $\mathcal{U}_f$, $\mathcal{U}_s$, $\mathcal{U}_i$, $\mathcal{U}_K$, and $\mathcal{U}_S$ all coincide such that one may speak of a standard upper power construction.

The remainder of chapter 20 is devoted to the proof of two theorems about the commuting of the standard lower construction $\mathcal{L}$ and the upper constructions $\mathcal{U}_S$ and $\mathcal{U}_i$ respectively. Both theorems coincide for continuous ground domains, and largely generalize the result of [FM90] where the commuting of $\mathcal{L}$ and $\mathcal{U}$ was shown for bounded complete algebraic ground domains using information systems. The proofs of our two theorems are completely different from each other and from the proof in [FM90]. The commuting of $\mathcal{L}$ and $\mathcal{U}_S$ is shown by topological arguments for $\mathcal{U}_S$-conform ground domains, whereas the commuting of $\mathcal{L}$ and $\mathcal{U}_i$ is shown by algebraic methods for *all* ground domains.

[2] Meanwhile, we found a domain where $\mathcal{U}_i$ differs from the other ones.

In chapter 21, we investigate the final **D**-construction $\mathcal{D}$ and its sub-constructions that are obtained by existential restriction to the sub-semirings **B**, $\overline{\mathbf{B}}$, and **C** of **D**. The restriction to **B** is $\mathcal{S}$, the sandwich power construction.

As the mixed power construction $\mathcal{M}$ is a sub-construction of $\mathcal{S}$, it should also be found among the sub-constructions of $\mathcal{D}$. In chapter 22, we present the conditions M and $\overline{M}$ that define the sub-construction $\mathcal{M}$ and its dual $\overline{\mathcal{M}}$. Moreover, the intersection of $\mathcal{M}$ and $\overline{\mathcal{M}}$ yields a sub-construction $\mathcal{C}$ of $\mathcal{D}$ that coincides with Plotkin's construction in case of continuous ground domains.

Chapter 23 contains some remarks about **B**- and **C**-constructions. $\mathcal{S}$ preserves bounded completeness, whereas $\mathcal{M}$ and $\mathcal{C}$ do not. In case of continuous ground domains, $\mathcal{M}$ is the initial **B**-construction, and $\mathcal{C}$ the initial **C**-construction. $\mathcal{S}$ is the final **B**-construction, whereas the final **C**-construction is degenerated and not among the sub-constructions of **D**.

The final chapter 24 introduces sandwich algebras and shows that sandwich power domains are free sandwich algebras if the ground domain is algebraic. I found this result before the algebraic theory of power domain constructions was fully developed. In contrast to many other results, it could not be merged into the general theory and is still standing for its own.

Chapter 17

Small semirings and their modules

In this chapter, we study the small semirings $\mathbf{L}$, $\mathbf{U}$, $\mathbf{C}$, $\mathbf{B}$, $\overline{\mathbf{B}}$, and $\mathbf{D}$ and the structure of the corresponding modules.

In section 17.2, we consider the modules of the three two-point semirings $\mathbf{L}$, $\mathbf{U}$, and $\mathbf{C}$. $\mathbf{C}$-modules are just commutative idempotent monoids. In these CI monoids, a logical order may be derived from addition by defining $x \sqsubseteq y$ iff $x + y = y$. This order is studied in section 17.3. Section 17.4 is concerned with the modules of the semirings $\mathbf{B}$, $\overline{\mathbf{B}}$, and $\mathbf{D}$. In particular, these three semirings are shown to be additive, and $\mathbf{B}$-modules are proved to be equivalent to the mix algebras of [Gun89b, Gun90]. Every $\mathbf{B}$-module contains a $\mathbf{U}$-module in a canonical way as shown in section 17.5.

We then leave the modules and turn to power constructions. In section 17.6, the general properties of power constructions for the small semirings are enumerated as they follow from the results of Part II.

The power domains belonging to the three semirings $\mathbf{L}$, $\mathbf{U}$, and $\mathbf{C}$ with carrier set $\{0, 1\}$ may be split into an 'empty part' around the empty set and a 'non-empty part' containing all singletons and closed w.r.t. union, extension, and limits (section 17.7). It is this non-empty part that was proposed in most of the original papers about the classical power domain constructions.

17.1 The considered semirings

The characteristic semiring of the *lower power construction* is $\mathbf{L} = \{0 < 1\}$ where $1+1 = 1$. In logical interpretation, $\mathbf{L}$ contains F and T where F is below T. Computationally, this means that a partial computation resulting in F may change this value to T if it proceeds. Hence, only computations resulting in T are terminated, whereas F means non-termination.

The characteristic semiring of the classical *upper power construction* of Smyth is $\mathbf{U} = \{1 < 0\}$ where $1 + 1 = 1$. Logically, T is below F in this semiring. Hence, only computations resulting in F are terminated, whereas T signals non-termination.

The characteristic semiring of the *convex power construction* of Plotkin is $\mathbf{C} = \{0, 1\}$ where $1 + 1 = 1$ and the two values are incomparable. Logically, T and F are both stable here. The semiring is discrete, whence computations with logical result cannot proceed. They have immediately to decide whether the result is T or F, and cannot change their 'opinion' afterwards.

A power construction with a more reasonable logic should have the Boolean domain $\mathbf{B} = \{\bot < [\mathsf{F}, \mathsf{T}]\}$ as semiring. Such constructions are called *set domain constructions* in [Hec90c]. The interpretation of $\bot$ is 'I do not (yet) know'. Computations with logical results start in this state which may change to F or T if the computation proceeds.

The *sandwich power construction* [BDW88] or *big set domain construction* [Hec90c] and the *mixed power construction* [Gun89b, Gun90] or *small set domain construction* [Hec90c] both have characteristic semiring $\mathbf{B}$ with parallel conjunction and disjunction.

Besides the semirings $\mathbf{L}$, $\mathbf{U}$, $\mathbf{C}$, and $\mathbf{B}$, we also consider the 'double semiring' $\mathbf{D} = \mathbf{L} \times \mathbf{U}$. The motivation to consider $\mathbf{D}$ is that the Boolean semiring $\mathbf{B}$ is a sub-semiring of $\mathbf{D}$.

The semiring $\mathbf{D} = \mathbf{L} \times \mathbf{U} = \{0 < 1\} \times \{1 < 0\}$ has four elements ordered as follows:

$$
\begin{tikzcd}
 & (1,0) & \\
(0,0) \arrow[ur] & & (1,1) \arrow[ul] \\
 & (0,1) \arrow[ul] \arrow[ur] &
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}
 & \mathsf{W} & \\
\mathsf{F} \arrow[ur] & & \mathsf{T} \arrow[ul] \\
 & \bot \arrow[ul] \arrow[ur] &
\end{tikzcd}
$$

The picture to the left shows a representation of $\mathbf{D}$ in terms of pairs of members of the lower and upper semiring. The picture to the right shows a logical interpretation of $\mathbf{D}$. $(0, 0)$, the neutral element of addition logically becomes F, the neutral element of disjunction. Correspondingly, $(1, 1)$ is interpreted as T. The least element $\bot$ denotes a state of ignorance. Every computation starts with $\bot$, and this result may be refined to either T or F when the computation proceeds. In contrast, W denotes a state of inconsistency: a computation returning W subsumes both T and F.

We now present tables for the semiring operations in $\mathbf{D}$. On the left hand side, they are depicted in terms of pairs, whereas on the right hand side, the logical interpretation is used.

$+$	$(0,1)$	$(0,0)$	$(1,1)$	$(1,0)$
$(0,1)$	$(0,1)$	$(0,1)$	$(1,1)$	$(1,1)$
$(0,0)$	$(0,1)$	$(0,0)$	$(1,1)$	$(1,0)$
$(1,1)$	$(1,1)$	$(1,1)$	$(1,1)$	$(1,1)$
$(1,0)$	$(1,1)$	$(1,0)$	$(1,1)$	$(1,0)$

$\vee$	$\bot$	F	T	W
$\bot$	$\bot$	$\bot$	T	T
F	$\bot$	F	T	W
T	T	T	T	T
W	T	W	T	W

$\cdot$	$(0,1)$	$(0,0)$	$(1,1)$	$(1,0)$
$(0,1)$	$(0,1)$	$(0,0)$	$(0,1)$	$(0,0)$
$(0,0)$	$(0,0)$	$(0,0)$	$(0,0)$	$(0,0)$
$(1,1)$	$(0,1)$	$(0,0)$	$(1,1)$	$(1,0)$
$(1,0)$	$(0,0)$	$(0,0)$	$(1,0)$	$(1,0)$

$\wedge$	$\bot$	F	T	W
$\bot$	$\bot$	F	$\bot$	F
F	F	F	F	F
T	$\bot$	F	T	W
W	F	F	W	W

Considering the right hand tables, one sees that the Boolean semiring $\mathbf{B} = \{\perp, \mathsf{F}, \mathsf{T}\}$ with parallel disjunction and conjunction is a sub-semiring of $\mathbf{D}$. Another sub-semiring of $\mathbf{D}$ is $\overline{\mathbf{B}} = \{\mathsf{F}, \mathsf{T}, \mathsf{W}\}$. Tables for its operations look like tables for the operations of $\mathbf{B}$ if $\perp$ is replaced by W. From a domain-theoretic viewpoint, $\overline{\mathbf{B}}$ is just the dual of $\mathbf{B}$. Another sub-semiring of $\mathbf{D}$ is $\mathbf{C} = \{\mathsf{F}, \mathsf{T}\}$, the intersection of $\mathbf{B}$ and $\overline{\mathbf{B}}$.

In contrast to $\mathbf{L}$ and $\mathbf{U}$, $\mathbf{D}$ admits negation. The continuous operation $\neg : [\mathbf{D} \to \mathbf{D}]$ exchanges the values of F and T, and maps $\perp$ and W to itself. Hence, the sub-semirings $\mathbf{B}$, $\overline{\mathbf{B}}$, and $\mathbf{C}$ are closed not only w.r.t. disjunction and conjunction, but also w.r.t. negation.

The six semirings $\mathbf{L}$, $\mathbf{U}$, $\mathbf{C}$, $\mathbf{B}$, $\overline{\mathbf{B}}$, and $\mathbf{D}$ are connected by semiring homomorphisms as indicated by the following figure:

$$\begin{tikzcd}
 & \overline{\mathbf{B}} \arrow[dr] & & \mathbf{L} \\
\mathbf{C} \arrow[ur] \arrow[dr] & & \mathbf{D} = \mathbf{L} \times \mathbf{U} \arrow[ur] \arrow[dr] & \\
 & \mathbf{B} \arrow[ur] & & \mathbf{U}
\end{tikzcd}$$

The homomorphisms from $\mathbf{C}$ to $\overline{\mathbf{B}}$ and $\mathbf{B}$, and those from $\overline{\mathbf{B}}$ and $\mathbf{B}$ to $\mathbf{D}$ are embeddings. The homomorphisms from $\mathbf{D}$ to $\mathbf{L}$ and $\mathbf{U}$ on the other hand are projections. All these homomorphisms are unique since the values of 0 and 1 are prescribed, and $\perp$ and W have to be mapped to common lower resp. upper bounds of 0 and 1.

If $\varphi : [R \to R']$ is a semiring homomorphism and $(M, R', \cdot)$ is an R'-module, then M may be turned into an R-module $(M, R, *)$ by the definition $r * m = \varphi r \cdot m$. Thus, all $\mathbf{L}$- and $\mathbf{U}$-modules are also $\mathbf{D}$-modules, all $\mathbf{D}$-modules are $\mathbf{B}$-modules as well as $\overline{\mathbf{B}}$-modules, and $\mathbf{B}$- and $\overline{\mathbf{B}}$-modules in turn are $\mathbf{C}$-modules.

In all semirings R we consider in this chapter, $1{+}1 = 1$ holds, whence $m{+}m = 1{\cdot}m{+}1{\cdot}m = (1+1)\cdot m = 1 \cdot m = m$ holds for all members m of all modules. Hence, all R-modules are commutative idempotent monoids, or shortly *CI monoids*.

17.2 L-, U-, and C-modules

The three two-valued semirings $\mathbf{L}$, $\mathbf{U}$, and $\mathbf{C}$ are all additive, i.e. all additive mappings between their modules are linear. The reason is that linearity of an additive function f may be shown by case analysis:

- $f(0 \cdot m) = f(0) = 0 = 0 \cdot f(m)$
- $f(1 \cdot m) = f(m) = 1 \cdot f(m)$

The modules of the two-valued semirings may be characterized without referring to multiplication:

Proposition 17.2.1

$\mathbf{C}$: M is a $\mathbf{C}$-module iff it is a CI monoid.

$\mathbf{L}$: M is an $\mathbf{L}$-module iff it is a CI monoid and 0 is its least element.

U : M is a **U**-module iff it is a CI monoid and 0 is its greatest element.

Proof: All modules are commutative monoids. As pointed out above, all **L**-, **U**-, and **C**-modules are CI monoids. In the **L**-case, 0 is least by $m = 1 \cdot m \geq 0 \cdot m = 0$. Analogously, 0 is greatest in **U**-modules.

Conversely, let M be a given CI monoid. A multiplication may be defined by $1 \cdot m = m$ and $0 \cdot m = 0$. This multiplication is monotonic in its left operand because of the additional axioms $m \geq 0$ and $m \leq 0$ in the cases **L** and **U** respectively. Since **C**, **L**, and **U** are finite, multiplication is continuous by Prop. 3.3.3. The module axioms $(r+s) \cdot m = r \cdot m + s \cdot m$ and $(r \cdot s) \cdot m = r \cdot (s \cdot m)$ may be verified by case analysis $r, s \in \{0, 1\}$. Idempotence is needed for the additive axiom in case $r = s = 1$. □

L- and **U**-modules may be characterized even without referring to addition.

Proposition 17.2.2

L : In every **L**-module, 0 is the least element and $a + b$ is the least upper bound of a and b. Conversely, every domain **X** with a least element $\perp$ and a least upper bound $x \sqcup y$ for every two points is an **L**-module.

In shorter terms: **L**-modules are just the complete domains, i.e. those in CC.

U : In every **U**-module, 0 is the greatest element and $a + b$ is the greatest lower bound of a and b. Conversely, every domain **X** with a greatest element $\top$ and a continuous greatest lower bound operation $\lambda(x, y).\, x \sqcap y$ is a **U**-module.

Proof: We first consider the case of the semiring **L**. By Prop. 17.2.1, 0 is the least element of an **L**-module M. From $0 \leq b$, $a = a + 0 \leq a + b$ follows. Similarly, $0 \leq a$ implies $b \leq a + b$. Thus, $a + b$ is an upper bound of a and b. Let u be an arbitrary upper bound. Then $a + b \leq u + u = u$ holds.

Thus, M is a domain with least element and least upper bound of every two points. Hence it is complete by Prop. 5.3.1.

Conversely, let **X** be a complete domain. The operation '$\sqcup$' in a complete domain is commutative, associative, idempotent, and has neutral element $\perp$. It is continuous by Prop. 5.3.5. Hence, $(\mathbf{X}, \sqcup, \perp)$ is a CI monoid with least element $\perp$. By Prop. 17.2.1, it is an **L**-module.

The proof for **U**-modules is analogous. Because the operation '$\sqcap$' may not be continuous, its continuity has to be explicitly stated. □

By this proposition, every lower power construction $\mathcal{L}$ creates complete domains $\mathcal{L}\mathbf{X}$ only. Thus, it preserves completeness (CC), bounded completeness (BC), FC, SC, and KC.

17.3 The logical order of CI monoids

All modules considered in this part of the thesis belong to one of the semirings **L**, **U**, **D**, **B**, $\overline{\mathbf{B}}$, or **C**. Hence, they are all **C**-modules i.e. CI monoids. Besides the a priori given domain order, CI monoids admit the definition of an alternative order '$\sqsubseteq$' that is derived from addition.

Definition 17.3.1 Let $(M, +, 0)$ be a CI monoid domain. Then we define $a \sqsubseteq b$ iff $a + b = b$. We call '$\sqsubseteq$' the *logical order* of M.

Proposition 17.3.2 The logical order of a CI monoid $(M, +, 0)$ is indeed an order. 0 is its least element, and $a + b$ is the logical least upper bound of a and b.

Proof:

- Reflexivity: $a \sqsubseteq a$ holds by idempotence $a + a = a$.
- Transitivity: $a \sqsubseteq b$ and $b \sqsubseteq c$ imply $a + b = b$ and $b + c = c$, whence $a + c = a + (b + c) = (a + b) + c = b + c = c$, i.e. $a \sqsubseteq c$.
- Antisymmetry: $a \sqsubseteq b$ and $b \sqsubseteq a$ imply $b = a + b = b + a = a$.
- Least element: $0 \sqsubseteq b$ follows from $0 + b = b$.
- Upper bound: Idempotence implies $a + (a + b) = (a + a) + b = a + b$, whence $a \sqsubseteq a + b$. By commutativity, $b \sqsubseteq a + b$ holds.
- Least upper bound: Assume $a \sqsubseteq u$ and $b \sqsubseteq u$. Then $(a+b)+u = a+(b+u) = a+u = u$, whence $a + b \sqsubseteq u$. □

By the proposition above, addition and neutral element of a CI monoid are uniquely determined by the logical order. Hence, a CI monoid may be described by providing its logical order instead of a table of values for its addition.

The addition of semiring **L** for instance is described by $0 < 1$ and $0 \sqsubset 1$, **U** by $1 < 0$ and $0 \sqsubset 1$, **C** by $[0, 1]$ and $0 \sqsubset 1$, **B** by $\perp < [\mathsf{F}, \mathsf{T}]$ and $\mathsf{F} \sqsubset \perp \sqsubset \mathsf{T}$. A strict disjunction would be characterized by $\mathsf{F} \sqsubset \mathsf{T} \sqsubset \perp$ instead. The additive part of semiring **D** is described by $\perp < [\mathsf{F}, \mathsf{T}] < \mathsf{W}$ and $\mathsf{F} \sqsubset [\perp, \mathsf{W}] \sqsubset \mathsf{T}$.

In ordinary CI monoids, the logical order '$\sqsubseteq$' and the domain order '$\leq$' are totally unrelated. In special cases, there is an relation.

Proposition 17.3.3

L : A CI monoid is an **L**-module iff domain order and logical order coincide ($a \sqsubseteq b$ iff $a \leq b$).

U : A CI monoid is a **U**-module iff domain order and logical order are opposite ($a \sqsubseteq b$ iff $a \geq b$).

Proof: If the orders coincide, then 0 is the least element w.r.t. '$\leq$', whence the monoid is an **L**-module by Prop. 17.2.2. If the orders are opposite, then 0 is greatest w.r.t. '$\leq$', whence the monoid is a **U**-module.

Conversely, in an **L**-module, addition is supremum, whence $a + b = b$ is equivalent to $a \sqcup b = b$, i.e. $a \leq b$. In **U**-modules, one gets $a \sqcap b = b$, i.e. $a \geq b$. □

A function f between two CI monoids is *logically monotonic* iff $a \sqsubseteq b$ implies $fa \sqsubseteq fb$.

Proposition 17.3.4

(1) Addition is logically monotonic, i.e. $a \sqsubseteq a'$ and $b \sqsubseteq b'$ imply $a + b \sqsubseteq a' + b'$.

(2) Every additive function is logically monotonic.

Proof:

(1) $(a+b)+(a'+b') = (a+a')+(b+b') = a'+b'$

(2) $a+b=b$ implies $fa+fb = f(a+b) = fb$ □

CI monoids are generally not logical domains, i.e. the logical supremum of a logically directed set may not exist. Even if directed logical suprema exist, they may not be preserved by additive functions. Thus, additive (domain) continuous functions need not be 'logically continuous'.

The semirings **L**, **U**, **C**, **D**, **B**, and $\overline{\mathbf{B}}$ have some special features in common. In all of these, multiplication is CI and $a+1=1$ holds, i.e. 1 is logically a greatest element. Due to the distributive laws and Prop. 17.3.4, multiplication in R as well as in all R-modules is logically monotonic. Hence, $a \cdot m \sqsubseteq 1 \cdot m = m$ holds in all R-modules. Within R, $u \sqsubseteq a, b$ implies $u = u \cdot u \sqsubseteq a \cdot b$, i.e. $a \cdot b$ is the logical greatest lower bound of a and b.

Proposition 17.3.5 Let R be any of the semirings **L**, **U**, **C**, **B**, $\overline{\mathbf{B}}$, and **D**. Then $0 \sqsubseteq r \sqsubseteq 1$ holds for all r in R. For every two points a and b of R, $a+b$ is the logical least upper bound, and $a \cdot b$ the logical greatest lower bound of a and b. In all R-modules M, $r \cdot m \sqsubseteq m$ holds for all r in R and m in M.

17.4 D-, B-, and $\overline{\mathbf{B}}$-modules

This section is concerned with a characterization of **D**-, **B**-, and $\overline{\mathbf{B}}$-modules that does not employ multiplication. Furthermore, we show that all additive functions between these modules are linear, i.e. **D**, **B**, and $\overline{\mathbf{B}}$ are additive semirings.

Because of the sub-semiring relations, every **D**-module is also a **B**-module and a $\overline{\mathbf{B}}$-module, and these modules in turn are **C**-modules. **C**-modules are just CI monoids by Prop. 17.2.1.

In **B**-modules M, we define a function $\_? : [M \to M]$ by $A? = \perp \cdot A$. Among others, '?' has the following properties:

(1) $A? = \perp \cdot A \leq 1 \cdot A = A$

(2) $A? = \perp \cdot A \leq 0 \cdot A = 0$

(3) $A + A? = 1 \cdot A + \perp \cdot A = (1+\perp) \cdot A = 1 \cdot A = A$

(4) $(A+B)? = \perp \cdot (A+B) = \perp \cdot A + \perp \cdot B = A? + B?$

In $\overline{\mathbf{B}}$-modules, we define a function $\_!$ by $A! = \mathsf{W} \cdot A$. It has properties analogous to those of '?'.

These facts motivate the following definition:

Definition 17.4.1 Let M be a CI monoid domain. A function $\_? : [M \to M]$ is a *lower approximation* iff it satisfies the following axioms:

$(A1)$ $A? \leq A$	$(A2)$ $A? \leq 0$
$(A3)$ $A + A? \geq A$	$(A4)$ $(A+B)? \leq A? + B?$

A function $\_! : [M \to M]$ is an *upper approximation* iff it satisfies

$(\overline{A}1)$ $A! \geq A$ $\quad$ $(\overline{A}2)$ $A! \geq 0$

$(\overline{A}3)$ $A + A! \leq A$ $\quad$ $(\overline{A}4)$ $(A+B)! \geq A! + B!$

In comparison to the properties of $\lambda A.\perp \cdot A$ and $\lambda A.\mathsf{W} \cdot A$ derived above, axioms (3) and (4) are weaker since the equality sign is replaced by '$\leq$' or '$\geq$'. In fact, the opposite inequations are redundant since they may be derived from the given axioms. These axioms are independent as some examples show.

For every axiom (Ai), there is a CI monoid M_i with an additional continuous operation $\_? : [M_i \to M_i]$ such that all axioms except (Ai) are satisfied.

(1) Example M_1 is given by the CI monoid $(\{\perp < \top\}, \sqcup, \perp)$. If identity is chosen for '?', then the axioms (A2) through (A4) are satisfied whereas (A1) is not because $\top? = \top \not\leq 0 = \perp$.

(2) Example M_2 is given by the CI monoid $(\{\perp < \top\}, \sqcap, \top)$ with the definition $A? = \top$ for all A. Then the axioms (A1), (A3), and (A4) hold because $\top$ is the neutral element of M_2. (A2) is violated because $\perp? = \top \not\leq \perp$.

(3) Example M_3 is given by the same CI monoid as M_2, but defining $A? = \perp$ for all A. Then the axioms (A1) and (A2) hold because $\perp$ is the least element of M_3. (A4) holds because of idempotence. (A3) is violated since $\top + \top? = \top \sqcap \perp = \perp \not\geq \top$.

(4) Example M_4 is more complex since we need four elements. The domain order is $\{\perp < [0, \square] < \top\}$ and the logical order is $\{0 \sqsubset \perp \sqsubset \square \sqsubset \top\}$. A little thought shows that the addition defined by this logical order is (domain) monotonic, whence continuous.

The operation '?' is defined by mapping $\top$ and 0 to 0, and $\perp$ and $\square$ to $\perp$. In shorter terms, $A?$ is the greatest lower bound of 0 and A. Hence, (A1) and (A2) immediately hold, and (A3) is easily verified. However (A4) fails since $(\square + \top)? = \top? = 0$, whereas $\square? + \top? = \perp + 0 = \perp$.

In the sequel, we study the general properties of a lower approximation $\_?$ in a CI monoid M.

(A1) $A? \leq A$ $\quad$ (A2) $A? \leq 0$

(A3) $A + A? \geq A$ $\quad$ (A4) $(A+B)? \leq A? + B?$

(T1) $A + B? \leq A$ $\quad$ since $A + B? \overset{A2}{\leq} A + 0 \overset{N}{=} A$

(T2) $A + A? = A$ $\quad$ by (A3) and (T1)

(T3) $0? = 0$ $\quad$ since $0 \overset{T2}{=} 0 + 0? \overset{N}{=} 0?$

(T4) $A?? = A?$ $\quad$ since $A?? \overset{A1}{\leq} A? \overset{T2}{=} A? + A?? \overset{T1}{\leq} A??$

(T5) $A? = A$ iff $A \leq 0$

Proof: '$\Rightarrow$' $A \overset{lhs}{=} A? \overset{A2}{\leq} 0$ $\quad$ '$\Leftarrow$' $A? \overset{A1}{\leq} A \overset{T2}{=} A + A? \overset{rhs}{\leq} 0 + A? \overset{N}{=} A?$

(T6) $X \leq 0$ and $X \leq A$ iff $X \leq A?$ $\quad$ i.e. $A?$ is the greatest lower bound of 0 and A.

Proof: '$\Rightarrow$' $X \leq 0$ implies $X = X?$ by (T5). $X \leq A$ implies $X? \leq A?$ by monotonicity of '?'. Together, $X \leq A?$ follows. $\quad$ '$\Leftarrow$' by (A1) and (A2).

(T7) $(A+B)? = A? + B?$

Proof: '$\leq$' is (A4). '$\geq$' is deduced by (T6) from $A? + B? \leq 0$ (by (A2) and (N)) and $A? + B? \leq A + B$ (by (A1)).

(T8) The three statements $A \leq A + B$ and $A? \leq B?$ and $A? \leq B$ are equivalent.

$$\begin{aligned} \text{Proof:}\quad (1) \Rightarrow (2):&\quad A? \overset{1}{\leq} (A+B)? \overset{T7}{=} A? + B? \overset{T1}{\leq} B? \\ (2) \Rightarrow (3):&\quad A? \overset{2}{\leq} B? \overset{A1}{\leq} B \\ (3) \Rightarrow (1):&\quad A \overset{T2}{=} A + A? \overset{3}{\leq} A + B \end{aligned}$$

(T9) $X \leq 0$ and $X \leq A$ and $A + X \geq A$ iff $X = A?$
Proof: '$\Leftarrow$' is immediate by (A1), (A2), and (A3).
'$\Rightarrow$': $X \leq 0$ and $X \leq A$ imply $X \leq A?$ by (T6). $A + X \geq A$ implies $A? \leq X$ by (T8).

(T10) Every CI monoid with a lower approximation $\_?$ is a $\mathbf{B}$-module.

Proof: We define $0 \cdot A = 0$, $1 \cdot A = A$, and $\perp \cdot A = A?$. By (A1) and (A2), this operation is monotonic in its $\mathbf{B}$-argument, whence it is continuous.

$r \cdot 0 = 0$:	(T3)
$r \cdot (A + B) = r \cdot A + r \cdot B$:	(T7)
$0 \cdot A = 0$:	immediate
$(r + s) \cdot A = r \cdot A + s \cdot A$:	by neutrality if $r = 0$ or $s = 0$, by idempotence if $r = s$, and by (T2) if $r = 1$ and $s = \perp$ or vice versa.
$1 \cdot A = A$:	immediate
$r \cdot (s \cdot A) = (r \cdot s) \cdot A$:	the only difficult case $r = s = \perp$ is handled by (T4).

(T11) Let M and M' be two CI monoids with lower approximations $\_?$ and $\_?'$ respectively. Then every additive morphism $f : [M \to M']$ satisfies $f(A?) = (fA)?'$.

Proof: For all $A \in M$, $A? \leq 0$, $A? \leq A$, and $A + A? \geq A$ hold by the axioms of '?'. These statements imply $f(A?) \leq 0$, $f(A?) \leq fA$, and $fA + f(A?) \geq fA$ respectively. By (T9), $f(A?) = (fA)?'$ follows.

In [Gun89b, Gun90], Gunter defined *mix algebras* by an axiom system consisting of (T7), (T4), (T2), (A1), and (T1). Because (T1) implies (A2) by choosing $A = 0$ and (T2) implies (A3) and (T7) implies (A4), his mix theory is equivalent with our theory of CI monoids with a lower approximation, and Gunter's mix algebras are just $\mathbf{B}$-modules.

(T9) is a particularly interesting theorem. It implies that the operation '?' is uniquely determined, i.e. for a given CI monoid, there is at most one choice for a lower approximation '?'. By (T6), the only candidate for '?' is $\lambda A.\, 0 \sqcap A$. Thus, to verify that a given CI monoid M is a $\mathbf{B}$-module, one has to check that for all A in M $0 \sqcap A$ exists, that the operation $\lambda A.\, 0 \sqcap A$ is continuous and satisfies axioms (A3) and (A4). By the choice $A? = 0 \sqcap A$, axioms (A1) and (A2) are obviously satisfied.

Translated into the language of $\mathbf{B}$-modules, $f(A?) = (fA)?$ reads $f(\perp \cdot A) = \perp \cdot fA$. Hence, (T11) means $\mathbf{B}$ is additive because the corresponding statements for 0 and 1 are trivial.

Since the axioms of upper approximations are just dual to those for lower ones, dual theorems hold for them, e.g. $0! = 0$ and $A!! = A!$. Hence, $A!$ is the least upper bound of 0 and A by the dual of (T6), there is at most one upper approximation in a given CI monoid by (T9), and $\overline{\mathbf{B}}$-modules are just CI monoids with an upper approximation. The dual of (T11) states that $\overline{\mathbf{B}}$ is additive.

Finally, we investigate how lower and upper approximations interact. In a CI monoid M with both a lower and an upper approximation '?' resp. '!', the following facts hold:

(F1) $A!? = A?! = 0$
By $(A2)$, $A!? \leq 0$ holds. By $(\overline{A}2)$, monotonicity of '?', and $(T1)$, $A!? \geq 0? = 0$ holds. $A?! = 0$ is shown analogously.

(F2) $A? + A! = A$
Employing first $(A1)$ and $(\overline{A}1)$ and then $(A3)$ and $(\overline{A}3)$, we obtain $A? + A! \leq A + A! \leq A$ and $A? + A! \geq A? + A \geq A$.

(F3) Every CI monoid with both a lower and an upper approximation $\_?$ resp. $\_!$ is a **D**-module.

The proof is similar to that of (T10). Some new arguments are needed: (F1) proves $r \cdot (s \cdot A) = (r \cdot s) \cdot A$ in case $r = \perp$ and $s = \mathsf{W}$ or vice versa since $\perp \cdot \mathsf{W} = 0$. (F2) proves $(r + s) \cdot A = r \cdot A + s \cdot A$ in the same case since $\perp + \mathsf{W} = 1$.

(F4) Additivity of **D** may be shown by combining (T11) and its dual.

The results of this section are summarized to the following theorem:

Theorem 17.4.2

(1) **B**-modules are CI monoids with a lower approximation.

(2) $\overline{\mathbf{B}}$-modules are CI monoids with an upper approximation.

(3) **D**-modules are CI monoids with both a lower and an upper approximation.

(4) All the semirings **B**, $\overline{\mathbf{B}}$, and **D** are additive.

As mentioned towards the end of section 17.1, all **L**- and **U**-modules are also **D**-modules. In case of an **L**-module M, the approximations are defined by $m? = 0$ and $m! = m$, whereas in case of a **U**-module, they are given by $m? = m$ and $m! = 0$.

17.5 The U-module contained in a B-module

Since the semiring **D** is the product of **L** and **U**, every **D**-module is isomorphic to the product of some **L**-module and some **U**-module by Th. 16.4.3. We now investigate whether every **B**-module may be embedded into the product of an **L**- and a **U**-module.

If $M = (M, +, 0, ?)$ is a **B**-module with its lower approximation '?', then one may define the subset

$$M? = \{m? \mid m \in M\} = \{m \in M \mid m? = m\} = \{m \in M \mid m \leq 0\}$$

The last equality is given by (T5) $m? = m$ iff $m \leq 0$, whereas the first equality is given by (T4) $m?? = m$.

The subset $M?$ contains 0 and is closed w.r.t. addition since $a, b \leq 0$ implies $a + b \leq 0 + 0 = 0$. Thus, $M?$ is a sub-monoid of M. It is a **U**-module since 0 is its greatest element. The mapping $\_? : [M \to M?]$ is additive by (T3) and (T7).

By dual arguments, every $\overline{\mathbf{B}}$-module M contains a sub-monoid $M!$ that is the image of the additive mapping '!'. Because of $M! = \{m \in M \mid m \geq 0\}$, the sub-monoid $M!$ is an $\mathbf{L}$-module.

If a $\mathbf{D}$-module M is given, then M is isomorphic to $M! \times M?$ by means of the pair of additive isomorphisms $\alpha m = (m!, m?)$ and $\beta(a, b) = a + b$. In fact, this result is only a special instance of Th. 16.4.3.

Since every $\mathbf{D}$-module is isomorphic to a product of an $\mathbf{L}$-module and a $\mathbf{U}$-module, one might believe that every $\mathbf{B}$-module be isomorphic to a sub-monoid of such a product. This is however wrong as an example shows.

Let $M = \{\perp < 0 < \top\}$ where addition is given by $\perp + x = x + \perp = \perp$, $0 + x = x + 0 = x$, and $x + x = x$ for all x in M. By case analysis, one may easily verify that M is a $\mathbf{B}$-module with $\perp? = \perp$ and $0? = \top? = 0$. Assume there are an $\mathbf{L}$-module M_1 and a $\mathbf{U}$-module M_2 such that there is an additive embedding $e : [M \to M_1 \times M_2]$. Then $e\,(0) = (0,\, 0)$ holds. Since 0 is least in M_1, $e\,(\perp) = (0,\, b)$ follows for some b in M_2. Dually, $e\,(\top) = (a,\, 0)$ holds for some a in M_1 since 0 is greatest in M_2. By additivity, $(0,\, b) = e\,(\perp) = e\,(\perp) + e\,(\top) = (0,\, b) + (a,\, 0) = (a,\, b)$ follows, whence $a = 0$, i.e. $e\,(\top) = e\,(0) = (0,\, 0)$. Thus, e is not an embedding.

The $\mathbf{B}$-module M of the example above cannot be obtained as a power domain for any ground domain $\mathbf{X}$ and any $\mathbf{B}$-construction $\mathcal{P}$. The reason is the linear mapping $ne : [\mathcal{P}\mathbf{X} \to \mathbf{B}]$, which maps singletons $\{\!|x|\!\}$ to 1. Since $\mathbf{X}$ is not empty, every power domain $\mathcal{P}\mathbf{X}$ contains at least one singleton. The example module M however does not admit any mapping to $\mathbf{B}$ whose range contains 1 since additive maps from M to $\mathbf{B}$ have to map 0 to 0, whence by monotonicity $\top$ is mapped to 0 and $\perp$ to 0 or $\perp$.

Hence, the discussion leaves open the following question:

Problem 13 Can every power domain obtained by a $\mathbf{B}$-construction be embedded into a product of an $\mathbf{L}$-module and a $\mathbf{U}$-module?

17.6 General properties of power constructions for the small semirings

Part II of the thesis allows to state some general remarks that are valid for all power constructions with characteristic semiring in $\mathcal{R} = \{\mathbf{L}, \mathbf{U}, \mathbf{C}, \mathbf{B}, \overline{\mathbf{B}}, \mathbf{D}\}$. All these semirings are obviously finite, commutative, and idempotent. They are all additive as shown in the previous sections. All semirings except $\mathbf{C}$ are non-discrete, and all except $\mathbf{C}$ and $\overline{\mathbf{B}}$ have a least element. Hence, we obtain the following statements for all R in $\mathcal{R}$:

(1) All R-constructions are idempotent ($A \uplus A = A$) by section 11.5.

(2) All R-constructions are commutative ($r \cdot A = A \cdot r$) by Prop. 11.12.2.

(3) All reduced R-constructions are symmetric by Prop. 14.3.3 (2).

(4) For $R \neq \mathbf{C}$: all R-constructions are faithful ($\{\!|x|\!\} \leq \{\!|y|\!\} \Rightarrow x \leq y$) by Cor. 11.9.3. The initial $\mathbf{C}$-construction is also faithful by Prop. 14.5.3 because $\mathbf{L}$ provides a non-discrete $\mathbf{C}$-module.

The final $\mathbf{C}$-construction and its core are not faithful by Prop. 15.7.2.

(5) All reduced and all final R-constructions preserve FIN, F-ALG, and F-CONT by the Propositions 14.3.3 (3) and 15.7.3.

(6) All initial R-constructions preserve ALG and CONT by Th. 14.5.2.

(7) For $R \in \{\mathbf{L}, \mathbf{U}, \mathbf{B}, \mathbf{D}\}$: all reduced and all final R-constructions yield power domains with a least element whenever the ground domain has a least element by Prop. 14.3.3 (4) and Prop. 15.7.4.

(8) The product of an **L**- and a **U**-construction results in a **D**-construction by Prop. 16.2.1. Conversely, every **D**-construction may be factorized into an **L**- and a **U**-construction by Cor. 16.6.3.

For all semirings R in $\mathcal{R}$, there are final constructions $\mathcal{P}_f^R$, their cores $\mathcal{P}_c^R$, and initial constructions $\mathcal{P}_i^R$. Besides these 18 constructions, some more constructions are obtained by restricting the **D**-constructions to the sub-semirings **B**, $\overline{\mathbf{B}}$, and **C** according to Th. 14.6.2. From the general theory, we only know the isomorphisms $\mathcal{P}_f^{\mathbf{D}} = \mathcal{P}_f^{\mathbf{L}} \times \mathcal{P}_f^{\mathbf{U}}$, $\mathcal{P}_c^{\mathbf{D}} = \mathcal{P}_c^{\mathbf{L}} \times \mathcal{P}_c^{\mathbf{U}}$, and $\mathcal{P}_i^{\mathbf{D}} = \mathcal{P}_i^{\mathbf{L}} \times \mathcal{P}_i^{\mathbf{U}}$ by the theorems 16.3.1, 16.5.1, and 16.5.3. The next chapters are devoted to the investigation of this host of power constructions and the task of finding more relations among them. For instance, we shall see $\mathcal{P}_f^{\mathbf{L}} = \mathcal{P}_c^{\mathbf{L}} = \mathcal{P}_i^{\mathbf{L}}$ and the coincidence of $\mathcal{P}_f^{\mathbf{B}}$ and the existential restriction of $\mathcal{P}_f^{\mathbf{D}}$ to **B**. Furthermore, we try to identify the known constructions among this set of constructions. Before doing so, we conclude this chapter by considering the role of the empty set in the power domains with two-valued semiring more closely.

17.7 The two parts of a power domain for L, U, or C

For this section, let $\mathcal{P}$ be an arbitrary power construction with a two-valued semiring R, i.e. R is **L**, **U**, or **C**. The power domains $\mathcal{P}\mathbf{X}$ are shown to fall into two disjoint parts, one containing the empty set, and the other one containing all singletons.

From section 10.5, we know the predicate of non-emptiness. ne is a linear morphism from $\mathcal{P}\mathbf{X}$ to the characteristic semiring R for all domains **X**. We refer to section 10.5 for its further properties. The semirings **L**, **U**, and **C** all have the same carrier set $\{0, 1\}$. Hence, we may define $\mathcal{P}^0\mathbf{X} = ne^{-1}[0]$ and $\mathcal{P}^1\mathbf{X} = ne^{-1}[1]$.

Proposition 17.7.1

(1) Both $\mathcal{P}^0\mathbf{X}$ and $\mathcal{P}^1\mathbf{X}$ are closed w.r.t. directed lubs.

