Part III

Special power constructions

In this part, we investigate how the five known power constructions — lower, upper, convex, mixed, and sandwich power construction — fit into the general algebraic framework. All these constructions have small semirings with at most 3 elements that share many common properties: their addition and multiplication is idempotent, they are all additive, and their modules may be described without referring to multiplication.

The three classical power constructions have two-valued semirings: $\mathbf{L} = \{0 < 1\}$ belongs to the lower construction, $\mathbf{U} = \{1 < 0\}$ belongs to the upper one, and $\mathbf{C} = \{0, 1\}$ belongs to the convex one. The semiring of the mixed and sandwich construction is $\mathbf{B} = \{\perp < [0, 1]\}$ with parallel disjunction as addition and parallel conjunction as multiplication.

To obtain a system of semirings that is better suited for our purposes, we also consider the semiring $\overline{\mathbf{B}} = \{[0, 1] < \top\}$ that is dual to $\mathbf{B}$, and the 'double' semiring $\mathbf{D} = \{\perp < [0, 1] < \top\}$. $\mathbf{C}$ is a sub-semiring of both $\mathbf{B}$ and its dual $\overline{\mathbf{B}}$, and $\mathbf{B}$ and $\overline{\mathbf{B}}$ are sub-semirings of $\mathbf{D}$. Thus, we are able to apply our results about sub-constructions of chapter 14. The 'double' semiring $\mathbf{D}$ is the product of $\mathbf{L}$ and $\mathbf{U}$ such that our results about product and factorization of power constructions (Ch. 16) can be applied.

In chapter 17, we investigate the structure of the modules of these semirings. We show that all these modules may be characterized without referring to multiplication. In particular, $\mathbf{C}$-modules are commutative idempotent monoids, and $\mathbf{B}$-modules are Gunter's mix algebras defined in [Gun89b, Gun90].

Chapter 18 is devoted to the $\mathbf{L}$-constructions. We show that there is a standard $\mathbf{L}$-construction $\mathcal{L}$ that is both initial and final. It coincides with the well known lower power construction in terms of Scott closed sets.

Chapter 19 deals with $\mathbf{U}$-constructions. For sober ground domains, the final $\mathbf{U}$-construction $\mathcal{U}_f$ is shown to be equivalent to the topological construction $\mathcal{U}_K$ in terms of compact upper sets. The initiality of this construction could however be shown for continuous ground domains only. Thus, one has to carefully distinguish between $\mathcal{U}_f$ and the initial $\mathbf{U}$-construction $\mathcal{U}_i$.

In chapter 20, a sub-construction $\mathcal{U}_s$ of $\mathcal{U}_f$ is identified. For sober ground domains, it is equivalent to a topological construction $\mathcal{U}_S$ in terms of strongly compact upper sets. All these constructions $\mathcal{U}_f$, $\mathcal{U}_s$, and $\mathcal{U}_i$ could not be proved to be isomorphic, although we do not know any ground domain where they differ.[2] For continuous ground domains only, $\mathcal{U}_f$, $\mathcal{U}_s$, $\mathcal{U}_i$, $\mathcal{U}_K$, and $\mathcal{U}_S$ all coincide such that one may speak of a standard upper power construction.

The remainder of chapter 20 is devoted to the proof of two theorems about the commuting of the standard lower construction $\mathcal{L}$ and the upper constructions $\mathcal{U}_S$ and $\mathcal{U}_i$ respectively. Both theorems coincide for continuous ground domains, and largely generalize the result of [FM90] where the commuting of $\mathcal{L}$ and $\mathcal{U}$ was shown for bounded complete algebraic ground domains using information systems. The proofs of our two theorems are completely different from each other and from the proof in [FM90]. The commuting of $\mathcal{L}$ and $\mathcal{U}_S$ is shown by topological arguments for $\mathcal{U}_S$-conform ground domains, whereas the commuting of $\mathcal{L}$ and $\mathcal{U}_i$ is shown by algebraic methods for *all* ground domains.

[2] Meanwhile, we found a domain where $\mathcal{U}_i$ differs from the other ones.

In chapter 21, we investigate the final $\mathbf{D}$-construction $\mathcal{D}$ and its sub-constructions that are obtained by existential restriction to the sub-semirings $\mathbf{B}$, $\overline{\mathbf{B}}$, and $\mathbf{C}$ of $\mathbf{D}$. The restriction to $\mathbf{B}$ is $\mathcal{S}$, the sandwich power construction.

As the mixed power construction $\mathcal{M}$ is a sub-construction of $\mathcal{S}$, it should also be found among the sub-constructions of $\mathcal{D}$. In chapter 22, we present the conditions M and $\overline{M}$ that define the sub-construction $\mathcal{M}$ and its dual $\overline{\mathcal{M}}$. Moreover, the intersection of $\mathcal{M}$ and $\overline{\mathcal{M}}$ yields a sub-construction $\mathcal{C}$ of $\mathcal{D}$ that coincides with Plotkin's construction in case of continuous ground domains.

Chapter 23 contains some remarks about $\mathbf{B}$- and $\mathbf{C}$-constructions. $\mathcal{S}$ preserves bounded completeness, whereas $\mathcal{M}$ and $\mathcal{C}$ do not. In case of continuous ground domains, $\mathcal{M}$ is the initial $\mathbf{B}$-construction, and $\mathcal{C}$ the initial $\mathbf{C}$-construction. $\mathcal{S}$ is the final $\mathbf{B}$-construction, whereas the final $\mathbf{C}$-construction is degenerated and not among the sub-constructions of $\mathbf{D}$.

The final chapter 24 introduces sandwich algebras and shows that sandwich power domains are free sandwich algebras if the ground domain is algebraic. I found this result before the algebraic theory of power domain constructions was fully developed. In contrast to many other results, it could not be merged into the general theory and is still standing for its own.

Chapter 17

Small semirings and their modules

In this chapter, we study the small semirings $\mathbf{L}$, $\mathbf{U}$, $\mathbf{C}$, $\mathbf{B}$, $\overline{\mathbf{B}}$, and $\mathbf{D}$ and the structure of the corresponding modules.

In section 17.2, we consider the modules of the three two-point semirings $\mathbf{L}$, $\mathbf{U}$, and $\mathbf{C}$. $\mathbf{C}$-modules are just commutative idempotent monoids. In these CI monoids, a logical order may be derived from addition by defining $x \sqsubseteq y$ iff $x + y = y$. This order is studied in section 17.3. Section 17.4 is concerned with the modules of the semirings $\mathbf{B}$, $\overline{\mathbf{B}}$, and $\mathbf{D}$. In particular, these three semirings are shown to be additive, and $\mathbf{B}$-modules are proved to be equivalent to the mix algebras of [Gun89b, Gun90]. Every $\mathbf{B}$-module contains a $\mathbf{U}$-module in a canonical way as shown in section 17.5.

We then leave the modules and turn to power constructions. In section 17.6, the general properties of power constructions for the small semirings are enumerated as they follow from the results of Part II.

The power domains belonging to the three semirings $\mathbf{L}$, $\mathbf{U}$, and $\mathbf{C}$ with carrier set $\{0, 1\}$ may be split into an 'empty part' around the empty set and a 'non-empty part' containing all singletons and closed w.r.t. union, extension, and limits (section 17.7). It is this non-empty part that was proposed in most of the original papers about the classical power domain constructions.

17.1 The considered semirings

The characteristic semiring of the *lower power construction* is $\mathbf{L} = \{0 < 1\}$ where $1{+}1 = 1$. In logical interpretation, $\mathbf{L}$ contains F and T where F is below T. Computationally, this means that a partial computation resulting in F may change this value to T if it proceeds. Hence, only computations resulting in T are terminated, whereas F means non-termination.

The characteristic semiring of the classical *upper power construction* of Smyth is $\mathbf{U} = \{1 < 0\}$ where $1 + 1 = 1$. Logically, T is below F in this semiring. Hence, only computations resulting in F are terminated, whereas T signals non-termination.

The characteristic semiring of the *convex power construction* of Plotkin is $\mathbf{C} = \{0, 1\}$ where $1 + 1 = 1$ and the two values are incomparable. Logically, T and F are both stable here. The semiring is discrete, whence computations with logical result cannot proceed. They have immediately to decide whether the result is T or F, and cannot change their 'opinion' afterwards.

A power construction with a more reasonable logic should have the Boolean domain $\mathbf{B} = \{\bot < [\mathsf{F}, \mathsf{T}]\}$ as semiring. Such constructions are called *set domain constructions* in [Hec90c]. The interpretation of $\bot$ is 'I do not (yet) know'. Computations with logical results start in this state which may change to F or T if the computation proceeds.

The *sandwich power construction* [BDW88] or *big set domain construction* [Hec90c] and the *mixed power construction* [Gun89b, Gun90] or *small set domain construction* [Hec90c] both have characteristic semiring **B** with parallel conjunction and disjunction.

Besides the semirings **L**, **U**, **C**, and **B**, we also consider the 'double semiring' $\mathbf{D} = \mathbf{L} \times \mathbf{U}$. The motivation to consider **D** is that the Boolean semiring **B** is a sub-semiring of **D**.

The semiring $\mathbf{D} = \mathbf{L} \times \mathbf{U} = \{0 < 1\} \times \{1 < 0\}$ has four elements ordered as follows:

$$\begin{tikzcd} & (1,0) & \\ (0,0) \arrow[ur] & & (1,1) \arrow[ul] \\ & (0,1) \arrow[ul] \arrow[ur] & \end{tikzcd} \qquad \begin{tikzcd} & \mathsf{W} & \\ \mathsf{F} \arrow[ur] & & \mathsf{T} \arrow[ul] \\ & \bot \arrow[ul] \arrow[ur] & \end{tikzcd}$$

The picture to the left shows a representation of **D** in terms of pairs of members of the lower and upper semiring. The picture to the right shows a logical interpretation of **D**. (0, 0), the neutral element of addition logically becomes F, the neutral element of disjunction. Correspondingly, (1, 1) is interpreted as T. The least element $\bot$ denotes a state of ignorance. Every computation starts with $\bot$, and this result may be refined to either T or F when the computation proceeds. In contrast, W denotes a state of inconsistency: a computation returning W subsumes both T and F.

We now present tables for the semiring operations in **D**. On the left hand side, they are depicted in terms of pairs, whereas on the right hand side, the logical interpretation is used.

+	(0, 1)	(0, 0)	(1, 1)	(1, 0)
(0, 1)	(0, 1)	(0, 1)	(1, 1)	(1, 1)
(0, 0)	(0, 1)	(0, 0)	(1, 1)	(1, 0)
(1, 1)	(1, 1)	(1, 1)	(1, 1)	(1, 1)
(1, 0)	(1, 1)	(1, 0)	(1, 1)	(1, 0)

$\vee$	$\bot$	F	T	W
$\bot$	$\bot$	$\bot$	T	T
F	$\bot$	F	T	W
T	T	T	T	T
W	T	W	T	W

$\cdot$	(0, 1)	(0, 0)	(1, 1)	(1, 0)
(0, 1)	(0, 1)	(0, 0)	(0, 1)	(0, 0)
(0, 0)	(0, 0)	(0, 0)	(0, 0)	(0, 0)
(1, 1)	(0, 1)	(0, 0)	(1, 1)	(1, 0)
(1, 0)	(0, 0)	(0, 0)	(1, 0)	(1, 0)

$\wedge$	$\bot$	F	T	W
$\bot$	$\bot$	F	$\bot$	F
F	F	F	F	F
T	$\bot$	F	T	W
W	F	F	W	W

Considering the right hand tables, one sees that the Boolean semiring $\mathbf{B} = \{\bot, \mathsf{F}, \mathsf{T}\}$ with parallel disjunction and conjunction is a sub-semiring of $\mathbf{D}$. Another sub-semiring of $\mathbf{D}$ is $\overline{\mathbf{B}} = \{\mathsf{F}, \mathsf{T}, \mathsf{W}\}$. Tables for its operations look like tables for the operations of $\mathbf{B}$ if $\bot$ is replaced by W. From a domain-theoretic viewpoint, $\overline{\mathbf{B}}$ is just the dual of $\mathbf{B}$. Another sub-semiring of $\mathbf{D}$ is $\mathbf{C} = \{\mathsf{F}, \mathsf{T}\}$, the intersection of $\mathbf{B}$ and $\overline{\mathbf{B}}$.

In contrast to $\mathbf{L}$ and $\mathbf{U}$, $\mathbf{D}$ admits negation. The continuous operation $\neg : [\mathbf{D} \to \mathbf{D}]$ exchanges the values of F and T, and maps $\bot$ and W to itself. Hence, the sub-semirings $\mathbf{B}$, $\overline{\mathbf{B}}$, and $\mathbf{C}$ are closed not only w.r.t. disjunction and conjunction, but also w.r.t. negation.

The six semirings $\mathbf{L}$, $\mathbf{U}$, $\mathbf{C}$, $\mathbf{B}$, $\overline{\mathbf{B}}$, and $\mathbf{D}$ are connected by semiring homomorphisms as indicated by the following figure:

$$\begin{tikzcd}
 & \overline{\mathbf{B}} \arrow[dr] & & \mathbf{L} \\
\mathbf{C} \arrow[ur] \arrow[dr] & & \mathbf{D} = \mathbf{L} \times \mathbf{U} \arrow[ur] \arrow[dr] & \\
 & \mathbf{B} \arrow[ur] & & \mathbf{U}
\end{tikzcd}$$

The homomorphisms from $\mathbf{C}$ to $\overline{\mathbf{B}}$ and $\mathbf{B}$, and those from $\overline{\mathbf{B}}$ and $\mathbf{B}$ to $\mathbf{D}$ are embeddings. The homomorphisms from $\mathbf{D}$ to $\mathbf{L}$ and $\mathbf{U}$ on the other hand are projections. All these homomorphisms are unique since the values of 0 and 1 are prescribed, and $\bot$ and W have to be mapped to common lower resp. upper bounds of 0 and 1.

If $\varphi : [R \to R']$ is a semiring homomorphism and $(M, R', \cdot)$ is an R'-module, then M may be turned into an R-module $(M, R, *)$ by the definition $r * m = \varphi r \cdot m$. Thus, all $\mathbf{L}$- and $\mathbf{U}$-modules are also $\mathbf{D}$-modules, all $\mathbf{D}$-modules are $\mathbf{B}$-modules as well as $\overline{\mathbf{B}}$-modules, and $\mathbf{B}$- and $\overline{\mathbf{B}}$-modules in turn are $\mathbf{C}$-modules.

In all semirings R we consider in this chapter, $1+1 = 1$ holds, whence $m+m = 1{\cdot}m+1{\cdot}m = (1+1)\cdot m = 1 \cdot m = m$ holds for all members m of all modules. Hence, all R-modules are commutative idempotent monoids, or shortly *CI monoids*.

17.2 L-, U-, and C-modules

The three two-valued semirings $\mathbf{L}$, $\mathbf{U}$, and $\mathbf{C}$ are all additive, i.e. all additive mappings between their modules are linear. The reason is that linearity of an additive function f may be shown by case analysis:

- $f(0 \cdot m) = f(0) = 0 = 0 \cdot f(m)$
- $f(1 \cdot m) = f(m) = 1 \cdot f(m)$

The modules of the two-valued semirings may be characterized without referring to multiplication:

Proposition 17.2.1

$\mathbf{C}$: M is a $\mathbf{C}$-module iff it is a CI monoid.

$\mathbf{L}$: M is an $\mathbf{L}$-module iff it is a CI monoid and 0 is its least element.

U : M is a **U**-module iff it is a CI monoid and 0 is its greatest element.