(2) θ is in $\mathcal{P}^0\mathbf{X}$.

(3) $\{\!|x|\!\}$ is in $\mathcal{P}^1\mathbf{X}$ for all x in **X**.

(4) Both $\mathcal{P}^0\mathbf{X}$ and $\mathcal{P}^1\mathbf{X}$ are closed w.r.t. union '$\uplus$'.

(5) Both $\mathcal{P}^0$ and $\mathcal{P}^1$ admit extension: If $f : [\mathbf{X} \to \mathcal{P}^i\mathbf{Y}]$ and A is in $\mathcal{P}^i\mathbf{X}$, then $ext\, f\, A$ is in $\mathcal{P}^i\mathbf{X}$ for $i = 0, 1$.

(6) In case $R = \mathbf{C}$: If A is in $\mathcal{P}^0\mathbf{X}$ and B in $\mathcal{P}^1\mathbf{X}$, then A and B are incomparable.

Proof:

(1) If D is a directed set in $\mathcal{P}^i\mathbf{X}$, then $ne\,(\bigsqcup D) = \bigsqcup ne\,[D] = \bigsqcup\{i\} = i$.

(2) $ne\,\theta = 0$.

(3) $ne\,\{\!|x|\!\} = 1$.

(4) If A and B are in $\mathcal{P}^i\mathbf{X}$, then $ne\,(A \uplus B) = ne\,A + ne\,B = i + i = i$ by idempotence.

(5) $ne\,(ext\,f\,A) = ext\,(\lambda x.\,ne\,(f x))\,A = ext\,(\lambda x.\,i)\,A = ext\,(\lambda x.\,1 \cdot i)\,A = ext\,(\lambda x.\,1)\,A \cdot i = ne\,A \cdot i = i \cdot i = i$

(6) If A were below B, then $0 = ne\,A \leq ne\,B = 1$ would hold. Analogously, A is not above B.

Whereas $\mathcal{P}^0\mathbf{X}$ contains nothing interesting than the empty set, $\mathcal{P}^1\mathbf{X}$ is almost a power domain; the only thing missing is the empty set. Because both $\mathcal{P}^0\mathbf{X}$ and $\mathcal{P}^1\mathbf{X}$ are non-empty and their respective elements are incomparable in case $R = \mathbf{C}$, $\mathbf{C}$-powerdomains never have a least element as a whole. $\mathcal{P}^1\mathbf{X}$ may however have a least element.

Proposition 17.7.2 Let $\mathcal{P}$ be reduced. Then $\mathcal{P}^0\mathbf{X} = \{\theta\}$ holds, and $\mathcal{P}^1\mathbf{X}$ has a least element, namely $\{\!|\bot|\!\}$, whenever $\mathbf{X}$ has a least element $\bot$.

Proof:

(1) Let $S = \{\theta\} \cup \mathcal{P}^1\mathbf{X}$. We show that S satisfies the properties of Def. 13.1.2. Then the core of $\mathcal{P}\mathbf{X}$, which is $\mathcal{P}\mathbf{X}$ itself, is a subset of S.

  (1) $\{\!|x|\!\} \in \mathcal{P}^1\mathbf{X} \subseteq S$.

  (2) $\theta \in S$.

  (3) If A and B are in $\mathcal{P}^1\mathbf{X}$, then $A \uplus B$ is also in it. If A is θ and B is in S, then $A \uplus B = B \in S$. Same for $B = \theta$.

  (4) If A is in S, then $r \cdot A$ is θ or A. In any case, it is in S.

  (5) Let D be a directed subset of S.

   Case $\mathbf{L}$: Either $D = \{\theta\}$ holds, or it contains a member of $\mathcal{P}^1\mathbf{X}$. Then $\bigsqcup D$ is in $\mathcal{P}^1\mathbf{X}$ since this set is upper.

   Case $\mathbf{U}$: Either $D \subseteq \mathcal{P}^1\mathbf{X}$ holds, then Prop. 17.7.1 (5) is employed. Or D contains θ, the greatest element of $\mathcal{P}\mathbf{X}$. Then $\bigsqcup D = \theta$ holds.

   Case $\mathbf{C}$: D is a subset of either $\{\theta\}$ or $\mathcal{P}^1\mathbf{X}$ because these two subsets of S have respectively uncomparable elements.

   In any case, $\bigsqcup D$ is in S.

(2) Let $S = \{\theta\} \cup T$, where $T = \{A \in \mathcal{P}^1\mathbf{X} \mid A \geq \{\!|\bot|\!\}\}$. Again, we show that S satisfies the core properties.

  (1) $\{\!|x|\!\} \geq \{\!|\bot|\!\}$ since $x \geq \bot$.

  (2) $\theta \in S$.

  (3) If A and B are in T, then $A \uplus B \geq \{\!|\bot|\!\} \uplus \{\!|\bot|\!\} = \{\!|\bot|\!\}$ holds by idempotence. If one of A and B is θ, then $A \uplus B$ is the other one.

  (4) $r \cdot A$ is θ or A.

  (5) The proof is similar to the proof of statement (5) in part (1). □

After having considered reduced constructions, we now turn to the final ones.

Proposition 17.7.3 Let $\mathcal{P}$ be final for the semiring R in $\{\mathbf{C}, \mathbf{L}, \mathbf{U}\}$. Then $\mathcal{P}^0\mathbf{X} = \{\theta\}$ holds for all ground domains $\mathbf{X}$.

Proof: For all $P : [[\mathbf{X} \to R] \stackrel{add}{\to} R]$, $ne\, P = ex\,(\lambda x.\, 1)\, P = P\,(\lambda x.\, 1)$ holds. Hence, we have to show that $P\,(\lambda x.\, 1) = 0$ implies $P = \theta$, i.e. $Pp = 0$ for all $p : [\mathbf{X} \to R]$.

From $1 = 1 + r$ for all r in R, we conclude $0 = P\,(\lambda x.\, 1) = P\,(\lambda x.\, 1 + p\, x) = P\,(\lambda x.\, 1) + Pp = 0 + Pp = Pp$ using additivity. □

In the literature, one often finds power constructions without empty set, i.e. $\mathcal{P}^1$ instead of $\mathcal{P}$. Plotkin's construction for instance is the non-empty part of $\mathcal{P}_i^{\mathbf{C}}$. In case of semiring $\mathbf{C}$, reduced power constructions $\mathcal{P}$ are particularly ugly because their power domains contain θ as a completely uncomparable point. The non-empty construction $\mathcal{P}^1$ then has the advantage to deliver power domains with least element whenever the ground domain has a least element.

For our general algebraic theory of power constructions however as developed in part II, the empty set cannot be dispensed with. Without the empty set, the characteristic semiring cannot be found since $\mathcal{P}^1\mathbf{1} = \{1\}$ holds without difference for all lower, upper, and convex power constructions.

Chapter 18

Lower power constructions

This chapter is concerned with the lower power constructions, i.e. the constructions with semiring $\mathbf{L} = \{0 < 1\}$. We start by considering the final lower construction $\mathcal{L}$. By Th. 15.1.1, it is given in terms of second order predicates. In the course of sections 18.1 and 18.2, we translate it into terms of *open grills*. The usual representation in terms of Scott closed sets is shown to be isomorphic in section 18.3. It provides enough structure to prove initiality of $\mathcal{L}$ (section 18.5). The domain theoretic properties are studied in the final section 18.6.

18.1 The final lower construction in terms of predicates

According to Th. 15.1.1, the final construction for semiring $\mathbf{L}$ is given in predicative form by $\mathcal{L}_f\mathbf{X} = [[\mathbf{X} \to \mathbf{L}] \stackrel{rlin}{\to} \mathbf{L}]$. By additivity of $\mathbf{L}$, every additive map between two $\mathbf{L}$-modules is linear, and we obtain $\quad \mathcal{L}_f\mathbf{X} = [[\mathbf{X} \to \mathbf{L}] \stackrel{add}{\to} \mathbf{L}]$.

As a domain, $\mathbf{L}$ has the carrier $\mathbf{2} = \{\bot < \top\}$. Addition in $\mathbf{L}$ is least upper bound '$\sqcup$' and 0 is $\bot$. Hence, one may also write $\quad \mathcal{L}_f\mathbf{X} = [[\mathbf{X} \to \mathbf{2}] \stackrel{\bot\sqcup}{\to} \mathbf{2}]$
indicating at the arrow what operations have to be preserved by the second order predicates.

In section 3.7, we showed that the domain $\Omega\mathbf{X}$ of open sets of $\mathbf{X}$ ordered by inclusion is isomorphic to the function space $[\mathbf{X} \to \mathbf{2}]$. The isomorphism is given by

$$O \in \Omega\mathbf{X} \;\mapsto\; p_O = \lambda x.\begin{cases} \top & \text{if } x \in O \\ \bot & \text{otherwise} \end{cases} \qquad \text{and}$$
$$p \in [\mathbf{X} \to \mathbf{2}] \;\mapsto\; O_p = p^{-1}[\top] = \{x \in \mathbf{X} \mid p\,x = \top\}$$

Its properties are summarized in the following table:

$\Omega\mathbf{X}$	$x \in O$	$\subseteq$	$\cup$	$\cap$	$\emptyset$	$\mathbf{X}$
$[\mathbf{X} \to \mathbf{2}]$	$p\,x = \top$	$\leq$	$\sqcup$	$\sqcap$	$\lambda x.\,\bot$	$\lambda x.\,\top$

Thus, $\Omega(\Omega\mathbf{X})$ is isomorphic to $[[\mathbf{X} \to \mathbf{2}] \to \mathbf{2}]$ by means of $\mathcal{E}\mathcal{O} = \lambda p.\, p_{\mathcal{O}} O_p$, or more elaborated

$$\mathcal{E}\mathcal{O} = \lambda p.\begin{cases} \top & \text{if } p^{-1}[\top] \in \mathcal{O} \\ \bot & \text{otherwise} \end{cases} \qquad \text{and its inverse} \qquad \varphi P = \{O \mid P\,p_O = \top\}.$$

Hence, $\mathcal{L}_f\mathbf{X}$ corresponds to a subset of $\Omega(\Omega\mathbf{X})$. $P\,(\lambda x.\bot) = \bot$ has to hold which translates into $\emptyset \notin \mathcal{G}$ where $\mathcal{G}$ is the open set of open sets corresponding to P. In addition, $P\,(p \sqcup q) = P\,p \sqcup P\,q$ has to hold, or equivalently $P\,(p \sqcup q) = \top$ iff $P\,p = \top$ or $P\,q = \top$. This translates into $O \cup O' \in \mathcal{G}$ iff $O \in \mathcal{G}$ or $O' \in \mathcal{G}$. The implication from right to left always holds since $\mathcal{G}$ is an upper set because it is open. Hence, only the implication from left to right matters. In analogy to a topological notion, we call open sets with these properties *grills*.

18.2 The lower power construction in terms of grills

An *open grill* of $\mathbf{X}$ is an open set $\mathcal{G}$ in $\Omega\mathbf{X}$ satisfying the two grill properties:

(1) $\emptyset$ is not in $\mathcal{G}$,

(2) Let O and O' be open sets in $\mathbf{X}$. If $O \cup O'$ is in $\mathcal{G}$, then at least one of O and O' is in $\mathcal{G}$.

Let $\mathcal{L}_\Gamma\mathbf{X}$ be the poset of open grills of $\mathbf{X}$ ordered by inclusion.

Theorem 18.2.1 $\mathcal{L}_f\mathbf{X}$ and $\mathcal{L}_\Gamma\mathbf{X}$ are isomorphic for all ground domains $\mathbf{X}$. The power operations for $\mathcal{L}_\Gamma\mathbf{X}$ are given by the following table:

$\mathcal{L}\mathbf{X}$	$[[\mathbf{X} \to \mathbf{L}] \overset{add}{\to} \mathbf{L}]$	$\mathcal{L}_\Gamma\mathbf{X}$
$A \leq B$	$\forall p : A\,p \leq B\,p$	$A \subseteq B$
$\bigsqcup\mathcal{D}$	$\lambda p.\, \bigsqcup_{D\in\mathcal{D}} Dp$	$\bigcup\mathcal{D}$
θ	$\lambda p.\, 0$	$\emptyset$
$A \uplus B$	$\lambda p.\, Ap + Bp$	$A \cup B$
$\{\!\vert x\vert\!\}$	$\lambda p.\, px$	$\mathcal{O}(x) = \{O \mid x \in O\}$
$ext\, f\, A$	$\lambda p.\, A\,(\lambda x.\, fxp)$	$\{O \mid \{x \mid O \in fx\} \in A\}$

Proof: Isomorphism and order are already known. One easily verifies that arbitrary unions of open grills are open grills again. Hence, $\bigcup\mathcal{D}$ is the lub of the directed set $\mathcal{D}$.

$\theta = \varphi\,(\lambda p.\, 0) = \{O \mid (\lambda p.\,\bot)\,p_O = \top\} = \emptyset$

$A \uplus B = \varphi\,(\lambda p.\, \mathcal{E}A\,p + \mathcal{E}B\,p) = \{O \mid \mathcal{E}A\,p_O \sqcup \mathcal{E}B\,p_O = \top\} = \{O \mid O \in A \text{ or } O \in B\} = A \cup B$

$\{\!\vert x\vert\!\} = \varphi\,(\lambda p.\, p\,x) = \{O \mid p_O\,x = \top\} = \{O \mid x \in O\} = \mathcal{O}(x)$

$$\begin{aligned}
ext\, f\, A &= \varphi\,(\lambda p.\, \mathcal{E}A\,(\lambda x.\, \mathcal{E}(fx)p)) \\
&= \{O \mid \mathcal{E}A\,(\lambda x.\, \mathcal{E}(fx)p_O) = \top\} \\
&= \{O \mid (\lambda x.\, \mathcal{E}(fx)p_O)^{-1}[\top] \in A\} \\
&= \{O \mid \{x \mid \mathcal{E}(fx)p_O = \top\} \in A\} \\
&= \{O \mid \{x \mid O \in fx\} \in A\} \qquad \Box
\end{aligned}$$

Summarizing, we see that the lower power domain in terms of open grills is quite unhandy, and the realization of the power operations, in particular of extension, is quite complex. Fortunately, we need neither show the continuity of *ext f* nor the validity of the power axioms for $\mathcal{L}_\Gamma$ since the isomorphism gives this for free.

Next, we derive formulae for some of the derived operations in $\mathcal{L}_\Gamma$.

$\mathcal{L}\mathbf{X}$	$[[\mathbf{X} \to \mathbf{L}] \stackrel{add}{\to} \mathbf{L}]$	$\mathcal{L}_\Gamma\mathbf{X}$
$map\, f\, A$	$\lambda p.\, A\,(p \circ f)$	$\{O \mid f^{-1}[O] \in A\}$
$\mathcal{E}\, A\, p$	$A\, p$	$\mathcal{E}\, A\, p = 1$ iff $p^{-1}[1] \in A$
$ne\, A$	$A\,(\lambda x.\, 1)$	$ne\, A = 1$ iff $\mathbf{X} \in A$ iff $A \neq \theta$
$filter\, q\, A$	$\lambda p.\, A\,(q \cdot p)$	$\{O \mid q^{-1}[1] \cap O \in A\}$
$U A$	$\lambda p.\, A\,(\lambda a.\, a\, p)$	$\{O \mid \{x \in \mathcal{L}_\Gamma\mathbf{X} \mid O \in x\} \in A\}$
$A \overrightarrow{\times} B$	$\lambda p.\, A\,(\lambda a.\, B\,(\lambda b.\, p\,(a,\, b)))$	$\{O \mid \{a \mid \{b \mid (a,\, b) \in O\} \in B\} \in A\}$
$A \overleftarrow{\times} B$	$\lambda p.\, B\,(\lambda b.\, A\,(\lambda a.\, p\,(a,\, b)))$	$\{O \mid \{b \mid \{a \mid (a,\, b) \in O\} \in A\} \in B\}$

Proof:

$$\begin{aligned} map\, f\, A &= ext\,(\iota \circ f)\, A \\ &= \{O \mid \{x \mid O \in \mathcal{O}(fx)\} \in A\} \\ &= \{O \mid \{x \mid fx \in O\} \in A\} \\ &= \{O \mid f^{-1}[O] \in A\} \end{aligned}$$

Abusing notation, one could write the last set as $(f^{-1})^{-1}[A]$.

The statement about $\mathcal{E}$ holds because of the correspondence between $\Omega\mathbf{X}$ and $[\mathbf{X} \to \mathbf{L}]$. For ne, note that $(\lambda x.\, 1)^{-1}[1] = \mathbf{X}$. Obviously, $\mathbf{X}$ is not contained in $\theta = \emptyset$. Conversely, if $A \neq \emptyset$, then A contains $\mathbf{X}$ because A is an upper set.

$$\begin{aligned} filter\, q\, A &= ext\,(\lambda x.\, qx \cdot \{\!|x|\!\})\, A \\ &= \{O \mid \{x \mid O \in (qx \cdot \mathcal{O}(x))\} \in A\} \end{aligned}$$

Note that $qx \cdot \mathcal{O} = \emptyset$ if $qx = 0$, and $= \mathcal{O}$ if $qx = 1$.

$$\begin{aligned} &= \{O \mid \{x \mid qx = 1 \wedge O \in \mathcal{O}(x)\} \in A\} \\ &= \{O \mid \{x \mid qx = 1 \wedge x \in O\} \in A\} \\ &= \{O \mid q^{-1}[1] \cap O \in A\} \end{aligned}$$

$$\begin{aligned} U A &= ext\; id\; A \\ &= \{O \in \Omega\mathbf{X} \mid \{X \in \mathcal{L}_\Gamma\mathbf{X} \mid O \in X\} \in A\} \end{aligned}$$

$$\begin{aligned} A \overrightarrow{\times} B &= ext\,(\lambda a.\, ext\,(\lambda b.\, \{\!|(a,\, b)|\!\})\, B)\, A \\ &= \{O \mid \{a \mid O \in ext\,(\lambda b.\, \{\!|(a,\, b)|\!\})\, B\} \in A\} \\ &= \{O \mid \{a \mid O \in \{O' \mid \{b \mid O' \in \{\!|(a,\, b)|\!\}\} \in B\} \in A\} \\ &= \{O \mid \{a \mid O \in \{O' \mid \{b \mid (a,\, b) \in O'\} \in B\} \in A\} \\ &= \{O \mid \{a \mid \{b \mid (a,\, b) \in O\} \in B\} \in A\} \end{aligned}$$

$A \overleftarrow{\times} B$ is handled analogously. □

Looking at the formulae for $A \overrightarrow{\times} B$ and $A \overleftarrow{\times} B$, it is not obvious whether they always equal or are sometimes different. We shall however see below that $\mathcal{L}$ is symmetric.

18.3 The lower power construction in terms of closed sets

In this section, we show that the common lower power construction in terms of closed sets is isomorphic to $\mathcal{L}_\Gamma \mathbf{X}$.

Proposition 18.3.1
$\mathcal{L}_\Gamma \mathbf{X}$ is isomorphic to the poset $\mathcal{L}_C \mathbf{X}$ of closed sets of $\mathbf{X}$ ordered by inclusion.

Proof: Given a closed set C, let $\mathcal{G}(C)$ be the set of all open sets of $\mathbf{X}$ that meet C. $\mathcal{G}(C)$ obviously does not contain $\emptyset$. If an open set O meets C, then every superset of O does so; hence, $\mathcal{G}(C)$ is an upper set of open sets. If a union of open sets meets C, then at least one of its constituents meets C. Thus, $\mathcal{G}(C)$ is in particular open and satisfies the implication involving $O \cup O'$. Summarizing, $\mathcal{G}(C)$ is an open grill for all closed sets C of $\mathbf{X}$.

The larger a closed set is, the more open sets meet it. Hence, $C \subseteq C'$ implies $\mathcal{G}(C) \subseteq \mathcal{G}(C')$. Conversely, assume $\mathcal{G}(C) \subseteq \mathcal{G}(C')$ holds, and let c be a point in C. Then every open environment of c is in $\mathcal{G}(C')$, i.e. meets C'. Thus, c is in $\mathsf{cl}\, C' = C'$ by Prop. 4.2.2. Summarizing, $C \subseteq C'$ holds iff $\mathcal{G}(C) \subseteq \mathcal{G}(C')$.

Finally, we have to show that the mapping $\mathcal{G}(.)$ is surjective. Let $\mathcal{G}$ be an open grill. Let U be the union of all open sets of $\mathbf{X}$ that are *not* in $\mathcal{G}$, and let C be its complement. U is open as union of open sets, whence C is closed. We claim $\mathcal{G} = \mathcal{G}(C)$.

The set $\mathcal{S} = \{O \text{ open} \mid O \notin \mathcal{G}\}$ is directed: It is not empty since $\emptyset$ is in it, and $O, O' \notin \mathcal{G}$ implies $O \cup O' \notin \mathcal{G}$. If U, the union of the directed set $\mathcal{S}$, were in $\mathcal{G}$, then one of the members of $\mathcal{S}$ would be in $\mathcal{G}$ as $\mathcal{G}$ is open. Thus, U is not in $\mathcal{G}$.

If an open set O meets C, then O is not a subset of U. Thus, $O \cup U$ is a proper superset of U. Hence, it is in $\mathcal{G}$ since U is the union of all open sets not in $\mathcal{G}$. $O \cup U \in \mathcal{G}$ and $U \notin \mathcal{G}$ imply $O \in \mathcal{G}$.

If O does not meet C, then O is a subset of U. If O were in $\mathcal{G}$, then U were in $\mathcal{G}$, too, as $\mathcal{G}$ is an upper set. The last two paragraphs together show $\mathcal{G}(C) = \mathcal{G}$. □

After establishing this isomorphism, we translate the power operations into terms of closed sets.

Theorem 18.3.2
The final lower power construction $\mathcal{L}_f$ is isomorphic to the power construction $\mathcal{L}_C$ given by

(1) $\mathcal{L}_C \mathbf{X} = \{C \subseteq \mathbf{X} \mid C \text{ is Scott closed}\}$ ordered by inclusion '$\subseteq$',

(2) $\bigsqcup_{i \in I} A_i = \mathsf{cl} \bigcup_{i \in I} A_i$ where 'cl' denotes Scott closure,

(3) $\theta = \emptyset$,

(4) $A \uplus B = A \cup B$,

(5) $\{\!|x|\!\} = \downarrow x$,

(6) $ext\, f\, A = \bigsqcup f[A] = \mathsf{cl} \bigcup f[A] = \mathsf{cl} \bigcup_{a \in A} fa$.

Proof:

(1) The isomorphism is already known (Prop. 18.3.1).

(2) Because $\mathsf{cl}\bigcup_{i\in I} A_i$ is the least closed superset of $\bigcup_{i\in I} A_i$.

(3) $\mathcal{G}(\emptyset) = \{O \mid O \cap \emptyset \neq \emptyset\} = \emptyset = \theta_\Gamma$.

(4) An open set meets $A \cup B$ iff it meets A or meets B. Hence, $\mathcal{G}(A \cup B) = \mathcal{G}(A) \cup \mathcal{G}(B) = \mathcal{G}(A) \uplus_\Gamma \mathcal{G}(B)$.

(5) An open set meets $\downarrow x$ iff it contains x, since it is upper. Hence, $\mathcal{G}(\downarrow x) = \{O \mid x \in O\} = \{\!|x|\!\}_\Gamma$.

(6) By Prop. 4.2.3, an open set meets $\mathsf{cl}\, S$ iff it meets S. Hence,

$$\begin{aligned}
\mathcal{G}(\mathsf{cl}\bigcup f[A]) &= \{O \mid O \cap \bigcup f[A] \neq \emptyset\}\\
&= \{O \mid \exists a \in A : O \cap fa \neq \emptyset\}\\
&= \{O \mid \exists a \in A : O \in \mathcal{G}(fa)\}\\
&= \{O \mid A \cap \{a \mid O \in \mathcal{G}(fa)\} \neq \emptyset\}\\
&= \{O \mid \{a \mid O \in \mathcal{G}(fa)\} \in \mathcal{G}(A)\}\\
&= \mathit{ext}_\Gamma\ (\mathcal{G}(.) \circ f)\ (\mathcal{G}(A))
\end{aligned}$$

Here, we have to make sure that $\{a \mid O \in \mathcal{G}(fa)\}$ is open. It is the inverse image by f of the open set $\{C' \in \mathcal{L}_C\mathbf{Y} \mid O \in \mathcal{G}(C')\} = \{C' \mid O \cap C' \neq \emptyset\}$.

These equations show that $\mathcal{G}(.)$ becomes a power isomorphism if the operations for closed sets are chosen as in the theorem. □

18.4 Derived operations

In this section, explicit formulae in terms of closed sets are presented for the derived power operations. These formulae all look quite natural.

- $\theta = \emptyset$,
- $A \uplus B = A \cup B$,
- $\{\!|x|\!\} = \downarrow x$,
- Combining union and singleton, one obtains $\{\!|x_1, \ldots, x_n|\!\} = \downarrow\{x_1, \ldots, x_n\}$.
- $\mathit{ext}\ f\ A = \mathsf{cl}\bigcup_{a\in A} fa$.
- $$\begin{aligned}
\mathit{map}\ f\ A &= \mathit{ext}\ (\lambda x.\, \{\!|fx|\!\})\ A\\
&= \mathsf{cl}\textstyle\bigcup_{a\in A} \downarrow(fa)\\
&= \mathsf{cl}\downarrow f[A] && \text{by Prop. 2.2.8}\\
&= \mathsf{cl}\, f[A] && \text{by Prop. 3.6.3 (1)}
\end{aligned}$$
- The big union is given by
  $U A = \mathit{ext}\ (\lambda x.\, x)\ A = \mathsf{cl}\bigcup_{a\in A} a = \mathsf{cl}\bigcup A$
- The formal Cartesian product coincides with the mathematical one:
  $$\begin{aligned}
A \overrightarrow{\times} B &= \mathit{ext}\ (\lambda a.\, \mathit{ext}\ (\lambda b.\, \{\!|(a, b)|\!\})\ B)\ A\\
&= \textstyle\bigsqcup_{a\in A} \bigsqcup_{b\in B} \downarrow\{(a, b)\}\\
&= \textstyle\bigsqcup_{c\in A\times B} \downarrow\{c\}\\
&= \mathsf{cl}\textstyle\bigcup_{c\in A\times B} \downarrow c\\
&= \mathsf{cl}\downarrow(A \times B) \ = \ A \times B
\end{aligned}$$

The very last equation holds because the product of closed sets is closed again by Prop. 4.8.1. $A \overleftarrow{\times} B$ also results in $A \times B$, whence $\mathcal{L}$ is symmetric.

- External left and right product coincide and are given by $0 \cdot A = \theta$ and $1 \cdot A = A$.
- For the filter operation, let $p : [\mathbf{X} \to \mathbf{L}]$ be a predicate.

$$\begin{aligned} \mathit{filter}\ p\ A &= \mathit{ext}\ (\lambda a.\, p\, a \cdot \{\!|a|\!\})\ A \;=\; \mathsf{cl} \textstyle\bigcup_{a \in A}(p\, a \cdot \downarrow a) \\ &= \mathsf{cl} \textstyle\bigcup \{\downarrow a \mid a \in A,\ p\, a = 1\} \;=\; \mathsf{cl}\ \{a \in A \mid p\, a = 1\} \;=\; \mathsf{cl}\,(A \cap p^{-1}[1]) \end{aligned}$$

By continuity of p, the set $p^{-1}[1]$ is open. Hence, the closure operator cannot be omitted.

- Let $p : [\mathbf{X} \to \mathbf{L}]$ be a predicate. Then

$$\mathcal{E}\, A\, p \;=\; \mathit{ext}\, p\, A \;=\; \mathsf{cl} \bigcup_{a \in A} p\, a$$

Identifying $\emptyset$ with 0 and $\{\diamond\}$ with 1, the set $\bigcup_{a \in A} p\, a$ is empty iff $p\, a = 0$ for all a in A, and it is $\{\diamond\}$ iff there is a in A with $p\, a = 1$. The closure operator is not needed since both $\emptyset$ and $\{\diamond\}$ are closed. Hence,

$$\mathcal{E}\, A\, p = \begin{cases} 1 & \text{if } \exists a \in A : p\, a = 1 \\ 0 & \text{otherwise} \end{cases}$$

This result again justifies the view of $\mathcal{E}$ as existential quantification.

- The non-empty predicate also is named properly in this case:

$$\mathit{ne}\ A = \mathcal{E}\ A\, (\lambda x.\, 1) = \begin{cases} 1 & \text{if } A \neq \emptyset \\ 0 & \text{otherwise} \end{cases}$$

18.5 Initiality of $\mathcal{L}$

Our next goal is to prove that the final construction for semiring $\mathbf{L}$ is also initial. To this end, we prove that the lower power domains in terms of closed sets are free $\mathbf{L}$-$\mathbf{X}$-modules.

Theorem 18.5.1 The final power construction for semiring $\mathbf{L}$ is also initial.

Proof: Let $\mathcal{L}\mathbf{X}$ be the lower power domain over $\mathbf{X}$ in terms of closed sets. As a power domain, it is an $\mathbf{L}$-module.

Let $f : [\mathbf{X} \to M]$ be a morphism from $\mathbf{X}$ to some $\mathbf{L}$-module M. Because M is a complete domain, one may define $\hat{f} A = \bigsqcup f[A]$ for every subset A of $\mathbf{X}$. We show three main properties of $\hat{f}$.

(1) $\hat{f}(\downarrow x) = f x$ follows from the monotonicity of f.

(2) $\hat{f}(\bigcup_{i \in I} A_i) = \bigsqcup_{i \in I} \hat{f} A_i$
Proof: $\hat{f}(\bigcup_{i \in I} A_i) \;=\; \bigsqcup f[\bigcup_{i \in I} A_i] \;=\; \bigsqcup \bigcup_{i \in I} f[A_i] \;=\; \bigsqcup_{i \in I} \bigsqcup f[A_i] \;=\; \bigsqcup_{i \in I} \hat{f} A_i$
Hence, $\hat{f}$ is in particular linear and monotonic w.r.t. '$\subseteq$'.

(3) $\hat{f}(\mathsf{cl}\, A) = \hat{f} A$ for all $A \subseteq \mathbf{X}$.
'$\geq$' follows from $\mathsf{cl}\, A \supseteq A$. For the opposite direction, let $y = \hat{f} A$, and assume there is $x \in \mathsf{cl}\, A$ with $f x \not\leq y$. Then $f x$ is in the open set $O = M \setminus \downarrow y$. By continuity of f, $O' = f^{-1}[O]$ is an open environment of x. Since x is in the closure of A, O' meets A, i.e.

there is a in A with $fa \in O$, i.e. $fa \not\leq y = \bigsqcup f[A]$. This contradiction implies $fx \leq y$ for all $x \in \mathsf{cl}\, A$, whence $\hat{f}(\mathsf{cl}\, A) \leq y$.

Now, we restrict $\hat{f}$ to $\bar{f} : \mathcal{L}\mathbf{X} \to M$. $\bar{f}$ is still linear and satisfies $\bar{f}\{\!|x|\!\} = fx$ by (1). It is continuous because

$$\bar{f}\,(\bigsqcup A_i) = \hat{f}(\mathsf{cl} \bigcup A_i) \overset{(3)}{=} \hat{f}(\bigcup A_i) \overset{(2)}{=} \bigsqcup \hat{f} A_i$$

Let $F_1, F_2 : [\mathcal{L}\mathbf{X} \to M]$ be two linear extensions of a morphism $f : [\mathbf{X} \to M]$. Then F_1 and F_2 coincide for cones $\downarrow x$. By linearity, they thus also coincide for all sets $\downarrow F$ where F is finite. If C is an arbitrary closed set, then $\mathcal{D} = \{\downarrow F \mid F \text{ is a finite subset of } C\}$ is directed and has union (and thus limit) C. Hence, $F_1 = F_2$ holds by continuity. □

18.6 Domain-theoretic properties

We already know that $\mathcal{L}\mathbf{X}$ is a complete domain no matter what $\mathbf{X}$ is, and that $\mathcal{L}$ preserves finiteness, algebraicity, and continuity. Hence, it also preserves all classes between algebraic and complete algebraic domains, e.g. the class of bounded complete algebraic domains and the class of finitely algebraic domains. In section 20.5, we show that $\mathcal{L}$ also preserves multi-algebraicity and multi-continuity.

Although we know that $\mathcal{L}\mathbf{X}$ is algebraic whenever $\mathbf{X}$ is, we do not know the base. It is provided by the following theorem:

Theorem 18.6.1 Let $\mathbf{X}$ be an algebraic domain with base $\mathbf{X}^0$. Then $\mathcal{L}\mathbf{X}$ is algebraic with base $\{\downarrow F \mid F \subseteq_f \mathbf{X}^0\}$.

Proof: First, we show that the sets $\downarrow F$ with $F \subseteq_f \mathbf{X}^0$ are isolated. They are closed by Prop. 3.6.3 (2).

Let F be a finite subset of $\mathbf{X}^0$, and let $\downarrow F \subseteq \bigsqcup_{i \in I} C_i = \mathsf{cl} \bigcup_{i \in I} C_i$. Since F consists of isolated points, $F \subseteq \bigcup_{i \in I} C_i$ follows by Prop. 6.1.3. Since F is finite, there is k in I with $F \subseteq C_k$ by Prop. 6.3.3. C_k being lower implies $\downarrow F \subseteq C_k$.

Now, we show that every closed set is the lub of a directed set of isolated closed sets. For given closed set C, let

$$\mathcal{D} = \{\downarrow F \mid F \subseteq_f C \cap \mathbf{X}^0\}$$

We claim $C = \mathsf{cl} \bigcup \mathcal{D}$.

The set $\mathcal{D}$ is directed because of $\downarrow F_1 \cup \downarrow F_2 = \downarrow (F_1 \cup F_2)$. The inclusion '$\supseteq$' certainly holds, because $F \subseteq C$ and C is closed. For the opposite inclusion, let x be a point of C. Since $\mathbf{X}$ is algebraic, there is a directed set $D' \subseteq \mathbf{X}$ with $x = \bigsqcup D'$. For every d in D', $\{d\} \subseteq_f C \cap \mathbf{X}^0$ holds, whence $D' \subseteq \bigcup \mathcal{D} \subseteq \mathsf{cl} \bigcup \mathcal{D}$. Because $\mathsf{cl} \bigcup \mathcal{D}$ is closed, x is a point of it. □

Chapter 19

Upper power constructions

This chapter is concerned with the upper power constructions, i.e. the constructions with semiring $\mathbf{U} = \{1 < 0\}$. Although this semiring looks as simple as the lower semiring $\mathbf{L} = \{0 < 1\}$, the situation here is much more complex. The theory is considerably harder than in the lower case, and nevertheless produces weaker results. For instance, we do not know whether initial and final upper power construction coincide.[1] There is some evidence that they do not; we shall provide hints at the appropriate places.

Again, we start by considering the final $\mathbf{U}$-construction. Its original form in terms of second order predicates is translated into terms of open filters in section 19.1. If the ground domain $\mathbf{X}$ is sober, then $\mathcal{U}\mathbf{X}$ may be translated further into a representation by compact upper sets (section 19.2). The domain-theoretic properties of these power domains are studied in section 19.3. In section 19.4, we attempt to prove the initiality of the final construction $\mathcal{U}$. In doing so, we point out why we did not succeed in finding a general proof such that a proof for continuous ground domains is the only result.

19.1 The upper construction $\mathcal{U}_f$ in terms of open filters

According to Th. 15.1.1, the final construction for semiring $\mathbf{U}$ is given in predicative form by $\mathcal{U}_f\mathbf{X} = [[\mathbf{X} \to \mathbf{U}] \overset{rlin}{\to} \mathbf{U}]$. By additivity of $\mathbf{U}$, every additive map between two $\mathbf{U}$-modules is linear, and we obtain $\qquad \mathcal{U}_f\mathbf{X} = [[\mathbf{X} \to \mathbf{U}] \overset{add}{\to} \mathbf{U}]$.

The carrier domain of $\mathbf{U}$ is $\mathbf{2} = \{\bot < \top\}$. It coincides with that of $\mathbf{L}$. In contrast to $\mathbf{L}$, addition in $\mathbf{U}$ is '$\sqcap$' and 0 is $\top$. Hence, one may also write $\mathcal{U}_f\mathbf{X} = [[\mathbf{X} \to \mathbf{2}] \overset{\top\sqcap}{\to} \mathbf{2}]$ indicating at the arrow what operations have to be preserved by the second order predicates.

Appealing again to the correspondence between predicates and open sets, $\mathcal{U}_f\mathbf{X}$ is isomorphic to a subset of $\Omega(\Omega\mathbf{X})$. $P\,(\lambda x.\,\top) = \top$ in $[[\mathbf{X} \to \mathbf{U}] \to \mathbf{U}]$ corresponds to $\mathbf{X} \in \mathcal{F}$ in $\Omega(\Omega\mathbf{X})$ where $\mathcal{F}$ is the set of open sets corresponding to P. In addition, $P\,(p \sqcap q) = P\,p \sqcap P\,q$ has to hold, or equivalently $P\,(p \sqcap q) = \top$ iff $P\,p = \top$ and $P\,q = \top$. This translates into $O \cap O' \in \mathcal{F}$ iff $O \in \mathcal{F}$ and $O' \in \mathcal{F}$. The implication from left to right always holds since $\mathcal{F}$ is an upper set because it is open. Hence, only the opposite implication matters. Comparing

[1] Meanwhile, they are shown to be different.

these two properties of $\mathcal{F}$ with Def. 8.6.1, one sees that $\mathcal{U}_f\mathbf{X}$ corresponds to the set of *open filters* of $\mathbf{X}$.

Open filters are open $\geq$-directed sets in the domain $(\Omega\mathbf{X}, \subseteq)$ of open sets of $\mathbf{X}$. The poset of all open filters of $\mathbf{X}$ ordered by inclusion '$\subseteq$' is denoted by $\mathcal{U}_\Phi\mathbf{X}$.

Arbitrary unions of open sets are open, and directed unions of directed sets are directed by Prop. 3.1.5. Hence, the union of a directed set of open filters is an open filter. This makes $\mathcal{U}_\Phi\mathbf{X}$ a domain with limits given by union. The isomorphism between $\mathcal{U}_f\mathbf{X}$ and $\mathcal{U}_\Phi\mathbf{X}$ is given by

$$\mathcal{E}\mathcal{O} = \lambda p. \begin{cases} \top & \text{if } p^{-1}[\top] \in \mathcal{O} \\ \bot & \text{otherwise} \end{cases} \qquad \text{and its inverse} \qquad \varphi P = \{O \mid P\, p_O = \top\}.$$

as pointed out in section 18.1.

Theorem 19.1.1 $\mathcal{U}_f\mathbf{X}$ and $\mathcal{U}_\Phi\mathbf{X}$ are isomorphic for all ground domains $\mathbf{X}$. The power operations for $\mathcal{U}_\Phi\mathbf{X}$ are given by the following table:

$\mathcal{U}\mathbf{X}$	$[[\mathbf{X} \to \mathbf{U}] \overset{add}{\to} \mathbf{U}]$	$\mathcal{U}_\Phi\mathbf{X}$
$A \leq B$	$\forall p : A\,p \leq B\,p$	$A \subseteq B$
$\bigsqcup \mathcal{D}$	$\lambda p.\, \bigsqcup_{D\in\mathcal{D}} Dp$	$\bigcup \mathcal{D}$
θ	$\lambda p.\, 0$	$\Omega\mathbf{X}$
$A \uplus B$	$\lambda p.\, Ap + Bp$	$A \cap B$
$\{\!\|x\|\!\}$	$\lambda p.\, p\,x$	$\mathcal{O}(x) = \{O \mid x \in O\}$
$ext\; f\; A$	$\lambda p.\, A\,(\lambda x.\, fxp)$	$\{O \mid \{x \mid O \in fx\} \in A\}$

Proof: The facts about order and limits are already known.

$\theta = \varphi\,(\lambda p.\, 0) = \{O \mid (\lambda p.\, \top)\, p_O = \top\} = \Omega\mathbf{X}$.