Proof: All modules are commutative monoids. As pointed out above, all **L**-, **U**-, and **C**-modules are CI monoids. In the **L**-case, 0 is least by $m = 1 \cdot m \geq 0 \cdot m = 0$. Analogously, 0 is greatest in **U**-modules.

Conversely, let M be a given CI monoid. A multiplication may be defined by $1 \cdot m = m$ and $0 \cdot m = 0$. This multiplication is monotonic in its left operand because of the additional axioms $m \geq 0$ and $m \leq 0$ in the cases **L** and **U** respectively. Since **C**, **L**, and **U** are finite, multiplication is continuous by Prop. 3.3.3. The module axioms $(r+s) \cdot m = r \cdot m + s \cdot m$ and $(r \cdot s) \cdot m = r \cdot (s \cdot m)$ may be verified by case analysis $r, s \in \{0, 1\}$. Idempotence is needed for the additive axiom in case $r = s = 1$. □

L- and **U**-modules may be characterized even without referring to addition.

Proposition 17.2.2

L : In every **L**-module, 0 is the least element and $a + b$ is the least upper bound of a and b. Conversely, every domain **X** with a least element $\perp$ and a least upper bound $x \sqcup y$ for every two points is an **L**-module.

In shorter terms: **L**-modules are just the complete domains, i.e. those in CC.

U : In every **U**-module, 0 is the greatest element and $a + b$ is the greatest lower bound of a and b. Conversely, every domain **X** with a greatest element $\top$ and a continuous greatest lower bound operation $\lambda(x, y).\, x \sqcap y$ is a **U**-module.

Proof: We first consider the case of the semiring **L**. By Prop. 17.2.1, 0 is the least element of an **L**-module M. From $0 \leq b$, $a = a + 0 \leq a + b$ follows. Similarly, $0 \leq a$ implies $b \leq a + b$. Thus, $a + b$ is an upper bound of a and b. Let u be an arbitrary upper bound. Then $a + b \leq u + u = u$ holds.

Thus, M is a domain with least element and least upper bound of every two points. Hence it is complete by Prop. 5.3.1.

Conversely, let **X** be a complete domain. The operation '$\sqcup$' in a complete domain is commutative, associative, idempotent, and has neutral element $\perp$. It is continuous by Prop. 5.3.5. Hence, $(\mathbf{X}, \sqcup, \perp)$ is a CI monoid with least element $\perp$. By Prop. 17.2.1, it is an **L**-module.

The proof for **U**-modules is analogous. Because the operation '$\sqcap$' may not be continuous, its continuity has to be explicitly stated. □

By this proposition, every lower power construction $\mathcal{L}$ creates complete domains $\mathcal{L}\mathbf{X}$ only. Thus, it preserves completeness (CC), bounded completeness (BC), FC, SC, and KC.

17.3 The logical order of CI monoids

All modules considered in this part of the thesis belong to one of the semirings **L**, **U**, **D**, **B**, $\overline{\mathbf{B}}$, or **C**. Hence, they are all **C**-modules i.e. CI monoids. Besides the a priori given domain order, CI monoids admit the definition of an alternative order '$\sqsubseteq$' that is derived from addition.

Definition 17.3.1 Let $(M, +, 0)$ be a CI monoid domain. Then we define $a \sqsubseteq b$ iff $a + b = b$. We call '$\sqsubseteq$' the *logical order* of M.

Proposition 17.3.2 The logical order of a CI monoid $(M, +, 0)$ is indeed an order. 0 is its least element, and $a + b$ is the logical least upper bound of a and b.

Proof:

- Reflexivity: $a \sqsubseteq a$ holds by idempotence $a + a = a$.
- Transitivity: $a \sqsubseteq b$ and $b \sqsubseteq c$ imply $a + b = b$ and $b + c = c$, whence $a + c = a + (b + c) = (a + b) + c = b + c = c$, i.e. $a \sqsubseteq c$.
- Antisymmetry: $a \sqsubseteq b$ and $b \sqsubseteq a$ imply $b = a + b = b + a = a$.
- Least element: $0 \sqsubseteq b$ follows from $0 + b = b$.
- Upper bound: Idempotence implies $a + (a + b) = (a + a) + b = a + b$, whence $a \sqsubseteq a + b$. By commutativity, $b \sqsubseteq a + b$ holds.
- Least upper bound: Assume $a \sqsubseteq u$ and $b \sqsubseteq u$. Then $(a+b)+u = a+(b+u) = a+u = u$, whence $a + b \sqsubseteq u$. □

By the proposition above, addition and neutral element of a CI monoid are uniquely determined by the logical order. Hence, a CI monoid may be described by providing its logical order instead of a table of values for its addition.

The addition of semiring **L** for instance is described by $0 < 1$ and $0 \sqsubset 1$, **U** by $1 < 0$ and $0 \sqsubset 1$, **C** by $[0, 1]$ and $0 \sqsubset 1$, **B** by $\bot < [\mathsf{F}, \mathsf{T}]$ and $\mathsf{F} \sqsubset \bot \sqsubset \mathsf{T}$. A strict disjunction would be characterized by $\mathsf{F} \sqsubset \mathsf{T} \sqsubset \bot$ instead. The additive part of semiring **D** is described by $\bot < [\mathsf{F}, \mathsf{T}] < \mathsf{W}$ and $\mathsf{F} \sqsubset [\bot, \mathsf{W}] \sqsubset \mathsf{T}$.

In ordinary CI monoids, the logical order '$\sqsubseteq$' and the domain order '$\leq$' are totally unrelated. In special cases, there is an relation.

Proposition 17.3.3

L : A CI monoid is an **L**-module iff domain order and logical order coincide ($a \sqsubseteq b$ iff $a \leq b$).

U : A CI monoid is a **U**-module iff domain order and logical order are opposite ($a \sqsubseteq b$ iff $a \geq b$).

Proof: If the orders coincide, then 0 is the least element w.r.t. '$\leq$', whence the monoid is an **L**-module by Prop. 17.2.2. If the orders are opposite, then 0 is greatest w.r.t. '$\leq$', whence the monoid is a **U**-module.

Conversely, in an **L**-module, addition is supremum, whence $a+b = b$ is equivalent to $a \sqcup b = b$, i.e. $a \leq b$. In **U**-modules, one gets $a \sqcap b = b$, i.e. $a \geq b$. □

A function f between two CI monoids is *logically monotonic* iff $a \sqsubseteq b$ implies $fa \sqsubseteq fb$.

Proposition 17.3.4

(1) Addition is logically monotonic, i.e. $a \sqsubseteq a'$ and $b \sqsubseteq b'$ imply $a + b \sqsubseteq a' + b'$.

(2) Every additive function is logically monotonic.

Proof:

(1) $(a+b)+(a'+b') = (a+a')+(b+b') = a'+b'$

(2) $a+b=b$ implies $fa+fb = f(a+b) = fb$ □

CI monoids are generally not logical domains, i.e. the logical supremum of a logically directed set may not exist. Even if directed logical suprema exist, they may not be preserved by additive functions. Thus, additive (domain) continuous functions need not be 'logically continuous'.

The semirings **L**, **U**, **C**, **D**, **B**, and $\overline{\mathbf{B}}$ have some special features in common. In all of these, multiplication is CI and $a+1=1$ holds, i.e. 1 is logically a greatest element. Due to the distributive laws and Prop. 17.3.4, multiplication in R as well as in all R-modules is logically monotonic. Hence, $a \cdot m \sqsubseteq 1 \cdot m = m$ holds in all R-modules. Within R, $u \sqsubseteq a, b$ implies $u = u \cdot u \sqsubseteq a \cdot b$, i.e. $a \cdot b$ is the logical greatest lower bound of a and b.