$A \uplus B = \varphi\,(\lambda p. \mathcal{E}A\,p + \mathcal{E}B\,p) = \{O \mid \mathcal{E}A\,p_O \sqcap \mathcal{E}B\,p_O = \top\} = \{O \mid O \in A \text{ and } O \in B\} = A \cap B$

The formulae for the operations $\{\!|.|\!\}$ and *ext* and their proofs look exactly as those in Th. 18.2.1. □

Next, we derive formulae for some of the derived operation in $\mathcal{U}_\Phi$.

$\mathcal{U}\mathbf{X}$	$[[\mathbf{X} \to \mathbf{U}] \overset{add}{\to} \mathbf{U}]$	$\mathcal{U}_\Phi\mathbf{X}$
$map\; f\; A$	$\lambda p.\, A\,(p \circ f)$	$\{O \mid f^{-1}[O] \in A\}$
$\mathcal{E}A\,p$	$A\,p$	$\mathcal{E}A\,p = 0$ iff $p^{-1}[0] \in A$
$ne\; A$	$A\,(\lambda x.\, 1)$	$ne\; A = 0$ iff $\emptyset \in A$ iff $A = \theta$
$filter\; q\; A$	$\lambda p.\, A\,(q \cdot p)$	$\{O \mid q^{-1}[0] \cup O \in A\}$
$U A$	$\lambda p.\, A\,(\lambda a.\, a\,p)$	$\{O \mid \{x \in \mathcal{U}_\Phi\mathbf{X} \mid O \in x\} \in A\}$
$A \overset{\rightarrow}{\times} B$	$\lambda p.\, A\,(\lambda a.\, B\,(\lambda b.\, p\,(a, b)))$	$\{O \mid \{a \mid \{b \mid (a, b) \in O\} \in B\} \in A\}$
$A \overset{\leftarrow}{\times} B$	$\lambda p.\, B\,(\lambda b.\, A\,(\lambda a.\, p\,(a, b)))$	$\{O \mid \{b \mid \{a \mid (a, b) \in O\} \in A\} \in B\}$

Proof: The formulae for *map*, U, '$\overset{\rightarrow}{\times}$', and '$\overset{\leftarrow}{\times}$' equal those in $\mathcal{L}_\Gamma$. Their proofs are similar to those in section 18.2.

The statement about $\mathcal{E}$ holds because of the correspondence between $\Omega\mathbf{X}$ and $[\mathbf{X} \to \mathbf{U}]$. For *ne*, note that $(\lambda x.\, 1)^{-1}[0] = \emptyset$. Obviously, $\emptyset$ is contained in $\theta = \Omega\mathbf{X}$. Conversely, if $\emptyset$ is in A, then all open sets O with $O \supseteq \emptyset$ are in A, and these are all.

$$\begin{aligned} \mathit{filter}\, q\, A &= \mathit{ext}\,(\lambda x.\, qx \cdot \{\!|x|\!\})\, A \\ &= \{O \mid \{x \mid O \in (qx \cdot \mathcal{O}(x))\} \in A\} \end{aligned}$$

Note that $qx \cdot \mathcal{O} = \Omega\mathbf{X}$ if $qx = 0$, and $= \mathcal{O}$ if $qx = 1$.

$$\begin{aligned} &= \{O \mid \{x \mid qx = 0 \vee O \in \mathcal{O}(x)\} \in A\} \\ &= \{O \mid \{x \mid qx = 0 \vee x \in O\} \in A\} \\ &= \{O \mid q^{-1}[0] \cup O \in A\} \qquad \square \end{aligned}$$

Looking at the formulae for $A \overset{\rightarrow}{\times} B$ and $A \overset{\leftarrow}{\times} B$, it is not obvious whether they always equal or are sometimes different.

Problem: Is the power construction $\mathcal{U}_\Phi = \mathcal{U}_f$ symmetric?

Summarizing, we see that the upper power domain $\mathcal{U}_\Phi\mathbf{X}$ is quite unhandy, and the realization of the power operations, in particular of extension, is quite complex. In the next sections, we demonstrate that one can do better for the large class of sober domains. The final upper power domain may be represented in terms of compact sets for this class.

19.2 The upper construction $\mathcal{U}_K$ in terms of compact upper sets

Following [Smy83], we define the upper power domain in terms of compact upper sets in this section. Unfortunately, this approach does not work out for all domains. The class of allowed domains however is quite large; it contains all sober domains and hence all continuous ones.

For an arbitrary domain $\mathbf{X}$, let $\mathcal{U}_K\mathbf{X}$ be the set of all compact upper sets of $\mathbf{X}$. For every compact set K, the set of open environments of K is an open filter by Prop. 8.6.3. Thus, there is a mapping $\mathcal{O}(.) : \mathcal{U}_K\mathbf{X} \to \mathcal{U}_\Phi\mathbf{X}$. By Prop. 4.4.4, for every two compact upper sets K and K', $K \supseteq K'$ is equivalent to $\mathcal{O}(K) \subseteq \mathcal{O}(K')$. Since we ordered $\mathcal{U}_\Phi\mathbf{X}$ by '$\subseteq$', we have to order $\mathcal{U}_K\mathbf{X}$ by '$\supseteq$'. Then we obtain that $\mathcal{O}(.) : \mathcal{U}_K\mathbf{X} \to \mathcal{U}_\Phi\mathbf{X}$ has the property $K \leq K'$ iff $\mathcal{O}(K) \leq \mathcal{O}(K')$. Unfortunately, $\mathcal{O}(.)$ is not always surjective.

Proposition 19.2.1 $\mathcal{U}_K\mathbf{X}$ and $\mathcal{U}_\Phi\mathbf{X}$ are isomorphic by $\mathcal{O}(.)$ iff $\mathbf{X}$ is sober.

Proof: By Th. 8.7.1, $\mathbf{X}$ is sober iff every open filter is comprising.

If $\mathcal{U}_K\mathbf{X}$ and $\mathcal{U}_\Phi\mathbf{X}$ are isomorphic by $\mathcal{O}(.)$, then every open filter F equals $\mathcal{O}(K)$ for some compact upper set K. Hence, it is comprising.

Conversely, if every open filter is comprising, then every open filter $\mathcal{O}$ equals $\mathcal{O}(\bigcap \mathcal{O})$, and $\bigcap \mathcal{O}$ is a compact upper set by Prop. 8.6.3. $\square$

Since there are non-sober domains, $\mathcal{U}_K\mathbf{X}$ and $\mathcal{U}_\Phi\mathbf{X}$ are not isomorphic for all domains $\mathbf{X}$.

To analyze the topology of $\mathcal{U}_K\mathbf{X}$, we employ the following Lemma.

Lemma 19.2.2 Let $\mathbf{X}$ be sober and O an open set of $\mathbf{X}$. Then $\mathcal{K}(O) = \{K \in \mathcal{U}_K\mathbf{X} \mid K \subseteq O\}$ is open in $\mathcal{U}_K\mathbf{X}$.

Proof: $\mathcal{K}(O)$ is an upper set since $K \geq K'$ implies $K \subseteq K' \subseteq O$.

By Th. 8.9.2, all sober domains have property K-RD. If $\mathcal{D}$ is a $\supseteq$-directed set of compact upper sets with $\bigsqcup\mathcal{D}$ in $\mathcal{K}(O)$, i.e. $\bigsqcup\mathcal{D} \subseteq O$, then $\bigcap\mathcal{D} \subseteq O$ follows since $K \supseteq \bigsqcup\mathcal{D}$ for all K in $\mathcal{D}$. By property K-RD, there is some K in $\mathcal{D}$ such that $K \subseteq O$, i.e. $K \in \mathcal{K}(O)$. $\square$

In the case of sober ground domains, we may translate the power operations of $\mathcal{U}_\Phi$ into the language of compact sets.

Theorem 19.2.3 Let $\mathbf{X}$ and $\mathbf{Y}$ be two sober ground domains. Then the power domain $\mathcal{U}_K\mathbf{X}$ described below is isomorphic to the final $\mathbf{U}$-powerdomain over $\mathbf{X}$.

(1) $\mathcal{U}_K\mathbf{X}$ is the set of all compact upper sets of $\mathbf{X}$ ordered by $K \leq K'$ iff $K \supseteq K'$

(2) $\bigsqcup\mathcal{K} = \bigcap\mathcal{K}$ for directed sets $\mathcal{K}$ in $\mathcal{U}_K\mathbf{X}$.

(3) $\theta = \emptyset$

(4) $A \uplus B = A \cup B$

(5) $\{\!|x|\!\} = \uparrow x$ for all $x \in \mathbf{X}$.

(6) If $f : [\mathbf{X} \to \mathcal{U}_K\mathbf{Y}]$ is continuous and A is in $\mathcal{U}_K\mathbf{X}$, then $\mathit{ext}\, f\, A = \bigcup_{a\in A} fa = \bigcup f[A]$.

All these operations are well defined and continuous.

Proof:

(1) follows from the discussion at the beginning of this section.

(2) Every sober domain has property K-RD by Th. 8.9.2. In domains with property K-RD, intersections of $\supseteq$-directed sets of compact upper sets are compact upper sets by Prop. 8.9.4. Hence, $\bigcap\mathcal{K}$ is in $\mathcal{U}_K\mathbf{X}$. It is the supremum of $\mathcal{K}$ by set-theoretic arguments.

(3) The empty set is a compact upper set. $\mathcal{O}(\emptyset) = \Omega\mathbf{X} = \theta_\Phi$ holds.

(4) $A\cup B$ is compact by Prop. 4.6.3. If O is an open set in $\mathcal{O}(A\cup B)$, then $O \supseteq A\cup B \supseteq A, B$ holds, whence O is in $\mathcal{O}(A) \cap \mathcal{O}(B)$. Conversely, if O is in the intersection, then $O \supseteq A$ and $O \supseteq B$ implies $O \supseteq A \cup B$. Thus, we get $\mathcal{O}(A\cup B) = \mathcal{O}(A) \cap \mathcal{O}(B) = \mathcal{O}(A) \uplus \mathcal{O}(B)$ by Th. 19.1.1.

(5) $\uparrow x$ is a compact upper set, and $\mathcal{O}(\uparrow x) = \mathcal{O}(x) = \{\!|x|\!\}_\Phi$ holds.

(6) Let $f : [\mathbf{X} \to \mathcal{U}_K\mathbf{Y}]$ be continuous and A in $\mathcal{U}_K\mathbf{X}$.

$$\begin{aligned}
\mathcal{O}(\textstyle\bigcup f[A]) &= \{O \in \Omega\mathbf{Y} \mid \textstyle\bigcup_{a\in A} fa \subseteq O\} \\
&= \{O \in \Omega\mathbf{Y} \mid \forall a \in A : fa \subseteq O\} \\
&= \{O \in \Omega\mathbf{Y} \mid A \subseteq \{x \mid fx \subseteq O\}\} \\
&= \{O \in \Omega\mathbf{Y} \mid \{x \in \mathbf{X} \mid O \in \mathcal{O}(fx)\} \in \mathcal{O}(A)\}
\end{aligned}$$

The set $\{x \mid fx \subseteq O\}$ is open because it is the inverse image of the open set $\mathcal{K}(O)$ by the continuous function f.

The set $\mathcal{O}(\bigcup f[A])$ is an open filter since we just demonstrated that it may be represented as $\mathit{ext}_\Phi\, (\mathcal{O}(.) \circ f)\, (\mathcal{O}(A))$. Hence, $\bigcup f[A]$ is a compact upper set by Prop. 8.6.3. $\square$

A direct topological proof of the compactness of $\bigcup f[A]$ is also possible, but would be much more tedious. The same remark is valid for a direct proof of the continuity of $ext\, f : \mathcal{U}_K\mathbf{X} \to \mathcal{U}_K\mathbf{Y}$. Both proofs are unnecessary because one may use that $ext\, f : \mathcal{U}_\Phi\mathbf{X} \to \mathcal{U}_\Phi\mathbf{Y}$ is well-defined and continuous. These facts are in turn inherited from the well-definedness and continuity of the operations in the final power construction defined in terms of functions of higher order.

The advantage of $\mathcal{U}_K$ over $\mathcal{U}_\Phi$ is that the operations look simpler. The main reason is that no longer sets of sets are considered, but simply sets of ground domain points.

Hence, the derived operations also become simpler:

- If $\mathbf{X}$ is sober such that $\mathcal{U}_K\mathbf{X}$ is sober again, then $U A = ext\, id\, A = \bigcup id[A] = \bigcup A$.

  Thus, $U A = \bigcup A$.
- Let $f : [\mathbf{X} \to \mathbf{Y}]$ where both $\mathbf{X}$ and $\mathbf{Y}$ are sober. Then $map\, f\, A = ext\,(\iota \circ f)\, A = \bigcup_{a\in A}\{\!|f a|\!\} = \bigcup_{a\in A} \uparrow f a = \uparrow \bigcup_{a\in A}\{f a\} = \uparrow f[A]$

  Thus, $map\, f\, A = \uparrow f[A]$.
- For $p : [\mathbf{X} \to \mathbf{U}]$ where $\mathbf{X}$ is sober, $ex\, p\, A = 1$ iff $\bigcup_{a\in A} f a = \{\!|\diamond|\!\} = \{\diamond\}$ iff $\exists a \in A : f a = \{\diamond\}$.

  Thus, $ex\, p\, A = 1$ iff there is a in A with $p\, a = 1$.
- Hence, $ne\, A = 1$ iff there is a in A with $(\lambda x.\, 1)\, a = 1$. This is equivalent to $A \neq \emptyset$.
- Let $p : [\mathbf{X} \to \mathbf{U}]$ where $\mathbf{X}$ is sober. Then $filter\, p\, A = ext\,(\lambda a.\, p\, a \cdot \{\!|a|\!\})\, A = \bigcup_{a\in A}(p\, a \cdot \uparrow a) = \bigcup_{a\in A,\, p\, a=1} \uparrow a = \uparrow\{a \in A \mid p\, a = 1\} = \uparrow(A \cap p^{-1}[1])$

  Thus, $filter\, p\, A = \uparrow\{a \in A \mid p\, a = 1\}$.
- Let $\star : [\mathbf{X} \times \mathbf{Y} \to \mathcal{U}_K\mathbf{Z}]$, where all three of $\mathbf{X}$, $\mathbf{Y}$, and $\mathbf{Z}$ are sober. Then

$$\begin{aligned} A \overrightarrow{\star} B &= ext\,(\lambda a.\, ext\,(\lambda b.\, a \star b)\, B)\, A \\ &= \textstyle\bigcup_{a\in A} ext\,(\lambda b.\, a \star b)\, B \\ &= \textstyle\bigcup_{a\in A} \bigcup_{b\in B}\, a \star b \\ &= \textstyle\bigcup_{(a,\, b)\in A\times B}\, a \star b \\ &= \textstyle\bigcup \star[A \times B] \end{aligned}$$

  $A \overleftarrow{\star} B$ gives the same result, whence the final $\mathcal{U}$-construction becomes symmetric when restricted to the class of sober domains.
- If $\mathbf{X}$ and $\mathbf{Y}$ are sober such that $\mathbf{X} \times \mathbf{Y}$ is sober too, then $A \overrightarrow{\times} B = A \overleftarrow{\times} B = \bigcup_{(a,\, b)\in A\times B}\{\!|(a,\, b)|\!\} = \uparrow(A \times B) = A \times B$ since the product of upper sets is an upper set.

The upper power construction in terms of compact upper sets $\mathcal{U}_K$ admits considerably simpler definitions of the basic and derived power operations than the final power construction $\mathcal{U}_\Phi$. This simplicity allows to prove the symmetry of $\mathcal{U}_K$ in contrast to $\mathcal{U}_\Phi$ where we did not succeed in proving or disproving symmetry.

Concerning sobriety and $\mathcal{U}_K$, we were not able to answer the following question:

Problem 14 If $\mathbf{X}$ is sober, is then $\mathcal{U}_K\mathbf{X}$ sober again?

19.3 Domain-theoretic properties of $\mathcal{U}_f$ and $\mathcal{U}_K$

In this section, we investigate how the final **U**-construction $\mathcal{U}$ copes with algebraicity and continuity.

Theorem 19.3.1 If $\mathbf{X}$ is multi-algebraic, then $\mathcal{U}_K\mathbf{X}$ is algebraic. Its base is the set of all finitary open sets of $\mathbf{X}$.

Proof: Finitary open sets A of $\mathbf{X}$ are isolated points in $\mathcal{U}_K\mathbf{X}$ since their upper cone $\{K \mid K \geq A\} = \mathcal{K}(A)$ is open in $\mathcal{U}_K\mathbf{X}$ by Lemma 19.2.2.

Let K be a compact upper set and let $\mathcal{D} = \mathcal{FO}(K)$ be the set of all finitary open supersets of K. By Th. 8.3.5, $\mathcal{D}$ is directed in $\mathcal{U}_K\mathbf{X}$ and its intersection (limit) is K. □

From the theorem, we may derive:

Corollary 19.3.2 If $\mathbf{X}$ is algebraic, then $\mathcal{U}_K\mathbf{X}$ is algebraic with base $\{\uparrow F \mid F \subseteq_f \mathbf{X}^0\}$.

Proof: By Th. 6.2.4, the open sets in an algebraic domain $\mathbf{X}$ are just the sets $\uparrow S$ where $S \subseteq \mathbf{X}^0$. Hence, the finitary open sets are just the sets $\uparrow F$ where F is a finite subset of the base. □

For continuity, a similar theorem holds.

Theorem 19.3.3 If $\mathbf{X}$ is multi-continuous, then $\mathcal{U}_K\mathbf{X}$ is reduced and continuous.

Proof: Let $K \in \mathcal{O}$ where $\mathcal{O}$ is open in $\mathcal{U}_K\mathbf{X}$. If $\mathcal{F}(K)$ is the set of finitary upper environments of K, then we know from Th. 8.3.2 that $\mathcal{F}(K)$ is $\supseteq$-directed with intersection (limit) K. Since $\mathcal{O}$ is open, there is some F in $\mathcal{F}(K)$ such that $F \in \mathcal{O}$. F in $\mathcal{F}(K)$ implies $K \subseteq O \subseteq F$ for some open set O of $\mathbf{X}$. By Lemma 19.2.2, $\mathcal{K}(O)$ is open in $\mathcal{U}_K\mathbf{X}$.

First, K is in $\mathcal{K}(O)$. Second, $B \in \mathcal{K}(O)$ implies $B \subseteq O \subseteq F$, i.e. $B \geq F$. Thus, $\mathcal{K}(O) \subseteq \uparrow\{F\}$ holds. Third, $F \in \mathcal{O}$ implies $\uparrow\{F\} \subseteq \mathcal{O}$. Summarizing, we obtain $K \in \mathcal{K}(O) \subseteq \uparrow\{F\} \subseteq \mathcal{O}$ as required for continuity.

As mentioned above, every K in $\mathcal{U}_K\mathbf{X}$ is the limit of a directed set of finitary upper sets. A finitary upper set $\uparrow\{e_1, \ldots, e_n\}$ is in the core of $\mathcal{U}_K\mathbf{X}$ since it equals $\{\!|e_1, \ldots, e_n|\!\}$. Thus, every K in $\mathcal{U}_K\mathbf{X}$ is in the core. □

The theorem leaves open the question:

Problem 15 Is the final **U**-construction reduced for all ground domains (or at least for all sober ground domains)?

19.4 The initial upper power construction $\mathcal{U}_i$

In this section, we try to prove the initiality of the final **U**-construction. Unfortunately, we are not able to perform the proof for a class of ground domains larger than CONT. On the other hand, we do not know of any counterexample such that the situation is quite unsatisfactory.[2]

[2] Meanwhile, such a counterexample was found.

In order to point out clearly where continuity is required in our proof, we start with an arbitrary domain and restrict it to a more special class when needed.

Let $\mathbf{X}$ be an arbitrary ground domain. We try to prove that $\mathcal{U}\mathbf{X}$ is initial, i.e. that it is a free $\mathbf{U}$-$\mathbf{X}$-module. To this end, let $(M, \sqcap, \top)$ be an arbitrary $\mathbf{U}$-module, and let $f : [\mathbf{X} \to M]$ be a morphism. We have to provide a linear map $\bar{f} : [\mathcal{U}\mathbf{X} \to M]$ with $\bar{f} \circ \iota = f$ and then show its uniqueness.

At first, we define a function f' on finite subsets of $\mathbf{X}$:

- For finite sets E, let $f'\{e_1, \ldots, e_n\} = fe_1 \sqcap \cdots \sqcap fe_n$.

This function is well defined because '$\sqcap$' is commutative and associative, and has a neutral element 0 in M that can be taken as the result of applying f' to $\emptyset$. f' has the following properties:

(1) $f'(\emptyset) = 0$

(2) $f'(A \cup B) = f'A \sqcap f'B$ for all finite sets A and B.

(3) $f'\{x\} = fx$ for all x in $\mathbf{X}$.

(4) If $\uparrow A \supseteq \uparrow B$, then $f'A \leq f'B$ for all finite sets A and B.

Properties (1) and (3) are obvious. For (2), idempotence of '$\sqcap$' is needed since some operands occurring once in $f'(A \cup B)$ may occur twice in $f'A \sqcap f'B$. The precondition of (4) means $B \subseteq \uparrow A$, whence for all b in B, there is $a(b)$ in A with $b \geq a(b)$. By monotonicity of f, $f'B = \sqcap_{b \in B} fb \geq \sqcap_{b \in B} f(a(b))$ holds. By idempotence of '$\sqcap$', the last term equals $\sqcap_{a \in a[B]} fa$. By adding the remaining elements of A, the last term can only become smaller. Hence, it is above $\sqcap_{a \in A} fa = f'A$.

Because of property (4), $\uparrow A = \uparrow B$ implies $f'A = f'B$. Hence, f' implies a function $\widehat{f}$ mapping all finitary upper sets of $\mathbf{X}$ into M. $\widehat{f}$ is defined by $\widehat{f}(\uparrow E) = f'E$ for finite sets E. Properties (1) through (4) of f' directly translate into corresponding properties of $\widehat{f}$:

(1) $\widehat{f}(\emptyset) = 0$

(2) $\widehat{f}(A \cup B) = \widehat{f}A \sqcap \widehat{f}B$ for all finitary upper sets A and B.

(3) $\widehat{f}(\uparrow x) = fx$

(4) If $A \supseteq B$, then $\widehat{f}A \leq \widehat{f}B$ for all finitary upper sets A and B.

The next step should be the extension of $\widehat{f}$ to all of $\mathcal{U}\mathbf{X}$. The problem with this is that we do not know much about the structure of $\mathcal{U}\mathbf{X}$ in general. If $\mathbf{X}$ is sober however, $\mathcal{U}\mathbf{X}$ is isomorphic to $\mathcal{U}_K\mathbf{X}$.

Let K be a compact upper set in $\mathbf{X}$. There are several completely different approaches to define $\bar{f}K$. First, one could consider the set $\overline{F}K$ of all values $\widehat{f}A$ where A is a finitary upper *subset* of K. $\overline{F}$ is downward directed since $A, B \subseteq K$ implies $A \cup B \subseteq K$, and $\widehat{f}(A \cup B)$ is the greatest lower bound of $\widehat{f}A$ and $\widehat{f}B$. Since $A \subseteq K$ means $A \geq K$ in $\mathcal{U}\mathbf{X}$, all members of $\overline{F}K$ are above the value $\bar{f}K$ to be defined. All this suggests to define $\bar{f}K$ to be the greatest lower bound of $\overline{F}K$. Unfortunately, we do not know whether such a greatest lower bound exists, and if so, whether $\bar{f}$ is continuous.

A second approach considers the set $\underline{F}K$ of all values $\widehat{f}A$ where A is a finitary upper *superset* of K. Since $A \supseteq K$ means $A \leq K$ in $\mathcal{U}\mathbf{X}$, all members of $\underline{F}K$ are below the value

$\bar{f}K$ to be defined. Unfortunately, we do not know in general whether $\underline{F}K$ contains any point at all.

If we assume $\mathbf{X}$ to be multi-continuous, then K is strongly compact by Prop. 8.3.3, and there is a finitary upper superset A of K in every environment of K. Hence, the set $\underline{F}K$ is not empty, and one may guess that it well approximates $\bar{f}K$. All this suggests to define $\bar{f}K$ as the least upper bound of $\underline{F}K$. However, we cannot generally prove the existence of such a least upper bound, because we cannot prove $\underline{F}K$ to be directed.

In the third approach, we consider the set $\underline{\underline{F}}K$ of all values $\hat{f}A$ where A is a finitary upper *environment* of K. To be more precise, let $\mathcal{F}(K)$ be the set of all finitary upper sets A such that there is an open set O with $K \subseteq O \subseteq A$. We first investigate this set more closely and then turn to $\underline{\underline{F}}K = \hat{f}[\mathcal{F}(K)] \subseteq M$.

Proposition 19.4.1 Let $\mathbf{X}$ be a sober domain. For all compact upper sets K of $\mathbf{X}$, let $\mathcal{F}(K)$ be the set of all finitary upper environments of K. Then

(1) $\emptyset$ is in $\mathcal{F}(\emptyset)$.

(2) $\mathcal{F}(K_1 \cup K_2) = \{A_1 \cup A_2 \mid A_i \in \mathcal{F}(K_i)\}$.

(3) $K \supseteq K'$ implies $\mathcal{F}(K) \subseteq \mathcal{F}(K')$.

(4) Let $\mathcal{K}$ be a directed set in $\mathcal{U}_K\mathbf{X}$. Then $\mathcal{F}(\bigcap \mathcal{K}) = \bigcup_{K \in \mathcal{K}} \mathcal{F}(K)$.

(5) If $\mathbf{X}$ is multi-continuous, then for all K in $\mathcal{U}_K\mathbf{X}$, the set $\mathcal{F}(K)$ is directed in $\mathcal{U}_K\mathbf{X}$ and $\bigcap \mathcal{F}(K) = K$ holds.

Proof:

(1) $\emptyset$ is a finitary upper open superset of $\emptyset$.

(2) Let A be a member of the left hand side. Then there is some open set O with $K_1 \cup K_2 \subseteq O \subseteq A$. This implies $K_i \subseteq O \subseteq A$, whence A is in both $\mathcal{F}(K_1)$ and $\mathcal{F}(K_2)$. Because of $A = A \cup A$, A is a member of the right hand side.

Conversely, if A is in the set on the right, then there are finitary upper sets A_i and open sets O_i such that $K_i \subseteq O_i \subseteq A_i$. This implies $K_1 \cup K_2 \subseteq O_1 \cup O_2 \subseteq A_1 \cup A_2 = A$.

(3) If $K \subseteq O \subseteq A$ for some open O, then $K' \subseteq K \subseteq O \subseteq A$.

(4) The union is a subset of $\mathcal{F}(\bigcap \mathcal{K})$ because of (3) and $K \supseteq \bigcap \mathcal{K}$ for all K in $\mathcal{K}$.

Assume $\bigcap \mathcal{K} \subseteq O \subseteq A$ for some open set O. Because sober domains are in K-RD, there is some K in $\mathcal{K}$ with $K \subseteq O$. Hence, A is in $\mathcal{F}(K)$.

(5) By Th. 8.3.2. □

We now define $\underline{\underline{F}}K = \hat{f}[\mathcal{F}(K)]$. If the ground domain is multi-continuous, $\mathcal{F}(K)$ is directed, whence $\underline{\underline{F}}K$ is directed, too, by monotonicity of $\hat{f}$. Hence, the set $\underline{\underline{F}}K$ has a least upper bound in M. Thus, we can define $\bar{f}K = \bigsqcup \underline{\underline{F}}K$. We have to show that this function is continuous, additive, and $\bar{f}(\uparrow x) = fx$ holds.

By property (1) of $\mathcal{F}(K)$, the value $\top = \widehat{f}\,\emptyset$ is in $\underline{\underline{F}}(\emptyset)$, whence $\bar{f}(\emptyset) = \top$. Using property (2) and continuity of '$\sqcap$' in M, one obtains

$$\begin{array}{rcl} \bar{f}(K_1 \cup K_2) & = & \bigsqcup \widehat{f}\,[\mathcal{F}(K_1 \cup K_2)] \\ & = & \bigsqcup \{\widehat{f}(A_1 \cup A_2) \mid A_i \in \mathcal{F}(K_i)\} \\ & = & \bigsqcup \{\widehat{f}A_1 \sqcap \widehat{f}A_2 \mid A_i \in \mathcal{F}(K_i)\} \\ & = & \bigsqcup \{\widehat{f}A_1 \mid A_1 \in \mathcal{F}(K_1)\} \sqcap \bigsqcup \{\widehat{f}A_2 \mid A_2 \in \mathcal{F}(K_2)\} \\ & = & \bar{f}K_1 \sqcap \bar{f}K_2 \end{array}$$

By property (3) of the Lemma, $K \leq K'$ implies $\mathcal{F}(K) \subseteq \mathcal{F}(K')$, whence $\widehat{f}\,[\mathcal{F}(K)] \subseteq \widehat{f}\,[\mathcal{F}(K')]$. By Prop. 2.4.7, $\bar{f}K \leq \bar{f}K'$ follows. Hence, $\bar{f}$ is monotonic.

Continuity of $\bar{f}$ is shown using part (4) of the Lemma. Let $\mathcal{K}$ be a directed set in $\mathcal{U}_K\mathbf{X}$.

$$\begin{array}{rcl} \bar{f}(\bigcap \mathcal{K}) & = & \bigsqcup \widehat{f}\,[\mathcal{F}(\bigcap \mathcal{K})] \;=\; \bigsqcup \widehat{f}\,[\bigcup_{K \in \mathcal{K}} \mathcal{F}(K)] \\ & = & \bigsqcup \bigcup_{K \in \mathcal{K}} \widehat{f}\,[\mathcal{F}(K)] \;=\; \bigsqcup_{K \in \mathcal{K}} \bigsqcup \widehat{f}\,[\mathcal{F}(K)] \;=\; \bigsqcup_{K \in \mathcal{K}} \bar{f}K \end{array}$$

Finally, we have to show $\bar{f}(\uparrow x) = fx$ for all x in $\mathbf{X}$. For all A in $\mathcal{F}(\uparrow x)$, $A \supseteq \uparrow x$ holds, whence $\widehat{f}A \subseteq \widehat{f}(\uparrow x) = fx$. Thus, $\bar{f}(\uparrow x) = \bigsqcup \widehat{f}\,[\mathcal{F}(\uparrow x)] \leq fx$ follows.

For the opposite direction, we assume $fx \in O$ for some open set O, and have to show $\bar{f}(\uparrow x) \in O$ by Lemma 4.3.1. $fx \in O$ implies $x \in O' = f^{-1}[O]$. By continuity of f, O' is an open environment of x. By multi-continuity, there are a finite set E and an open set O'' with $x \in O'' \subseteq \uparrow E \subseteq O'$. Thus, $\uparrow E$ is in $\mathcal{F}(\uparrow x)$, and $fe \in O$ holds for all e in E. Unfortunately, this does not generally imply $\widehat{f}(\uparrow E) = \bigsqcap_{e \in E} fe \in O$. Thus, the proof cannot be completed.

If $\mathbf{X}$ is even continuous, then the set E may be assumed to be a singleton. Thus, $\widehat{f}(\uparrow E) = fe$ is in O, whence $\bar{f}(\uparrow x) = \bigsqcup \widehat{f}\,[\mathcal{F}(\uparrow x)]$ is in O, and the proof is completed.

Summarizing, we obtain

Theorem 19.4.2 For continuous ground domain $\mathbf{X}$, initial and final upper power domain over $\mathbf{X}$ (and thus all reduced upper power domains over $\mathbf{X}$) coincide.

As indicated at the beginning of this section, this theorem is not as general as one might wish. We saw that the corresponding proof for multi-continuous ground domain only fails in proving $\bar{f}(\uparrow x) \geq fx$.

Chapter 20

More about L- and U-constructions

In this chapter, we present some more results about **L**- and **U**-constructions. According to the (non-)results of the previous chapter, one has to carefully distinguish between the initial **U**-construction $\mathcal{U}_i$ and the final **U**-construction $\mathcal{U}_f$. In particular, it is possible that $\mathcal{U}_f$ is not reduced. Indeed, we were able to identify a sub-construction $\mathcal{U}_s$ of $\mathcal{U}_f$ (section 20.1). For sober ground domains, its power domains may be represented in terms of strongly compact sets $\mathcal{U}_S\mathbf{X}$. Unfortunately, we were not able to prove that $\mathcal{U}_s$ is a proper sub-construction of $\mathcal{U}_f$.

For sober ground domains, $\mathcal{U}_f\mathbf{X}$ and $\mathcal{U}_K\mathbf{X}$, and also $\mathcal{U}_s\mathbf{X}$ and $\mathcal{U}_S\mathbf{X}$ are isomorphic. In section 20.2, we identify the maximal domain of definition of $\mathcal{U}_K$ and $\mathcal{U}_S$ and show that they are partial power constructions of their own independent from whether they coincide with $\mathcal{U}_f$ and $\mathcal{U}_s$ respectively. The properties of $\mathcal{U}_S$ are investigated in sections 20.3 and 20.4. The analysis of the topology of $\mathcal{U}_S\mathbf{X}$ leads to the class of $\mathcal{U}_S$-conform domains that contains all multi-continuous domains.

In section 20.5, we show by topological reasoning that $\mathcal{L}\mathcal{U}_S\mathbf{X}$ and $\mathcal{U}_S\mathcal{L}\mathbf{X}$ are isomorphic for all $\mathcal{U}_S$-conform ground domains $\mathbf{X}$. In the final section 20.6, we then show by algebraic reasoning that $\mathcal{L}\mathcal{U}_i\mathbf{X}$ and $\mathcal{U}_i\mathcal{L}\mathbf{X}$ are isomorphic for *all* ground domains $\mathbf{X}$. Both results largely generalize the paper [FM90] where the isomorphism was shown for bounded complete algebraic domains $\mathbf{X}$ using information systems.

20.1 The U-construction $\mathcal{U}_s$ in terms of strong filters

Since we were not able to show the final **U**-construction $\mathcal{U}_f$ to be reduced, there might be some sub-constructions of $\mathcal{U}_f$. Indeed, we found a sub-construction $\mathcal{U}_s$ defined in terms of *strong filters*. It is open whether $\mathcal{U}_f$ and $\mathcal{U}_s$ coincide for all ground domains.

For sober ground domains, open filters correspond one-to-one to compact upper sets. The *strong* filters are defined such that they correspond to *strongly* compact upper sets.

Definition 20.1.1 An open filter $\mathcal{F}$ is *strong* iff for all O in $\mathcal{F}$, there is a finite set $E \subseteq O$ such that $\mathcal{O}(E) \subseteq \mathcal{F}$.

In the sequel, we verify that the power operations of $\mathcal{U}_f \cong \mathcal{U}_\Phi$ preserve the 'strength' of filters, i.e. that the strong filters induce a sub-construction $\mathcal{U}_s$ of $\mathcal{U}_f$. The power operations of $\mathcal{U}_f$ are given by Th. 19.1.1 in terms of filters. Remember that open filters are open sets of open sets of $\mathbf{X}$ that contain $\mathbf{X}$ and are closed w.r.t. intersection. They are ordered by set inclusion.

- If $\mathcal{D}$ is a directed set of open filters, then $\bigsqcup \mathcal{D}$ is $\bigcup \mathcal{D}$. Assume all members of $\mathcal{D}$ are strong. If O is in $\bigcup \mathcal{D}$, then O is in $\mathcal{F}$ for some $\mathcal{F}$ in $\mathcal{D}$. Since $\mathcal{F}$ is strong, there is a finite set $E \subseteq O$ such that $\mathcal{O}(E) \subseteq \mathcal{F} \subseteq \bigcup \mathcal{D}$.
- θ is given by $\Omega\mathbf{X} = \mathcal{O}(\emptyset)$ and $\{\!|x|\!\}$ by $\mathcal{O}(\{x\})$. We generalize both cases to $\mathcal{O}(E)$ for finite set E.

  If O is in $\mathcal{O}(E)$, then $E \subseteq O$ and $\mathcal{O}(E) \subseteq \mathcal{O}(E)$ hold. Hence, all open filters $\mathcal{O}(E)$ for finite E are strong.
- For two filters $\mathcal{F}_1$ and $\mathcal{F}_2$, $\mathcal{F}_1 \uplus \mathcal{F}_2$ is given by $\mathcal{F}_1 \cap \mathcal{F}_2$. If O is a member of this intersection, then $O \in \mathcal{F}_i$ holds for $i = 1, 2$. If the filters $\mathcal{F}_i$ are strong, then there are finite sets $E_i \subseteq O$ such that $\mathcal{O}(E_i) \subseteq \mathcal{F}_i$. $E = E_1 \cup E_2$ is a finite subset of O. Let O' be a member of $\mathcal{O}(E)$. Then $O' \supseteq E \supseteq E_i$ follows for $i = 1, 2$, whence $O' \in \mathcal{O}(E_i) \subseteq \mathcal{F}_i$, i.e. $O' \in \mathcal{F}_1 \cap \mathcal{F}_2$.
- For $f : [\mathbf{X} \to \mathcal{U}_s\mathbf{Y}]$ and A in $\mathcal{U}_s\mathbf{X}$, *ext* $f\,A$ is given by $B = \{O \mid \{x \mid O \in fx\} \in A\}$. If O is in B, then $O' = \{x \mid O \in fx\}$ is in A. Since A is strong, there is a finite subset E' of O' such that $\mathcal{O}(E') \subseteq A$. By $E' \subseteq O'$, set O is in fe for all $e \in E'$. Since fe is strong, there are finite sets $F_e \subseteq O$ such that $\mathcal{O}(F_e) \subseteq fe$. The union $F = \bigcup_{e \in E'} F_e$ of these sets is a finite subset of O.

  We have to show $\mathcal{O}(F) \subseteq B$. Let P be a member of $\mathcal{O}(F)$. $P \supseteq F \supseteq F_e$ for all e in E' implies $P \in \mathcal{O}(F_e) \subseteq fe$. Hence, $P' = \{x \mid P \in fx\}$ is a superset of E', whence $P' \in \mathcal{O}(E') \subseteq A$. Thus, P is in B.

The arguments above prove the following theorem:

Theorem 20.1.2 The strong filters form a sub-construction $\mathcal{U}_s$ of $\mathcal{U}_f$.

The next proposition shows that strong filters are to open filters as strongly compact sets are to compact sets.

Proposition 20.1.3
Let K be a compact upper set. Then $\mathcal{O}(K)$ is a strong filter iff K is strongly compact.

Proof: $\mathcal{O}(K)$ is strong iff $\forall O \in \mathcal{O}(K)\; \exists E \subseteq_f O : \mathcal{O}(E) \subseteq \mathcal{O}(K)$.
By definition of $\mathcal{O}(.)$, $O \in \mathcal{O}(K)$ is equivalent to $K \subseteq O$. By Lemma 4.4.4, $\mathcal{O}(E) \subseteq \mathcal{O}(K)$ is equivalent to $\uparrow E \supseteq K$. Hence, we get the equivalent formula

$$\forall O \supseteq K\; \exists E \text{ finite} : K \subseteq \uparrow E \subseteq O$$

This is the definition of strong compactness. □

By Prop. 19.2.1, we know that $\mathcal{U}_f\mathbf{X}$ and $\mathcal{U}_K\mathbf{X}$ are isomorphic iff $\mathbf{X}$ is sober. Here, $\mathcal{U}_K\mathbf{X}$ is the set of all compact upper sets of $\mathbf{X}$.

Proposition 20.1.4 If $\mathbf{X}$ is sober, then $\mathcal{U}_s\mathbf{X}$ is isomorphic to the set $\mathcal{U}_S\mathbf{X}$ of all strongly compact upper sets of $\mathbf{X}$ ordered by inverse inclusion '$\supseteq$'.

Proof: We have to show surjectivity of $\mathcal{O}(.) : \mathcal{U}_S\mathbf{X} \to \mathcal{U}_s\mathbf{X}$. If $\mathcal{F}$ is a strong filter, then by sobriety $\mathcal{F} = \mathcal{O}(K)$ holds for some compact set K. By Prop. 20.1.3, K is strongly compact. □

In contrast to Prop. 19.2.1, we cannot show the equivalence to sobriety here. Thus, the following question remains open:

Problem 16 Is sobriety equivalent to $\mathcal{U}_s\mathbf{X} \cong \mathcal{U}_S\mathbf{X}$?

The two constructions $\mathcal{U}_f$ and $\mathcal{U}_s$ coincide for multi-continuous ground domains by Prop. 8.3.3. We do however not know whether they always coincide.

Problem 17 For which domains are all open filters strong?

The construction $\mathcal{U}_s$ is intuitively 'more reduced' than the larger construction $\mathcal{U}_f$. From $\mathcal{U}_f$, we know reducedness for multi-continuous ground domains only; by Th. 8.3.2, $K = \bigcap \mathcal{F}(K)$ holds in this case. $\mathcal{U}_s$ is known to be reduced for domains in FC & SOB in addition since every strongly compact set is a directed intersection of finitary upper sets by Prop. 5.4.4.

Problem 18 For what ground domains $\mathbf{X}$ is $\mathcal{U}_s\mathbf{X}$ reduced?

20.2 The topological U-constructions $\mathcal{U}_K$ and $\mathcal{U}_S$

In case of a sober ground domain $\mathbf{X}$, the final **U**-powerdomain $\mathcal{U}_f\mathbf{X}$ is isomorphic to the **U**-powerdomain $\mathcal{U}_K\mathbf{X}$ in terms of compact upper sets. For non-sober $\mathbf{X}$, $\mathcal{U}_f\mathbf{X}$ and $\mathcal{U}_K\mathbf{X}$ are not isomorphic but $\mathcal{U}_K\mathbf{X}$ might still define a **U**-construction. We show however that $\mathcal{U}_K\mathbf{X}$ is not useful for all domains $\mathbf{X}$. Instead, we show that it forms a sub-powerdomain of $\mathcal{U}_f\mathbf{X}$ just for the class K-RD of domains with the K-Rudin property.

Analogously, one may consider the set $\mathcal{U}_S\mathbf{X}$ of strongly compact upper sets. It forms a sub-powerdomain of $\mathcal{U}_s\mathbf{X}$ for domains in S-RD.

We know that the mapping $\mathcal{O}(.) : \mathcal{U}_K\mathbf{X} \to \mathcal{U}_f\mathbf{X}$ cuts down to $\mathcal{O}(.) : \mathcal{U}_S\mathbf{X} \to \mathcal{U}_s\mathbf{X}$ by Prop. 20.1.3. $K \leq K'$ is equivalent to $\mathcal{O}(K) \subseteq \mathcal{O}(K')$. Hence, $\mathcal{U}_K\mathbf{X}$ is isomorphic to a subdomain of $\mathcal{U}_f\mathbf{X}$ by $\mathcal{O}(.)$ iff $\mathcal{U}_K\mathbf{X}$ is a domain and $\mathcal{O}(.)$ is continuous.

Definition 20.2.1

A domain $\mathbf{X}$ is *$\mathcal{U}_K$-admitting* iff $\mathcal{U}_K\mathbf{X}$ is a domain and $\mathcal{O}(.) : \mathcal{U}_K\mathbf{X} \to \mathcal{U}_f\mathbf{X}$ is continuous.

A domain $\mathbf{X}$ is *$\mathcal{U}_S$-admitting* iff $\mathcal{U}_S\mathbf{X}$ is a domain and $\mathcal{O}(.) : \mathcal{U}_S\mathbf{X} \to \mathcal{U}_f\mathbf{X}$ is continuous.

There are some equivalent formulations of the $\mathcal{U}_K$-admitting property:

Theorem 20.2.2 For a domain $\mathbf{X}$, the following statements are equivalent:

(1) $\mathbf{X}$ is $\mathcal{U}_K$-admitting.

(2) $\mathcal{U}_K\mathbf{X}$ is a domain, and for all open sets O in $\mathbf{X}$, the set $\mathcal{K}(O) = \{K \in \mathcal{U}_K\mathbf{X} \mid K \subseteq O\}$ is open in $\mathcal{U}_K\mathbf{X}$.

(3) For every open set O in $\mathbf{X}$ and every directed set $\mathcal{K}$ in $\mathcal{U}_K\mathbf{X}$, $O \supseteq \bigcap \mathcal{K}$ implies there is some K in $\mathcal{K}$ with $O \supseteq K$. This is property K-RD.

In this case, the limit of a directed set $\mathcal{K}$ in $\mathcal{U}_K\mathbf{X}$ is its intersection $\bigcap\mathcal{K}$.

An analogous statement holds for $\mathcal{U}_S$-admitting, S-RD, and strongly compact sets.

In the sequel, part (3) will be the main instrument to cope with when investigating $\mathcal{U}_K\mathbf{X}$.

Proof: We start by deriving (3) from (1). Assume $\mathbf{X}$ is $\mathcal{U}_K$-admitting, and let $\mathcal{K}$ be a directed set in it. Then $\mathcal{K}$ has a limit K, and $\bigcap\mathcal{K} \supseteq K$ holds because '$\supseteq$' is the order in $\mathcal{U}_K\mathbf{X}$. By continuity of $\mathcal{O}(.)$, $\mathcal{O}(K) = \bigcup_{A\in\mathcal{K}}\mathcal{O}(A)$ holds. Let O be an open set in $\mathbf{X}$ with $O \supseteq \bigcap\mathcal{K}$. Then $K \subseteq \bigcap\mathcal{K} \subseteq O$ whence $O \in \mathcal{O}(K)$. Thus, there is A in $\mathcal{K}$ with $O \in \mathcal{O}(A)$ i.e. $O \supseteq A$.

The fact $\bigsqcup\mathcal{K} = \bigcap\mathcal{K}$ follows from (3) because of Prop. 8.9.4.

Now, we show the equivalence of (2) and (3). The set $\mathcal{K}(O)$ occurring in (2) is always an upper set in $\mathcal{U}_K$ since $K \leq K'$ and $K \in \mathcal{K}(O)$ imply $K' \subseteq K \subseteq O$, whence $K' \in \mathcal{K}(O)$. Hence, $\mathcal{K}(O)$ is open iff $\bigsqcup\mathcal{K} \subseteq O$ implies $A \subseteq O$ for some A in $\mathcal{K}$. (3) directly implies this, since $\bigsqcup\mathcal{K} = \bigcap\mathcal{K}$. Conversely, if (2) holds, then $\bigcap\mathcal{K} \subseteq O$ implies $\bigsqcup\mathcal{K} \subseteq O$ and further $A \subseteq O$ for some A in $\mathcal{K}$.

Finally, we show that (3) implies (1). (3) implies the addition, i.e. $\bigsqcup\mathcal{K}$ exists and is $\bigcap\mathcal{K}$. Thus, $\mathcal{U}_K\mathbf{X}$ is a domain. For continuity of $\mathcal{O}(.)$, we have to show $\mathcal{O}(\bigcap\mathcal{K}) = \bigcup_{A\in\mathcal{K}}\mathcal{O}(A)$. '$\supseteq$' holds since $A \subseteq O$ implies $\bigcap\mathcal{K} \subseteq A \subseteq O$. '$\subseteq$' holds by (3): if $\bigcap\mathcal{K} \subseteq O$, then $A \subseteq O$ for some $A \in \mathcal{K}$.

The proof for strong compactness is completely analogous. We did not use the specific properties of compact sets in the proof above. □

If $\mathbf{X}$ is $\mathcal{U}_K$-admitting, then $\mathcal{U}_K\mathbf{X}$ is isomorphic to a sub-domain of $\mathcal{U}_\Phi\mathbf{X}$ by $\mathcal{O}(.)$. This sub-domain is closed w.r.t. the power operations in $\mathcal{U}_\Phi\mathbf{X}$. Hence, $\mathcal{U}_K\mathbf{X}$ may also be considered an upper power domain over $\mathbf{X}$ if the operations in $\mathcal{O}[\mathcal{U}_K\mathbf{X}] \subseteq \mathcal{U}_\Phi\mathbf{X}$ are translated back into operations in $\mathcal{U}_K\mathbf{X}$.

Theorem 20.2.3 $\mathcal{U}_K$ is a partial upper power construction defined for the class K-RD of all $\mathcal{U}_K$-admitting domains. The power domains and operations are given by

(1) $\mathcal{U}_K\mathbf{X}$ is the set of all compact upper sets of $\mathbf{X}$.

(2) $K \leq K'$ iff $K \supseteq K'$

(3) $\bigsqcup\mathcal{K} = \bigcap\mathcal{K}$ for directed sets $\mathcal{K}$ in $\mathcal{U}_K\mathbf{X}$.

(4) $\theta = \emptyset$

(5) $A \uplus B = A \cup B$

(6) $\{\!|x|\!\} = \uparrow x$ for all $x \in \mathbf{X}$.

(7) If $f : [\mathbf{X} \to \mathcal{U}_K\mathbf{Y}]$ is continuous and A is in $\mathcal{U}_K\mathbf{X}$, then $\mathit{ext}\, f\, A = \bigcup_{a\in A} fa = \bigcup f[A]$.

All these operations are well defined and continuous.

$\mathcal{U}_S$ is also a partial power construction. It is defined for the class S-RD of all $\mathcal{U}_S$-admitting domains. The power operations are like those above.

Proof: The proof of Th. 19.2.3 did not make specific use of sobriety or compactness. Hence, it may also be taken to prove the theorem at hand. □

Now, we turn to the completeness properties of $\mathcal{U}_K\mathbf{X}$ and $\mathcal{U}_S\mathbf{X}$.

Theorem 20.2.4 If $\mathbf{X}$ is in KC & K-RD, then $\mathcal{U}_K\mathbf{X}$ is in CC. If $\mathbf{X}$ is in SC & S-RD, then $\mathcal{U}_S\mathbf{X}$ is in CC.

Proof: We show the statement for $\mathcal{U}_K$. The statement for $\mathcal{U}_S$ is shown analogously.

Property KC means that the whole of $\mathbf{X}$ is compact and the intersection of two upper cones is compact. We have to show that $\mathcal{U}\mathbf{X}$ has a least element and lubs of every two points. Obviously, $\mathbf{X}$ itself is the least element.

For every two points a and b of $\mathbf{X}$, let $a * b = \uparrow a \cap \uparrow b$. By KC, '$*$' is a map from $\mathbf{X} \times \mathbf{X}$ to $\mathcal{U}\mathbf{X}$. It is continuous because $\uparrow a \cap \uparrow \bigsqcup D = \bigcap_{d \in D}(\uparrow a \cap \uparrow d)$ holds. By double extension, we get $A \Cap B = \mathit{ext}\,(\lambda a.\, \mathit{ext}\,(\lambda b.\, a * b)\, B)\, A$. Well-definedness of the power operations in $\mathcal{U}_K\mathbf{X}$ yields compactness of $A \Cap B$. We claim $A \Cap B = A \cap B$, i.e. $A \Cap B$ is the least upper bound of A and B.

By Th. 19.2.3, $A \Cap B = \bigcup_{a \in A} \bigcup_{b \in B} \uparrow a \cap \uparrow b$ holds. If x is a member of this set, then x is above some point a of A and also above some b of B. Hence, x is in $A \cap B$. Conversely, if y is a point of the intersection, then y is in both A and B, and y in $\uparrow y \cap \uparrow y$ holds. $\square$

By this proposition, initial and final **U**-construction preserve the classes BC & ALG and BC & CONT since they coincide with $\mathcal{U}_K$ and $\mathcal{U}_S$ for continuous ground domains.

We were not able to decide whether the constructions $\mathcal{U}_K$ and $\mathcal{U}_S$ may be iterated:

Problem 19 If $\mathbf{X}$ is $\mathcal{U}_K$-admitting, is then $\mathcal{U}_K\mathbf{X}$ $\mathcal{U}_K$-admitting again? If $\mathbf{X}$ is $\mathcal{U}_S$-admitting, is then $\mathcal{U}_S\mathbf{X}$ $\mathcal{U}_S$-admitting again?

20.3 Open sets and additive functions in $\mathcal{U}_S$

In this section, we state and prove some properties of the **U**-construction $\mathcal{U}_S$ in terms of strongly compact upper sets. It is defined for the class S-RD of $\mathcal{U}_S$-admitting domains.

By Th. 20.2.2, the sets $\mathcal{K}_S(O) = \{S \in \mathcal{U}_S\mathbf{X} \mid S \subseteq O\}$ are open in $\mathcal{U}_S\mathbf{X}$ if O is open in $\mathbf{X}$. In the next Lemma, we characterize all open sets of $\mathcal{U}_S\mathbf{X}$ that are obtained by $\mathcal{K}_S(.)$.

Lemma 20.3.1 Let $\mathbf{X}$ be $\mathcal{U}_S$-admitting. For a subset $\mathcal{O}$ of $\mathcal{U}_S\mathbf{X}$, the following two statements are equivalent:

(1) There is an open set O of $\mathbf{X}$ such that $\mathcal{O} = \mathcal{K}_S(O)$.

(2) $\mathcal{O}$ is open and closed w.r.t. binary union.

Proof: The sets $\mathcal{K}_S(O)$ are open by Th. 20.2.2. They are closed w.r.t. union since $S_1, S_2 \subseteq O$ implies $S_1 \cup S_2 \subseteq O$.

Let conversely $\mathcal{O}$ be an open set of $\mathcal{U}_S\mathbf{X}$ that is closed w.r.t. union. Let $O = \iota^{-1}[\mathcal{O}] = \{x \in \mathbf{X} \mid \uparrow x \in \mathcal{O}\}$. O is open as inverse image of $\mathcal{O}$ by ι. We claim $\mathcal{O} = \mathcal{K}_S(O)$.

Let S be in $\mathcal{O}$. x in S implies $\uparrow x \subseteq S$, i.e. $S \leq \uparrow x$. Hence, $\uparrow x$ is in $\mathcal{O}$, whence x is in O. Thus, $S \subseteq O$ holds. Conversely, $S \subseteq O$ implies $S \subseteq \uparrow E \subseteq O$ for some finite set E by strong compactness. By $\uparrow E \subseteq O$, every e in E is in O, whence $\uparrow e \in \mathcal{O}$ for all e in E. Thus, $S \geq \uparrow E = \bigcup_{e \in E} \uparrow e \in \mathcal{O}$ because $\mathcal{O}$ is closed w.r.t. finite union. $\square$

Since we used strong compactness explicitly in this proof, an analogous proof for $\mathcal{U}_K$ is impossible. We may however prove a similar criterion where 'binary union' is replaced by 'arbitrary union'. This criterion would not be quite useful however.

The criterion for $\mathcal{U}_S$ may be used to show that $\mathcal{U}_S$ has unique additive extensions.

Lemma 20.3.2 Let $\mathbf{X}$ and $\mathbf{Y}$ be two $\mathcal{U}_S$-admitting domains. If $F : [\mathcal{U}_S\mathbf{X} \to \mathcal{U}_S\mathbf{Y}]$ is additive, then $FS = \bigcap\{FA \mid A \in \mathcal{F}'(S)\}$ holds for all S in $\mathcal{U}_S\mathbf{X}$ where $\mathcal{F}'(S)$ is the set of all finitary upper supersets of S.

Proof: If $A \supseteq S$, then $FA \supseteq FS$. Thus, FS is a subset of the intersection.

Conversely, we have to show $T = \bigcap\{FA \mid A \in \mathcal{F}'(S)\} \subseteq FS$. Applying Lemma 4.4.4, we have to show that $FS \subseteq O$ implies $T \subseteq O$ for all open sets O of $\mathbf{Y}$. Let $FS \subseteq O$. Then FS is in $\mathcal{K}_S(O)$, whence S is in the open set $\mathcal{O} = F^{-1}[\mathcal{K}_S(O)]$. $\mathcal{O}$ is closed w.r.t. binary union since F is additive: $A, B \in \mathcal{O}$ implies $FA, FB \in \mathcal{K}_S(O)$, whence $F(A \cup B) = FA \cup FB \in \mathcal{K}_S(O)$, whence $A \cup B \in \mathcal{O}$. By Lemma 20.3.1, there is some open set O' of $\mathbf{X}$ such that $\mathcal{O} = \mathcal{K}_S(O')$. $S \subseteq O'$ implies $S \subseteq A \subseteq O'$ for some finitary upper set A by strong compactness of S. $A \subseteq O'$ means $FA \subseteq O$, whence $T \subseteq O$. □

The Lemma shows that additive morphisms from $\mathcal{U}_S\mathbf{X}$ to $\mathcal{U}_S\mathbf{Y}$ only depend on their values on finitary upper sets. Thus, we get

Theorem 20.3.3 Let $\mathbf{X}$ and $\mathbf{Y}$ be two $\mathcal{U}_S$-admitting domains. If $F, G : [\mathcal{U}_S\mathbf{X} \to \mathcal{U}_S\mathbf{Y}]$ are two additive functions with $F \circ \iota = G \circ \iota$, then $F = G$ follows.

Proof: The precondition means $F(\uparrow x) = G(\uparrow x)$ for all x in $\mathbf{X}$. By additivity, $FA = GA$ holds for all finitary upper sets A. By Lemma 20.3.2, $FS = GS$ follows for all S in $\mathcal{U}_S\mathbf{X}$. □

An analogous theorem for $\mathcal{U}_K$ could not be proved.

20.4 $\mathcal{U}_S$ and domain classes

In this section, we consider the relations between $\mathcal{U}_S$ and the domain classes of (multi-)algebraicity and (multi-)continuity. We also introduce a novel class, the class of $\mathcal{U}_S$-conform domains.

Theorem 20.4.1 Let $\mathbf{X}$ be $\mathcal{U}_S$-admitting.

(1) $\mathcal{U}_S\mathbf{X}$ is algebraic iff $\mathbf{X}$ is multi-algebraic.

(2) $\mathcal{U}_S\mathbf{X}$ is continuous iff $\mathbf{X}$ is multi-continuous.

Hence, $\mathcal{U}_S$ preserves ALG, M-ALG, CONT, and M-CONT.

Proof: The implications from right to left are already provided by Th. 19.3.1 and Th. 19.3.3. These two theorems were formulated for $\mathcal{U}_K$. For multi-continuous $\mathbf{X}$, $\mathcal{U}_K\mathbf{X}$ and $\mathcal{U}_S\mathbf{X}$ coincide by Prop. 8.3.3.

Let now $\mathcal{U}_S\mathbf{X}$ be continuous. Let x be a point of an open set O of $\mathbf{X}$. Then $\uparrow x \in \mathcal{K}_S(O)$ whence by continuity of $\mathcal{U}_S\mathbf{X}$ there are an open set $\mathcal{O}$ of $\mathcal{U}_S\mathbf{X}$ and a member A of $\mathcal{U}_S\mathbf{X}$ such that $\uparrow x \in \mathcal{O} \subseteq \uparrow\{A\} \subseteq \mathcal{K}_S(O)$. Let $O' = \iota^{-1}[\mathcal{O}]$. By continuity of ι, O' is an open set of $\mathbf{X}$.

First, $\iota x = \uparrow x \in \mathcal{O}$ implies $x \in O'$. Second, $y \in O'$ means $\uparrow y \in \mathcal{O} \subseteq \uparrow\{A\}$, whence $\uparrow y \geq A$, i.e. $\uparrow y \subseteq A$. Thus, we obtain $O' \subseteq A$. Third, $\uparrow\{A\} \subseteq \mathcal{K}_S(O)$ implies $A \in \mathcal{K}_S(O)$, whence $A \subseteq O$. Summarizing, we obtain $x \in O' \subseteq A \subseteq O$. A being strongly compact implies the existence of a finitary set F such that we finally get $x \in O' \subseteq A \subseteq F \subseteq O$.

Finally, let $\mathcal{U}_S\mathbf{X}$ be algebraic. Let x be a point of an open set O of $\mathbf{X}$. Then $\uparrow x \in \mathcal{K}_S(O)$ whence by algebraicity of $\mathcal{U}_S\mathbf{X}$ there is a member A of $\mathcal{U}_S\mathbf{X}$ such that $\uparrow\{A\}$ is open in $\mathcal{U}_S\mathbf{X}$ and $\uparrow x \in \uparrow\{A\} \subseteq \mathcal{K}_S(O)$. A point a is in A iff $\uparrow a \subseteq A$ iff $\uparrow a \geq A$ iff $\uparrow a \in \uparrow\{A\}$. Thus, $A = \iota^{-1}[\uparrow\{A\}]$ is open in $\mathbf{X}$. Since it is also strongly compact, it is finitary. $\uparrow x \in \uparrow\{A\} \subseteq \mathcal{K}_S(O)$ implies $x \in A \subseteq O$ as required. □

$\mathcal{U}_S$-admission allows to map open sets of the ground domain to open sets of the power domain by means of $\mathcal{K}_S(.)$ (see Th. 20.2.2). If even the contrary is possible we speak of $\mathcal{U}_S$-conformity.

Definition 20.4.2 A domain $\mathbf{X}$ is *$\mathcal{U}_S$-conform* iff it is $\mathcal{U}_S$-admitting, and for all $S \in \mathcal{O}$ where $\mathcal{O}$ is an open set of $\mathcal{U}_S\mathbf{X}$, there is an open set O in $\mathbf{X}$ such that $S \in \mathcal{K}_S(O) \subseteq \mathcal{O}$.

Topologically speaking, $\mathcal{U}_S$-conformity means that the Scott topology of $\mathcal{U}_S\mathbf{X}$ has as a base the sets $\mathcal{K}_S(O)$ where O is open in $\mathbf{X}$.

Theorem 20.4.3 Every multi-continuous domain is $\mathcal{U}_S$-conform.

Proof:
Every multi-continuous domain is in S-RD, i.e. $\mathcal{U}_S$-admitting. For $\mathcal{U}_S$-conformity, let $A \in \mathcal{O}$ where $\mathcal{O}$ is open in $\mathcal{U}_S\mathbf{X}$. From Th. 8.3.2, we know that A is the supremum of the directed set $\mathcal{F}(A)$ of finitary upper environments of A. Since $\mathcal{O}$ is open, there is some $F \in \mathcal{F}(A)$ with $F \in \mathcal{O}$. From $F \in \mathcal{F}(A)$, we get an open set O with $A \subseteq O \subseteq F$. Then $A \in \mathcal{K}_S(O)$ holds, and $B \in \mathcal{K}_S(O)$ implies $B \subseteq O \subseteq F$, i.e. $B \geq F$, whence $B \in \mathcal{O}$. Thus, we obtain $A \in \mathcal{K}(O) \subseteq \mathcal{O}$. □

Problem 20 If $\mathbf{X}$ is $\mathcal{U}_S$-conform, is then $\mathcal{U}_S\mathbf{X}$ again $\mathcal{U}_S$-conform?

Problem 21 What are the relations between $\mathcal{U}_S$-conformity and sobriety?

20.5 $\mathcal{L}$ and $\mathcal{U}_S$ may be exchanged

In [FM90], it was shown that $\mathcal{U}(\mathcal{L}\mathbf{X})$ and $\mathcal{L}(\mathcal{U}\mathbf{X})$ are isomorphic for bounded complete algebraic ground domains $\mathbf{X}$. The longish proof was done by means of the theory of information systems. We were able to largely generalize this statement up to the class of $\mathcal{U}_S$-conform domains using $\mathcal{U}_S$ for $\mathcal{U}$. Our proof is still longish, but completely differs from the proof in [FM90] in that it is done by topological means. In section 20.6, we shall show by algebraic means that $\mathcal{L}$ and the initial **U**-construction $\mathcal{U}_i$ may also be exchanged.

Theorem 20.5.1 If $\mathbf{X}$ is $\mathcal{U}_S$-conform, then $\mathcal{U}_S\mathcal{L}\mathbf{X}$ and $\mathcal{L}\mathcal{U}_S\mathbf{X}$ are isomorphic.

Before we are going to prove this theorem, we note that it allows to elegantly prove the preservation of M-ALG and M-CONT under $\mathcal{L}$.

Theorem 20.5.2 $\mathcal{L}$ preserves M-ALG and M-CONT.

Proof: If $\mathbf{X}$ is multi-algebraic, then it is $\mathcal{U}_S$-conform by Th. 20.4.3, and thus Th. 20.5.1 applies. $\mathcal{U}_S\mathbf{X}$ is algebraic by Th. 20.4.1, whence $\mathcal{L}\mathcal{U}_S\mathbf{X}$ is algebraic by Th. 18.6.1. By isomorphism, $\mathcal{U}_S\mathcal{L}\mathbf{X}$ is algebraic, too, and by Th. 20.4.1, $\mathcal{L}\mathbf{X}$ is multi-algebraic. The proof for M-CONT is analogous. □

The proof of Th. 20.5.1 proceeds in several steps. The precondition of $\mathcal{U}$-conformity[1] is not needed until the very last step. Hence, we start by assuming that $\mathbf{X}$ is only $\mathcal{U}$-admitting. Then, $\mathcal{L}\mathcal{U}\mathbf{X}$ is a domain. In contrast, $\mathcal{U}\mathcal{L}\mathbf{X}$ is a domain even for arbitrary $\mathbf{X}$ since $\mathcal{L}\mathbf{X}$ is complete and complete domains are $\mathcal{U}$-admitting by Th. 8.9.3.

Step 1: A monotonic function $\varphi : \mathcal{U}\mathcal{L}\mathbf{X} \to \mathcal{L}\mathcal{U}\mathbf{X}$

For a strongly compact upper set $\mathcal{K}$ of $\mathcal{L}\mathbf{X}$, we define $\varphi\mathcal{K} = \{K \in \mathcal{U}\mathbf{X} \mid C \cap K \neq \emptyset \; \forall C \in \mathcal{K}\}$. We have to show that $\varphi\mathcal{K}$ is a closed subset of $\mathcal{U}\mathbf{X}$. It is a lower set since $K' \leq K$ means $K' \supseteq K$, and thus, $C \cap K \neq \emptyset$ implies $C \cap K' \neq \emptyset$.

Let $(K_i)_{i \in I}$ be a directed family in $\varphi\mathcal{K}$. If we assume $\bigcap_{i\in I} K_i \notin \varphi\mathcal{K}$, then there is a member C of $\mathcal{K}$ such that $C \cap \bigcap_{i \in I} K_i = \emptyset$. Hence, $\bigcap_{i\in I} K_i \subseteq \mathsf{co}\, C$ holds, whence $K_k \subseteq \mathsf{co}\, C$ for some $k \in I$ by $\mathcal{U}$-admission. Here, the closedness of C was used. $K_k \subseteq \mathsf{co}\, C$ means $K_k \cap C = \emptyset$, whence K_k is not in $\varphi\mathcal{K}$. Because of this contradiction, $\bigcap_{i\in I} K_i$ is in $\varphi\mathcal{K}$.

$\mathcal{K} \leq \mathcal{K}'$ means $\mathcal{K} \supseteq \mathcal{K}'$. If K is in $\varphi\mathcal{K}$, then K meets all members of $\mathcal{K}$, and thus it also meets all members of $\mathcal{K}'$. Thus, $\mathcal{K} \leq \mathcal{K}'$ implies $\varphi\mathcal{K} \leq \varphi\mathcal{K}'$.

Step 2: A monotonic function $\psi : \mathcal{L}\mathcal{U}\mathbf{X} \to \mathcal{U}\mathcal{L}\mathbf{X}$

For a closed set $\mathcal{C}$ of $\mathcal{U}\mathbf{X}$, we define $\psi\mathcal{C} = \{C \in \mathcal{L}\mathbf{X} \mid K \cap C \neq \emptyset \; \forall K \in \mathcal{C}\}$.[2] We have to show that $\psi\mathcal{C}$ is a strongly compact upper set of $\mathcal{L}\mathbf{X}$.

For a strongly compact upper set K, we define $\psi'(K) = \{C \in \mathcal{L}\mathbf{X} \mid K \cap C \neq \emptyset\}$. We first show $\psi'(K) \in \mathcal{U}\mathcal{L}\mathbf{X}$ and later conclude $\psi\mathcal{C} \in \mathcal{U}\mathcal{L}\mathbf{X}$.

We claim $\psi' = map\, \iota_{\mathcal{L}}$.[3] Since the latter maps from $\mathcal{U}\mathbf{X}$ to $\mathcal{U}\mathcal{L}\mathbf{X}$, we get $\psi'(K) \in \mathcal{U}\mathcal{L}\mathbf{X}$. $map\, \iota_{\mathcal{L}}\, K = \uparrow(\iota_{\mathcal{L}}[K]) = \uparrow\{\downarrow x \mid x \in K\}$ holds. Let C be a member of this set. Then $C \geq \downarrow x$, i.e. $C \supseteq \downarrow x$, for some $x \in K$, whence $x \in C \cap K$. Conversely, assume there is some x in $C \cap K$. Then $\downarrow x \subseteq C$ since C is closed, whence $C \geq \downarrow x$ where $x \in K$.

Now, $\psi\mathcal{C} = \bigcap_{K \in \mathcal{C}} \psi'(K)$ holds. Since $\mathcal{L}\mathbf{X}$ is complete, $\mathcal{U}\mathcal{L}\mathbf{X}$ is also complete by Th. 20.2.4, and all suprema are given by intersection. Hence, arbitrary intersections of members of $\mathcal{U}\mathcal{L}\mathbf{X}$ are back in $\mathcal{U}\mathcal{L}\mathbf{X}$ again.

$\mathcal{C} \leq \mathcal{C}'$ means $\mathcal{C} \subseteq \mathcal{C}'$. Thus, whenever a set A meets all members of $\mathcal{C}'$, then it meets all members of $\mathcal{C}$. Hence, $\psi\mathcal{C}' \subseteq \psi\mathcal{C}$ holds, i.e. $\psi\mathcal{C} \leq \psi\mathcal{C}'$. Thus, ψ is monotonic.

Step 3: Relations between φ and ψ

Let $\mathcal{K}$ be a member of $\mathcal{U}\mathcal{L}\mathbf{X}$, and let C in turn be a member of $\mathcal{K}$. By definition of φ, C meets all members of $\varphi\mathcal{K}$. By definition of ψ, C is then in $\psi(\varphi\mathcal{K})$. Thus, we get $\mathcal{K} \subseteq \psi(\varphi\mathcal{K})$, or in terms of order $\mathcal{K} \geq \psi(\varphi\mathcal{K})$, or by abstraction $\psi \circ \varphi \leq id$.

[1] We drop the index 'S' of $\mathcal{U}_S$ in this proof.

[2] Interestingly, the definitions φ and ψ only differ in the domain and co-domain.

[3] This was the way I found the mapping ψ. The mapping φ may be analogously derived.

Analogously, $\mathcal{C} \subseteq \varphi(\psi\mathcal{C})$ holds for all $\mathcal{C}$ in $\mathcal{LU}\mathbf{X}$. This means $\mathcal{C} \leq \varphi(\psi\mathcal{C})$, whence $\varphi \circ \psi \geq id$. As an aside, this means that φ is an upper adjoint and preserves all infima, whereas ψ is a lower adjoint and preserves all suprema, i.e. it is in particular continuous.

The two relations $\psi \circ \varphi \leq id$ and $\varphi \circ \psi \geq id$ imply $\varphi \circ \psi \circ \varphi = (\varphi \circ \psi) \circ \varphi \geq \varphi$ and $\varphi \circ \psi \circ \varphi = \varphi \circ (\psi \circ \varphi) \leq \varphi$, whence $\varphi \circ \psi \circ \varphi = \varphi$. Similarly, $\psi \circ \varphi \circ \psi = \psi$ holds. In Step 4, we show that φ is injective, whence $\varphi \circ \psi \circ \varphi = \varphi$ implies $\psi \circ \varphi = id$. In the final Step 5, we show that ψ is injective, too, whence also $\varphi \circ \psi = id$.

Step 4: φ is injective: $\varphi\mathcal{K} \leq \varphi\mathcal{K}'$ implies $\mathcal{K} \leq \mathcal{K}'$

Let $\mathcal{K}$ and $\mathcal{K}'$ be two strongly compact upper sets of $\mathcal{L}\mathbf{X}$. $\varphi\mathcal{K} \leq \varphi\mathcal{K}'$ means $\varphi\mathcal{K} \subseteq \varphi\mathcal{K}'$. We have to conclude $\mathcal{K} \leq \mathcal{K}'$, i.e. $\mathcal{K} \supseteq \mathcal{K}'$. Applying Lemma 4.4.4, we assume $\mathcal{K} \subseteq \mathcal{O}$ where $\mathcal{O}$ is an open set of $\mathcal{L}\mathbf{X}$, and have to show $\mathcal{K}' \subseteq \mathcal{O}$.

Since $\mathcal{K}$ is strongly compact, $\mathcal{K} \subseteq \mathcal{O}$ implies there is a finitary set $\mathcal{F}$ in $\mathcal{L}\mathbf{X}$ such that $\mathcal{K} \subseteq \mathcal{F} \subseteq \mathcal{O}$. As $\mathcal{F}$ is finitary, there is a finite set $\mathcal{E}$ of closed sets $E_1, \ldots, E_n$ such that $\mathcal{F} = \uparrow\mathcal{E}$. We show $\mathcal{K}' \subseteq \mathcal{F}$, whence $\mathcal{K}' \subseteq \mathcal{O}$ immediately follows by $\mathcal{F} \subseteq \mathcal{O}$.

Let C be a member of $\mathcal{K}'$. C is closed, whence its complement $O = \mathsf{co}\, C$ is open. The sets $O_i = \mathsf{co}\, E_i$ are also open. With these notions, we first show an auxiliary property: $\mathcal{K}(O) \subseteq \mathcal{K}(O_1) \cup \cdots \cup \mathcal{K}(O_n)$. This property will later be used to show that there is some k such that $O \subseteq O_k$. By complementing, this implies $C \supseteq E_k$, i.e. $C \geq E_k$ for some k, whence $C \in \uparrow\mathcal{E} = \mathcal{F}$.

Proof of $\mathcal{K}(O) \subseteq \mathcal{K}(O_1) \cup \cdots \cup \mathcal{K}(O_n)$:
Let K be a member of $\mathcal{K}(O)$, i.e. a strongly compact upper subset of O. Then K does not meet O's complement C, whence K is not in $\varphi\mathcal{K}'$. Thus, K is neither in $\varphi\mathcal{K}$ because of the precondition $\varphi\mathcal{K} \subseteq \varphi\mathcal{K}'$. Hence, there is a member A of $\mathcal{K}$ such that $A \cap K = \emptyset$. Because of $\mathcal{K} \subseteq \uparrow\mathcal{E}$, there is some E_i in $\mathcal{E}$ such that $A \geq E_i$, i.e. $A \supseteq E_i$. Hence, $E_i \cap K = \emptyset$ also holds, whence $K \subseteq O_i$, i.e. $K \in \mathcal{K}(O_i)$.

Proof of $O \subseteq O_k$ for some k:
Assume $O \not\subseteq O_k$ for all k. Then for all k there is some x_k in O with $x_k \notin O_k$. Let $K = \uparrow\{x_1, \ldots, x_n\} \subseteq O$. By finiteness, this is a strongly compact set. Hence, $K \in \mathcal{K}(O)$, whence there is some i such that $K \subseteq O_i$, i.e. in particular $x_i \in K \subseteq O_i$. Thus, we derived a contradiction.

Step 4 is now concluded. Notice that we did not use $\mathcal{U}$-conformity so far, i.e. all results proved till now are valid for general $\mathcal{U}$-admitting ground domains $\mathbf{X}$.

Step 5: ψ is injective: $\psi\mathcal{C} \leq \psi\mathcal{C}'$ implies $\mathcal{C} \leq \mathcal{C}'$

Let $\mathcal{C}$ and $\mathcal{C}'$ be two closed sets of $\mathcal{U}\mathbf{X}$. $\psi\mathcal{C} \leq \psi\mathcal{C}'$ means $\psi\mathcal{C} \supseteq \psi\mathcal{C}'$. We have to conclude $\mathcal{C} \leq \mathcal{C}'$, i.e. $\mathcal{C} \subseteq \mathcal{C}'$.

Let K be in $\mathcal{C}$, and assume K is not in $\mathcal{C}'$. Then K is in the complement $\mathcal{O}$ of $\mathcal{C}'$. Since $\mathcal{O}$ is open in $\mathcal{U}\mathbf{X}$, there is an open set O of $\mathbf{X}$ such that $K \in \mathcal{K}(O) \subseteq \mathcal{O}$ by $\mathcal{U}$-conformity. $K \subseteq O$ implies $K \cap \mathsf{co}\, O = \emptyset$. Hence, $\mathsf{co}\, O$ is not in $\psi\mathcal{C}$, whence it is in $\psi\mathcal{C}'$ neither. This means there is some $K' \in \mathcal{C}'$ such that $K' \cap \mathsf{co}\, O = \emptyset$. Thus, $K' \subseteq O$, whence $K' \in \mathcal{K}(O) \subseteq \mathcal{O}$ contradicting $K' \in \mathcal{C}'$. □

20.6 $\mathcal{L}$ and $\mathcal{U}_i$ may be exchanged

In this section, we investigate algebraic structures that are both **L**- and **U**-modules. The theory of these structures helps to prove that the initial **L**-construction $\mathcal{L}$ and the initial **U**-construction $\mathcal{U} = \mathcal{U}_i$ commute for *every* ground domain $\mathbf{X}$. In contrast to the proof for $\mathcal{L}$ and $\mathcal{U}_S$ in the previous section, this proof will be mainly performed by algebraic methods. Thus, it has a completely different structure. The proved theorem is also unrelated to Th. 20.5.1 unless somebody succeeds in proving $\mathcal{U}_S$ to be initial.

The proof of $\mathcal{L}\mathcal{U}\mathbf{X} \cong \mathcal{U}\mathcal{L}\mathbf{X}$ is done by introducing the algebraic theory of **L**-**U**-modules and showing that both $\mathcal{L}\mathcal{U}\mathbf{X}$ and $\mathcal{U}\mathcal{L}\mathbf{X}$ are free **L**-**U**-modules over $\mathbf{X}$. **L**-**U**-modules are domains that are **L**-modules as well as **U**-modules such that the respective additions distribute over each other.

Definition 20.6.1 A domain $\mathbf{M}$ is an **L**-**U**-*module* iff

(1) There are a least element $\perp$ and a least upper bound $a \sqcup b$ for every two points a and b.

(2) There are a greatest element $\top$ and a greatest lower bound $a \sqcap b$ for every two points a and b. The operation '$\sqcap$' is continuous.

(3) $a \sqcup (b_1 \sqcap b_2) = (a \sqcup b_1) \sqcap (a \sqcup b_2)$ and $a \sqcap (b_1 \sqcup b_2) = (a \sqcap b_1) \sqcup (a \sqcap b_2)$ hold for all a, b_1, b_2 in $\mathbf{M}$.

An **L**-**U**-**X**-*module* is a pair $(\mathbf{M}, \eta)$ of an **L**-**U**-module $\mathbf{M}$ and a morphism $\eta : [\mathbf{X} \to \mathbf{M}]$.

A function $f : [\mathbf{M} \to \mathbf{M}']$ between two **L**-**U**-modules is **L**-**U**-*linear* iff it is both **L**-linear and **U**-linear, i.e. $f(\perp) = \perp$, $f(\top) = \top$, $f(a \sqcup b) = fa \sqcup fb$, and $f(a \sqcap b) = fa \sqcap fb$ hold.

A function f between two **L**-**U**-**X**-modules $(\mathbf{M}, \eta)$ and $(\mathbf{M}', \eta')$ is **L**-**U**-**X**-*linear* iff in addition $f \circ \eta = \eta'$ holds.

By Prop. 17.2.2, part (1) of the definition states that $\mathbf{M}$ be an **L**-module, whereas part (2) requires it to be a **U**-module. Continuity of '$\sqcup$' need not be explicitly required because of Prop. 5.3.5. The two distributivities in (3) are equivalent to each other as known from lattice theory. Thus, one need only check one of them in every case. The definitions involving $\mathbf{X}$ are analogues of the definition of R-$\mathbf{X}$-modules.

We start the development of the **L**-**U**-theory by two theorems about the generation of **L**-**U**-modules.

Theorem 20.6.2 If $\mathbf{X}$ is a **U**-module, then $\mathcal{L}\mathbf{X}$ is an **L**-**U**-module and the singleton map is **U**-linear, i.e. $\{\!|\top|\!\} = \top$ and $\{\!|a \sqcap b|\!\} = \{\!|a|\!\} \sqcap \{\!|b|\!\}$ hold.

Proof: We use the explicit representation of $\mathcal{L}\mathbf{X}$ in terms of closed sets. $\{\!|\top|\!\} = {\downarrow}\top = \mathbf{X}$ is the greatest element of $\mathcal{L}\mathbf{X}$.