Proposition 17.3.5 Let R be any of the semirings **L**, **U**, **C**, **B**, $\overline{\mathbf{B}}$, and **D**. Then $0 \sqsubseteq r \sqsubseteq 1$ holds for all r in R. For every two points a and b of R, $a+b$ is the logical least upper bound, and $a \cdot b$ the logical greatest lower bound of a and b. In all R-modules M, $r \cdot m \sqsubseteq m$ holds for all r in R and m in M.

17.4 D-, B-, and $\overline{\mathbf{B}}$-modules

This section is concerned with a characterization of **D**-, **B**-, and $\overline{\mathbf{B}}$-modules that does not employ multiplication. Furthermore, we show that all additive functions between these modules are linear, i.e. **D**, **B**, and $\overline{\mathbf{B}}$ are additive semirings.

Because of the sub-semiring relations, every **D**-module is also a **B**-module and a $\overline{\mathbf{B}}$-module, and these modules in turn are **C**-modules. **C**-modules are just CI monoids by Prop. 17.2.1.

In **B**-modules M, we define a function $\_? : [M \to M]$ by $A? = \perp \cdot A$. Among others, '?' has the following properties:

(1) $A? = \perp \cdot A \leq 1 \cdot A = A$

(2) $A? = \perp \cdot A \leq 0 \cdot A = 0$

(3) $A + A? = 1 \cdot A + \perp \cdot A = (1+\perp) \cdot A = 1 \cdot A = A$

(4) $(A+B)? = \perp \cdot (A+B) = \perp \cdot A + \perp \cdot B = A? + B?$

In $\overline{\mathbf{B}}$-modules, we define a function $\_!$ by $A! = \mathsf{W} \cdot A$. It has properties analogous to those of '?'.

These facts motivate the following definition:

Definition 17.4.1 Let M be a CI monoid domain. A function $\_? : [M \to M]$ is a *lower approximation* iff it satisfies the following axioms:

$(A1)$ $A? \leq A$	$(A2)$ $A? \leq 0$
$(A3)$ $A + A? \geq A$	$(A4)$ $(A+B)? \leq A? + B?$

A function $\_! : [M \to M]$ is an *upper approximation* iff it satisfies

$(\overline{A}1)$ $A! \geq A$ $\qquad$ $(\overline{A}2)$ $A! \geq 0$

$(\overline{A}3)$ $A + A! \leq A$ $\qquad$ $(\overline{A}4)$ $(A+B)! \geq A! + B!$

In comparison to the properties of $\lambda A.\perp \cdot A$ and $\lambda A.\mathsf{W} \cdot A$ derived above, axioms (3) and (4) are weaker since the equality sign is replaced by '$\leq$' or '$\geq$'. In fact, the opposite inequations are redundant since they may be derived from the given axioms. These axioms are independent as some examples show.

For every axiom (Ai), there is a CI monoid M_i with an additional continuous operation $\_? : [M_i \to M_i]$ such that all axioms except (Ai) are satisfied.

(1) Example M_1 is given by the CI monoid $(\{\perp < \top\}, \sqcup, \perp)$. If identity is chosen for '?', then the axioms (A2) through (A4) are satisfied whereas (A1) is not because $\top? = \top \not\leq 0 = \perp$.

(2) Example M_2 is given by the CI monoid $(\{\perp < \top\}, \sqcap, \top)$ with the definition $A? = \top$ for all A. Then the axioms (A1), (A3), and (A4) hold because $\top$ is the neutral element of M_2. (A2) is violated because $\perp? = \top \not\leq \perp$.

(3) Example M_3 is given by the same CI monoid as M_2, but defining $A? = \perp$ for all A. Then the axioms (A1) and (A2) hold because $\perp$ is the least element of M_3. (A4) holds because of idempotence. (A3) is violated since $\top + \top? = \top \sqcap \perp = \perp \not\geq \top$.

(4) Example M_4 is more complex since we need four elements. The domain order is $\{\perp < [0, \square] < \top\}$ and the logical order is $\{0 \sqsubset \perp \sqsubset \square \sqsubset \top\}$. A little thought shows that the addition defined by this logical order is (domain) monotonic, whence continuous.

The operation '?' is defined by mapping $\top$ and 0 to 0, and $\perp$ and $\square$ to $\perp$. In shorter terms, $A?$ is the greatest lower bound of 0 and A. Hence, (A1) and (A2) immediately hold, and (A3) is easily verified. However (A4) fails since $(\square + \top)? = \top? = 0$, whereas $\square? + \top? = \perp + 0 = \perp$.

In the sequel, we study the general properties of a lower approximation $\_?$ in a CI monoid M.

(A1) $A? \leq A$ $\qquad$ (A2) $A? \leq 0$

(A3) $A + A? \geq A$ $\qquad$ (A4) $(A+B)? \leq A? + B?$

(T1) $A + B? \leq A$ $\quad$ since $A + B? \overset{A2}{\leq} A + 0 \overset{N}{=} A$

(T2) $A + A? = A$ $\quad$ by (A3) and (T1)

(T3) $0? = 0$ $\quad$ since $0 \overset{T2}{=} 0 + 0? \overset{N}{=} 0?$

(T4) $A?? = A?$ $\quad$ since $A?? \overset{A1}{\leq} A? \overset{T2}{=} A? + A?? \overset{T1}{\leq} A??$

(T5) $A? = A$ iff $A \leq 0$

Proof: '$\Rightarrow$' $A \overset{lhs}{=} A? \overset{A2}{\leq} 0$ $\quad$ '$\Leftarrow$' $A? \overset{A1}{\leq} A \overset{T2}{=} A + A? \overset{rhs}{\leq} 0 + A? \overset{N}{=} A?$

(T6) $X \leq 0$ and $X \leq A$ iff $X \leq A?$ $\quad$ i.e. $A?$ is the greatest lower bound of 0 and A.

Proof: '$\Rightarrow$' $X \leq 0$ implies $X = X?$ by (T5). $X \leq A$ implies $X? \leq A?$ by monotonicity of '?'. Together, $X \leq A?$ follows. $\quad$ '$\Leftarrow$' by (A1) and (A2).

(T7) $(A+B)? = A? + B?$

Proof: '$\leq$' is (A4). '$\geq$' is deduced by (T6) from $A? + B? \leq 0$ (by (A2) and (N)) and $A? + B? \leq A + B$ (by (A1)).

(T8) The three statements $A \le A + B$ and $A? \le B?$ and $A? \le B$ are equivalent.

$$\begin{aligned} \text{Proof: } & (1) \Rightarrow (2): & A? \overset{1}{\le} (A+B)? \overset{T7}{=} A? + B? \overset{T1}{\le} B? \\ & (2) \Rightarrow (3): & A? \overset{2}{\le} B? \overset{A1}{\le} B \\ & (3) \Rightarrow (1): & A \overset{T2}{=} A + A? \overset{3}{\le} A + B \end{aligned}$$

(T9) $X \le 0$ and $X \le A$ and $A + X \ge A$ iff $X = A?$
Proof: '$\Leftarrow$' is immediate by (A1), (A2), and (A3).
'$\Rightarrow$': $X \le 0$ and $X \le A$ imply $X \le A?$ by (T6). $A + X \ge A$ implies $A? \le X$ by (T8).

(T10) Every CI monoid with a lower approximation $\_?$ is a $\mathbf{B}$-module.

Proof: We define $0 \cdot A = 0$, $1 \cdot A = A$, and $\perp \cdot A = A?$. By (A1) and (A2), this operation is monotonic in its $\mathbf{B}$-argument, whence it is continuous.

$r \cdot 0 = 0$:	(T3)
$r \cdot (A + B) = r \cdot A + r \cdot B$:	(T7)
$0 \cdot A = 0$:	immediate
$(r + s) \cdot A = r \cdot A + s \cdot A$:	by neutrality if $r = 0$ or $s = 0$, by idempotence if $r = s$, and by (T2) if $r = 1$ and $s = \perp$ or vice versa.
$1 \cdot A = A$:	immediate
$r \cdot (s \cdot A) = (r \cdot s) \cdot A$:	the only difficult case $r = s = \perp$ is handled by (T4).

(T11) Let M and M' be two CI monoids with lower approximations $\_?$ and $\_?'$ respectively. Then every additive morphism $f : [M \to M']$ satisfies $f(A?) = (fA)?'$.

Proof: For all $A \in M$, $A? \le 0$, $A? \le A$, and $A + A? \ge A$ hold by the axioms of '?'. These statements imply $f(A?) \le 0$, $f(A?) \le fA$, and $fA + f(A?) \ge fA$ respectively. By (T9), $f(A?) = (fA)?'$ follows.

In [Gun89b, Gun90], Gunter defined *mix algebras* by an axiom system consisting of (T7), (T4), (T2), (A1), and (T1). Because (T1) implies (A2) by choosing $A = 0$ and (T2) implies (A3) and (T7) implies (A4), his mix theory is equivalent with our theory of CI monoids with a lower approximation, and Gunter's mix algebras are just $\mathbf{B}$-modules.

(T9) is a particularly interesting theorem. It implies that the operation '?' is uniquely determined, i.e. for a given CI monoid, there is at most one choice for a lower approximation '?'. By (T6), the only candidate for '?' is $\lambda A. 0 \sqcap A$. Thus, to verify that a given CI monoid M is a $\mathbf{B}$-module, one has to check that for all A in M $0 \sqcap A$ exists, that the operation $\lambda A. 0 \sqcap A$ is continuous and satisfies axioms (A3) and (A4). By the choice $A? = 0 \sqcap A$, axioms (A1) and (A2) are obviously satisfied.

Translated into the language of $\mathbf{B}$-modules, $f(A?) = (fA)?$ reads $f(\perp \cdot A) = \perp \cdot fA$. Hence, (T11) means $\mathbf{B}$ is additive because the corresponding statements for 0 and 1 are trivial.

Since the axioms of upper approximations are just dual to those for lower ones, dual theorems hold for them, e.g. $0! = 0$ and $A!! = A!$. Hence, $A!$ is the least upper bound of 0 and A by the dual of (T6), there is at most one upper approximation in a given CI monoid by (T9), and $\overline{\mathbf{B}}$-modules are just CI monoids with an upper approximation. The dual of (T11) states that $\overline{\mathbf{B}}$ is additive.

Finally, we investigate how lower and upper approximations interact. In a CI monoid M with both a lower and an upper approximation '?' resp. '!', the following facts hold:

(F1) $A!? = A?! = 0$
By $(A2)$, $A!? \leq 0$ holds. By $(\overline{A}2)$, monotonicity of '?', and $(T1)$, $A!? \geq 0? = 0$ holds. $A?! = 0$ is shown analogously.

(F2) $A? + A! = A$
Employing first $(A1)$ and $(\overline{A}1)$ and then $(A3)$ and $(\overline{A}3)$, we obtain $A? + A! \leq A + A! \leq A$ and $A? + A! \geq A? + A \geq A$.

(F3) Every CI monoid with both a lower and an upper approximation $\_?$ resp. $\_!$ is a **D**-module.

The proof is similar to that of (T10). Some new arguments are needed: (F1) proves $r \cdot (s \cdot A) = (r \cdot s) \cdot A$ in case $r = \bot$ and $s = \mathsf{W}$ or vice versa since $\bot \cdot \mathsf{W} = 0$. (F2) proves $(r + s) \cdot A = r \cdot A + s \cdot A$ in the same case since $\bot + \mathsf{W} = 1$.

(F4) Additivity of **D** may be shown by combining (T11) and its dual.

The results of this section are summarized to the following theorem:

Theorem 17.4.2

(1) **B**-modules are CI monoids with a lower approximation.

(2) $\overline{\mathbf{B}}$-modules are CI monoids with an upper approximation.

(3) **D**-modules are CI monoids with both a lower and an upper approximation.

(4) All the semirings **B**, $\overline{\mathbf{B}}$, and **D** are additive.

As mentioned towards the end of section 17.1, all **L**- and **U**-modules are also **D**-modules. In case of an **L**-module M, the approximations are defined by $m? = 0$ and $m! = m$, whereas in case of a **U**-module, they are given by $m? = m$ and $m! = 0$.

17.5 The U-module contained in a B-module

Since the semiring **D** is the product of **L** and **U**, every **D**-module is isomorphic to the product of some **L**-module and some **U**-module by Th. 16.4.3. We now investigate whether every **B**-module may be embedded into the product of an **L**- and a **U**-module.

If $M = (M, +, 0, ?)$ is a **B**-module with its lower approximation '?', then one may define the subset

$$M? = \{m? \mid m \in M\} = \{m \in M \mid m? = m\} = \{m \in M \mid m \leq 0\}$$

The last equality is given by (T5) $m? = m$ iff $m \leq 0$, whereas the first equality is given by (T4) $m?? = m$.

The subset $M?$ contains 0 and is closed w.r.t. addition since $a, b \leq 0$ implies $a + b \leq 0 + 0 = 0$. Thus, $M?$ is a sub-monoid of M. It is a **U**-module since 0 is its greatest element. The mapping $\_? : [M \to M?]$ is additive by (T3) and (T7).

By dual arguments, every $\overline{\mathbf{B}}$-module M contains a sub-monoid $M!$ that is the image of the additive mapping '!'. Because of $M! = \{m \in M \mid m \geq 0\}$, the sub-monoid $M!$ is an $\mathbf{L}$-module.

If a $\mathbf{D}$-module M is given, then M is isomorphic to $M! \times M?$ by means of the pair of additive isomorphisms $\alpha m = (m!, m?)$ and $\beta(a, b) = a + b$. In fact, this result is only a special instance of Th. 16.4.3.

Since every $\mathbf{D}$-module is isomorphic to a product of an $\mathbf{L}$-module and a $\mathbf{U}$-module, one might believe that every $\mathbf{B}$-module be isomorphic to a sub-monoid of such a product. This is however wrong as an example shows.

Let $M = \{\perp < 0 < \top\}$ where addition is given by $\perp + x = x + \perp = \perp$, $0 + x = x + 0 = x$, and $x + x = x$ for all x in M. By case analysis, one may easily verify that M is a $\mathbf{B}$-module with $\perp? = \perp$ and $0? = \top? = 0$. Assume there are an $\mathbf{L}$-module M_1 and a $\mathbf{U}$-module M_2 such that there is an additive embedding $e : [M \to M_1 \times M_2]$. Then $e\,(0) = (0, 0)$ holds. Since 0 is least in M_1, $e\,(\perp) = (0, b)$ follows for some b in M_2. Dually, $e\,(\top) = (a, 0)$ holds for some a in M_1 since 0 is greatest in M_2. By additivity, $(0, b) = e\,(\perp) = e\,(\perp) + e\,(\top) = (0, b) + (a, 0) = (a, b)$ follows, whence $a = 0$, i.e. $e\,(\top) = e\,(0) = (0, 0)$. Thus, e is not an embedding.

The $\mathbf{B}$-module M of the example above cannot be obtained as a power domain for any ground domain $\mathbf{X}$ and any $\mathbf{B}$-construction $\mathcal{P}$. The reason is the linear mapping $ne : [\mathcal{P}\mathbf{X} \to \mathbf{B}]$, which maps singletons $\{\!|x|\!\}$ to 1. Since $\mathbf{X}$ is not empty, every power domain $\mathcal{P}\mathbf{X}$ contains at least one singleton. The example module M however does not admit any mapping to $\mathbf{B}$ whose range contains 1 since additive maps from M to $\mathbf{B}$ have to map 0 to 0, whence by monotonicity $\top$ is mapped to 0 and $\perp$ to 0 or $\perp$.

Hence, the discussion leaves open the following question:

Problem 13 Can every power domain obtained by a $\mathbf{B}$-construction be embedded into a product of an $\mathbf{L}$-module and a $\mathbf{U}$-module?