From $\sqcap : [\mathbf{X} \times \mathbf{X} \to \mathbf{X}]$, we get the operation '$\Cap$' by double extension:

$$\begin{aligned} A \Cap B &= ext\,(\lambda a.\, ext\,(\lambda b.\, \{\!|a \sqcap b|\!\})\, B)\, A \\ &= \mathsf{cl} \textstyle\bigcup_{a \in A} \mathsf{cl} \bigcup_{b \in B} {\downarrow}(a \sqcap b) \\ &= \mathsf{cl} \textstyle\bigcup_{a \in A} \bigcup_{b \in B} \{a \sqcap b\} \\ &= \mathsf{cl}\, \{a \sqcap b \mid a \in A,\, b \in B\} \end{aligned}$$

We claim that the latter set equals $A \cap B$. If a is in A and b in B, then $a \sqcap b$ is in $A \cap B$ since A and B are lower. Conversely, if x is in $A \cap B$, then x is in both A and B, and $x = x \sqcap x$ holds.

Thus, we derived $A \Cap B = A \cap B = A \sqcap B$. The reason for this derivation was to show continuity of '$\sqcap$' in $\mathcal{L}\mathbf{X}$. The operators '$\sqcap$' = '$\cap$' and '$\sqcup$' = '$\cup$' distribute by set theory. $\{\!|a|\!\} \Cap \{\!|b|\!\} = \{\!|a \sqcap b|\!\}$ holds because '$\Cap$' is the double extension of '$\sqcap$'. $\square$

The next theorem is the dual of Th. 20.6.2. Its proof is more complex since the initial **U**-powerdomains $\mathcal{U}\mathbf{X}$ are not known explicitly in general.

Theorem 20.6.3 If $\mathbf{X}$ is an **L**-module, then $\mathcal{U}\mathbf{X}$ is an **L**-**U**-module and the singleton map is **L**-linear, i.e. $\{\!|\bot|\!\} = \bot$ and $\{\!|a \sqcup b|\!\} = \{\!|a|\!\} \sqcup \{\!|b|\!\}$ hold.

Proof: By Prop. 14.3.3 (4), $\mathcal{U}\mathbf{X}$ has a least element, namely $\bot_{\mathbf{U}} \cdot \{\!|\bot_{\mathbf{X}}|\!\} = 1 \cdot \{\!|\bot|\!\} = \{\!|\bot|\!\}$. By double extension, we raise the operation '$\sqcup$' of $\mathbf{X}$ to

$$A \Cap B = ext\,(\lambda a.\, ext\,(\lambda b.\, \{\!|a \sqcup b|\!\})\, B)\, A$$

Then $\{\!|a|\!\} \Cap \{\!|b|\!\} = \{\!|a \sqcup b|\!\}$ holds, and '$\Cap$' is additive in both arguments. Thus, we are done when $A \Cap B = A \sqcup B$ is proved. Additivity of '$\Cap$' implies the distributive laws in connection with '$\Cup$' = '$\sqcap$'.

We show $A \Cap B = A \sqcup B$ in four steps.

Step 1: $A \Cap B \geq B$

$$\begin{aligned} A \Cap B &\geq ext\,(\lambda a.\, ext\,(\lambda b.\, \{\!|b|\!\})\, B)\, A \\ &\overset{(S3)}{=} ext\,(\lambda a.\, B)\, A \\ &\geq ext\,(\lambda a.\, B)\, \{\!|\bot|\!\} \\ &\overset{(P3)}{=} (\lambda a.\, B)\, \bot \;=\; B \end{aligned}$$

Step 2: $A \Cap B \geq A$

$$\begin{aligned} A \Cap B &\geq ext\,(\lambda a.\, ext\,(\lambda b.\, \{\!|a|\!\})\, B)\, A \\ &\geq ext\,(\lambda a.\, ext\,(\lambda b.\, \{\!|a|\!\})\, \{\!|\bot|\!\})\, A \\ &\overset{(P3)}{=} ext\,(\lambda a.\, \{\!|a|\!\})\, A \\ &\overset{(S3)}{=} A \end{aligned}$$

Step 3: $A \Cap A = A$

We show that the set $S = \{A \in \mathcal{U}\mathbf{X} \mid A \Cap A = A\}$ admits all **U**-**X**-module operations, whence $S = \mathcal{U}\mathbf{X}$ follows by reducedness of $\mathcal{U}\mathbf{X}$.

- θ is in S, since $\theta \Cap \theta = ext\,(\ldots)\,\theta = \theta$ holds by (P1).
- All singletons are in S because of $\{\!|c|\!\} \Cap \{\!|c|\!\} = \{\!|c \sqcup c|\!\} = \{\!|c|\!\}$.
- Limits of directed subsets of S are in S by continuity of '$\Cap$'.
- Let A and B be two members of S. By additivity of '$\Cap$', we obtain

  $$\begin{aligned} (A \Cup B) \Cap (A \Cup B) &= (A \Cap A) \Cup (A \Cap B) \Cup (B \Cap A) \Cup (B \Cap B) \\ &= A \Cup (A \Cap B) \Cup (B \Cap A) \Cup B \end{aligned}$$

  Since '$\Cup$' is greatest lower bound[4] and we already showed $A \leq A \Cap B$ and $B \leq B \Cap A$, the result simplifies to $A \Cup B$. Thus, $A \Cup B$ is back in S again.

[4]It is this argument that makes the proof specific for **L**- and **U**-constructions.

Step 4: $A \Cap B = A \sqcup B$.

By $A \Cap B \geq A, B$, the formal set $A \Cap B$ is an upper bound of A and B. If both A and B are below some C, then $A \Cap B \leq C \Cap C = C$ follows from monotonicity of '$\Cap$' and step 3.□

At next, we show that extension preserves linearity of the opposite kind.

Theorem 20.6.4

(1) Let $\mathbf{X}$ be a $\mathbf{U}$-module, $\mathbf{M}$ an $\mathbf{L}$-$\mathbf{U}$-module, and $f : [\mathbf{X} \to \mathbf{M}]$ a $\mathbf{U}$-linear map. Then its (unique) extension $\overline{f} : [\mathcal{L}\mathbf{X} \to \mathbf{M}]$ is $\mathbf{L}$-$\mathbf{U}$-linear.

(2) Let $\mathbf{X}$ be an $\mathbf{L}$-module, $\mathbf{M}$ an $\mathbf{L}$-$\mathbf{U}$-module, and $f : [\mathbf{X} \to \mathbf{M}]$ an $\mathbf{L}$-linear map. Then its (unique) extension $\overline{f} : [\mathcal{U}\mathbf{X} \to \mathbf{M}]$ is $\mathbf{L}$-$\mathbf{U}$-linear.

Proof: We start by proving (1). As an extension generated by $\mathcal{L}$, $\overline{f}$ is $\mathbf{L}$-linear, i.e. $\overline{f}(\perp) = \perp$ and $\overline{f}(A \sqcup B) = \overline{f}(A) \sqcup \overline{f}(B)$ hold.

By $\mathbf{U}$-linearity of f, we get $\overline{f}\{\!|\top|\!\} = f(\top) = \top$. The last equation to show is $\overline{f}(A \sqcap B) = \overline{f}A \sqcap \overline{f}B$. Let F be defined by $F(A, B) = \overline{f}(A \sqcap B)$ and G by $G(A, B) = \overline{f}A \sqcap \overline{f}B$. By Th. 20.6.2, $\mathcal{L}\mathbf{X}$ is an $\mathbf{L}$-$\mathbf{U}$-module as $\mathbf{M}$ is. By the distributive law in part (3) of the definition of $\mathbf{L}$-$\mathbf{U}$-modules, '$\sqcap$' is $\mathbf{L}$-linear in both arguments, whence both F and G are $\mathbf{L}$-linear in both arguments. They coincide for singletons:

$$\begin{aligned}
\overline{f}(\{\!|a|\!\} \sqcap \{\!|b|\!\}) &= \overline{f}\{\!|a \sqcap b|\!\} && \text{by } \mathbf{U}\text{-linearity of } \{\!|.|\!\} \\
&= f(a \sqcap b) && \text{since } \overline{f} \text{ extends } f \\
&= fa \sqcap fb && \text{by } \mathbf{U}\text{-linearity of } f \\
&= \overline{f}\{\!|a|\!\} \sqcap \overline{f}\{\!|b|\!\}
\end{aligned}$$

Thus, F and G coincide by initiality of $\mathcal{L}$.

Now, part (1) is completely proved. Part (2) may be proved analogously since the proof of part (1) is completely algebraic, i.e. it does not involve any order-theoretic or topological arguments. □

After these preliminaries, we now come to the main theorem.

Theorem 20.6.5

(1) For every ground domain $\mathbf{X}$, $(\mathcal{L}\mathcal{U}\mathbf{X},\ \iota_{\mathcal{L}} \circ \iota_{\mathcal{U}})$ is an initial $\mathbf{L}$-$\mathbf{U}$-$\mathbf{X}$-module, i.e. there is exactly one $\mathbf{L}$-$\mathbf{U}$-$\mathbf{X}$-linear map to every $\mathbf{L}$-$\mathbf{U}$-$\mathbf{X}$-module.

(2) For every ground domain $\mathbf{X}$, $(\mathcal{U}\mathcal{L}\mathbf{X},\ \iota_{\mathcal{U}} \circ \iota_{\mathcal{L}})$ is also an initial $\mathbf{L}$-$\mathbf{U}$-$\mathbf{X}$-module.

(3) For every ground domain $\mathbf{X}$, the iterated power domains $\mathcal{L}\mathcal{U}\mathbf{X}$ and $\mathcal{U}\mathcal{L}\mathbf{X}$ are isomorphic by an $\mathbf{L}$-$\mathbf{U}$-linear morphism that maps $\{\!|\{\!|x|\!\}_{\mathcal{U}}|\!\}_{\mathcal{L}}$ to $\{\!|\{\!|x|\!\}_{\mathcal{L}}|\!\}_{\mathcal{U}}$.

Proof: We first show (1). Let $(\mathbf{M}, f)$ be a given $\mathbf{L}$-$\mathbf{U}$-$\mathbf{X}$-module. Then $f : [\mathbf{X} \to \mathbf{M}]$ holds. Since $\mathbf{M}$ is a $\mathbf{U}$-module, there is a $\mathbf{U}$-linear extension $\widehat{f} : [\mathcal{U}\mathbf{X} \to \mathbf{M}]$. Since $\mathbf{M}$ is also an $\mathbf{L}$-module, there is an extension $\overline{f} : [\mathcal{L}\mathcal{U}\mathbf{X} \to \mathbf{M}]$ of $\widehat{f}$. By Th. 20.6.4, $\overline{f}$ is $\mathbf{L}$-$\mathbf{U}$-linear. Because of its generation by extending twice, $\overline{f} \circ \iota_{\mathcal{L}} \circ \iota_{\mathcal{U}} = \widehat{f} \circ \iota_{\mathcal{U}} = f$ holds. Hence, $\overline{f}$ is $\mathbf{L}$-$\mathbf{U}$-$\mathbf{X}$-linear from $(\mathcal{L}\mathcal{U}\mathbf{X},\ \iota_{\mathcal{L}} \circ \iota_{\mathcal{U}})$ to $(\mathbf{M}, f)$.

Let F_1 and F_2 be two such $\mathbf{L}$-$\mathbf{U}$-$\mathbf{X}$-linear maps, i.e. $F_i \circ \iota_{\mathcal{L}} \circ \iota_{\mathcal{U}} = f$ holds. We have to show $F_1 = F_2$. Let $F_i' = F_i \circ \iota_{\mathcal{L}}$. The mapping $\iota_{\mathcal{L}}$ is $\mathbf{U}$-linear by Th. 20.6.2. Thus, F_i' are two $\mathbf{U}$-linear maps from $\mathcal{U}\mathbf{X}$ to $\mathbf{M}$ with $F_i' \circ \iota_{\mathcal{U}} = f$. By initiality of $\mathcal{U}$, $F_1' = F_2'$ follows. Thus,

F_1 and F_2 are two **L**-linear maps from $\mathcal{L}\mathcal{U}\mathbf{X}$ to $\mathbf{M}$ with $F_1 \circ \iota_{\mathcal{L}} = F_2 \circ \iota_{\mathcal{L}}$. By initiality of $\mathcal{L}$, $F_1 = F_2$ follows.

Part (2) is proved analogously. Parts (1) and (2) together imply part (3). $\square$

Chapter 21

Subconstructions of the final D-construction

In this chapter, we investigate the sub-constructions of the final $\mathbf{D}$-construction $\mathcal{D}$. Existential restriction to the three sub-semirings $\mathbf{B}$, $\overline{\mathbf{B}}$, and $\mathbf{C}$ of $\mathbf{D}$ gives us three sub-constructions, which we call $\mathcal{S}$, $\overline{\mathcal{S}}$, and $\mathcal{T}$ respectively. The name $\mathcal{S}$ was chosen since $\mathcal{S}$ generalizes the sandwich power construction of Buneman [BDW88] that is defined for algebraic ground domain only. $\overline{\mathcal{S}}$ belongs to $\overline{\mathbf{B}}$ as $\mathcal{S}$ belongs to $\mathbf{B}$. The name $\mathcal{T}$ has no deeper meaning except that the letter '$\mathcal{T}$' is the successor of '$\mathcal{S}$'.

The topics of the chapter are prepared by studying the logical properties of predicates with values in $\mathbf{D}$ in section 21.1. Then, the final $\mathbf{D}$-construction $\mathcal{D}$ and its operations are translated into topological terms in section 21.2 for sober ground domains.

$\mathcal{S}$, the existential restriction of $\mathcal{D}$ to $\mathbf{B}$ is introduced algebraically in section 21.3. In the case of sober ground domains, it is translated into topological terms in section 21.4. Its defining condition, the sandwich condition, can be drastically simplified provided the ground domain is in the class KC & M-CONT (see section 21.5). The sandwich power domains for algebraic ground domains are studied in section 21.6.

In section 21.7, the dual sandwich construction $\overline{\mathcal{S}}$ is introduced as the existential restriction of $\mathcal{D}$ to the semiring $\overline{\mathbf{B}}$. In case of sober ground domains, its defining condition is translated into the language of topology and largely simplified. The restriction $\mathcal{T}$ to semiring $\mathbf{C}$ and intersections of all these sub-constructions of $\mathcal{D}$ are dealt with in the final section 21.8.

21.1 D-predicates

By Th. 16.3.1, the final $\mathbf{D}$-construction $\mathcal{D}$ is isomorphic to the product of the final $\mathbf{L}$-construction $\mathcal{L}$ and the final $\mathbf{U}$-construction $\mathcal{U}$:

$$\mathcal{D}\mathbf{X} = [[\mathbf{X} \to \mathbf{D}] \overset{add}{\to} \mathbf{D}] = [[\mathbf{X} \to \mathbf{L}] \overset{add}{\to} \mathbf{L}] \times [[\mathbf{X} \to \mathbf{U}] \overset{add}{\to} \mathbf{U}]$$

Although the second equality is only an isomorphism, we do not write down the isomorphisms explicitly for simplification. Instead, we directly apply pairs of functions to pairs of predicates subsuming an equality $(P^L,\, P^U)\,(p^L,\, p^U) = (P^L p^L,\, P^U p^U)$.

Due to the equation $\mathbf{D} = \mathbf{L} \times \mathbf{U}$, the predicate domains $[\mathbf{X} \to \mathbf{D}]$ and $[\mathbf{X} \to \mathbf{L}] \times [\mathbf{X} \to \mathbf{U}]$ are also isomorphic. Again, we assume equality here, i.e. a pair (p^L, p^U) in $[\mathbf{X} \to \mathbf{L}] \times [\mathbf{X} \to \mathbf{U}]$ may be directly applied to a point x resulting in $(p^L, p^U)\,x = (p^L x, p^U x)$.

In section 17.1, we mentioned negation $\neg : [\mathbf{D} \to \mathbf{D}]$ that maps F to T, T to F, and $\perp$ and W to themselves. In terms of $\mathbf{L} \times \mathbf{U}$, $(0, 0)$ is mapped to $(1, 1)$ and vice versa, whereas $(0, 1)$ and $(1, 0)$ are mapped to themselves. Assuming two operations $\neg_L : [\mathbf{L} \to \mathbf{U}]$ and $\neg_U : [\mathbf{U} \to \mathbf{L}]$ which both exchange 0 and 1, one obtains $\neg(l, u) = (\neg_U u, \neg_L l)$. Note that $\neg_L$ and $\neg_U$ are inverse order isomorphisms between $\mathbf{L} = \{0 < 1\}$ and $\mathbf{U} = \{1 < 0\}$.

The operations $\neg_L$ and $\neg_U$ may also be used to convert $\mathbf{L}$-predicates into $\mathbf{U}$-predicates and vice versa. For a given $\mathbf{L}$-predicate $p : [\mathbf{X} \to \mathbf{L}]$, $\widetilde{p} = \neg_L \circ p : [\mathbf{X} \to \mathbf{U}]$ provides a $\mathbf{U}$-predicate. Conversely, a $\mathbf{U}$-predicate $q : [\mathbf{X} \to \mathbf{U}]$ is transformed into the $\mathbf{L}$-predicate $\widetilde{q} = \neg_U \circ q : [\mathbf{X} \to \mathbf{L}]$. De Morgan's formulae $\widetilde{p_1 + p_2} = \widetilde{p_1} \cdot \widetilde{p_2}$ and $\widetilde{p_1 \cdot p_2} = \widetilde{p_1} + \widetilde{p_2}$ are easily verified for two $\mathbf{L}$-predicates p_1 and p_2 as well as for two $\mathbf{U}$-predicates. The negation of a $\mathbf{D}$-predicate $(p, q) : [\mathbf{X} \to \mathbf{D}] = [\mathbf{X} \to \mathbf{L}] \times [\mathbf{X} \to \mathbf{U}]$ then becomes $\neg(p, q) = (\widetilde{q}, \widetilde{p})$.

21.2 $\mathcal{D}$ in topological terms

By Th. 16.3.1, the final $\mathbf{D}$-construction $\mathcal{D}$ is the product of $\mathcal{L}$ and $\mathcal{U}$. For sober ground domain, we know topological representations of $\mathcal{L}$ and $\mathcal{U}$. Hence, $\mathcal{D}$ may also be represented in topological terms:

- $\mathcal{D}\mathbf{X}$ is the set of all pairs (C, K) of a closed subset C of $\mathbf{X}$ with a compact upper set K of $\mathbf{X}$.
- $(C, K) \le (C', K')$ iff $C \subseteq C'$ and $K \supseteq K'$.
- The limit of a directed family $(C_i, K_i)_{i \in I}$ is $(\mathsf{cl} \bigcup_{i \in I} C_i, \bigcap_{i \in I} K_i)$.
- $\theta = (\emptyset, \emptyset)$
- $(C, K) \uplus (C', K') = (C \cup C', K \cup K')$
- $\{\!|x|\!\} = (\downarrow x, \uparrow x)$
- For $f : [\mathbf{X} \to \mathcal{D}\mathbf{Y}]$, $\mathit{ext}\, f\, A = (\mathsf{cl} \bigcup f^L[A^L], \bigcup f^U[A^U])$.

For extension and the derived operations below, we adopt the convention to denote the lower component of a pair A by A^L and the upper component by A^U. Similarly, a function $f : [\mathbf{X} \to \mathcal{D}\mathbf{Y}]$ is identified with a pair $(f^L, f^U) : [\mathbf{X} \to \mathcal{L}\mathbf{Y}] \times [\mathbf{X} \to \mathcal{U}\mathbf{Y}]$.

- $\mathit{map}\, f\, A = (\mathsf{cl}\, f^L[A^L], \uparrow f^U[A^U])$
- $A \overset{\rightarrow}{\times} B = A \overset{\leftarrow}{\times} B = (A^L \times B^L, A^U \times B^U)$
  This equation shows that $\mathcal{D}$ becomes symmetric when restricted to a class of domains closed w.r.t. Cartesian product and contained in the class of sober domains. The largest such class we know of is the class of multi-continuous domains by Prop. 8.4.2 and Th. 8.8.1.

- $r \cdot (C, K) = \begin{cases} (C, K) & \text{if } r = \mathsf{T} \\ (\emptyset, \emptyset) & \text{if } r = \mathsf{F} \\ (\emptyset, K) & \text{if } r = \bot \\ (C, \emptyset) & \text{if } r = \mathsf{W} \end{cases}$

  For the last two cases, remember $\bot = (1, 0)$ and $\mathsf{W} = (0, 1)$.

- $$\begin{aligned} \textit{filter}\ p\ A &= (\mathsf{cl}\,\{a \in A^L \mid p^L\, a = 1\},\ \uparrow\!\{a \in A^U \mid p^U\, a = 1\}) \\ &= (\mathsf{cl}\,\{a \in A^L \mid p\, a \geq \mathsf{T}\},\ \uparrow\!\{a \in A^U \mid p\, a \leq \mathsf{T}\}) \end{aligned}$$

21.3 $\mathcal{S}$ — the existential restriction of $\mathcal{D}$ to B

Since $\mathbf{B}$ is a sub-semiring of $\mathbf{D}$, we know from Th. 14.6.2 that the construction $\mathcal{S}$ defined by

- $\mathcal{S}\mathbf{X} = \{A \in \mathcal{D}\mathbf{X} \mid \forall p \in [\mathbf{X} \to \mathbf{B}] : Ap \in \mathbf{B}\}$

is a power construction with characteristic semiring $\mathbf{B}$. In Th. 21.6.4 below, we shall see that $\mathcal{S}$ is a generalization of the *sandwich power construction* defined in [BDW88] for algebraic ground domains and investigated further in [Gun89b, Gun90, Hec90c, Hec90a]. In anticipation of the theorem, we chose the abbreviation $\mathcal{S}$ and call the domain $\mathcal{S}\mathbf{X}$ *sandwich power domain* and its elements *sandwiches*. Consequently, the condition restricting $\mathcal{D}\mathbf{X}$ to $\mathcal{S}\mathbf{X}$ is called *sandwich condition* or shorter *condition S*. Since $\mathcal{D}\mathbf{X}$ is isomorphic to the product $\mathcal{L}\mathbf{X} \times \mathcal{U}\mathbf{X}$, we sometimes call its elements *pairs*.

We start the investigation of $\mathcal{S}$ by illustrating the location of $\mathcal{S}\mathbf{X}$ inside of $\mathcal{D}\mathbf{X}$:

Proposition 21.3.1
$\mathcal{S}\mathbf{X}$ is a closed subset of $\mathcal{D}\mathbf{X}$. Hence, every pair below a sandwich is a sandwich.

Proof: The sandwich power domain is characterized by the condition $Ap \in \mathbf{B}$ for all $p : [\mathbf{X} \to \mathbf{B}]$. Let $A' \leq A$. Then for all $p : [\mathbf{X} \to \mathbf{B}]$, $A'p \leq Ap$ holds. Since $\mathbf{B}$ is a lower set in $\mathbf{D}$, $Ap \in \mathbf{B}$ implies $A'p \in \mathbf{B}$. Thus, $\mathcal{S}\mathbf{X}$ is a lower set in $\mathcal{D}\mathbf{X}$. It is closed since it is a sub-domain. Or more directly: since $(\bigsqcup_{i\in I} A_i)\, p = \bigsqcup_{i \in I}(A_i p)$ and $\mathbf{B}$ is closed in $\mathbf{D}$. □

In the sequel, we shall transform the sandwich condition gradually. We start by turning the universal quantification into an existential one.

$$\begin{aligned} \forall p \in [\mathbf{X} \to \mathbf{B}] : A\, p \in \mathbf{B} \quad & \text{iff} \quad \forall p \in [\mathbf{X} \to \mathbf{D}] : ((\forall x \in \mathbf{X} : p\, x \in \mathbf{B}) \Rightarrow A\, p \in \mathbf{B}) \\ & \text{iff} \quad \forall p \in [\mathbf{X} \to \mathbf{D}] : (A\, p = \mathsf{W} \Rightarrow \exists x \in \mathbf{X} : p\, x = \mathsf{W}) \end{aligned}$$

This formula may be interpreted such that $\mathcal{S}\mathbf{X}$ consists of all consistent second order predicates in $\mathcal{D}\mathbf{X}$. A consistent second order predicate does not create inconsistencies by itself. If it results in an inconsistency ($Ap = \mathsf{W}$), then its argument already was inconsistent ($p\, x = \mathsf{W}$ for some x).

By splitting the pairs into components, we obtain further by $\mathsf{W} = (1, 0)$:

$$\begin{aligned} \text{iff} \quad & \forall p^L \in [\mathbf{X} \to \mathbf{L}],\ p^U \in [\mathbf{X} \to \mathbf{U}] : \\ & \qquad (A^L\, p^L = 1 \text{ and } A^U\, p^U = 0 \Rightarrow \exists x \in \mathbf{X} : p^L\, x = 1 \text{ and } p^U\, x = 0) \end{aligned}$$

Renaming p^L into p and p^U into q, the conclusion looks like $p\,x = 1$ and $q\,x = 0$ for some x. It may be more elegantly formulated. Negating q, one gets $p\,x = 1$ and $\widetilde{q}\,x = 1$ for some x. Both p and $\widetilde{q}$ are **L**-predicates and thus may be multiplied. We equivalently obtain $(p \cdot \widetilde{q})\,x = 1$ for some x, or $p \cdot \widetilde{q} \neq \underline{0}$ where $\underline{0}$ is the constant **L**-predicate $\lambda x.\, 0$. Thus, we finally get for (A^L, A^U) in $\mathcal{D}\mathbf{X}$:

Proposition 21.3.2

$$(A^L, A^U) \in \mathcal{S}\mathbf{X} \text{ iff } \forall p : [\mathbf{X} \to \mathbf{L}],\, q : [\mathbf{X} \to \mathbf{U}] : (A^L\,p = 1 \text{ and } A^U\,q = 0 \text{ implies } p \cdot \widetilde{q} \neq \underline{0})$$

21.4 $\mathcal{S}$ in topological terms

In this section, we first want to represent $\mathcal{S}$ in terms of open grills and filters. In case of a sober ground domain, we then translate the grills into closed sets and the filters into compact upper sets.

For the translation into grills and filters, we use the condition in the form

$$(A^L\,p^L = 1 \text{ and } A^U\,p^U = 0 \Rightarrow \exists x \in \mathbf{X} : p^L\,x = 1 \text{ and } p^U\,x = 0)$$

Translating 0 and 1 into $\perp$ and $\top$, we obtain

$$(A^L\,p^L = \top \text{ and } A^U\,p^U = \top \Rightarrow \exists x \in \mathbf{X} : p^L\,x = \top \text{ and } p^U\,x = \top)$$

since $\top_{\mathbf{L}} = 1$ and $\top_{\mathbf{U}} = 0$.

Let $\mathcal{G}$ be the open grill belonging to A^L and $\mathcal{O}$ the open filter belonging to A^U. To complete the translation to set notation, we represent the predicates p^L and p^U by open sets O^L and O^U. Then $p^L\,x = \top$ means $x \in O^L$, and same for O^U. Similarly, $A^L p^L = \top$ means $O^L \in \mathcal{G}$, and $A^U p^U = \top$ becomes $O^U \in \mathcal{O}$. Hence, we obtain

$$\begin{aligned}
(\mathcal{G}, \mathcal{O}) \in \mathcal{S}\mathbf{X} \quad & \text{iff} \quad \forall O^L, O^U \in \Omega\mathbf{X} : \\
& \qquad\qquad (O^L \in \mathcal{G} \text{ and } O^U \in \mathcal{O} \Rightarrow \exists x \in \mathbf{X} : x \in O^L \text{ and } x \in O^U) \\
& \text{iff} \quad \forall O^L \in \mathcal{G},\, O^U \in \mathcal{O} : O^L \cap O^U \neq \emptyset
\end{aligned}$$

Formulating this in English words, one obtains

Proposition 21.4.1 The sandwich power domain $\mathcal{S}\mathbf{X}$ over some ground domain $\mathbf{X}$ is isomorphic to the set of all pairs $(\mathcal{G}, \mathcal{O})$ of an open grill and an open filter of $\mathbf{X}$ where every member of $\mathcal{G}$ meets all members of $\mathcal{O}$.

For sober ground domain $\mathbf{X}$, one can go one step further and translate the open filters into compact upper sets K. The translation of open grills into closed sets C is always possible. $O^L \in \mathcal{G}$ becomes $C \cap O^L \neq \emptyset$, and $O^U \in \mathcal{O}$ becomes $K \subseteq O^U$.

Hence, the restriction translates into: for all open sets O^L and O^U, if C meets O^L and $K \subseteq O^U$ then O^L meets O^U. For fixed C and O^U, the following holds:

Every open set meeting C meets O^U
  iff every open environment of every point of C meets O^U
  iff every point of C is in the closure of O^U by Prop. 4.2.2
  iff $C \subseteq \mathsf{cl}\, O^U$

Hence, one obtains

Theorem 21.4.2 The sandwich power domain $\mathcal{S}\mathbf{X}$ over a sober ground domain $\mathbf{X}$ is isomorphic to the set of all pairs (C, K) of a closed set C and a compact upper set K such that for all open sets O with $K \subseteq O$ the inclusion $C \subseteq \mathsf{cl}\, O$ holds. Order and power operations are inherited from $\mathcal{D}$ (see section 21.2).

Two remarks seem to be appropriate. First, the condition '$K \subseteq O$ implies $C \subseteq \mathsf{cl}\, O$' looks quite strange, and it is not obvious how it could have been found without considering the second order predicates. Thus, this strange topological condition is motivated by a natural condition on second order predicates. Second, if we had defined a power domain construction directly as in the theorem above, we would have been forced to verify that each power operation respects the topological criterion. This would have been a non-trivial task, in particular for the extension functional.

Before we investigate the restriction '$K \subseteq O$ implies $C \subseteq \mathsf{cl}\, O$' more closely in the next section, we briefly consider existential quantification in $\mathcal{S}\mathbf{X}$.

- $ex\, p\, A = (ex_L\, p^L A^L,\; ex_U\, p^U A^U)$ for $p : [\mathbf{X} \to \mathbf{B}]$.

  The pair on the right hand side denotes a Boolean. We want to translate this expression into ordinary Boolean notation.

  The result is $\mathsf{T} = (1, 1)$ iff $ex_L\; p^L A^L = 1$. The second component does not matter because $(1, 0)$ is not a legal Boolean. The equation $ex_L\; p^L A^L = 1$ in turn is equivalent to the existence of $x \in A^L$ such that $p^L x = 1$. For this x, $p^U x$ cannot be 0 since 0 is not a legal Boolean. Thus, $p^L x = 1$ is equivalent to $p\, x = \mathsf{T}$.

  Similarly, the result is $\mathsf{F} = (0, 0)$ iff $ex_U\, p^U A^U = 0$. This in turn is equivalent to the non-existence of $x \in A^L$ such that $p^U x = 1$, i.e. $p^U x$ is 0 for all $x \in A^U$. Since $(1, 0)$ is not a legal Boolean, this is equivalent to $p\, x = \mathsf{F}$.

  Thus, $ex\; p\; A = \begin{cases} \mathsf{T} & \text{if } \exists\, u \in A^L : p\, u = \mathsf{T} \\ \mathsf{F} & \text{if } \forall\, v \in A^U : p\, v = \mathsf{F} \\ \bot & \text{otherwise} \end{cases}$

  The sandwich condition guarantees that the first two cases exclude each other.

- The formula for ne is derived from that for ex:

  $$ne\; A = ex\; (\lambda x.\, \mathsf{T})\; A = \begin{cases} \mathsf{T} & \text{if } A^L \neq \emptyset \\ \mathsf{F} & \text{if } A^U = \emptyset \\ \bot & \text{otherwise} \end{cases}$$

21.5 Simplification of the sandwich condition

In this section, we try to simplify the restriction '$K \subseteq O$ open implies $C \subseteq \mathsf{cl}\, O$' for pairs (C, K) of a closed and a compact upper set. A drastic simplification of this condition is achieved for a special class of ground domains. An example is presented showing that general simplification is impossible however.

Theorem 21.5.1 Let $\mathbf{X}$ be a multi-continuous domain in class KC. Then for all closed $C \subseteq \mathbf{X}$ and all compact upper sets $K \subseteq \mathbf{X}$, the following two statements are equivalent:

(1) The sandwich condition: If $K \subseteq O$ for some open set O, then $C \subseteq \mathsf{cl}\, O$.

(2) $C \subseteq \downarrow K$.

Property KC was introduced by Prop. 5.5.2 and means that the intersection of two upper cones is compact. All finitely continuous domains are continuous and satisfy this condition (Prop. 7.2.2), i.e. they are covered by the theorem above.

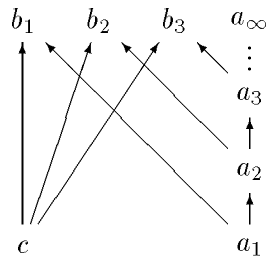
Proof: If $C \subseteq \downarrow K$ holds, then $K \subseteq O$ implies $C \subseteq \downarrow K \subseteq \downarrow O \subseteq \mathsf{cl}\, O$.

The opposite direction is much more difficult. A multi-continuous domain is sober, whence in K-RD. Thus, the pre-condition $\mathbf{X}$ in KC & K-RD of the lemmas 8.10.1 and 8.10.2 is satisfied.

Let C be a closed set and K be a compact upper set satisfying the sandwich condition. Let $\mathcal{F}$ be the set of all finitary upper sets F such that there is an open set O with $K \subseteq O \subseteq F$. From Th. 8.3.2, we know that $\mathcal{F}$ is a $\supseteq$-directed set whose intersection is K.

For all sets $F \in \mathcal{F}$, $K \subseteq O \subseteq F$ holds, whence $C \subseteq \mathsf{cl}\, O \subseteq \mathsf{cl}\, F$ follows from the sandwich condition. Lemma 8.10.1 implies $\mathsf{cl}\, F = \downarrow F$ if $\mathbf{X}$ is chosen as the closed set in the Lemma. Hence, $C \subseteq \bigcap_{F \in \mathcal{F}} \downarrow F = \downarrow \bigcap_{F \in \mathcal{F}} F = \downarrow K$ applying Lemma 8.10.2. $\Box$

Property KC is important for the validity of the theorem. Without it, the theorem does not hold even for algebraic ground domains. Consider the following example. Let $\mathbf{X} = \{a_1, a_2, a_3, \ldots, a_\infty, b_1, b_2, b_3, \ldots, c\}$. There is no point b_∞. The a-points form an ascending sequence: $a_1 < a_2 < \cdots < a_\infty$, whereas the b-points are incomparable. Every a-point is below the corresponding b-point: $a_n < b_n$. The remaining point c is below all b-points, but not below any a-point, not even below a_∞. The domain is visualized by the following picture:



This domain is algebraic. Property KC is not satisfied since $\uparrow c \cap \uparrow a_1 = \{b_1, b_2, \ldots\}$ is an infinite discrete open set covered by the infinite number of open sets $\{b_i\}$, whence it is not compact. (Interestingly, the additional inequality $c \leq a_\infty$ would establish property KC, but would destroy algebraicity on the other hand.) Let $C = \downarrow c = \{c\}$ and let $K = \uparrow a_\infty = \{a_\infty\}$. C and K satisfy the sandwich condition although $C \subseteq \downarrow K$, i.e. $c \leq a_\infty$ does not hold. Every open set with $K \subseteq O$, i.e. $a_\infty \in O$, contains some point a_n with $n < \infty$. Hence, it contains b_n because it is an upper set, whence c is in $\downarrow O$ by $c \leq b_n$. Thus, c is more than ever in $\mathsf{cl}\, O$. $\Box$

21.6 $\mathcal{S}$ for algebraic ground domains

For algebraic ground domain, the sandwich power domain is shown to be algebraic again. The sandwich condition looks particularly simple for its isolated points. This allows to show that our sandwich power domain generalizes the sandwich power domain of [BDW88].

If $\mathbf{X}$ is algebraic, then both $\mathcal{L}\mathbf{X}$ and $\mathcal{U}\mathbf{X}$ are algebraic. Their bases are given by the sets of all $\downarrow F$ and $\uparrow F$ respectively for finite subsets of $\mathbf{X}^0$. By Prop. 8.4.2, $\mathcal{D}\mathbf{X}$ is algebraic, and by Prop. 6.1.6, its base is the set of all pairs $(\downarrow E, \uparrow F)$ where E and F are finite subsets of $\mathbf{X}^0$. These pairs are also isolated in $\mathcal{S}\mathbf{X}$ provided they satisfy the sandwich condition because $\mathcal{S}\mathbf{X}$ is a sub-domain of $\mathcal{D}\mathbf{X}$. Every point in $\mathcal{D}\mathbf{X}$ is the directed limit of such pairs. Since all pairs below a sandwich are sandwiches again by Prop. 21.3.1, every point of $\mathcal{S}\mathbf{X}$ is the limit of a directed set of isolated sandwiches. Thus, we obtain

Proposition 21.6.1
The sandwich power domain over an algebraic ground domain is algebraic. Its base is the set of all sandwiches $(\downarrow E, \uparrow F)$ where E and F are finite subsets of $\mathbf{X}^0$.

The sandwich criterion simplifies drastically for such isolated pairs:

Lemma 21.6.2 Let $\mathbf{X}$ be a domain. If E and F are finite sets of isolated points of $\mathbf{X}$, then $(\downarrow E, \uparrow F)$ satisfies the sandwich condition iff $E \subseteq \downarrow\uparrow F$.

Proof: Assume $E \subseteq \downarrow\uparrow F$. Then for every $O \supseteq \uparrow F$, $\downarrow E \subseteq \downarrow\uparrow F \subseteq \downarrow O \subseteq \mathsf{cl}\, O$ holds.

For the opposite, note that $\uparrow F$ is open since F consists of isolated points. Thus, the sandwich condition implies $E \subseteq \downarrow E \subseteq \mathsf{cl} \uparrow F$. Since E consists of isolated points, Prop. 6.1.3 yields $E \subseteq \downarrow\uparrow F$. □

The representation of the base of $\mathcal{S}\mathbf{X}$ may even be further simplified choosing suitable sets E and F.

Lemma 21.6.3 Let $\mathbf{X}$ be algebraic and let E and F be finite subsets of $\mathbf{X}^0$ with $E \subseteq \downarrow\uparrow F$. Then there is a finite subset F' of $\mathbf{X}^0$ with $\uparrow F = \uparrow F'$ and $E \subseteq \downarrow F'$.

Proof: Since $E \subseteq \downarrow\uparrow F$, for every $e \in E$ there is some point $x_e \in \mathbf{X}$ and some point $f_e \in F$ such that $e \leq x_e \geq f_e$. By Prop. 6.4.1, the points x_e may be assumed to be in the base $\mathbf{X}^0$. With $E' = \{x_e \mid e \in E\}$, we define $F' = E' \cup F$. E' is a finite subset of $\mathbf{X}^0$, whence F' also is. All points e in E are below x_e in F', whence $E \subseteq \downarrow F'$ follows. $\uparrow F \subseteq \uparrow F'$ immediately follows from $F \subseteq F'$. For the opposite inclusion, x_e is above f_e for all e in E, whence $E' \subseteq \uparrow F$ whence $F' \subseteq \uparrow F$. □

Summarizing, we obtain the following theorem:

Theorem 21.6.4 For algebraic ground domain $\mathbf{X}$, our sandwich power domain over $\mathbf{X}$ is algebraic and coincides with the sandwich power domain of [BDW88] and [Gun89b, Gun90]. Its base is the set of all pairs $(\downarrow E, \uparrow F)$ with $E \subseteq \downarrow\uparrow F$, or equivalently the set of all pairs $(\downarrow E, \uparrow F)$ with $E \subseteq \downarrow F$, where in both cases E and F are finite subsets of $\mathbf{X}^0$.

Proof: For the comparison with the sandwich power domain in [BDW88, Gun89b, Gun90] notice that the authors of these papers write the sandwiches the other way round, i.e. the lower set to the right. Correcting this and translating notation, the paper [Gun89b] defines the sandwich power domain to be the ideal completion of all pairs (E, F) of finite subsets of $\mathbf{X}^0$ such that there is a finite subset G of $\mathbf{X}^0$ with $E \subseteq \downarrow G$ and $G \subseteq \uparrow F$. This directly implies $E \subseteq \downarrow\uparrow F$, and conversely, G may be chosen as the set E' in the proof of Lemma 21.6.3. These pairs are pre-ordered by $(E, F) \preceq (E', F')$ iff $\downarrow E \subseteq \downarrow E'$ and $\uparrow F \supseteq \uparrow F'$. Hence, the poset of equivalence classes of this pre-ordered set is just our base as presented in the theorem. □

Since the existential restriction to $\mathbf{B}$ of the final double power construction properly generalizes the sandwich power construction to all domains, we are allowed to denote it by $\mathcal{S}$ as we have already done throughout the preceding sections.

21.7 $\overline{\mathcal{S}}$ — the existential restriction to $\overline{\mathbf{B}}$

Instead of restricting $\mathcal{D}$ to the semiring $\mathbf{B} = \{\bot, \mathsf{F}, \mathsf{T}\}$, we may also restrict it to the dual semiring $\overline{\mathbf{B}} = \{\mathsf{F}, \mathsf{T}, \mathsf{W}\}$. The result is denoted by $\overline{\mathcal{S}}$. By Th. 14.6.2, it is defined by

- $\overline{\mathcal{S}}\mathbf{X} = \{A \in \mathcal{D}\mathbf{X} \mid \forall p \in [\mathbf{X} \to \overline{\mathbf{B}}] : A\,p \in \overline{\mathbf{B}}\}$

It is a power construction with characteristic semiring $\overline{\mathbf{B}}$. In contrast to $\mathcal{S}$, this construction was never proposed in the literature. Hence, there is no established name for its members. We sometimes call them *dual sandwiches*. The condition restricting $\mathcal{D}$ to $\overline{\mathcal{S}}$ is called *condition* $\overline{S}$.

We start the investigation of $\overline{\mathcal{S}}$ by illustrating the location of $\overline{\mathcal{S}}\mathbf{X}$ inside of $\mathcal{D}\mathbf{X}$:

Proposition 21.7.1 $\overline{\mathcal{S}}\mathbf{X}$ is an upper subset of $\mathcal{D}\mathbf{X}$. Hence, every pair above a dual sandwich is a dual sandwich.

Proof: Let $A \leq A'$ where A is in $\overline{\mathcal{S}}\mathbf{X}$. Then for all $p : [\mathbf{X} \to \overline{\mathbf{B}}]$, $Ap \leq A'p$ holds. Since $\overline{\mathbf{B}}$ is an upper set in $\mathbf{D}$, $Ap \in \overline{\mathbf{B}}$ implies $A'p \in \overline{\mathbf{B}}$. Thus, $\overline{\mathcal{S}}\mathbf{X}$ is an upper set in $\mathcal{D}\mathbf{X}$. □

Condition $\overline{S}$ for $A = (A^L, A^U)$ may be transformed as follows:

$$\forall p \in [\mathbf{X} \to \overline{\mathbf{B}}] : A\,p \in \overline{\mathbf{B}} \quad \text{iff} \quad \forall p \in [\mathbf{X} \to \mathbf{D}] : (A\,p = \bot \Rightarrow \exists x \in \mathbf{X} : p\,x = \bot)$$

This formula may be interpreted such that $\overline{\mathcal{S}}\mathbf{X}$ consists of all omniscient second order predicates of $\mathcal{D}\mathbf{X}$. An omniscient second order predicate does not create ignorance by itself. If it results in a state of ignorance, $(Ap = \bot)$, then its argument already did not know everything $(p\,x = \bot$ for some $x)$.

By splitting the pairs into components, we obtain further by $\bot = (0, 1)$:

$$\begin{aligned} \text{iff} \quad & \forall p^L \in [\mathbf{X} \to \mathbf{L}], p^U \in [\mathbf{X} \to \mathbf{U}] : \\ & (A^L\,p^L = 0 \text{ and } A^U\,p^U = 1 \Rightarrow \exists x \in \mathbf{X} : p^L\,x = 0 \text{ and } p^U\,x = 1) \end{aligned}$$

In transforming further, we employ the same trick as we did for condition S. Renaming p^L into p and p^U into q, the conclusion $p\,x = 0$ and $q\,x = 1$ for some x is equivalent to $\widetilde{p} \cdot q \neq \underline{0}$ in terms of $\mathbf{U}$-predicates. Thus, we finally get for (A^L, A^U) in $\mathcal{D}\mathbf{X}$:

Proposition 21.7.2

$$(A^L, A^U) \in \overline{\mathcal{S}}\mathbf{X} \text{ iff } \forall p : [\mathbf{X} \to \mathbf{L}],\, q : [\mathbf{X} \to \mathbf{U}] : (A^L\, p = 0 \text{ and } A^U\, q = 1 \text{ implies } \widetilde{p} \cdot q \neq \underline{0})$$

Next, we go on to open grills $\mathcal{G}$ and open filters $\mathcal{O}$. We use the condition

$$(A^L\, p^L = 0 \text{ and } A^U\, p^U = 1 \Rightarrow \exists x \in \mathbf{X} : p^L\, x = 0 \text{ and } p^U\, x = 1)$$

The predicates p^L and p^U are replaced by open sets O^L and O^U. All atomic components of the formula above are just negations of the corresponding components in case of the condition S. Hence

$$\begin{array}{lll} (\mathcal{G}, \mathcal{O}) \in \overline{\mathcal{S}}\mathbf{X} & \text{iff} & \forall O^L, O^U \in \Omega\mathbf{X} : \\ & & \quad (O^L \notin \mathcal{G} \text{ and } O^U \notin \mathcal{O} \Rightarrow \exists x \in \mathbf{X} : x \notin O^L \text{ and } x \notin O^U) \\ & \text{iff} & \forall O^L, O^U \in \Omega\mathbf{X} : \\ & & \quad (O^L \cup O^U = \mathbf{X} \Rightarrow O^L \in \mathcal{G} \text{ or } O^U \in \mathcal{O}) \end{array}$$

For sober ground domain $\mathbf{X}$, we translate the open filters $\mathcal{O}$ into compact upper sets K, and the open grills $\mathcal{G}$ into closed sets C. For pairs (C, K) of a closed set and a compact upper set, condition $\overline{S}$ then becomes

$$O^L \cup O^U = \mathbf{X} \Rightarrow C \cap O^L \neq \emptyset \text{ or } K \subseteq O^U$$

We claim that the last implication is equivalent to '$C \subseteq O^U \Rightarrow K \subseteq O^U$'. Let $C \subseteq O^U$, and let O^L be the complement of C. Then $C \cap O^L = \emptyset$ holds, and $C \subseteq O^U$ implies $O^L \cup O^U = \mathbf{X}$. By condition $\overline{S}$, one may conclude $K \subseteq O^U$.

Conversely, assume $O^L \cup O^U = \mathbf{X}$ and $C \cap O^L = \emptyset$. Then $C = C \cap \mathbf{X} = C \cap (O^L \cup O^U) = C \cap O^U$, whence $C \subseteq O^U$. By the new condition, $K \subseteq O^U$ follows.

We now simplified $\overline{S}$ to the condition that $C \subseteq O$ implies $K \subseteq O$ for all open sets O. By Lemma 4.4.4, this is equivalent to $K \subseteq {\uparrow}C$.

Theorem 21.7.3 The sandwich power domain $\overline{\mathcal{S}}\mathbf{X}$ over a sober ground domain $\mathbf{X}$ is isomorphic to the set of all pairs (C, K) of a closed set C and a compact upper set K such that $K \subseteq {\uparrow}C$ holds. Order and power operations are inherited from $\mathcal{D}$ (see section 21.2).

The inherent asymmetry of domains effects that condition $\overline{S}$ may always be simplified to $K \subseteq {\uparrow}C$. Condition S however can be simplified to the comparable form $C \subseteq {\downarrow}K$ in special cases only.

An isolated pair $({\downarrow}E, {\uparrow}F)$ where E and F are finite subsets of $\mathbf{X}^0$ is a dual sandwich iff $F \subseteq {\uparrow}{\downarrow}E$. Later, we shall see that $\overline{\mathcal{S}}\mathbf{X}$ is algebraic whenever $\mathbf{X}$ is algebraic. The proof of this claim is postponed to section 22.10 where it is handled together with some other algebraicity proofs.

21.8 $\mathcal{T}$ and some intersections

The third sub-semiring $\mathbf{C}$ of $\mathbf{D}$ also gives rise to a sub-construction of $\mathcal{D}$. We call it $\mathcal{T}$; this name has no deeper meaning. $\mathcal{T}$ is defined by

- $\mathcal{T}\mathbf{X} = \{P \in \mathcal{D}\mathbf{X} \mid \forall p \in [\mathbf{X} \to \mathbf{C}] : Pp \in \mathbf{C}\}$

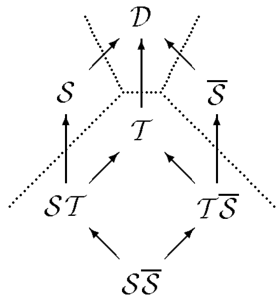
From Th. 14.6.2, we know that $\mathcal{T}$ is a power construction with characteristic semiring $\mathbf{C}$. We do not transform the defining condition of $\mathcal{T}$ further since the outcome would not be very interesting.

Since the intersection of sub-constructions is a sub-construction again, we obtain some more sub-constructions of $\mathcal{D}$ by intersecting $\mathcal{S}$, $\overline{\mathcal{S}}$, and $\mathcal{T}$. We denote them by $\mathcal{S}\overline{\mathcal{S}} = \mathcal{S} \cap \overline{\mathcal{S}}$, $\mathcal{S}\mathcal{T} = \mathcal{S} \cap \mathcal{T}$, and $\mathcal{T}\overline{\mathcal{S}} = \mathcal{T} \cap \overline{\mathcal{S}}$.The intersection of all three is $\mathcal{S}\overline{\mathcal{S}}$ because of the following inclusion statement:

Proposition 21.8.1 For all ground domains $\mathbf{X}$, $\mathcal{S}\overline{\mathcal{S}}\mathbf{X} \subseteq \mathcal{T}\mathbf{X}$ holds.

Proof: Let A be in $\mathcal{S}\overline{\mathcal{S}}\mathbf{X}$, and let $p : [\mathbf{X} \to \mathbf{C}]$. Then p is also in $[\mathbf{X} \to \mathbf{B}]$ and $[\mathbf{X} \to \overline{\mathbf{B}}]$, whence $A\,p$ is in both $\mathbf{B}$ and $\overline{\mathbf{B}}$ as A is in both $\mathcal{S}\mathbf{X}$ and $\overline{\mathcal{S}}\mathbf{X}$. Thus, $A\,p \in \mathbf{C}$. □

The mutual inclusions of these power constructions are visualized by the following diagram.



The dotted lines separate the constructions with different characteristic semirings. $\mathcal{D}$ has semiring $\mathbf{D}$, $\mathcal{S}$ has $\mathbf{B}$, $\overline{\mathcal{S}}$ has $\overline{\mathbf{B}}$, and the constructions $\mathcal{T}$, $\mathcal{S}\mathcal{T}$, $\mathcal{T}\overline{\mathcal{S}}$, and $\mathcal{S}\overline{\mathcal{S}}$ have semiring $\mathbf{C}$. All these constructions are shown to be different in section 22.8.

Chapter 22

The conditions M and $\overline{\mathrm{M}}$

In the previous chapter, we introduced all sub-constructions of the final **D**-construction $\mathcal{D}$ that are obtained by existential restriction to sub-semirings of **D** and intersection. In this chapter, we present some smaller sub-constructions of $\mathcal{D}$ characterized by additional logical conditions on the second order predicates. The need for smaller constructions is also due to the fact that the sandwich construction and its dual are not reduced (see section 22.1).

The additional logical conditions are prepared by studying the notions of lower and upper implication of $\mathcal{D}$-predicates in section 22.2. In section 22.3, we establish the logical conditions M and $\overline{M}$ on second order predicates and show that they induce two power domain constructions that we call $\mathcal{M}$ and $\overline{\mathcal{M}}$.

The defining condition M of the construction $\mathcal{M}$ is transformed in section 22.4, translated into topological terms in section 22.5 for sober ground domains, and drastically simplified in section 22.6 for ground domains in KC & M-CONT. The dual mixed construction $\overline{\mathcal{M}}$ is considered algebraically and topologically in section 22.7.

By intersecting the sub-constructions of $\mathcal{D}$ introduced so far, we obtain some more sub-constructions. All these 14 sub-constructions are summarized in section 22.8 and shown to be different even for small finite ground domains.

The most interesting sub-construction that is obtained by intersection is $\mathcal{C}$, the intersection of $\mathcal{M}$ and $\overline{\mathcal{M}}$. In section 22.9, we show that every member of $\mathcal{M}\mathbf{X}$ is a union of a member of $\mathcal{C}\mathbf{X}$ and a member of $\mathcal{M}\mathbf{X}$ that is below the empty set.

The case of an algebraic ground domain is considered in section 22.10. In this case, $\mathcal{S}$ is the known sandwich construction of [BDW88], $\mathcal{M}$ the mixed construction of [Gun89b, Gun90], and $\mathcal{C}$ Plotkin's construction of [Plo76].

22.1 Are sandwich power domains reduced?

We now start to investigate the question whether the sandwich power construction is reduced, and if not what its core is. One reason for being not reduced could be the fact that the sandwich power construction is the existential restriction of the *final* **D**-construction that in turn is built using the *final* **U**-construction. If $\mathcal{U}_f$ happens to be not reduced (an open question), then this could make the sandwich power construction non-reduced. The effect would however be small; we know that $\mathcal{U}_f\mathbf{X}$ is reduced for multi-continuous $\mathbf{X}$.

Independently from this uncertain effect, a second reason for 'junk' in the sandwich power domains could be the weakness of the sandwich condition. The condition was derived as the weakest possible condition that lets existential quantification w.r.t. $\mathbf{B}$ be well defined. There might well be stronger conditions that are also respected by all power operations. This is indeed true; we shall soon meet such a stronger condition.

To be precise, sandwich power domains are generally not reduced, not even for the simplest ground domains. As an example, we consider the ground domain $\mathbf{2} = \{\perp < \top\}$. Since this domain is finite, all lower sets are closed, and all upper sets are compact. Hence, both $\mathcal{L}\mathbf{2}$ and $\mathcal{U}\mathbf{2}$ contain 3 elements, and therefore, $\mathcal{D}\mathbf{2}$ has 9 elements. They are listed in the following table. The order is such that the lower left corner contains the least element, and elements increase when going to the right or upward.

	$\emptyset$	$\{\!\mid\,\mid\!\} = \theta$	$-$	$-$
$\uparrow$	$\{\top\}$	$\perp \cdot \{\!\mid\top\mid\!\}$	$*$	$\{\!\mid\top\mid\!\}$
$\mathcal{U}\mathbf{2}$	$\{\perp, \top\}$	$\perp \cdot \{\!\mid\perp\mid\!\}$	$\{\!\mid\perp\mid\!\}$	$\{\!\mid\perp, \top\mid\!\}$
		$\emptyset$	$\{\perp\}$	$\{\perp, \top\}$
		$\mathcal{L}\mathbf{2}$	$\longrightarrow$	

The two places marked by '$-$' do not contain sandwiches. Since $\mathbf{2}$ is finite, Th. 21.5.1 holds, and (C, K) is a sandwich iff $C \subseteq {\downarrow}K$. This is not true if K is empty and C is not empty. Hence, the sandwich power domain $\mathcal{S}\mathbf{2}$ has 7 elements. From these, 6 can be built up using the power operations θ, '$\uplus$', and '$\cdot$', as indicated in the table. The remaining sandwich $S = (\{\perp\}, \{\top\})$ at position '$*$' cannot be built such way. It is neither θ nor $\{\!\mid x\mid\!\}$ nor $\perp \cdot \{\!\mid x\mid\!\}$ for some x since all these sandwiches occur on other positions in the table. It also cannot be built by union: if $A \uplus B = (\{\perp\}, \{\top\})$, then $A^L \cup B^L = \{\perp\}$ holds. Hence, one of these, say A^L, equals $\{\perp\}$. Because of $A^U \cup B^U = \{\top\}$, $A^U \subseteq \{\top\}$ holds. A^U cannot be empty, because $(\{\perp\}, \emptyset)$ is not a sandwich. Hence $A = (\{\perp\}, \{\top\}) = S$. Because all is finite, S can neither be obtained as limit of a directed set of the remaining 6 sandwiches. Thus, S is not in the core of $\mathcal{S}\mathbf{2}$.

Summarizing, we found that $\mathcal{S}\mathbf{2}$ has 7 elements, whereas its core only has 6. The unreachable sandwich S is characterized by the fact that its lower part is not empty, but does not meet its upper part.

Unfortunately, there is no easy way to see what condition distinguishes the 6 sandwiches in the core from the one not in the core. By studying various sandwich power domains for more complex ground domains, the correct condition was found. We call it *mix condition* or condition M because Th. 22.10.3 will show that it restricts the sandwich power domain to the *mixed power domain* defined in [Gun89b, Gun90] for algebraic ground domains.

22.2 Lower and upper implication for D-predicates

The definition of condition M and its dual, condition $\overline{M}$ is prepared by investigating the logic of $\mathbf{D}$ more closely.

To obtain a concise presentation, we always work with $\mathbf{D}$-*predicates* $[\mathbf{X} \to \mathbf{D}]$ in the sequel where $\mathbf{X}$ is an arbitrary domain. All results are valid for $\mathbf{D}$ itself also since $\mathbf{D}$ is isomorphic to $[\mathbf{1} \to \mathbf{D}]$. $[\mathbf{X} \to \mathbf{D}]$ is isomorphic to $[\mathbf{X} \to \mathbf{L}] \times [\mathbf{X} \to \mathbf{U}]$, whence a in $[\mathbf{X} \to \mathbf{D}]$ may be written as pair (a^L, a^U).

In addition to the logical operations of disjunction '+', conjunction '$\cdot$', and negation '$\neg$', we introduce a kind of difference: $a - b = a \cdot \neg b$. It is mainly used as a notational abbreviation. The following relations are easily verified:

Proposition 22.2.1 For all $\mathbf{D}$-predicates a and b:

(1) $(a+b)^L = a^L + b^L$ and $(a+b)^U = a^U + b^U$

(2) $(a \cdot b)^L = a^L \cdot b^L$ and $(a \cdot b)^U = a^U \cdot b^U$

(3) $(\neg a)^L = \widetilde{a^U}$ and $(\neg a)^U = \widetilde{a^L}$ (cf. section 21.1)

(4) $(a-b)^L = a^L \cdot \widetilde{b^U}$ and $(a-b)^U = a^U \cdot \widetilde{b^L}$.

Statement (4) is an immediate consequence of statements (2) and (3). It shows that difference '$-$' combines the lower and upper components of its arguments. This property explains the usage of difference in condition M and $\overline{M}$ as will be seen below.

The next proposition claims the equivalence of various conditions. They are coined as lower and upper implication.

Proposition 22.2.2 For $\mathbf{D}$-predicates, the following equivalences hold:

(1) $a^L \le b^L$ iff $a + b \le b$. In this case, we say that a and b are in the relation of *lower implication* '$\overset{L}{\mapsto}$'.

(2) $a^U \ge b^U$ iff $a + b \ge b$. In this case, we say that a and b are in the relation of *upper implication* '$\overset{U}{\mapsto}$'.

Proof:

(1) By part (1) of Prop. 22.2.1, $a + b \le b$ holds iff $a^L + b^L \le b^L$ and $a^U + b^U \le b^U$. Since the inequality involving 'U' is a tautology, it can be dropped. Hence, $a+b \le b$ iff $a^L + b^L \le b^L$. This inequality is equivalent to $a^L \le b^L$.

(2) By part (1) of Prop. 22.2.1, $a + b \ge b$ holds iff $a^L + b^L \ge b^L$ and $a^U + b^U \ge b^U$. Here, the inequality with 'L' is a tautology. $a^U + b^U \ge b^U$ is equivalent to $a^U \ge b^U$. □

Lower and upper implication enjoy some properties that are needed in the next section.

Proposition 22.2.3 Let $X = L$ or U in the following.

(1) The relation '$\overset{X}{\mapsto}$' is reflexive and transitive.

(2) If $a \overset{X}{\mapsto} a'$ and $b \overset{X}{\mapsto} b'$, then $a + b \overset{X}{\mapsto} a' + b'$.

(3) $(a+b) - (a'+b') \overset{X}{\mapsto} (a - a') + (b - b')$.

(4) If P is an additive second order predicate, then $a \overset{X}{\mapsto} b$ implies $Pa \overset{X}{\mapsto} Pb$.

(5) If $(a_i)_{i\in I}$ and $(b_i)_{i\in I}$ are directed families of $\mathbf{D}$-predicates with $a_i \overset{X}{\mapsto} b_i$ for all $i \in I$, then $\bigsqcup_{i\in I} a_i \overset{X}{\mapsto} \bigsqcup_{i\in I} b_i$.

(6) $\perp \cdot a \overset{L}{\mapsto} \perp \cdot b$ no matter what $\mathbf{D}$-predicates a and b are. Similarly, $\mathsf{W} \cdot a \overset{U}{\mapsto} \mathsf{W} \cdot b$.

Proof:

(1) Immediate by definition.

(2) $a^L \le a'^L$ and $b^L \le b'^L$ implies $(a+b)^L = a^L + b^L \le a'^L + b'^L = (a'+b')^L$. The proof for '$\overset{U}{\mapsto}$' is similarly simple.

(3) For '$\overset{L}{\mapsto}$': $((a+b)-(a'+b'))^L = (a^L + b^L) \cdot \widetilde{a'^U + b'^U} = (a^L + b^L) \cdot \widetilde{a'^U} \cdot \widetilde{b'^U} = (a^L \cdot \widetilde{a'^U} \cdot \widetilde{b'^U}) + (b^L \cdot \widetilde{a'^U} \cdot \widetilde{b'^U}) \le (a^L \cdot \widetilde{a'^U}) + (b^L \cdot \widetilde{b'^U}) = (a-a')^L + (b-b')^L = ((a-a')+(b-b'))^L$.

Here, '$\le$' holds since $p \cdot q \le p$ holds for L-predicates p and q. The proof for '$\overset{U}{\mapsto}$' is analogous; here, '$\ge$' holds since $p \cdot q \ge p$ holds for U-predicates.

(4) For '$\overset{L}{\mapsto}$': $a \overset{L}{\mapsto} b$ implies $a + b \le b$, whence $Pa + Pb = P(a+b) \le Pb$, i.e. $Pa \overset{L}{\mapsto} Pb$. The proof for '$\overset{U}{\mapsto}$' is similar.

(5) We use the equivalence $a \overset{L}{\mapsto} b$ iff $a^L \le b^L$. If $(a_i)_{i\in I}$ is directed, then $(a_i^L)_{i\in I}$ is directed, too. $a_i^L \le b_i^L$ implies $(\bigsqcup_{i\in I} a_i)^L = \bigsqcup_{i\in I} a_i^L \le \bigsqcup_{i\in I} b_i^L = (\bigsqcup_{i\in I} b_i)^L$. The proof for '$\overset{U}{\mapsto}$' is analogous.

(6) $(\perp \cdot a)^L = 0 \cdot a^L = 0$ and also $(\perp \cdot b)^L = 0$. Similarly, $(\mathsf{W} \cdot a)^U = 0 \cdot a^U = 0$. □

22.3 The conditions M and $\overline{M}$

After the preliminaries of the previous section, we are now able to define the conditions M and $\overline{M}$:

Definition 22.3.1

A in $\mathcal{D}\mathbf{X}$ satisfies condition M iff $Ap - Aq \overset{L}{\mapsto} A(p-q)$ for all predicates $p, q : [\mathbf{X} \to \mathbf{D}]$.

A in $\mathcal{D}\mathbf{X}$ satisfies condition $\overline{M}$ iff $Ap - Aq \overset{U}{\mapsto} A(p-q)$ for all predicates $p, q : [\mathbf{X} \to \mathbf{D}]$.

These definitions may seem to be ad hoc and badly motivated. We shall however see in Th. 22.10.3 that condition M restricts the final $\mathbf{D}$-powerdomain $\mathcal{D}\mathbf{X}$ to the mixed power domain $\mathcal{M}\mathbf{X}$ of [Gun89b, Gun90] if the ground domain $\mathbf{X}$ is algebraic, and further restricting $\mathcal{M}\mathbf{X}$ by means of condition $\overline{M}$ results in Plotkin's power construction.

In order to establish the power constructions $\mathcal{M}\mathbf{X}$ and $\overline{\mathcal{M}}\mathbf{X}$, we have to show that the power operations preserve the restrictions. Using the generic Prop. 22.2.3, the proofs for M and $\overline{M}$ are completely analogous. We formulate it for M.

- $\theta = \lambda p.\, 0$, whence $\theta p - \theta q = 0 - 0 = 0 = \theta(p-q)$. By reflexivity of lower and upper implication (Prop. 22.2.3 (1)), $\theta \in \mathcal{M}\mathbf{X}$ and $\theta \in \overline{\mathcal{M}}\mathbf{X}$ follow.
- $\{\!|x|\!\} = \lambda p.\, p\,x$, whence $\{\!|x|\!\}\, p - \{\!|x|\!\}\, q = p\,x - q\,x = (p-q)\,x = \{\!|x|\!\}\,(p-q)$. $p\,x - q\,x = (p-q)\,x$ holds since all logical operations are defined pointwise on predicates. Reflexivity again yields the desired result.

- For A, B in $\mathcal{M}\mathbf{X}$,

$$\begin{array}{rcll} (A \uplus B)\,p - (A \uplus B)\,q &=& (A\,p + B\,p) - (A\,q + B\,q) & \\ &\overset{L}{\mapsto}& (A\,p - A\,q) + (B\,p - B\,q) & \text{by Prop. 22.2.3 (3)} \\ &\overset{L}{\mapsto}& A\,(p-q) + B\,(p-q) & \\ && \quad \text{since } A,\ B \text{ in } \mathcal{M}\mathbf{X} \text{ by Prop. 22.2.3 (2)} & \\ &=& (A \uplus B)\,(p-q) & \end{array}$$

- For $f : [\mathbf{X} \to \mathcal{M}\mathbf{Y}]$ and A in $\mathcal{M}\mathbf{X}$,

$$\begin{array}{rcll} (\mathit{ext}\,f\,A)\,p - (\mathit{ext}\,f\,A)\,q &=& A\,(\lambda x.\,fxp) - A\,(\lambda x.\,fxq) & \\ &\overset{L}{\mapsto}& A\,(\lambda x.\,fxp - fxq) & \text{since } A \text{ in } \mathcal{M}\mathbf{X} \\ &\overset{L}{\mapsto}& A\,(\lambda x.\,fx\,(p-q)) & \\ && \quad \text{since } fx \text{ in } \mathcal{M}\mathbf{X} \text{ by Prop. 22.2.3 (4); } A \text{ is additive} & \\ &=& (\mathit{ext}\,f\,A)\,(p-q) & \end{array}$$

- If $(A_i)_{i\in I}$ is a directed family in $\mathcal{M}\mathbf{X}$, then both $(A_i p - A_i q)_{i\in I}$ and $(A_i\,(p-q))_{i\in I}$ are directed families with $A_i p - A_i q \overset{L}{\mapsto} A\,(p-q)$ for all $i \in I$. By Prop. 22.2.3 (5), $A\,p - A\,q \overset{L}{\mapsto} A\,(p-q)$ follows where $A = \bigsqcup_{i\in I} A_i$.

- Finally, we show two facts about the external product: $\bot \cdot A$ satisfies M for all A in $\mathcal{D}\mathbf{X}$ — even those that do not satisfy M. Analogously, $\mathsf{W} \cdot A$ satisfies $\overline{M}$ for all A.

  The first statement holds since $\bot \cdot Ap - \bot \cdot Aq = \bot \cdot Ap \cdot \neg(\bot \cdot Aq) \overset{L}{\mapsto} \bot \cdot A\,(p-q)$ by Prop. 22.2.3 (6).

22.4 $\mathcal{M}$ — the mixed power domain construction

In this section, we investigate the power domain construction derived from $\mathcal{D}\mathbf{X}$ by means of condition M. We abbreviate this construction by $\mathcal{M}$ since it will turn out to be a generalization of Gunter's mixed power domain construction [Gun89b, Gun90] that was defined for algebraic ground domains only. Hence, we chose the names *condition M* or *mix condition*. The elements of $\mathcal{M}\mathbf{X}$ are called *mixes*.

$$\mathcal{M}\mathbf{X} = \{P : [[\mathbf{X} \to \mathbf{D}] \overset{add}{\to} \mathbf{D}] \mid \forall p,\, q \in [\mathbf{X} \to D] : Pp - Pq \overset{L}{\mapsto} P\,(p-q)\}$$

In the previous section, we have proved that $\mathcal{M}$ is a power domain construction. Its characteristic semiring is $\mathbf{B}$ since it contains all three of $\mathsf{T} = \{\!|\diamond|\!\}$, $\mathsf{F} = \theta$, and $\bot = \bot \cdot \{\!|\diamond|\!\}$, and since it does not contain W because the mix condition implies the sandwich condition as we shall see below.

In the sequel, we want to translate the mix condition into topological terms. This is done in analogy to the sandwich power construction. The first step leads to pairs of open grills and open filters, and the second step to pairs of closed sets and compact upper sets. In the course of this translation, we also prove that condition M implies condition S, i.e. the mixed power domains are subsets of the sandwich power domains.

Let $p = (p^L, p^U)$ and $q = (q^L, q^U)$ be two predicates. For $A = (A^L, A^U)$, the mix condition may then be transformed using the facts collected in Prop. 22.2.1.

$$\begin{array}{lll} A\,p - A\,q \overset{L}{\mapsto} A\,(p-q) & \text{iff} & (A\,p - A\,q)^L \le (A\,(p-q))^L \\ & \text{iff} & (A\,p - A\,q)^L = 1 \Rightarrow (A\,(p-q))^L = 1 \\ & \text{iff} & A^L p^L \cdot \widetilde{A^U q^U} = 1 \Rightarrow A^L(p^L \cdot \widetilde{q^U}) = 1 \\ & \text{iff} & A^L p^L = 1 \text{ and } A^U q^U = 0 \Rightarrow A^L(p^L \cdot \widetilde{q^U}) = 1 \end{array}$$

Thus, we finally obtain:

Proposition 22.4.1

$(A^L, A^U) \in \mathcal{M}\mathbf{X}$ iff

$\forall p : [\mathbf{X} \to \mathbf{L}],\ q : [\mathbf{X} \to \mathbf{U}] : A^L p = 1$ and $A^U q = 0$ implies $A^L(p \cdot \widetilde{q}) = 1$

From this equivalence, one may easily conclude $\mathcal{M}\mathbf{X} \subseteq \mathcal{S}\mathbf{X}$. The precondition $A^L p = 1$ and $A^U q = 0$ of the sandwich condition in the form of Prop. 21.3.2 equals that of the mix condition. By the mix condition, $A^L(p \cdot \widetilde{q}) = 1$ follows. If $p \cdot \widetilde{q}$ were $\underline{0}$, then $A^L(p \cdot \widetilde{q}) = 0$ would hold because of additivity of A^L. Thus, the mix condition implies $p \cdot \widetilde{q} \neq \underline{0}$.

22.5 $\mathcal{M}$ in topological terms

In this section, we translate condition M into topological terms. At first, we translate the predicates to open sets. p becomes O^L and q becomes O^U. Then $p \cdot \widetilde{q} = p \sqcap \widetilde{q}$ corresponds to $O^L \cap O^U$. The lower second order predicate A^L is translated into an open grill $\mathcal{G}$, and the upper one into an open filter $\mathcal{O}$. We remember $A^L p = 1 = \top$ iff $O^L \in \mathcal{G}$, and $A^U q = 0 = \top$ iff $O^U \in \mathcal{O}$. Thus, we obtain

$$(\mathcal{G}, \mathcal{O}) \in \mathcal{M}\mathbf{X} \text{ iff } \forall O^L \in \mathcal{G}, O^U \in \mathcal{O} : O^L \cap O^U \in \mathcal{G}$$

For sober ground domain $\mathbf{X}$, one can go one step further and translate the open grills $\mathcal{G}$ into closed sets C, and the open filters $\mathcal{O}$ into compact upper sets K. $O \in \mathcal{G}$ then becomes $C \cap O \neq \emptyset$, and $O' \in \mathcal{O}$ becomes $K \subseteq O'$. Hence, the mix condition translates into:
for all open sets O and O', if C meets O and $K \subseteq O'$ then $C \cap O \cap O' \neq \emptyset$.
For fixed C and O', the following holds:

Every open set meeting C meets $C \cap O'$
- iff every open environment of every point of C meets $C \cap O'$
- iff every point of C is in the closure of $C \cap O'$ by Prop. 4.2.2
- iff $C \subseteq \mathsf{cl}\,(C \cap O')$.

Hence, one obtains

Theorem 22.5.1 The mixed power domain $\mathcal{M}\mathbf{X}$ over a sober ground domain $\mathbf{X}$ is isomorphic to the set of all pairs (C, K) of a closed set C and a compact upper set K such that for all open sets O with $K \subseteq O$ the inclusion $C \subseteq \mathsf{cl}\,(C \cap O)$ holds. The order is given by $(C, K) \le (C', K')$ iff $C \subseteq C'$ and $K \supseteq K'$. The power operations are defined as in $\mathcal{D}\mathbf{X}$ (see 21.2).

22.6 Simplification of the mix condition

In this section, we investigate the mix restriction '$K \subseteq O$ open implies $C \subseteq \mathsf{cl}\,(C \cap O)$' for pairs (C, K) of a closed and a compact upper set. As with the sandwich condition, we present a drastic simplification for a special class of ground domains.

Theorem 22.6.1 Let $\mathbf{X}$ be a multi-continuous domain in class KC. Then for all closed $C \subseteq \mathbf{X}$ and all compact upper sets $K \subseteq \mathbf{X}$, the following two statements are equivalent:

(1) The mix condition: If $K \subseteq O$ for some open set O, then $C \subseteq \mathsf{cl}\,(C \cap O)$.

(2) $C \subseteq \downarrow(C \cap K)$.

Condition KC was introduced by Prop. 5.5.2 and means that the intersection of two upper cones is compact. All finitely continuous domains are continuous and satisfy this condition (Prop. 7.2.2), i.e. they are covered by the theorem above.

Proof: If $C \subseteq \downarrow(C \cap K)$ holds, then $K \subseteq O$ implies $C \subseteq \downarrow(C \cap K) \subseteq \downarrow(C \cap O) \subseteq \mathsf{cl}\,(C \cap O)$.

As pointed out in the proof of Th. 21.5.1, the preconditions of the lemmas 8.10.1 and 8.10.2 are satisfied.

Let C be a closed set and K a compact upper set satisfying the mix condition. Let $\mathcal{F}$ be the set of all finitary upper sets F such that there is an open set O with $K \subseteq O \subseteq F$. From Th. 8.3.2, we know that $\mathcal{F}$ is a $\supseteq$-directed set whose intersection is K.

For all sets $F \in \mathcal{F}$, $K \subseteq O \subseteq F$ holds, whence $C \subseteq \mathsf{cl}\,(C \cap O) \subseteq \mathsf{cl}\,(C \cap F)$ follows from the mix condition. Lemma 8.10.1 implies $\mathsf{cl}\,(C \cap F) = \downarrow(C \cap F)$. Hence, $C \subseteq \bigcap_{F \in \mathcal{F}} \downarrow(C \cap F) = \downarrow \bigcap_{F \in \mathcal{F}}(C \cap F) = \downarrow(C \cap K)$ applying Lemma 8.10.2. $\square$

In contrast to the situation with the sandwich power domain, we do not know whether condition KC is important for the validity of the theorem. The example following Th. 21.5.1 does not matter here since the sandwich occurring in it does not satisfy the mix condition.

22.7 $\overline{\mathcal{M}}$ — the dual mixed power domain construction

In this section, we investigate the power domain construction $\overline{\mathcal{M}}$ derived from $\mathcal{D}$ by means of condition $\overline{M}$. The elements of $\overline{\mathcal{M}}\mathbf{X}$ are called *dual mixes*. In section 22.3, we have proved that $\overline{\mathcal{M}}$ is a power domain construction. Its characteristic semiring is $\overline{\mathbf{B}}$ since it contains all three of $\mathsf{T} = \{\!|\diamond|\!\}$, $\mathsf{F} = \theta$, and $\mathsf{W} = \mathsf{W} \cdot \{\!|\diamond|\!\}$, and condition $\overline{M}$ implies condition $\overline{S}$ as we shall see below.

Let $p = (p^L, p^U)$ and $q = (q^L, q^U)$ be two predicates. For $A = (A^L, A^U)$, the condition may then be transformed using the facts collected in Prop. 22.2.1.

$$\begin{array}{lll} A\,p - A\,q \overset{U}{\mapsto} A\,(p-q) & \text{iff} & (A\,p - A\,q)^U \geq (A\,(p-q))^U \\ & \text{iff} & (A\,p - A\,q)^U = 1 \Rightarrow (A\,(p-q))^U = 1 \\ & \text{iff} & A^U p^U \cdot \widetilde{A^L q^L} = 1 \Rightarrow A^U(p^U \cdot \widetilde{q^L}) = 1 \\ & \text{iff} & A^U p^U = 1 \text{ and } A^L q^L = 0 \Rightarrow A^U(p^U \cdot \widetilde{q^L}) = 1 \end{array}$$

Thus, we finally obtain:

Proposition 22.7.1

$$(A^L, A^U) \in \overline{\mathcal{M}}\mathbf{X} \text{ iff}$$
$$\forall p : [\mathbf{X} \to \mathbf{U}],\, q : [\mathbf{X} \to \mathbf{L}] : A^U p = 1 \text{ and } A^L q = 0 \text{ implies } A^U(p \cdot \widetilde{q}) = 1$$

From this equivalence, one may easily conclude $\overline{\mathcal{M}}\mathbf{X} \subseteq \overline{\mathcal{S}}\mathbf{X}$. If $p \cdot \widetilde{q}$ were $\underline{0}$, then $A^U(p \cdot \widetilde{q}) = 0$ would hold by additivity.

The transformation of condition $\overline{M}$ proceeds by translating the predicates to open sets. p becomes O^U and q becomes O^L. Then $p \cdot \widetilde{q} = p \sqcup \widetilde{q}$ corresponds to $O^U \cup O^L$. The lower second order predicate A^L is translated into an open grill $\mathcal{G}$, and the upper one into an open filter $\mathcal{O}$. All equations in the formula above are just negations of those in Prop. 22.4.1. Thus, we obtain

$$\begin{array}{lll} (\mathcal{G}, \mathcal{O}) \in \overline{\mathcal{M}}\mathbf{X} & \text{iff} & O^L \notin \mathcal{G} \text{ and } O^U \notin \mathcal{O} \Rightarrow O^L \cup O^U \notin \mathcal{O} \\ & \text{iff} & O^L \cup O^U \in \mathcal{O} \Rightarrow O^L \in \mathcal{G} \text{ or } O^U \in \mathcal{O} \end{array}$$

For sober ground domain $\mathbf{X}$, we translate open grills $\mathcal{G}$ into closed sets C and open filters $\mathcal{O}$ into compact upper sets K.

$$(C, K) \in \overline{\mathcal{M}}\mathbf{X} \quad \text{iff} \quad K \subseteq O^L \cup O^U \Rightarrow C \cap O^L \neq \emptyset \text{ or } K \subseteq O^U$$

Let C' be the complement of C. Then we claim that the condition above is equivalent to '$K \subseteq C' \cup O \Rightarrow K \subseteq O$'. The old condition implies the new one since $C \cap C' = \emptyset$ and C' is open. The new condition implies the old one since $C \cap O^L = \emptyset$ implies $O^L \subseteq C'$ whence $K \subseteq O^L \cup O^U \subseteq C' \cup O^U$.

To simplify further, note that $K \subseteq C' \cup O$ is equivalent to $C \cap K \subseteq O$. Thus, we obtain $(C, K) \in \overline{\mathcal{M}}\mathbf{X}$ iff $C \cap K \subseteq O$ implies $K \subseteq O$ for all open sets O. By Lemma 4.4.4, this is equivalent to $K \subseteq \uparrow(C \cap K)$.