17.6 General properties of power constructions for the small semirings

Part II of the thesis allows to state some general remarks that are valid for all power constructions with characteristic semiring in $\mathcal{R} = \{\mathbf{L}, \mathbf{U}, \mathbf{C}, \mathbf{B}, \overline{\mathbf{B}}, \mathbf{D}\}$. All these semirings are obviously finite, commutative, and idempotent. They are all additive as shown in the previous sections. All semirings except $\mathbf{C}$ are non-discrete, and all except $\mathbf{C}$ and $\overline{\mathbf{B}}$ have a least element. Hence, we obtain the following statements for all R in $\mathcal{R}$:

(1) All R-constructions are idempotent ($A \uplus A = A$) by section 11.5.

(2) All R-constructions are commutative ($r \cdot A = A \cdot r$) by Prop. 11.12.2.

(3) All reduced R-constructions are symmetric by Prop. 14.3.3 (2).

(4) For $R \neq \mathbf{C}$: all R-constructions are faithful ($\{\!|x|\!\} \leq \{\!|y|\!\} \Rightarrow x \leq y$) by Cor. 11.9.3. The initial $\mathbf{C}$-construction is also faithful by Prop. 14.5.3 because $\mathbf{L}$ provides a non-discrete $\mathbf{C}$-module.

The final $\mathbf{C}$-construction and its core are not faithful by Prop. 15.7.2.

(5) All reduced and all final R-constructions preserve FIN, F-ALG, and F-CONT by the Propositions 14.3.3 (3) and 15.7.3.

(6) All initial R-constructions preserve ALG and CONT by Th. 14.5.2.

(7) For $R \in \{\mathbf{L}, \mathbf{U}, \mathbf{B}, \mathbf{D}\}$: all reduced and all final R-constructions yield power domains with a least element whenever the ground domain has a least element by Prop. 14.3.3 (4) and Prop. 15.7.4.

(8) The product of an **L**- and a **U**-construction results in a **D**-construction by Prop. 16.2.1. Conversely, every **D**-construction may be factorized into an **L**- and a **U**-construction by Cor. 16.6.3.

For all semirings R in $\mathcal{R}$, there are final constructions $\mathcal{P}^R_f$, their cores $\mathcal{P}^R_c$, and initial constructions $\mathcal{P}^R_i$. Besides these 18 constructions, some more constructions are obtained by restricting the **D**-constructions to the sub-semirings **B**, $\overline{\mathbf{B}}$, and **C** according to Th. 14.6.2. From the general theory, we only know the isomorphisms $\mathcal{P}^{\mathbf{D}}_f = \mathcal{P}^{\mathbf{L}}_f \times \mathcal{P}^{\mathbf{U}}_f$, $\mathcal{P}^{\mathbf{D}}_c = \mathcal{P}^{\mathbf{L}}_c \times \mathcal{P}^{\mathbf{U}}_c$, and $\mathcal{P}^{\mathbf{D}}_i = \mathcal{P}^{\mathbf{L}}_i \times \mathcal{P}^{\mathbf{U}}_i$ by the theorems 16.3.1, 16.5.1, and 16.5.3. The next chapters are devoted to the investigation of this host of power constructions and the task of finding more relations among them. For instance, we shall see $\mathcal{P}^{\mathbf{L}}_f = \mathcal{P}^{\mathbf{L}}_c = \mathcal{P}^{\mathbf{L}}_i$ and the coincidence of $\mathcal{P}^{\mathbf{B}}_f$ and the existential restriction of $\mathcal{P}^{\mathbf{D}}_f$ to **B**. Furthermore, we try to identify the known constructions among this set of constructions. Before doing so, we conclude this chapter by considering the role of the empty set in the power domains with two-valued semiring more closely.

17.7 The two parts of a power domain for L, U, or C

For this section, let $\mathcal{P}$ be an arbitrary power construction with a two-valued semiring R, i.e. R is **L**, **U**, or **C**. The power domains $\mathcal{P}\mathbf{X}$ are shown to fall into two disjoint parts, one containing the empty set, and the other one containing all singletons.

From section 10.5, we know the predicate of non-emptiness. ne is a linear morphism from $\mathcal{P}\mathbf{X}$ to the characteristic semiring R for all domains **X**. We refer to section 10.5 for its further properties. The semirings **L**, **U**, and **C** all have the same carrier set $\{0, 1\}$. Hence, we may define $\mathcal{P}^0\mathbf{X} = ne^{-1}[0]$ and $\mathcal{P}^1\mathbf{X} = ne^{-1}[1]$.

Proposition 17.7.1

(1) Both $\mathcal{P}^0\mathbf{X}$ and $\mathcal{P}^1\mathbf{X}$ are closed w.r.t. directed lubs.

(2) θ is in $\mathcal{P}^0\mathbf{X}$.

(3) $\{\!|x|\!\}$ is in $\mathcal{P}^1\mathbf{X}$ for all x in **X**.

(4) Both $\mathcal{P}^0\mathbf{X}$ and $\mathcal{P}^1\mathbf{X}$ are closed w.r.t. union '$\uplus$'.

(5) Both $\mathcal{P}^0$ and $\mathcal{P}^1$ admit extension: If $f : [\mathbf{X} \to \mathcal{P}^i\mathbf{Y}]$ and A is in $\mathcal{P}^i\mathbf{X}$, then $ext\, f\, A$ is in $\mathcal{P}^i\mathbf{X}$ for $i = 0, 1$.

(6) In case $R = \mathbf{C}$: If A is in $\mathcal{P}^0\mathbf{X}$ and B in $\mathcal{P}^1\mathbf{X}$, then A and B are incomparable.

Proof:

(1) If D is a directed set in $\mathcal{P}^i\mathbf{X}$, then $ne\,(\bigsqcup D) = \bigsqcup ne\,[D] = \bigsqcup\{i\} = i$.

(2) $ne\,\theta = 0$.

(3) $ne\,\{\!|x|\!\} = 1$.

(4) If A and B are in $\mathcal{P}^i\mathbf{X}$, then $ne\,(A \uplus B) = ne\,A + ne\,B = i + i = i$ by idempotence.

(5) $ne\,(ext\,f\,A) = ext\,(\lambda x.\,ne\,(f x))\,A = ext\,(\lambda x.\,i)\,A = ext\,(\lambda x.\,1 \cdot i)\,A = ext\,(\lambda x.\,1)\,A \cdot i = ne\,A \cdot i = i \cdot i = i$

(6) If A were below B, then $0 = ne\,A \le ne\,B = 1$ would hold. Analogously, A is not above B.

Whereas $\mathcal{P}^0\mathbf{X}$ contains nothing interesting than the empty set, $\mathcal{P}^1\mathbf{X}$ is almost a power domain; the only thing missing is the empty set. Because both $\mathcal{P}^0\mathbf{X}$ and $\mathcal{P}^1\mathbf{X}$ are non-empty and their respective elements are incomparable in case $R = \mathbf{C}$, $\mathbf{C}$-powerdomains never have a least element as a whole. $\mathcal{P}^1\mathbf{X}$ may however have a least element.

Proposition 17.7.2 Let $\mathcal{P}$ be reduced. Then $\mathcal{P}^0\mathbf{X} = \{\theta\}$ holds, and $\mathcal{P}^1\mathbf{X}$ has a least element, namely $\{\!|\bot|\!\}$, whenever $\mathbf{X}$ has a least element $\bot$.

Proof:

(1) Let $S = \{\theta\} \cup \mathcal{P}^1\mathbf{X}$. We show that S satisfies the properties of Def. 13.1.2. Then the core of $\mathcal{P}\mathbf{X}$, which is $\mathcal{P}\mathbf{X}$ itself, is a subset of S.

  (1) $\{\!|x|\!\} \in \mathcal{P}^1\mathbf{X} \subseteq S$.

  (2) $\theta \in S$.

  (3) If A and B are in $\mathcal{P}^1\mathbf{X}$, then $A \uplus B$ is also in it. If A is θ and B is in S, then $A \uplus B = B \in S$. Same for $B = \theta$.

  (4) If A is in S, then $r \cdot A$ is θ or A. In any case, it is in S.

  (5) Let D be a directed subset of S.

   Case $\mathbf{L}$: Either $D = \{\theta\}$ holds, or it contains a member of $\mathcal{P}^1\mathbf{X}$. Then $\bigsqcup D$ is in $\mathcal{P}^1\mathbf{X}$ since this set is upper.

   Case $\mathbf{U}$: Either $D \subseteq \mathcal{P}^1\mathbf{X}$ holds, then Prop. 17.7.1 (5) is employed. Or D contains θ, the greatest element of $\mathcal{P}\mathbf{X}$. Then $\bigsqcup D = \theta$ holds.

   Case $\mathbf{C}$: D is a subset of either $\{\theta\}$ or $\mathcal{P}^1\mathbf{X}$ because these two subsets of S have respectively uncomparable elements.

   In any case, $\bigsqcup D$ is in S.

(2) Let $S = \{\theta\} \cup T$, where $T = \{A \in \mathcal{P}^1\mathbf{X} \mid A \ge \{\!|\bot|\!\}\}$. Again, we show that S satisfies the core properties.

  (1) $\{\!|x|\!\} \ge \{\!|\bot|\!\}$ since $x \ge \bot$.

  (2) $\theta \in S$.

  (3) If A and B are in T, then $A \uplus B \ge \{\!|\bot|\!\} \uplus \{\!|\bot|\!\} = \{\!|\bot|\!\}$ holds by idempotence. If one of A and B is θ, then $A \uplus B$ is the other one.

  (4) $r \cdot A$ is θ or A.

  (5) The proof is similar to the proof of statement (5) in part (1). □

After having considered reduced constructions, we now turn to the final ones.

Proposition 17.7.3 Let $\mathcal{P}$ be final for the semiring R in $\{\mathbf{C}, \mathbf{L}, \mathbf{U}\}$. Then $\mathcal{P}^0\mathbf{X} = \{\theta\}$ holds for all ground domains $\mathbf{X}$.

Proof: For all $P : [[\mathbf{X} \to R] \stackrel{add}{\to} R]$, $ne\,P = ex\,(\lambda x.\,1)\,P = P\,(\lambda x.\,1)$ holds. Hence, we have to show that $P\,(\lambda x.\,1) = 0$ implies $P = \theta$, i.e. $Pp = 0$ for all $p : [\mathbf{X} \to R]$.

From $1 = 1 + r$ for all r in R, we conclude $0 = P\,(\lambda x.\,1) = P\,(\lambda x.\,1 + p\,x) = P\,(\lambda x.\,1) + Pp = 0 + Pp = Pp$ using additivity. □

In the literature, one often finds power constructions without empty set, i.e. $\mathcal{P}^1$ instead of $\mathcal{P}$. Plotkin's construction for instance is the non-empty part of $\mathcal{P}_i^{\mathbf{C}}$. In case of semiring $\mathbf{C}$, reduced power constructions $\mathcal{P}$ are particularly ugly because their power domains contain θ as a completely uncomparable point. The non-empty construction $\mathcal{P}^1$ then has the advantage to deliver power domains with least element whenever the ground domain has a least element.

For our general algebraic theory of power constructions however as developed in part II, the empty set cannot be dispensed with. Without the empty set, the characteristic semiring cannot be found since $\mathcal{P}^1\mathbf{1} = \{1\}$ holds without difference for all lower, upper, and convex power constructions.

Chapter 18

Lower power constructions

This chapter is concerned with the lower power constructions, i.e. the constructions with semiring $\mathbf{L} = \{0 < 1\}$. We start by considering the final lower construction $\mathcal{L}$. By Th. 15.1.1, it is given in terms of second order predicates. In the course of sections 18.1 and 18.2, we translate it into terms of *open grills*. The usual representation in terms of Scott closed sets is shown to be isomorphic in section 18.3. It provides enough structure to prove initiality of $\mathcal{L}$ (section 18.5). The domain theoretic properties are studied in the final section 18.6.

18.1 The final lower construction in terms of predicates

According to Th. 15.1.1, the final construction for semiring $\mathbf{L}$ is given in predicative form by $\mathcal{L}_f\mathbf{X} = [[\mathbf{X} \to \mathbf{L}] \overset{rlin}{\to} \mathbf{L}]$. By additivity of $\mathbf{L}$, every additive map between two $\mathbf{L}$-modules is linear, and we obtain $\mathcal{L}_f\mathbf{X} = [[\mathbf{X} \to \mathbf{L}] \overset{add}{\to} \mathbf{L}]$.

As a domain, $\mathbf{L}$ has the carrier $\mathbf{2} = \{\perp < \top\}$. Addition in $\mathbf{L}$ is least upper bound '$\sqcup$' and 0 is $\perp$. Hence, one may also write $\mathcal{L}_f\mathbf{X} = [[\mathbf{X} \to \mathbf{2}] \overset{\perp\sqcup}{\to} \mathbf{2}]$ indicating at the arrow what operations have to be preserved by the second order predicates.

In section 3.7, we showed that the domain $\Omega\mathbf{X}$ of open sets of $\mathbf{X}$ ordered by inclusion is isomorphic to the function space $[\mathbf{X} \to \mathbf{2}]$. The isomorphism is given by

$$O \in \Omega\mathbf{X} \;\mapsto\; p_O = \lambda x. \begin{cases} \top & \text{if } x \in O \\ \perp & \text{otherwise} \end{cases} \qquad \text{and}$$
$$p \in [\mathbf{X} \to \mathbf{2}] \;\mapsto\; O_p = p^{-1}[\top] = \{x \in \mathbf{X} \mid p\,x = \top\}$$

Its properties are summarized in the following table:

$\Omega\mathbf{X}$	$x \in O$	$\subseteq$	$\cup$	$\cap$	$\emptyset$	$\mathbf{X}$
$[\mathbf{X} \to \mathbf{2}]$	$p\,x = \top$	$\leq$	$\sqcup$	$\sqcap$	$\lambda x.\perp$	$\lambda x.\top$

Thus, $\Omega(\Omega\mathbf{X})$ is isomorphic to $[[\mathbf{X} \to \mathbf{2}] \to \mathbf{2}]$ by means of $\mathcal{E}\mathcal{O} = \lambda p.p_{\mathcal{O}}O_p$, or more elaborated

$$\mathcal{E}\mathcal{O} = \lambda p. \begin{cases} \top & \text{if } p^{-1}[\top] \in \mathcal{O} \\ \perp & \text{otherwise} \end{cases} \qquad \text{and its inverse} \qquad \varphi P = \{O \mid P\,p_O = \top\}.$$

Hence, $\mathcal{L}_f\mathbf{X}$ corresponds to a subset of $\Omega(\Omega\mathbf{X})$. $P\,(\lambda x.\bot) = \bot$ has to hold which translates into $\emptyset \notin \mathcal{G}$ where $\mathcal{G}$ is the open set of open sets corresponding to P. In addition, $P\,(p \sqcup q) = P\,p \sqcup P\,q$ has to hold, or equivalently $P\,(p \sqcup q) = \top$ iff $P\,p = \top$ or $P\,q = \top$. This translates into $O \cup O' \in \mathcal{G}$ iff $O \in \mathcal{G}$ or $O' \in \mathcal{G}$. The implication from right to left always holds since $\mathcal{G}$ is an upper set because it is open. Hence, only the implication from left to right matters. In analogy to a topological notion, we call open sets with these properties *grills*.

18.2 The lower power construction in terms of grills

An *open grill* of $\mathbf{X}$ is an open set $\mathcal{G}$ in $\Omega\mathbf{X}$ satisfying the two grill properties:

(1) $\emptyset$ is not in $\mathcal{G}$,

(2) Let O and O' be open sets in $\mathbf{X}$. If $O \cup O'$ is in $\mathcal{G}$, then at least one of O and O' is in $\mathcal{G}$.