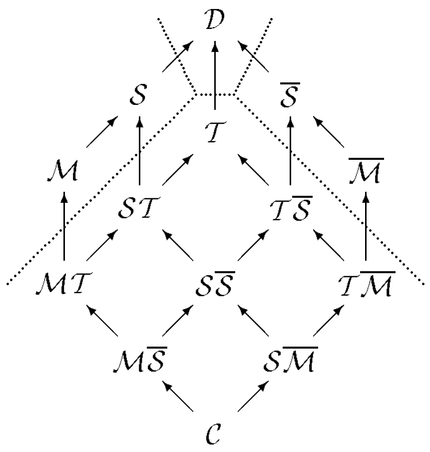
Theorem 22.7.2 The dual mixed power domain $\overline{\mathcal{M}}\mathbf{X}$ over a sober ground domain $\mathbf{X}$ is isomorphic to the set of all pairs (C, K) of a closed set C and a compact upper set K such that $K \subseteq \uparrow(C \cap K)$ holds.

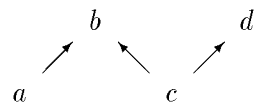
22.8 Other sub-constructions of $\mathcal{D}$

The final $\mathbf{D}$-construction $\mathcal{D}$ has 5 'atomic' sub-constructions obtained by restricting $\mathcal{D}$ using the conditions S, $\overline{S}$, M, $\overline{M}$, and T. The resulting constructions $\mathcal{S}$ and $\mathcal{M}$ have semiring $\mathbf{B}$, whereas the dual constructions $\overline{\mathcal{S}}$ and $\overline{\mathcal{M}}$ have semiring $\overline{\mathbf{B}}$, and $\mathcal{T}$ has semiring $\mathbf{C}$. For all ground domains $\mathbf{X}$, the inclusions $\mathcal{M}\mathbf{X} \subseteq \mathcal{S}\mathbf{X}$, $\overline{\mathcal{M}}\mathbf{X} \subseteq \overline{\mathcal{S}}\mathbf{X}$ and $\mathcal{S}\mathbf{X} \cap \overline{\mathcal{S}}\mathbf{X} \subseteq \mathcal{T}\mathbf{X}$ hold.

Further sub-constructions of $\mathcal{D}$ result from intersections of the constructions above. Because of the subset relations, 8 constructions are obtained. The most interesting one is $\mathcal{C} = \mathcal{M} \cap \overline{\mathcal{M}}$ because it coincides with Plotkin's power construction for algebraic ground domains.

The relative order of the constructions is indicated by the following diagram where the arrows denote subset relations. All constructions below $\mathcal{T}$ have semiring $\mathbf{C}$, $\mathcal{M}$ and $\mathcal{S}$ have semiring $\mathbf{B}$, $\overline{\mathcal{M}}$ and $\overline{\mathcal{S}}$ have $\overline{\mathbf{B}}$, and $\mathcal{D}$ has $\mathbf{D}$ (see the dotted lines).





$\mathbf{X}$

Examples that all these constructions differ are obtained when using the simple ground domain $\mathbf{X} = \{a < b > c < d\}$. For finite ground domains, the simple formulae for conditions S and M apply. A pair (C, K) satisfies S iff $C \subseteq \downarrow K$, $\overline{S}$ iff $K \subseteq \uparrow C$, M iff $C \subseteq \downarrow(C \cap K)$, and $\overline{M}$ iff $K \subseteq \uparrow(C \cap K)$. Since $\mathbf{X}$ admits only two predicates $[\mathbf{X} \to \mathbf{C}]$, namely $\lambda x.\, \mathsf{F}$ and $\lambda x.\, \mathsf{T}$, one easily deduces $(C, K) \in \mathcal{T}\mathbf{X}$ iff both C and K are not empty or else $C = K = \emptyset$.

- $(\{a\}, \{a, b\})$ is in $\mathcal{C}\mathbf{X}$.
- $(\{a, b, c\}, \{b, d\})$ is in $\mathcal{M}\overline{\mathcal{S}}\mathbf{X}$, but not in $\overline{\mathcal{M}}\mathbf{X}$.
- $(\{a, c\}, \{b, c, d\})$ is in $\mathcal{S}\overline{\mathcal{M}}\mathbf{X}$, but not in $\mathcal{M}\mathbf{X}$.
- $(\{a\}, \{b\})$ is in $\mathcal{S}\overline{\mathcal{S}}\mathbf{X}$, but neither in $\mathcal{M}\mathbf{X}$ nor in $\overline{\mathcal{M}}\mathbf{X}$.
- $(\{a\}, \{a, b, d\})$ is in $\mathcal{M}\mathcal{T}\mathbf{X}$, but not in $\overline{\mathcal{S}}\mathbf{X}$.
- $(\{a, c, d\}, \{d\})$ is in $\mathcal{T}\overline{\mathcal{M}}\mathbf{X}$, but not in $\mathcal{S}\mathbf{X}$.
- $(\emptyset, \{d\})$ is in $\mathcal{M}\mathbf{X}$, but neither in $\mathcal{T}\mathbf{X}$ nor in $\overline{\mathcal{S}}\mathbf{X}$.
- $(\{a\}, \emptyset)$ is in $\overline{\mathcal{M}}\mathbf{X}$, but neither in $\mathcal{T}\mathbf{X}$ nor in $\mathcal{S}\mathbf{X}$.
- $(\{a\}, \{d\})$ is in $\mathcal{T}\mathbf{X}$, but neither in $\mathcal{S}\mathbf{X}$ nor in $\overline{\mathcal{S}}\mathbf{X}$.

22.9 The $\mathcal{M}$-$\mathcal{C}$-Theorem

Our next effort in this chapter is the derivation of a strong relation between the sub-constructions $\mathcal{M}$ and $\mathcal{C}$ of the final $\mathbf{D}$-construction $\mathcal{D}$.

Theorem 22.9.1 For every ground domain $\mathbf{X}$ holds: For every m in $\mathcal{M}\mathbf{X}$, there is some c in $\mathcal{C}\mathbf{X}$ such that $m = m? \uplus c$.

Proof: Let $m = (L, U)$ where L is in the final $\mathbf{L}$-powerdomain $\mathcal{L}\mathbf{X}$ and U is in the final $\mathbf{U}$-powerdomain $\mathcal{U}\mathbf{X}$. Then $m? = \bot \cdot m = (0, 1) \cdot (L, U) = (\theta, U)$ holds.

Let S be the set of all $\mathbf{L}$-predicates p with $Lp = 0$. $\underline{0}$ is in S by additivity of L. If p_1 and p_2 are in S, then $p_1 + p_2$ is in S by $L\,(p_1 + p_2) = L\,p_1 + L\,p_2 = 0 + 0 = 0$. Since addition in $\mathbf{L}$ is least upper bound, this shows that S is directed. Let s be its supremum. By continuity, $L\,s = 0$ holds.

The **L**-predicate s induces a **U**-predicate $\widetilde{s}$. Let $U' = \mathit{filter}\,\widetilde{s}\,U$. Since 1 is least in **U**, $\widetilde{s} \geq \underline{1}$ holds, whence $U' \geq \mathit{filter}\,\underline{1}\,U = U$ follows. In **U**-powerdomains, union is greatest lower bound, whence $U \uplus U' = U$. Thus, with $c = (L, U')$, we obtain $m? \uplus c = (\theta, U) \uplus (L, U') = (\theta \uplus L, U \uplus U') = (L, U) = m$ as required.

We still have to show c in $\mathcal{C}\mathbf{X}$. $(\mathit{filter}\,\widetilde{s}\,U)\,q = U\,(\widetilde{s} \cdot q)$ holds as computed in section 15.6. To show condition M for $c = (L, U')$, we have to derive $L\,(p \cdot \widetilde{q}) = 1$ from $L\,p = 1$ and $U'\,q = 0$. The precondition implies $L\,p = 1$ and $U\,(\widetilde{s} \cdot q) = 0$. Since $m = (L, U)$ satisfies condition M, we conclude $1 = L\,(p \cdot \widetilde{\widetilde{s} \cdot q}) = L\,(p \cdot (s + \widetilde{q})) = L\,(p \cdot s) + L\,(p \cdot \widetilde{q}) \leq L\,s + L\,(p \cdot \widetilde{q}) = L\,(p \cdot \widetilde{q})$. The '$\leq$' relation holds since '$\cdot$' is greatest lower bound in **L**. $L\,s$ is 0 as mentioned above.

To show condition $\overline{M}$ for $c = (L, U')$, we have to derive $U'\,(p \cdot \widetilde{q}) = 1$ from $U'\,p = 1$ and $L\,q = 0$. Equivalently, we have to derive $U\,(p \cdot \widetilde{s} \cdot \widetilde{q}) = 1$ from $U\,(p \cdot \widetilde{s}) = 1$ and $L\,q = 0$. $L\,q = 0$ implies $q \leq s$ since s is the supremum of all predicates that are mapped to 0 by L. Since addition in **L** is lub, $q + s = s$ follows, whence $\widetilde{s} = \widetilde{q + s} = \widetilde{q} \cdot \widetilde{s}$. Thus $U\,(p \cdot \widetilde{s}) = 1$ implies $U\,(p \cdot \widetilde{s} \cdot \widetilde{q}) = 1$. □

An analogous $\overline{\mathcal{M}}$-$\mathcal{C}$-theorem cannot be proved analogously since the first step, i.e. the construction of s, does not work out. In the world of **U**-predicates, the set of all p with $U\,p = 0$ is downward directed, and its infimum may not exist. Here, the inherent asymmetry of domains matters.

22.10 The case of an algebraic ground domain

Next, we consider the case of an algebraic ground domain. We claim that for all $\mathcal{R} \in \{\mathcal{S}, \overline{\mathcal{S}}, \mathcal{M}, \overline{\mathcal{M}}, \mathcal{S}\overline{\mathcal{S}}, \mathcal{M}\overline{\mathcal{S}}, \mathcal{S}\overline{\mathcal{M}}, \mathcal{C}\}$, the power domain $\mathcal{R}\mathbf{X}$ is algebraic if the ground domain is, and the base of $\mathcal{R}\mathbf{X}$ equals the part of the base of $\mathcal{D}\mathbf{X}$ which is within $\mathcal{R}\mathbf{X}$, i.e. $(\mathcal{R}\mathbf{X})^0 = (\mathcal{D}\mathbf{X})^0 \cap \mathcal{R}\mathbf{X}$. The power constructions involving $\mathcal{T}$ are excluded from this claim because they are of minor interest. To prove the claim, we show two lemmata.

Lemma 22.10.1 Every pair in $(\mathcal{D}\mathbf{X})^0 \cap \mathcal{R}\mathbf{X}$ is isolated in $\mathcal{R}\mathbf{X}$.

Proof: Let A be a member of $\mathcal{R}\mathbf{X}$ that is isolated in $\mathcal{D}\mathbf{X}$, and let $A \leq \bigsqcup_{\mathcal{R}\mathbf{X}} \mathcal{D}$ where $\mathcal{D}$ is a directed set in $\mathcal{R}\mathbf{X}$. Because $\mathcal{R}\mathbf{X}$ is a sub-domain of $\mathcal{D}\mathbf{X}$, lubs of directed sets w.r.t. $\mathcal{R}\mathbf{X}$ coincide with lubs w.r.t. $\mathcal{D}\mathbf{X}$. Hence $A \leq D$ for some D in $\mathcal{D}$ since A is isolated in $\mathcal{D}\mathbf{X}$. □

The second Lemma is less trivial.

Lemma 22.10.2 Let P be a member of $\mathcal{R}\mathbf{X}$, and let A be an isolated point of $\mathcal{D}\mathbf{X}$ below P. Then there is an isolated point B in $\mathcal{D}\mathbf{X}$ that lies within $\mathcal{R}\mathbf{X}$ and is between A and P.

$$A \in (\mathcal{D}\mathbf{X})^0,\ P \in \mathcal{R}\mathbf{X},\ A \leq P \Rightarrow \exists B \in (\mathcal{D}\mathbf{X})^0 \cap \mathcal{R}\mathbf{X} : A \leq B \leq P.$$

Before we are going to prove this lemma, we show that the two lemmata imply algebraicity. Let P be in $\mathcal{R}\mathbf{X}$. Then let $\mathcal{A} = \{A \in (\mathcal{D}\mathbf{X})^0 \mid A \leq P\}$ and $\mathcal{B} = \{B \in (\mathcal{D}\mathbf{X})^0 \cap \mathcal{R}\mathbf{X} \mid B \leq P\}$. Since $\mathcal{D}\mathbf{X}$ is algebraic, $\mathcal{A}$ is directed with lub P. Obviously, $\mathcal{B} \subseteq \mathcal{A}$ holds, and Lemma 22.10.2 implies $\mathcal{A} \subseteq {\downarrow}\mathcal{B}$. Thus, $\mathcal{A}$ and $\mathcal{B}$ are cofinal, whence $\mathcal{B}$ is also directed with the same lub P by Prop. 3.1.8. Lemma 22.10.1 states that $\mathcal{B}$ is a set of isolated points in $\mathcal{R}\mathbf{X}$.

Proof of the Lemma:
We have to show the claim for each $\mathcal{R}$ separately. Generally, $A = (\downarrow E, \uparrow F)$ holds where E and F are finite subsets of $\mathbf{X}^0$, and $P = (C, K)$ where C is closed, K is a compact upper set, and $E \subseteq C$ and $K \subseteq \uparrow F$ hold because of $A \leq P$. Two finite subsets E' and F' of $\mathbf{X}^0$ are to be found that satisfy the conditions of $\mathcal{R}$ and lie between A and P, i.e. $E \subseteq \downarrow E'$, $E' \subseteq C$, and $K \subseteq \uparrow F' \subseteq \uparrow F$ have to hold.

$\mathcal{S}$: A itself is in $\mathcal{S}\mathbf{X}$ since $\mathcal{S}\mathbf{X}$ is lower in $\mathcal{D}\mathbf{X}$. (Indeed, we already proved the algebraicity of $\mathcal{S}\mathbf{X}$ in section 21.5.)

$\overline{\mathcal{S}}$: We know $E \subseteq C$, $K \subseteq \uparrow F$, and $K \subseteq \uparrow C$ because of condition $\overline{S}$.

Let $\mathcal{E} = \{E' \subseteq C \mid E' \subseteq_{fin} \mathbf{X}^0, E \subseteq E'\}$. $\mathcal{E}$ contains E and is closed w.r.t. union. Hence, it is $\subseteq$-directed. $\bigcup \mathcal{E} = C$ holds, whence $\uparrow C = \uparrow \bigcup \mathcal{E} = \bigcup \uparrow [\mathcal{E}]$. The set $\uparrow[\mathcal{E}]$ is a $\subseteq$-directed collection of open sets that cover K since $K \subseteq \uparrow C$. By compactness of K, there is E' in $\mathcal{E}$ with $K \subseteq \uparrow E'$. $E \subseteq E' \subseteq C$ holds as required.

$K \subseteq \uparrow E'$ and $K \subseteq \uparrow F$ implies $K \subseteq \uparrow E' \cap \uparrow F$. This set is open as intersection of two open sets. Since K is compact and $\mathbf{X}$ is algebraic, there is a finite set of isolated points F' with $K \subseteq \uparrow F' \subseteq \uparrow E' \cap \uparrow F$. Then $K \subseteq \uparrow F' \subseteq \uparrow F$ holds as required, and $\uparrow F' \subseteq \uparrow E' \subseteq \uparrow\downarrow E'$ shows that $(\downarrow E', \uparrow F')$ satisfies $\overline{S}$.

$\mathcal{S}\overline{\mathcal{S}}$: $A \leq P \in \mathcal{S}\overline{\mathcal{S}}\mathbf{X} \subseteq \overline{\mathcal{S}}\mathbf{X}$ implies the existence of B in $(\mathcal{D}\mathbf{X})^0 \cap \overline{\mathcal{S}}\mathbf{X}$ with $A \leq B \leq P$ as we have seen above. Since $\mathcal{S}\mathbf{X}$ is a lower set in $\mathcal{D}\mathbf{X}$, $B \in \mathcal{S}\mathbf{X}$ also holds.

$\mathcal{M}$: $(\downarrow E, \uparrow F) \leq (C, K)$ means $E \subseteq C$ and $K \subseteq \uparrow F$. $\uparrow F$ is open by Prop. 6.2.4, whence we obtain $C \subseteq \mathsf{cl}(C \cap \uparrow F)$ by the mix property of (C, K). This implies $E \subseteq \mathsf{cl}(C \cap \uparrow F)$, whence even $E \subseteq \downarrow(C \cap \uparrow F)$ follows by Prop. 6.1.3.

Hence, for all e in E, there is c_e in C and f_e in F such that $e \leq c_e \geq f_e$. As $\mathbf{X}$ is algebraic, there are isolated points g_e with $e, f_e \leq g_e \leq c_e$. Let $E' = \{g_e \mid e \in E\}$. Then E' is a finite subset of $\mathbf{X}^0$.

$e \leq g_e \in E'$ for all e in E implies $E \subseteq \downarrow E'$. $g_e \leq c_e \in C$ for all e in E implies $E' \subseteq \downarrow C = C$. $g_e \geq f_e$ for all e in E implies $E' \subseteq \uparrow F$, whence $(\downarrow E', \uparrow F)$ is a mix because it implies $E' = E' \cap \uparrow F \subseteq \mathsf{cl}(\downarrow E' \cap \uparrow F)$.

$\overline{\mathcal{M}}$: $K \subseteq \uparrow F$ and condition $\overline{M}$ imply $K \subseteq \uparrow(C \cap K) \subseteq \uparrow(C \cap \uparrow F) = \uparrow(C \cap F)$. The last equality holds since C is lower. Let $F' = C \cap F$. Then $K \subseteq \uparrow F' \subseteq \uparrow F$ holds as required.

Let $E' = E \cup F'$. Then $E \subseteq E' \subseteq C$ holds since $F' \subseteq C$. Finally, $F' \subseteq E'$ implies $F' = E' \cap F' \subseteq \uparrow(\downarrow E' \cap \uparrow F')$.

$\mathcal{S}\overline{\mathcal{M}}$: Follows from '$\overline{\mathcal{M}}$' as '$\mathcal{S}\overline{\mathcal{S}}$' follows from '$\overline{\mathcal{S}}$' since $\mathcal{S}\mathbf{X}$ is a lower set.

$\mathcal{M}\overline{\mathcal{S}}$: $A \leq P \in \mathcal{M}\overline{\mathcal{S}}\mathbf{X} \subseteq \overline{\mathcal{S}}\mathbf{X}$ implies a point B_1 in $(\mathcal{D}\mathbf{X})^0 \cap \overline{\mathcal{S}}\mathbf{X}$ with $A \leq B_1 \leq P$. Then $P \in \mathcal{M}\mathbf{X}$ implies B in $(\mathcal{D}\mathbf{X})^0 \cap \mathcal{M}\mathbf{X}$ with $B_1 \leq B \leq P$. Because $\overline{\mathcal{S}}\mathbf{X}$ is an upper set in $\mathcal{D}\mathbf{X}$, B is also in $\overline{\mathcal{S}}\mathbf{X}$.

$\mathcal{C}$: As in '$\overline{\mathcal{M}}$', $K \subseteq \uparrow(C \cap F)$ holds, whence $K \subseteq \uparrow F' \subseteq \uparrow F$ where $F' = C \cap F$. By defining E' as in '$\mathcal{M}$', $E \subseteq \downarrow E' \subseteq C$ holds. $E' \subseteq \uparrow F'$ holds since $c_e \geq g_e \geq f_e$, i.e. $f_e \in F'$.

Now let $G = E' \cup F'$. We claim that $(\downarrow G, \uparrow G)$ is the desired pair. $E \subseteq \downarrow E' \subseteq \downarrow G$ holds, and $G \subseteq C$ since $F' \subseteq C$. In addition, $K \subseteq \uparrow F' \subseteq \uparrow G$ holds and also $G \subseteq \uparrow F' \subseteq \uparrow F$ since $E' \subseteq \uparrow F'$.

$(\downarrow G, \uparrow G)$ is in $\mathcal{C}\mathbf{X}$ since $G \subseteq \downarrow G \cap \uparrow G$ whence conditions M and $\overline{M}$ follow. □

The proof above not only shows the algebraicity of $\mathcal{R}\mathbf{X}$, but also provides nice representations for the bases of these power domains. The most important ones are included in the following theorem.

Theorem 22.10.3 All the constructions $\mathcal{D}$, $\mathcal{S}$, $\mathcal{M}$, $\overline{\mathcal{S}}$, $\overline{\mathcal{M}}$, $\mathcal{S}\overline{\mathcal{S}}$, $\mathcal{S}\overline{\mathcal{M}}$, $\mathcal{M}\overline{\mathcal{S}}$, and $\mathcal{C}$ preserve algebraicity, continuity, finiteness, finite algebraicity, and finite continuity of the ground domain. In case of algebraic ground domain, their bases are the intersections of the base of $\mathcal{D}$ with the respective power domains. The base of $\mathcal{D}\mathbf{X}$ is the set of all pairs $(\downarrow E, \uparrow F)$ where E and F are finite subsets of the base of $\mathbf{X}$. In particular, the bases of the most important constructions are characterized as follows:

$$\begin{array}{llllll} \mathcal{S}: & E \subseteq \downarrow F & \overline{\mathcal{S}}: & F \subseteq \uparrow E & & \\ \mathcal{M}: & E \subseteq F & \overline{\mathcal{M}}: & F \subseteq E & \mathcal{C}: & E = F \end{array}$$

Proof: Preservation of algebraicity was shown above. Preservation of continuity follows by the theory of retracts. All these constructions are sub-constructions of $\mathcal{D}$, whence they preserve finiteness as $\mathcal{D}$ does.

The characterization of the base of $\mathcal{S}$ is taken over from Th. 21.6.4, whereas '$\overline{\mathcal{S}}$', '$\overline{\mathcal{M}}$', and '$\mathcal{C}$' follow from the proof of Lemma 22.10.2. For $\mathcal{M}$, this Lemma only provided $E \subseteq \uparrow F$. Given $(\downarrow E, \uparrow F)$ with $E \subseteq \uparrow F$, let $F' = E \cup F$. Then $\uparrow F' = \uparrow E \cup \uparrow F = \uparrow F$ holds since $E \subseteq \uparrow F$, and $E \subseteq F'$ obviously holds. This implies the form in the theorem. □

We already know that our sandwich power construction $\mathcal{S}$ generalizes the one in [BDW88] and [Gun89b, Gun90] that was defined for algebraic ground domain only. The original characterization $E \subseteq \uparrow F$ — $E \geq^\sharp F$ in Gunter's notation — shows that our mixed power construction generalizes Gunter's [Gun89b, Gun90].

The base of $\mathcal{C}\mathbf{X}$ is the set of all pairs $(\downarrow F, \uparrow F)$ where F is a finite subset of $\mathbf{X}^0$. The intersection of $\downarrow F$ and $\uparrow F$ is the convex hull $\updownarrow F$ of F. It suffices to recover $\downarrow F$ and $\uparrow F$ since $\downarrow F = \downarrow\updownarrow F$ and $\uparrow F = \uparrow\updownarrow F$. The ordering of these convex sets is given by $\updownarrow F \leq \updownarrow F'$ iff $\updownarrow F \subseteq \downarrow\updownarrow F'$ and $\updownarrow F' \subseteq \uparrow\updownarrow F$. This is the Egli-Milner ordering. Hence, $\mathcal{C}\mathbf{X}$ equals Plotkin's power domain for algebraic ground domain.

Chapter 23

Properties of B- and C-constructions

In this chapter, we first consider the preservation of bounded completeness under some **B**- and **C**-constructions. The sandwich construction $\mathcal{S}$ and the non-empty part of $\mathcal{S}\overline{\mathcal{S}}$ are shown to preserve bounded completeness, whereas the smaller constructions $\mathcal{M}$ and $\mathcal{C}$ do not (section 23.1).

In section 23.2, we investigate initiality and reducedness of the constructions $\mathcal{M}$ and $\mathcal{C}$. For algebraic ground domains, they are known to be initial. This property carries over to continuous ground domains by Th. 14.4.3. For multi-continuous ground domains, we furthermore prove the reducedness of $\mathcal{M}$.

Then, we consider the final power constructions for semirings **B** and **C**, and investigate whether they are sub-constructions of the final **D**-construction $\mathcal{D}$. In the case of semiring **B**, the answer is yes: the final **B**-construction is just the existential restriction $\mathcal{S}$ of $\mathcal{D}$ to **B** (section 23.3). In the case of semiring **C**, the answer is no: the final **C**-construction is not faithful (section 23.4). Hence, it cannot be a sub-construction of any **D**-construction since all these constructions are faithful. In section 23.5, we reveal some more awkward properties of the final **C**-construction. If the ground domain is an infinite discrete domain, then linear functions are not uniquely determined by their values on singletons. In section 23.6, we briefly consider the core of the final **C**-construction.

23.1 Notes on bounded completeness

The sandwich power construction $\mathcal{S}$ and the **C**-construction $\mathcal{S}\overline{\mathcal{S}}$ have the advantage to preserve bounded completeness.

Proposition 23.1.1 If $\mathbf{X}$ is bounded complete and sober, then $\mathcal{S}\mathbf{X}$ is bounded complete.
Thus, $\mathcal{S}$ preserves BC & ALG and BC & CONT.

Proof: By Prop. 20.2.4, $\mathcal{U}\mathbf{X}$ is complete in this case. The lub of two members K_1 and K_2 is given by $K_1 \cap K_2$. $\mathcal{L}\mathbf{X}$ is complete for all ground domains; the lub of C_1 and C_2 is $C_1 \cup C_2$.

If $\mathbf{X}$ has a least element $\bot$, then $\mathcal{D}\mathbf{X}$ has the least element $\bot\cdot\{\!|\bot|\!\} = (0,\, 1)\cdot(\downarrow\bot,\, \uparrow\bot) = (\emptyset,\, \mathbf{X})$. This pair obviously satisfies the sandwich condition, whence it is also the least element of $\mathcal{S}\mathbf{X}$.

If two sandwiches $S_1 = (C_1,\, K_1)$ and $S_2 = (C_2,\, K_2)$ are bounded by some sandwich S', then the pair $S = (C_1 \cup C_2,\, K_1 \cap K_2)$ is the least upper bound of S_1 and S_2 in $\mathcal{D}\mathbf{X}$. S is a sandwich by Prop. 21.3.1 since it is below S'. Thus, it is the lub of S_1 and S_2 in $\mathcal{S}\mathbf{X}$. $\square$

The **C**-construction $\mathcal{S}\overline{\mathcal{S}}$ given by the intersection of $\mathcal{S}$ and $\overline{\mathcal{S}}$ does not preserve bounded completeness since its power domains have no least element. They fall into two incomparable parts: an empty part mapped to 0 by ne, and a non-empty part $\mathcal{S}\overline{\mathcal{S}}^1\mathbf{X}$ mapped to 1 (cf. section 17.7). We claim that the non-empty part preserves bounded completeness.

Proposition 23.1.2 If $\mathbf{X}$ is bounded complete and sober, then $\mathcal{S}\overline{\mathcal{S}}^1\mathbf{X}$ is bounded complete. Thus, $\mathcal{S}\overline{\mathcal{S}}^1$ preserves BC & ALG and BC & CONT.

Proof: If $\bot$ is the least element of $\mathbf{X}$, then the pair $\{\!|\bot|\!\} = (\downarrow\bot,\, \uparrow\bot) = (\{\bot\},\, \mathbf{X})$ is in $\mathcal{S}\overline{\mathcal{S}}\mathbf{X}$ since $\mathcal{S}\overline{\mathcal{S}}$ is a sub-construction of $\mathcal{D}$. It is in $\mathcal{S}\overline{\mathcal{S}}^1\mathbf{X}$ since all singletons are mapped to 1 by ne. Let $(C,\, K)$ be an arbitrary member of $\mathcal{S}\overline{\mathcal{S}}^1\mathbf{X}$. $(\{\bot\},\, K)$ is below $(C,\, K)$ iff C is not empty. If C were empty, then K would also be empty by the dual sandwich condition $K \subseteq \uparrow C$. $(\emptyset,\, \emptyset) = \theta$ is however mapped to 0 by ne. Thus, $\{\!|\bot|\!\}$ is the least element of $\mathcal{S}\overline{\mathcal{S}}^1\mathbf{X}$.
Let S_1 and S_2 be two members of $\mathcal{S}\overline{\mathcal{S}}^1\mathbf{X}$ that are bounded by some S' in $\mathcal{S}\overline{\mathcal{S}}^1\mathbf{X}$. Then S_1 and S_2 are also bounded in $\mathcal{S}\mathbf{X}$, whence they have a lub S in $\mathcal{S}\mathbf{X}$ by Prop. 23.1.1. Since $S \geq S_1$ and S_1 is in $\mathcal{S}\overline{\mathcal{S}}\mathbf{X}$, S is in $\mathcal{S}\overline{\mathcal{S}}\mathbf{X}$ because $\overline{\mathcal{S}}\mathbf{X}$ is an upper set in $\mathcal{D}\mathbf{X}$ by Prop. 21.7.1. $ne\, S \geq ne\, S_1 = 1$ implies $S \in \mathcal{S}\overline{\mathcal{S}}^1\mathbf{X}$. $\square$

In contrast to the sandwich power domains, the power domains $\mathcal{M}\mathbf{X}$ and $\mathcal{C}\mathbf{X}$ are not necessarily bounded complete, even in the case of finite bounded complete ground domain $\mathbf{X}$. The example is identical with Plotkin's example for his power domain.

Let $\mathbf{X} = \mathbf{B} \times \mathbf{B}$. To be concise, we write the domain members as xy instead of $(x,\, y)$, e.g. $\bot\bot$, TF. Since this domain is finite, all its lower subsets are closed and all subsets are compact. Thus, Th. 22.6.1 applies and condition M reads $(L,\, U) \in \mathcal{M}\mathbf{X}$ iff $L \subseteq \downarrow(L \cap U)$. Condition $\overline{M}$ is $U \subseteq \uparrow(L \cap U)$.

Let $U = \{\mathsf{T}\bot,\, \mathsf{F}\bot\}$ and $V = \{\bot\mathsf{T},\, \bot\mathsf{F}\}$, and let $Y = \{\mathsf{TT},\, \mathsf{FF}\}$ and $Z = \{\mathsf{TF},\, \mathsf{FT}\}$. Then consider the corresponding formal sets $A = \{\!|\mathsf{T}\bot,\, \mathsf{F}\bot|\!\} = (\downarrow U,\, \uparrow U)$, $B = (\downarrow V,\, \uparrow V)$, $C = (\downarrow Y,\, \uparrow Y)$ and $D = (\downarrow Z,\, \uparrow Z)$. A, B, C, and D are members of $\mathcal{C}\mathbf{X}$ since they satisfy the criterion for the base of $\mathcal{C}\mathbf{X}$ of Th. 22.10.3. It is easy to verify that both of A and B are below both of C and D.

A least upper bound X of A and B, and also a greatest lower bound X of C and D must lie in between, i.e. $A,\, B \leq X \leq C,\, D$. This means $A^U,\, B^U \supseteq X^U \supseteq C^U,\, D^U$, whence $A^U \cap B^U \supseteq X^U \supseteq C^U \cup D^U$, or $\uparrow U \cap \uparrow V \supseteq X^U \supseteq \uparrow Y \cup \uparrow Z$. Since $\uparrow U \cap \uparrow V$ equals $\uparrow Y \cup \uparrow Z$, X^U is uniquely determined to be $\{\mathsf{TT},\, \mathsf{TF},\, \mathsf{FT},\, \mathsf{FF}\}$.
Analogously, $A^L \cup B^L \subseteq X^L \subseteq C^L \cap D^L$ holds. This determines X^L since both sides are equal. Thus, we obtain $X = (\{\bot\bot,\, \mathsf{T}\bot,\, \mathsf{F}\bot,\, \bot\mathsf{T},\, \bot\mathsf{F}\},\, \{\mathsf{TT},\, \mathsf{TF},\, \mathsf{FT},\, \mathsf{FF}\})$, i.e. X^L and X^U are disjoint, although they are not empty. Thus, X is neither a member of $\mathcal{M}\mathbf{X}$ nor a member

of $\overline{\mathcal{M}}\mathbf{X}$ (nevertheless, it is in $\mathcal{S}\overline{\mathcal{S}}\mathbf{X}$). This shows that A and B have no lub, and C and D no glb in $\mathcal{M}\mathbf{X}$, $\overline{\mathcal{M}}\mathbf{X}$, and $\mathcal{C}\mathbf{X}$.

Since $\mathcal{S}$ preserves bounded completeness but is not reduced, whereas $\mathcal{M}$ is reduced but does not preserve BC, one might believe that the power domains $\mathcal{S}\mathbf{X}$ are reduced if one restricts attention to bounded complete sub-powerdomains. This belief is wrong since for instance $\mathcal{M}\mathbf{2}$ and $\mathcal{S}\mathbf{2}$ differ as pointed out in section 22.1, but are both bounded complete. Nevertheless, $\mathcal{S}$ might be a minimal $\mathbf{B}$-construction that preserves BC & ALG.

23.2 Notes on initiality and reducedness

In [Gun89b], Gunter proves the initiality of $\mathcal{M}$ in case of an algebraic ground domain $\mathbf{X}$. Hence, $\mathcal{M}\mathbf{X}$ and $\mathcal{P}_i^{\mathbf{B}}\mathbf{X}$ are isomorphic for all algebraic ground domains $\mathbf{X}$. By Th. 14.4.3, this isomorphism extends to all continuous ground domains. Similarly, the initiality of $\mathcal{C}^1\mathbf{X}$ was claimed in [HP79] for algebraic ground domain $\mathbf{X}$ w.r.t. the class of $\mathbf{C}$-modules without 0. One easily verifies that the addition of the empty set to $\mathcal{C}^1\mathbf{X}$ does not matter if only $\mathbf{C}$-modules according to our definition are considered. Applying Th. 14.4.3 once more, we obtain

Theorem 23.2.1 For continuous ground domains, $\mathcal{M}$ and the initial $\mathbf{B}$-construction coincide as well as $\mathcal{C}$ and the initial $\mathbf{C}$-construction.

In the remainder of this section, we show that $\mathcal{M}\mathbf{X}$ is reduced for multi-continuous $\mathbf{X}$. We do not know whether it is not reduced for some strange ground domains. If $\mathcal{M}$ is not reduced, it differs from its core $\mathcal{M}^c$. Then $\mathcal{M}$ and $\mathcal{M}^c$ will be two different generalizations of Gunter's mixed power construction that have to be carefully distinguished. If in addition $\mathcal{M}^c$ and the initial $\mathbf{B}$-construction differ,[1] one gets even a third generalization.

The proof of $\mathcal{M}^c\mathbf{X} = \mathcal{M}\mathbf{X}$ for multi-continuous $\mathbf{X}$ proceeds in several steps. In each step, we prove a larger class of mixes to be contained in the core. The first steps also work in the larger class of sober domains.

Proposition 23.2.2 Let $\mathbf{X}$ be a sober ground domain. If E and F are finite sets of $\mathbf{X}$ such that $E \subseteq \uparrow F$, then $(\downarrow E, \uparrow F)$ is in $\mathcal{M}^c\mathbf{X}$.

Proof: First of all, $(\downarrow E, \uparrow F)$ is a mix since $E \subseteq \uparrow F$ implies $E = E \cap \uparrow F \subseteq \mathsf{cl}\,(\downarrow E \cap O)$ for all supersets O of $\uparrow F$. Let $E = \{e_1, \ldots, e_n\}$ and $F = \{f_1, \ldots, f_m\}$. We claim $(\downarrow E, \uparrow F) = \{\!|e_1, \ldots, e_n, f_1?, \ldots, f_m?|\!\}$.

Since $\{\!|e|\!\} = (\downarrow e, \uparrow e)$ and $\{\!|f?|\!\} = (\emptyset, \uparrow f)$, the right hand side equals $(\downarrow E \cup \emptyset, \uparrow E \cup \uparrow F)$. $\uparrow E \cup \uparrow F$ is $\uparrow F$ since $E \subseteq \uparrow F$.

$\{\!|e_1, \ldots, e_n, f_1?, \ldots, f_m?|\!\}$ is a finite union (by means of '$\uplus$') of singletons and uncertain singletons. In module theory, one would say it is a finite linear combination of singletons. Hence, it is in $\mathcal{M}^c\mathbf{X}$. □

In the next step, we make the lower part arbitrary, whereas the upper part remains finitary.

[1] Meanwhile, they are shown to differ.

Proposition 23.2.3 Let $\mathbf{X}$ be a sober ground domain. If C is a closed set and F a finitary upper set such that $C \subseteq \mathsf{cl}\,(C \cap F)$, then (C, F) is in $\mathcal{M}^c\mathbf{X}$.

Proof: Let $\mathcal{E}$ be the set of all finite subsets of $C \cap F$, and let $\mathcal{D} = \{(\downarrow E, F) \mid E \in \mathcal{E}\}$. Because of $E \subseteq F$, this set is a subset of $\mathcal{M}^c\mathbf{X}$ by Prop. 23.2.2. It is directed, since $E_1, E_2 \subseteq_f C \cap F$ implies $E_1 \cup E_2 \subseteq_f C \cap F$, and $(\downarrow(E_1 \cup E_2), F)$ is the lub of $(\downarrow E_i, F)$.

We claim that (C, F) is the limit of $\mathcal{D}$. We have to show $\mathsf{cl} \bigcup \downarrow[\mathcal{E}] = C$. All members of $\mathcal{E}$ are subsets of C, whence '$\subseteq$' follows. For the opposite, notice that $\mathcal{E}$ contains all singleton subsets of $C \cap F$, whence $\bigcup \mathcal{E} = C \cap F$. Thus, $\mathsf{cl} \bigcup \downarrow[\mathcal{E}] = \mathsf{cl} \bigcup \mathcal{E} = \mathsf{cl}\,(C \cap F)$ follows. $\mathsf{cl}\,(C \cap F) \supseteq C$ holds by the precondition.

(C, F) is in $\mathcal{M}^c\mathbf{X}$ since it is the lub of a directed set of core elements. □

In the final step, we generalize the upper set.

Proposition 23.2.4 Let $\mathbf{X}$ be a multi-continuous ground domain. Then every mix (C, K) belongs to $\mathcal{M}^c\mathbf{X}$.

Proof: Let $\mathcal{F}$ be the set of all finitary upper environments of K. From Th. 8.3.2, we know that $\mathcal{F}$ is directed in $\mathcal{U}_f\mathbf{X}$ with lub K. Let $\mathcal{D} = \{(C, F) \mid F \in \mathcal{F}\}$. The members of this set satisfy the preconditions of Prop. 23.2.3: if F is in $\mathcal{F}$, then there is an open set O with $K \subseteq O \subseteq F$. Because (C, K) is a mix, $C \subseteq \mathsf{cl}\,(C \cap O) \subseteq \mathsf{cl}\,(C \cap F)$ holds.

Thus, $\mathcal{D}$ is a subset of $\mathcal{M}^c\mathbf{X}$. Since $\mathcal{F}$ is directed with lub K, $\mathcal{D}$ is directed, too, with lub (C, K). Thus, (C, K) is in $\mathcal{M}^c\mathbf{X}$. □

Summarizing, we obtain

Theorem 23.2.5 For multi-continuous ground domain $\mathbf{X}$, $\mathcal{M}^c\mathbf{X} = \mathcal{M}\mathbf{X}$ holds.

23.3 The final B-construction

The Boolean semiring $\mathbf{B}$ is a sub-semiring of the double semiring $\mathbf{D}$. Generally, the existential restriction of the final construction for the larger semiring is completely different from the final power construction of the sub-semiring. An example will be presented in section 23.4. In the case of $\mathbf{B}$ and $\mathbf{D}$ however, these two constructions happen to coincide. Hence, we claim

Theorem 23.3.1 The final $\mathbf{B}$-construction is isomorphic to the existential restriction of the final $\mathbf{D}$-construction, i.e. the sandwich power construction: $\mathcal{P}_f^{\mathbf{B}} = \mathcal{S}$.

Proof: We have to establish an isomorphism between $\mathcal{P}_f^{\mathbf{B}}\mathbf{X} = [[\mathbf{X} \to \mathbf{B}] \stackrel{add}{\to} \mathbf{B}]$ and $\mathcal{S}\mathbf{X} = \{P \in [[\mathbf{X} \to \mathbf{D}] \stackrel{add}{\to} \mathbf{D}] \mid \forall p : [\mathbf{X} \to \mathbf{B}] : Pp \in \mathbf{B}\}$. An obvious choice is the restriction and co-restriction $\mathcal{R}$ of functions in $\mathcal{S}\mathbf{X}$ to arguments in $[\mathbf{X} \to \mathbf{B}]$ and results in $\mathbf{B}$. Since the power operations in $\mathcal{S}\mathbf{X}$ are inherited from those in $\mathcal{D}_f$, restriction $\mathcal{R}$ coincides with existential quantification in $\mathcal{S}\mathbf{X}$, whence it is a power homomorphism as indicated in chapter 15. Employing Prop. 12.2.1, we only have to show that it is bijective and its inverse is monotonic, then it is a power isomorphism.

From Prop. 16.3.1, we know that $\mathcal{D}_f\mathbf{X}$ and $\mathcal{L}_f\mathbf{X} \times \mathcal{U}_f\mathbf{X}$ are isomorphic. Hence, every $P : [[\mathbf{X} \to \mathbf{D}] \stackrel{add}{\to} \mathbf{D}]$ may be considered a pair (P^L, P^U) where $P^L : [[\mathbf{X} \to \mathbf{L}] \stackrel{add}{\to} \mathbf{L}]$ and $P^U : [[\mathbf{X} \to \mathbf{U}] \stackrel{add}{\to} \mathbf{U}]$. Accordingly, every function from somewhere to $\mathbf{D}$ may be considered a pair of functions to $\mathbf{L}$ and $\mathbf{U}$ respectively.

Let P and Q be two members of $\mathcal{S}\mathbf{X}$ such that $\mathcal{R}P \le \mathcal{R}Q$. We have to show $P \le Q$. For every $f : [\mathbf{X} \to \mathbf{D}]$ resp. $(f^L, f^U) : [\mathbf{X} \to \mathbf{L}] \times [\mathbf{X} \to \mathbf{U}]$, $Pf = P(f^L, f^U) = P(f^L, \underline{0}) + P(\underline{0}, f^U)$ where $\underline{0} = \lambda x.\, 0$ holds by additivity of P. The second summand is handled easily: function $(\underline{0}, f^U)$ can produce the values $(0, 0) = 0$ and $(0, 1) = \perp$ only. Both are in $\mathbf{B}$, whence by $\mathcal{R}P \le \mathcal{R}Q$ the inequality $P(\underline{0}, f^U) \le Q(\underline{0}, f^U)$ follows. The first summand cannot be handled this way since its argument may produce $(1, 0) = \mathsf{W}$. We consider the argument $(f^L, \underline{1})$ instead. It can produce the values $(0, 1) = \perp$ and $(1, 1) = 1$ which are both in $\mathbf{B}$. Hence, $(P^L f^L, P^U \underline{1}) = P(f^L, \underline{1}) \le Q(f^L, \underline{1}) = (Q^L f^L, Q^U \underline{1})$ holds, whence $P^L f^L \le Q^L f^L$ follows. Since $P^U \underline{0} = 0 = Q^U \underline{0}$, one obtains by combination $P(f^L, \underline{0}) \le Q(f^L, \underline{0})$. Returning to the sum, this proves $P(f^L, f^U) \le Q(f^L, f^U)$. Thus, we have shown $P \le Q$.

We now know that $\mathcal{R}$ is injective and that its inverse — if it exists — is monotonic. Hence, only surjectivity of $\mathcal{R}$ remains to be shown. Let $Q : [[\mathbf{X} \to \mathbf{B}] \stackrel{add}{\to} \mathbf{B}]$ be given. We now consider $\mathbf{B}$ as subset of $\mathbf{D} = \mathbf{L} \times \mathbf{U}$, i.e. consisting of pairs. Hence, one may assume $Q = (Q_1, Q_2)$ where $Q_1 : [[\mathbf{X} \to \mathbf{B}] \stackrel{add}{\to} \mathbf{L}]$ and $Q_2 : [[\mathbf{X} \to \mathbf{B}] \stackrel{add}{\to} \mathbf{U}]$. We show that Q_1 does not depend on the second component of its argument, and correspondingly, Q_2 does not depend on the first.

(1) $Q_1(f^L, f^U) = Q_1(f^L, \underline{1})$ for all $(f^L, f^U) : [\mathbf{X} \to \mathbf{B}]$.

This statement makes sense because $(f^L, \underline{1})$ always maps to $\mathbf{B}$ as we saw above. In $\mathbf{U}$, 1 is least, whence $Q_1(f^L, f^U) \ge Q_1(f^L, \underline{1})$ by monotonicity. By additivity of Q_1, $Q_1(f^L, \underline{1}) = Q_1(f^L, f^U) + Q_1(f^L, \underline{1}) \ge Q_1(f^L, f^U)$ since $1 = x + 1$ in $\mathbf{U}$ and $x \ge 0$ in $\mathbf{L}$.

(2) $Q_2(f^L, f^U) = Q_2(\underline{0}, f^U)$ for all $(f^L, f^U) : [\mathbf{X} \to \mathbf{B}]$.

This statement makes sense because $(\underline{0}, f^U)$ always maps to $\mathbf{B}$ as pointed out above. 0 is the least element of $\mathbf{L}$, whence $Q_2(f^L, f^U) \ge Q_2(\underline{0}, f^U)$ by monotonicity. By additivity of Q_2, $Q_2(f^L, f^U) = Q_2(f^L, f^U) + Q_2(\underline{0}, f^U) \le Q_2(\underline{0}, f^U)$ since $x = x + 0$ in $\mathbf{L}$ and $x \le 0$ in $\mathbf{U}$.

For $f^U : [\mathbf{X} \to \mathbf{U}]$, $(\underline{0}, f^U)$ maps from $\mathbf{X}$ to $\mathbf{B}$ whence the definition $P^U f^U = Q_2(\underline{0}, f^U)$ makes sense. Similarly, for $f^L : [\mathbf{X} \to \mathbf{L}]$, $(f^L, \underline{1})$ is in $[\mathbf{X} \to \mathbf{B}]$ whence $P^L f^L = Q_1(f^L, \underline{1})$ may be defined. P^L and P^U are maps in $[[\mathbf{X} \to \mathbf{L}] \to \mathbf{L}]$ and $[[\mathbf{X} \to \mathbf{U}] \to \mathbf{U}]$ respectively. In the sequel, we show their additivity. Thus, they may be combined to $P = (P^L, P^U)$ in $[[\mathbf{X} \to \mathbf{D}] \stackrel{add}{\to} \mathbf{D}]$. We then have to show $\mathcal{R}P = Q$; this automatically implies $P \in \mathcal{S}\mathbf{X}$.

$P^U \underline{0} = Q_2(\underline{0}, \underline{0}) = 0$ and $P^U(f+g) = P^U f + P^U g$ hold by additivity of Q because $(f+g, \underline{0}) = (f, \underline{0}) + (g, \underline{0})$.
$P^L(f + g) = P^L f + P^L g$ holds by additivity of Q because $(f + g, \underline{1}) = (f, \underline{1}) + (g, \underline{1})$. $P^L \underline{0} = Q_1(\underline{0}, \underline{1}) = Q_1(\underline{0}, \underline{0}) = 0$ holds; here statement (1) from above is applied.

Finally, we have to show $\mathcal{R}P = Q$, i.e. for $f : [\mathbf{X} \to \mathbf{B}]$, $Pf = Qf$ holds. Applying statements (1) and (2) from above, we obtain $Pf = (P^L f^L, P^U f^U) = (Q_1(f^L, \underline{1}), Q_2(\underline{0}, f^U)) = (Q_1(f^L, f^U), Q_2(f^L, f^U)) = Q(f^L, f^U) = Qf$. □

The theorem allows a slight simplification of the definition of $\mathcal{M}$. In section 22.3, we defined

- $\mathcal{M}\mathbf{X} = \{P \in [[\mathbf{X} \to \mathbf{D}] \stackrel{add}{\to} \mathbf{D}] \mid Pp - Pq \geq \mathsf{T} \Rightarrow P(p-q) \geq \mathsf{T}\ \forall p,\, q : [\mathbf{X} \to \mathbf{D}]\}$

Now, we know $\mathcal{M}\mathbf{X} \subseteq \mathcal{S}\mathbf{X} = [[X \to \mathbf{B}] \stackrel{add}{\to} \mathbf{B}]$. Hence, one may replace all occurrences of $\mathbf{D}$ by $\mathbf{B}$ in the formula above. For values b in $\mathbf{B}$, $b \geq \mathsf{T}$ is equivalent to $b = \mathsf{T}$. Thus, one obtains

- $\mathcal{M}\mathbf{X} = \{P \in [[\mathbf{X} \to \mathbf{B}] \stackrel{add}{\to} \mathbf{B}] \mid Pp - Pq = \mathsf{T} \Rightarrow P(p-q) = \mathsf{T}\ \forall p,\, q : [\mathbf{X} \to \mathbf{B}]\}$

23.4 The final C-construction

In this section, we investigate the final $\mathbf{C}$-construction $\mathcal{P}_f^{\mathbf{C}}$. For the sake of simplicity, we abbreviate it by $\mathcal{P}_f$ in the sequel.

The properties of $\mathcal{P}_f$ are quite awkward.

Proposition 23.4.1

(1) $\mathcal{P}_f\mathbf{X}$ is discrete for all ground domains $\mathbf{X}$.

(2) If $x \leq x'$ holds in $\mathbf{X}$, then $\{\!|x|\!\} = \{\!|x'|\!\}$ holds in $\mathcal{P}_f\mathbf{X}$.

(3) If $\mathbf{X}$ has a least element $\perp$, then $\mathcal{P}_f\mathbf{X}$ has exactly two elements: θ and $\{\!|\perp|\!\}$. They are incomparable.

Proof:

(1) $P \leq Q$ holds iff $Pp \leq Qp$ for all $p : [\mathbf{X} \to \mathbf{C}]$. Since Pp and Qp are in $\mathbf{C}$, and $\mathbf{C}$ is discrete, $Pp = Qp$ follows for all p, whence $P = Q$.

(2) $x \leq x'$ implies $\{\!|x|\!\} \leq \{\!|x'|\!\}$ by monotonicity. By (1), $\{\!|x|\!\} = \{\!|x'|\!\}$ follows.

(3) If $f : [\mathbf{X} \to \mathbf{C}]$, then $fx \geq f\perp$ holds for all x in $\mathbf{X}$. As $\mathbf{C}$ is discrete, $fx = f\perp$ follows. Hence, $[\mathbf{X} \to \mathbf{C}]$ has exactly two elements: $\lambda x.\, 0$ and $\lambda x.\, 1$. Every additive second order predicate must map $\lambda x.0$ to 0; it only has the choice to map $\lambda x.1$. Thus, $[[\mathbf{X} \to \mathbf{C}] \stackrel{add}{\to} \mathbf{C}]$ has at most two elements. Conversely, it has at least two elements, namely θ and $\{\!|\perp|\!\}$. They are different, even incomparable, because they are distinguished by ne. $\square$

By this proposition, $\mathcal{P}_f$ is not faithful. It maps all ground domains with a least element — no matter how large they might be — to a power domain of just two elements. Probably, this is the reason why this construction never was proposed in the literature.

23.5 C-constructions on discrete ground domains

One might guess that $\mathcal{P}_f\mathbf{X}$ always be quite small. This however is not true: discrete ground domains on the contrary are mapped to huge power domains. Remember we already know two $\mathbf{C}$-constructions defined on the class of discrete domains: the set of arbitrary subsets $\mathcal{P}_{set}$ and the set of finite subsets $\mathcal{P}_{fin}$. The resulting power domains are always ordered discretely. Since functions from a discrete domain are continuous always, continuity is of no concern in this section.

Proposition 23.5.1 For discrete X, $\mathcal{P}_{fin}X$ may be embedded into $\mathcal{P}_{set}X$, and this may in turn be embedded into $\mathcal{P}_f X$. Both embeddings are surjective iff X is finite.

Proof: The statements about $\mathcal{P}_{fin}X$ and $\mathcal{P}_{set}X$ are obvious. Abbreviating the latter by $\mathcal{P}X$, we have to provide an embedding from $\mathcal{P}X$ into $\mathcal{P}_f X$. By section 15.5, existential quantification is the only power homomorphism from $\mathcal{P}$ to $\mathcal{P}_f$. We have to show $\mathcal{E}A = \mathcal{E}B$ implies $A = B$ for A, B in $\mathcal{P}X$.

For a in X, let $f_a : X \to \mathbf{C}$ be defined such that $f_a x = 1$ iff $a = x$, and $= 0$ otherwise. Then $\mathcal{E}Af_a = 1$ iff there is x in A with $a = x$ iff $a \in A$. Hence, $\mathcal{E}A = \mathcal{E}B$ implies $\mathcal{E}Af_a = 1$ iff $\mathcal{E}Bf_a = 1$, whence $a \in A$ iff $a \in B$, i.e. $A = B$.

Finally, we have to show $\mathcal{E}$ is surjective iff X is finite. Let X be finite. For $P : (X \to \mathbf{C}) \stackrel{add}{\to} \mathbf{C}$, let $A = \{a \in X \mid Pf_a = 1\}$. We claim $\mathcal{E}A = P$. Let $p : X \to \mathbf{C}$, and let $B = \{b \in X \mid p\,b = 1\}$. Then $p = \sum_{b \in B} f_b$ holds; this sum is well-defined since X and more than ever B are finite. By additivity of P, $Pp = \sum_{b \in B} Pf_b$ holds. This value is 1 iff there is b in B with $Pf_b = 1$ iff there is b in B with $b \in A$ iff there is b in A with $p\,b = 1$ iff $\mathcal{E}Ap = 1$.

Now let X be infinite. We have to show that $\mathcal{E}$ is not surjective. Let $P : (X \to \mathbf{C}) \to \mathbf{C}$ be defined such that Pp is 0 iff $p^{-1}[1]$ is finite. This function is additive since $P(\lambda x.\, 0)$ is 0, and $P\,(p+q)$ is 0 iff $(p+q)^{-1}[1]$ is finite iff $p^{-1}[1] \cup q^{-1}[1]$ is finite iff both $p^{-1}[1]$ and $q^{-1}[1]$ are finite iff $Pp = 0$ and $Pq = 0$ iff $Pp + Pq = 0$. Assume there is a set A with $\mathcal{E}A = P$. For all a in A, $\mathcal{E}Af_a = 1$ holds, whereas $Pf_a = 0$ because $f_a^{-1}[1] = \{a\}$. Hence, A is empty. Thus, $\mathcal{E}A\,p = 0$ for all $p : X \to \mathbf{C}$. $P(\lambda x.\, 1)$ however is 1 since $(\lambda x.\, 1)^{-1}[1] = X$ is infinite. □

By this proposition, we see that $\mathcal{P}_f X$ is even larger than the power set of X if X is an infinite discrete domain. In section 15.8, we wondered whether all final power constructions have unique extension, and claimed the answer be no. An example is provided by the final **C**-construction.

Proposition 23.5.2 For discrete ground domain X, $\mathcal{P}_{fin}X$ is the initial **C**-powerdomain over X. The cores of $\mathcal{P}_{set}X$ and of $\mathcal{P}_f X$ are both (isomorphic to) $\mathcal{P}_{fin}X$. $\mathcal{P}_{set}X$ and $\mathcal{P}_f X$ have unique extensions iff X is finite.

Proof: Let M be a not necessarily discrete **C**-module, i.e. a commutative idempotent monoid, and let $f : X \to M$ be a function. Then define $ext\, f\, A = \sum_{a \in A} f a$ for all finite subsets of X. By idempotence, $ext\, f$ is additive. $ext\, f\, \{x\} = fx$ obviously holds. Because of the required additivity, $ext\, f$ is unique with these properties.

Because of Prop. 23.5.1, $\mathcal{P}_{fin}X$ is contained in both of $\mathcal{P}_{set}X$ and $\mathcal{P}_f X$. Hence, it is their core because it is reduced.

For finite X, all three considered power domains are isomorphic by Prop. 23.5.1. Since $\mathcal{P}_{fin}X$ is initial, the remaining two also have unique extensions.

Let X be infinite. We define two different linear functions from $\mathcal{P}_{set}X$ to $\mathbf{C}$ that coincide for singletons. Let FA be 0 for all $A \subseteq X$, and let GA be 0 iff A is finite, and 1 otherwise. F obviously is additive, and G is so since a binary union is finite iff its constituents are. $F\{x\} = G\{x\} = 0$ holds for all x in X, whereas $FX = 0$ and $GX = 1$ since X is infinite.

This example cannot be taken directly for $\mathcal{P}_f X$ because it is not obvious how to define G for all members of $\mathcal{P}_f X$. We call $A \in \mathcal{P}_f X$ *finite* iff there are $x_1, \ldots, x_k \in X$, such that

$A = \{\!|x_1, \ldots, x_k|\!\}$, and we call $B \in \mathcal{P}_f X$ *sub-finite* iff there is an arbitrary C and a finite A such that $B \uplus C = A$. Because of $A \uplus A = A$, every finite set is sub-finite. After these preliminaries, we define

$$FA = 0 \text{ for all } A \in \mathcal{P}_f X$$
$$GA = \begin{cases} 0 & \text{if } A \text{ is sub-finite} \\ 1 & \text{otherwise} \end{cases}$$

F is obviously linear. For G, some more work is needed. $G\theta = 0$ holds since θ is sub-finite. For linearity, we then have to show $G(A \uplus B) = GA + GB$. $G(A \uplus B) = 0$ holds iff $A \uplus B$ is sub-finite, and $GA + GB = 0$ holds iff $GA = 0$ and $GB = 0$ iff A and B are sub-finite. Hence, we have to show $A \uplus B$ is sub-finite iff A and B are sub-finite.

If A and B are sub-finite, there are arbitrary U and V and finite C and D such that $A \uplus U = C$ and $B \uplus V = D$ whence $A \uplus B \uplus U \uplus V = C \uplus D$ whence $A \uplus B$ is sub-finite because the union of finite members of $\mathcal{P}_f X$ is finite again. Conversely, if $A \uplus B$ is sub-finite then there are some U and a finite C with $A \uplus B \uplus U = C$. This equation may be directly interpreted such that A and B are sub-finite.

The linear mappings F and G coincide for singletons; both map all singletons $\{\!|x|\!\}$ to 0 because singletons are sub-finite. Nevertheless, F and G are different. This is shown by presenting an A in $\mathcal{P}_f X$ that is not sub-finite.

Let A be defined by $A\,p = \begin{cases} 0 & \text{if } p\,x = 0 \text{ for all } x \in X \\ 1 & \text{otherwise} \end{cases}$

Obviously, $A\,(\lambda x.\,0) = 0$ holds. A is linear by the following derivation:

$$\begin{array}{lll} A\,(p+q) = 0 & \text{iff} & (p+q)\,x = 0 \text{ for all } x \\ & \text{iff} & px + qx = 0 \text{ for all } x \\ & \text{iff} & (px = 0 \text{ and } qx = 0) \text{ for all } x \\ & \text{iff} & (px = 0 \text{ for all } x) \text{ and } (qx = 0 \text{ for all } x) \\ & \text{iff} & A\,p = 0 \text{ and } A\,q = 0 \\ & \text{iff} & A\,p + A\,q = 0 \end{array}$$

Finally, we show that A is not sub-finite. Assume there are U in $\mathcal{P}_f X$ and $x_1, \ldots, x_k \in X$ such that $A \uplus U = \{\!|x_1, \ldots, x_k|\!\}$. Let x' be a point of X that is not among $x_1, \ldots, x_k$, and let $p : X \to \mathbf{C}$ be defined such that x' is mapped to 1 and all other points are mapped to 0. Then $(A \uplus U)\,p = A\,p + U\,p = 1 + U\,p = 1$ holds, whereas $\{\!|x_1, \ldots, x_k|\!\}\,p = p\,x_1 + \cdots + p\,x_k = 0$ — a contradiction. Hence, $FA = 0$, but $GA = 1$. □

23.6 The core of the final C-construction

For ground domains with least element, $\mathcal{P}_f^{\mathbf{C}}\mathbf{X}$ has only two elements: θ and $\{\!|\bot|\!\}$ by Prop. 23.4.1 (3). Thus, it is reduced and coincides with its core. This provides an example for a reduced power construction that is not faithful. For infinite discrete ground domains however, $\mathcal{P}_f^{\mathbf{C}}\mathbf{X}$ and its core $\mathcal{P}_c^{\mathbf{C}}\mathbf{X}$ are far from being equal. The core of $\mathcal{P}_f^{\mathbf{C}}\mathbf{X}$ is initial in this case.

The lower semiring $\mathbf{L}$ is a commutative idempotent monoid, i.e. a $\mathbf{C}$-module. It is not discrete, whence by Prop. 14.5.3, the initial $\mathbf{C}$-construction $\mathcal{P}_i^{\mathbf{C}}$ is faithful. Thus, $\mathcal{P}_i^{\mathbf{C}}\mathbf{X}$ and $\mathcal{P}_c^{\mathbf{C}}\mathbf{X}$ are not isomorphic if $\mathbf{X}$ is a domain with bottom and at least one further element.

Hence, we have seen that the three autochthonous $\mathbf{C}$-constructions $\mathcal{P}_i^{\mathbf{C}}$, $\mathcal{P}_c^{\mathbf{C}}$, and $\mathcal{P}_f^{\mathbf{C}}$ considerably differ. The difference shows up even for complete algebraic ground domains.

Chapter 24

Sandwich theory

In case of the power constructions $\mathcal{L}$, $\mathcal{U}$, $\mathcal{C}$, and $\mathcal{M}$, the power domain over a continuous ground domain is algebraically characterized as the free (initial) R-module with appropriate semiring R. In Gunter's papers, the freedom of the sandwich power domains is an open question. Our results show that they cannot be characterized as free modules of some semiring since $\mathcal{M}$ and $\mathcal{S}$ share the same semiring, but nevertheless differ.

The chapter at hand shows that the sandwich power domains are also free algebras belonging to an algebraic theory that however cannot be seen as theory of R-modules for some semiring R. This *sandwich theory* involves a partial operation with a strange meaning. The problem is that all reasonable operations on sandwiches seem to preserve the mix condition, whence only 'unreasonable' operations remain.

The chapter at hand consists of only two sections: in section 24.1, the axioms and theorems of the sandwich theory are introduced, and in section 24.2, the sandwich power domains are shown to be free sandwich algebras if the ground domain is algebraic.

24.1 Axioms and theorems of the sandwich theory

To define sandwich algebras, we use an operation that *marries* sandwiches to each other i.e. $(A^L, A^U) \infty (B^L, B^U) = (A^L, B^U)$. A problem is that the partners may not harmonize such that the marriage is not durable. This means that '∞' is a partial operation only. The exact definition for the sandwich power domain is

$$(A^L, A^U) \infty (B^L, B^U) = \begin{cases} (A^L, B^U) & \text{if } (A^L, B^U) \in \mathcal{S}\mathbf{X} \\ \textit{undefined} & \text{otherwise} \end{cases}$$

'∞' becomes a total continuous operation if '*undefined*' is added to the sandwich power domain as an artificial top element. This is done in the following definition which enumerates the axioms of a sandwich algebra.

Definition 24.1.1 (Sandwich algebras)
A *sandwich algebra* $(\mathbf{P}, +, 0, \infty)$ is a commutative idempotent monoid domain $(\mathbf{P}, +, 0)$

with an additional continuous operation $\_ \circledcirc \_ : [\mathbf{P} \times \mathbf{P} \to \mathbf{P} \cup \{\mathit{undefined}\}]$ where $x \leq \mathit{undefined}$ for all $x \in \mathbf{P}$ satisfying the following 4 axioms

$(A1)$ $A \circledcirc A = A$

$(A2)$ $A \circledcirc B \leq (A + A') \circledcirc B$ $\qquad$ $(A3)$ $A \circledcirc B \geq A \circledcirc (B + B')$

$(A4)$ If $A \circledcirc B$ and $A' \circledcirc B'$ are defined, then $(A + A') \circledcirc (B + B') = (A \circledcirc B) + (A' \circledcirc B')$

A mapping f between two sandwich algebras is a *sandwich homomorphism* iff it is continuous and additive and satisfies $f(A \circledcirc B) = (fA) \circledcirc (fB)$ whenever $A \circledcirc B$ is defined, i.e. in $\mathbf{P}$.

One may easily verify that sandwich power domains become sandwich algebras by defining '$\circledcirc$' as indicated above.

As for the mix theory, some theorems may be derived from the axioms. In the sequel, idempotence is abbreviated by (I) and neutrality of 0 by (N).

(T1) If $A \circledcirc B$ and $A \circledcirc B'$ are defined, then $A \circledcirc (B + B') = (A \circledcirc B) + (A \circledcirc B')$
Proof: let $A' = A$ in (A4) and apply (I).

(T2) If $A \circledcirc B$ and $A' \circledcirc B$ are defined, then $(A + A') \circledcirc B = (A \circledcirc B) + (A' \circledcirc B)$
Proof: let $B' = B$ in (A4) and apply (I).

(T3) If $A \circledcirc B$ is defined, then $(A + B) \circledcirc B = (A \circledcirc B) + B$ and $A \circledcirc (A + B) = A + (A \circledcirc B)$.
Proof: the first equation by (T2) and (A1), the second one by (T1) and (A1).

(T4) $A \circledcirc B$ and B are both $\leq (A + B) \circledcirc B$.
Proof: $B = B \circledcirc B$ by (A1), then apply (A2) in both cases.

(T5) $A \circledcirc (A + B) \leq$ both of A and $A \circledcirc B$.
Proof: $A = A \circledcirc A$ by (A1), then apply (A3) in both cases.

(T6) $0 \circledcirc B \leq B$ $\quad$ since $0 \circledcirc B \overset{T4}{\leq} (0 + B) \circledcirc B \overset{N}{=} B \circledcirc B \overset{A1}{=} B$

(T7) $0 \circledcirc B \leq 0$ $\quad$ since $0 \circledcirc B \overset{N}{=} 0 \circledcirc (0 + B) \overset{T5}{\leq} 0$

(T8) Every sandwich algebra is a mix algebra (a $\mathbf{B}$-module) by the definition $B? = 0 \circledcirc B$.
Proof: (T6) means $B? \leq B$ and (T7) means $B? \leq 0$. These inequations also show that $B?$ is defined for all B. (T3) implies $B? + B = (0 \circledcirc B) + B = (0 + B) \circledcirc B = B$ by (N) and (A1). (T1) gives $(A + B)? = A? + B?$.

(T9) $A \leq A + B$ iff $0 \circledcirc A \leq 0 \circledcirc B$ iff for all C: $C \circledcirc A \leq C \circledcirc B$.
Proof: '(1) $\Rightarrow$ (3)': $C \circledcirc A \overset{1}{\leq} C \circledcirc (A + B) \overset{A3}{\leq} C \circledcirc B$
(3) obviously implies (2), and (2) implies (1) by (T8) and theorem (T8) of the mix theory.

(T10) $A + B \leq B$ iff for all C: $A \circledcirc C \leq B \circledcirc C$.
Proof: '$\Rightarrow$': $A \circledcirc C \overset{A2}{\leq} (A + B) \circledcirc C \overset{lhs}{\leq} B \circledcirc C$
'$\Leftarrow$': $A + B \overset{A1}{=} (A + B) \circledcirc (A + B) \overset{T2}{=} (A \circledcirc (A + B)) + (B \circledcirc (A + B))$. (T2) is applicable since both marriages to the right are defined due to (T5) viz. $A \circledcirc (A + B) \leq A$ and $B \circledcirc (A + B) \leq B$. The precondition with $C = A + B$ may be applied to the last term resulting in $A + B \leq (B \circledcirc (A + B)) + (B \circledcirc (A + B)) \overset{I}{=} B \circledcirc (A + B) \overset{T5}{\leq} B$.

(T11) $A \leq A+B \geq B$ iff for all C: $C \circledcirc A = C \circledcirc B$.
This is deduced directly from (T9). We call A and B in this case *upper equivalent* and write $A \approx_U B$. This relation obviously is an equivalence relation. For the sandwich power domain, one can show $A \approx_U B$ iff $A^U = B^U$, hence the name.

(T12) $A \geq A+B \leq B$ iff for all C: $A \circledcirc C = B \circledcirc C$.
This is deduced directly from (T10). We call A and B in this case *lower equivalent* and write $A \approx_L B$. For the sandwich power domain, one can show $A \approx_L B$ iff $A^L = B^L$, hence the name.

(T13) If $A \circledcirc B$ is defined, then it is in the intersection of the lower class of A and the upper class of B, i.e. $A \approx_L A \circledcirc B \approx_U B$.
Proof: The lower equivalence by (T3) and (T5), and the upper one by (T3) and (T4).

(T14) For given A and B, there is an X such that $A \approx_L X \approx_U B$ iff $A \circledcirc B$ is defined. In this case, $A \circledcirc B = X$ holds. $A \circledcirc B$ is undefined iff there is no such X.
Proof: If $A \circledcirc B$ is defined, there is such an X by (T13). Conversely, assume such an X. Note that (T11) and (T12) hold no matter whether the married couples involving 'C' are defined or not. Hence, $A \circledcirc B \stackrel{T12}{=} X \circledcirc B \stackrel{T11}{=} X \circledcirc X \stackrel{A1}{=} X$, i.e. $A \circledcirc B$ is defined and equals X.

Similar to mix algebras, (T14) implies that the operation '$\circledcirc$' is uniquely determined in a given sandwich algebra, i.e. for given commutative idempotent monoid domain, there is at most one choice for the operation '$\circledcirc$' to turn it into a sandwich algebra. Another important consequence is the following theorem:

Theorem 24.1.2 An additive mapping between two sandwich algebras is automatically a sandwich homomorphism.

Proof: Let $f : [\mathbf{X} \to \mathbf{Y}]$ be a (continuous) additive map between the two sandwich algebras $\mathbf{X}$ and $\mathbf{Y}$. Let A and B be members of $\mathbf{X}$ such that $A \circledcirc B$ is defined. Then $A \approx_L A \circledcirc B \approx_U B$ holds by (T13). Using the equivalence of $S \approx_L T$ with $S \geq S+T \leq T$ and of $S \approx_U T$ with $S \leq S+T \geq T$, one obtains by monotonicity and linearity of f the relations $fA \approx_L f(A \circledcirc B) \approx_U fB$. (T14) then implies $fA \circledcirc fB = f(A \circledcirc B)$. □

24.2 Sandwich power domains as free sandwich algebras

In this section, we show that the sandwich power domains are free sandwich algebras:

Theorem 24.2.1 For every domain $\mathbf{X}$, the sandwich power domain $\mathcal{S}\mathbf{X}$ is a sandwich domain. If $\mathbf{X}$ is algebraic, then for every arbitrary sandwich domain $\mathbf{Y}$ and every continuous map $f : [\mathbf{X} \to \mathbf{Y}]$, there is a unique additive continuous map $\bar{f} : [\mathcal{S}\mathbf{X} \to \mathbf{Y}]$ with $\bar{f}\{\!|x|\!\} = fx$. $\bar{f}$ is a sandwich homomorphism.

Proof: For a finite set of isolated points $E \subseteq \mathbf{X}$, we define $f'E = \sum_{e \in E} fe$. The isolated sandwiches are given by $(\downarrow E, \uparrow F)$ with $E \subseteq \downarrow\uparrow F$. We define for them $\bar{f}(\downarrow E, \uparrow F) = f'E \circledcirc f'F$.

We first show that the married couple on the right hand side is always defined. If E is empty, $f'E \circledcirc f'F = 0 \circledcirc f'F \leq 0$ is defined by (T7). Let e be some member of E. By $E \subseteq \downarrow\uparrow F$, there are points y and x such that $x \in F$ and $e, x \leq y$. Then $fe \otimes fx \leq fy \otimes fy = fy$ is defined by (A1). By (A3), $fe \otimes f'F = fe \otimes (fx + f'(F \setminus \{x\})) \leq fe \otimes fx$ holds, whence $fe \otimes f'F$ is defined for all e in E. Applying (T2) iteratedly, we obtain $f'E \otimes f'F = (\sum_{e \in E} fe) \otimes f'F = \sum_{e \in E}(fe \otimes f'F)$ is defined.

We also have to show that $\bar{f}$ is well-defined, i.e. independent from the specific choice of E and F. We do this by showing its monotonicity. Well-definedness then follows from antisymmetry in $\mathbf{Y}$.

Assume $(\downarrow E, \uparrow F) \leq (\downarrow E', \uparrow F')$ holds, i.e. $E \subseteq \downarrow E'$ and $F' \subseteq \uparrow F$. We have to show $f'E \otimes f'F \leq f'E' \otimes f'F'$. $E \subseteq \downarrow E'$ means there is a member $y(x)$ in E' for every x in E such that $y(x) \geq x$. Thus, we obtain

$$\begin{array}{rcl} f'E \otimes f'F & = & \sum_{x \in E} fx \otimes f'F \\ & \leq & \sum_{x \in E} f(y(x)) \otimes f'F \\ & \overset{(I)}{=} & \sum_{y \in y[E]} fy \otimes f'F \\ & \overset{(A2)}{\leq} & \sum_{y \in E'} fy \otimes f'F \;=\; f'E' \otimes f'F \end{array}$$

Next, $F' \subseteq \uparrow F$ means there is a member $y(x)$ in F for every x in F' such that $y(x) \leq x$. Thus, we obtain

$$\begin{array}{rcl} f'E' \otimes f'F' & = & f'E' \otimes \sum_{x \in F'} fx \\ & \geq & f'E' \otimes \sum_{x \in F'} f(y(x)) \\ & \overset{(I)}{=} & f'E' \otimes \sum_{y \in y[F']} fy \\ & \overset{(A3)}{\geq} & f'E' \otimes \sum_{y \in F} fy \;=\; f'E' \otimes f'F \end{array}$$

Combining both inequations results in $f'E \otimes f'F \leq f'E' \otimes f'F'$ as required.

Now, we know that $\bar{f}$ is well-defined and monotonic. Next, we show its additivity.

$$\bar{f}\theta \;=\; \bar{f}(\emptyset, \emptyset) \;=\; f'(\emptyset) \otimes f'(\emptyset) \;=\; 0 \otimes 0 \;=\; 0$$

holds using (A1).

$$\begin{array}{rcl} \bar{f}(A \uplus B) & = & \bar{f}(A^L \cup B^L,\, A^U \cup B^U) \\ & = & (f'A^L + f'B^L) \otimes (f'A^U + f'B^U) \\ & \overset{(A4)}{=} & (f'A^L \otimes f'A^U) + (f'B^L \otimes f'B^U) \\ & = & \bar{f}A + \bar{f}B \end{array}$$

shows the additivity of $\bar{f}$.

$\bar{f}$ can be extended from the base of $\mathcal{S}\mathbf{X}$ to the whole of $\mathcal{S}\mathbf{X}$ resulting in a continuous additive map $\overline{f} : [\mathcal{S}\mathbf{X} \to \mathbf{Y}]$.

Uniqueness is shown by using the fact $(\downarrow E, \uparrow F) = (\downarrow E, \uparrow E) \otimes (\downarrow F, \uparrow F) = \{\!|e \mid e \in E|\!\} \otimes \{\!|y \mid y \in F|\!\}$ and theorem 24.1.2. □

Corollary 24.2.2

If $\mathbf{X}$ is algebraic, then for every continuous map $f : [\mathbf{X} \to \mathcal{S}\mathbf{Y}]$, there is a unique additive continuous extension $\bar{f} : [\mathcal{S}\mathbf{X} \to \mathcal{S}\mathbf{Y}]$ with $\bar{f} \circ \{\!|.|\!\} = f$. It is explicitly given by $\bar{f} = \mathit{ext}\, f$ where *ext* is defined as in section 21.2. □

This corollary is particularly nice because it neither mentions sandwich theory nor marrying, i.e. it is independent from this partial and semantically dubious operation. It shows that even power constructions which are far from being reduced may have unique extensions.

Bibliography

[Bar80] H.P. Barendregt. *The Lambda Calculus — Its Syntax and Semantics*. North Holland, 1980.

[BDW88] P. Buneman, S.B. Davidson, and A. Watters. A semantics for complex objects and approximate queries. Internal Report MS-CIS-87-99, University of Pennsylvania, October 1988. Also in: 7th ACM Principles of Database Systems.

[Bro82] M. Broy. A fixed point approach to applicative multiprogramming. In M. Broy and G. Schmidt, editors, *Theoretical Foundations of Programming Methodology*. Reidel, 1982.

[FM90] K.E. Flannery and J.J. Martin. The Hoare and Smyth power domain constructors commute under composition. *Journal of Computer and System Sciences*, 40:125–135, 1990.

[GLS83] G. Gierz, J.D. Lawson, and A. Stralka. Quasicontinuous posets. *Houston Journal of Mathematics*, 9:191–208, 1983.

[Gun87] C.A. Gunter. Universal profinite domains. *Information and Computation*, 72:1–30, 1987.

[Gun89a] C.A. Gunter. A logical interpretation of powerdomains. Internal report without number and date, University of Pennsylvania, before September 1989.

[Gun89b] C.A. Gunter. The mixed powerdomain. Internal Report MS-CIS-89-77, Logic & Computation 18, University of Pennsylvania, December 1989.

[Gun90] C.A. Gunter. Relating total and partial correctness interpretations of nondeterministic programs. In P. Hudak, editor, *Principles of Programming Languages (POPL '90)*, pages 306–319. ACM, 1990.

[Hec90a] R. Heckmann. Power domains and second order predicates. Study Note S.1.6-SN-25.0, Universität des Saarlandes, PROSPECTRA Project, February 1990.

[Hec90b] R. Heckmann. Power domain constructions. Technical Report A 16/90, Universität des Saarlandes, August 1990.

[Hec90c] R. Heckmann. Set domains. In N. Jones, editor, *ESOP '90*, pages 177–196. *Lecture Notes in Computer Science 432*, Springer-Verlag, 1990.

[Hec91] R. Heckmann. Power domain constructions. *Science of Computer Programming*, 1991. to appear.

[HM81] K. Hofmann and M. Mislove. Local compactness and continuous lattices. In Banaschewski and Hoffmann, editors, *Continuous Lattices, Bremen 1979. Lecture Notes in Mathematics 871*, Springer-Verlag, 1981.

[Hoo87] R. Hoofman. Powerdomains. Technical Report RUU-CS-87-23, Rijksuniversiteit Utrecht, November 1987.

[HP79] M.C.B Hennessy and G.D. Plotkin. Full abstraction for a simple parallel programming language. In J. Becvar, editor, *Foundations of Computer Science*, pages 108–120. *Lecture Notes in Computer Science 74*, Springer-Verlag, 1979.

[Joh81] P. Johnstone. Scott is not always sober. In Banaschewski and Hoffmann, editors, *Continuous Lattices, Bremen 1979. Lecture Notes in Mathematics 871*, Springer-Verlag, 1981.

[Joh82] P. Johnstone. *Stone Spaces.* Cambridge University Press, 1982.

[Jun88] A. Jung. *Cartesian Closed Categories of Domains.* PhD thesis, FB Mathematik, Technische Hochschule Darmstadt, 1988.

[Law88] J.D. Lawson. The versatile continuous order. In Michael G. Main, A. Melton, Michael Mislove, and D. Schmidt, editors, *Mathematical Foundations of Programming Language Semantics (MFPLS '87)*, pages 565–622. *Lecture Notes in Computer Science 298*, Springer-Verlag, 1988.

[Mai85] M.G. Main. Free constructions of powerdomains. In A. Melton, editor, *Mathematical Foundations of Programming Semantics*, pages 162–183. *Lecture Notes in Computer Science 239*, Springer-Verlag, 1985.

[Plo76] G.D. Plotkin. A powerdomain construction. *SIAM Journal on Computing*, 5(3):452–487, 1976.

[Rud80] Mary E. Rudin. Directed sets which converge. In McAuley and Rao, editors, *General Topology and Modern Analysis (Riverside)*, pages 305–307, 1980.

[Smy78] M.B. Smyth. Power domains. *Journal of Computer and System Sciences*, 16:23–36, 1978.

[Smy83] M.B. Smyth. Power domains and predicate transformers: A topological view. In J. Diaz, editor, *ICALP '83*, pages 662–676. *Lecture Notes in Computer Science 154*, Springer-Verlag, 1983.

Index

Introduction

The index consists of two parts: a *notation index* and a *word index*. The notation index contains symbols such as '⚭' or '$\ll$', words and abbreviations in special fonts, e.g. '*ext*' and 'CONT'. The symbols are ordered by visual appearance and related meaning, whereas the abbreviations are ordered primarily by font, and secondarily alphabetically. The word index is to be used to look for (more or less) usual English words.

Notation index

Word index