Let $\mathcal{L}_\Gamma\mathbf{X}$ be the poset of open grills of $\mathbf{X}$ ordered by inclusion.

Theorem 18.2.1 $\mathcal{L}_f\mathbf{X}$ and $\mathcal{L}_\Gamma\mathbf{X}$ are isomorphic for all ground domains $\mathbf{X}$. The power operations for $\mathcal{L}_\Gamma\mathbf{X}$ are given by the following table:

$\mathcal{L}\mathbf{X}$	$[[\mathbf{X} \to \mathbf{L}] \overset{add}{\to} \mathbf{L}]$	$\mathcal{L}_\Gamma\mathbf{X}$
$A \leq B$	$\forall p : A\,p \leq B\,p$	$A \subseteq B$
$\bigsqcup \mathcal{D}$	$\lambda p.\, \bigsqcup_{D\in\mathcal{D}} Dp$	$\bigcup \mathcal{D}$
θ	$\lambda p.\, 0$	$\emptyset$
$A \uplus B$	$\lambda p.\, Ap + Bp$	$A \cup B$
$\{\!\|x\|\!\}$	$\lambda p.\, px$	$\mathcal{O}(x) = \{O \mid x \in O\}$
$ext\,f\,A$	$\lambda p.\, A\,(\lambda x.\, fxp)$	$\{O \mid \{x \mid O \in fx\} \in A\}$

Proof: Isomorphism and order are already known. One easily verifies that arbitrary unions of open grills are open grills again. Hence, $\bigcup \mathcal{D}$ is the lub of the directed set $\mathcal{D}$.

$\theta = \varphi\,(\lambda p.\, 0) = \{O \mid (\lambda p.\, \bot)\, p_O = \top\} = \emptyset$

$A \uplus B = \varphi\,(\lambda p. \mathcal{E} A\, p + \mathcal{E} B\, p) = \{O \mid \mathcal{E} A\, p_O \sqcup \mathcal{E} B\, p_O = \top\} = \{O \mid O \in A \text{ or } O \in B\} = A \cup B$

$\{\!|x|\!\} = \varphi\,(\lambda p.\, p\, x) = \{O \mid p_O\, x = \top\} = \{O \mid x \in O\} = \mathcal{O}(x)$

$$\begin{aligned} ext\,f\,A &= \varphi\,(\lambda p.\, \mathcal{E} A\,(\lambda x.\, \mathcal{E}(fx)p)) \\ &= \{O \mid \mathcal{E} A\,(\lambda x.\, \mathcal{E}(fx)p_O) = \top\} \\ &= \{O \mid (\lambda x.\, \mathcal{E}(fx)p_O)^{-1}[\top] \in A\} \\ &= \{O \mid \{x \mid \mathcal{E}(fx)p_O = \top\} \in A\} \\ &= \{O \mid \{x \mid O \in fx\} \in A\} \qquad \square \end{aligned}$$

Summarizing, we see that the lower power domain in terms of open grills is quite unhandy, and the realization of the power operations, in particular of extension, is quite complex. Fortunately, we need neither show the continuity of *ext f* nor the validity of the power axioms for $\mathcal{L}_\Gamma$ since the isomorphism gives this for free.

Next, we derive formulae for some of the derived operations in $\mathcal{L}_\Gamma$.

$\mathcal{L}\mathbf{X}$	$[[\mathbf{X} \to \mathbf{L}] \stackrel{add}{\to} \mathbf{L}]$	$\mathcal{L}_\Gamma\mathbf{X}$
$map\ f\ A$	$\lambda p.\, A\,(p \circ f)$	$\{O \mid f^{-1}[O] \in A\}$
$\mathcal{E}\,A\,p$	$A\,p$	$\mathcal{E}\,A\,p = 1$ iff $p^{-1}[1] \in A$
$ne\ A$	$A\,(\lambda x.\, 1)$	$ne\ A = 1$ iff $\mathbf{X} \in A$ iff $A \neq \theta$
$filter\ q\ A$	$\lambda p.\, A\,(q \cdot p)$	$\{O \mid q^{-1}[1] \cap O \in A\}$
$U\,A$	$\lambda p.\, A\,(\lambda a.\, a\,p)$	$\{O \mid \{x \in \mathcal{L}_\Gamma\mathbf{X} \mid O \in x\} \in A\}$
$A \overrightarrow{\times} B$	$\lambda p.\, A\,(\lambda a.\, B\,(\lambda b.\, p\,(a,\, b)))$	$\{O \mid \{a \mid \{b \mid (a,\, b) \in O\} \in B\} \in A\}$
$A \overleftarrow{\times} B$	$\lambda p.\, B\,(\lambda b.\, A\,(\lambda a.\, p\,(a,\, b)))$	$\{O \mid \{b \mid \{a \mid (a,\, b) \in O\} \in A\} \in B\}$

Proof:

$$\begin{aligned} map\ f\ A &= ext\,(\iota \circ f)\,A \\ &= \{O \mid \{x \mid O \in \mathcal{O}(f x)\} \in A\} \\ &= \{O \mid \{x \mid f x \in O\} \in A\} \\ &= \{O \mid f^{-1}[O] \in A\} \end{aligned}$$

Abusing notation, one could write the last set as $(f^{-1})^{-1}[A]$.

The statement about $\mathcal{E}$ holds because of the correspondence between $\Omega\mathbf{X}$ and $[\mathbf{X} \to \mathbf{L}]$. For ne, note that $(\lambda x.\, 1)^{-1}[1] = \mathbf{X}$. Obviously, $\mathbf{X}$ is not contained in $\theta = \emptyset$. Conversely, if $A \neq \emptyset$, then A contains $\mathbf{X}$ because A is an upper set.

$$\begin{aligned} filter\ q\ A &= ext\,(\lambda x.\, qx \cdot \{\!|x|\!\})\,A \\ &= \{O \mid \{x \mid O \in (qx \cdot \mathcal{O}(x))\} \in A\} \end{aligned}$$

Note that $qx \cdot \mathcal{O} = \emptyset$ if $qx = 0$, and $= \mathcal{O}$ if $qx = 1$.

$$\begin{aligned} &= \{O \mid \{x \mid qx = 1 \wedge O \in \mathcal{O}(x)\} \in A\} \\ &= \{O \mid \{x \mid qx = 1 \wedge x \in O\} \in A\} \\ &= \{O \mid q^{-1}[1] \cap O \in A\} \end{aligned}$$

$$\begin{aligned} U\,A &= ext\ id\ A \\ &= \{O \in \Omega\mathbf{X} \mid \{X \in \mathcal{L}_\Gamma\mathbf{X} \mid O \in X\} \in A\} \end{aligned}$$

$$\begin{aligned} A \overrightarrow{\times} B &= ext\,(\lambda a.\, ext\,(\lambda b.\, \{\!|(a,\, b)|\!\})\,B)\,A \\ &= \{O \mid \{a \mid O \in ext\,(\lambda b.\, \{\!|(a,\, b)|\!\})\,B\} \in A\} \\ &= \{O \mid \{a \mid O \in \{O' \mid \{b \mid O' \in \{\!|(a,\, b)|\!\}\} \in B\} \in A\} \\ &= \{O \mid \{a \mid O \in \{O' \mid \{b \mid (a,\, b) \in O'\} \in B\} \in A\} \\ &= \{O \mid \{a \mid \{b \mid (a,\, b) \in O\} \in B\} \in A\} \end{aligned}$$

$A \overleftarrow{\times} B$ is handled analogously. □

Looking at the formulae for $A \overrightarrow{\times} B$ and $A \overleftarrow{\times} B$, it is not obvious whether they always equal or are sometimes different. We shall however see below that $\mathcal{L}$ is symmetric.

18.3 The lower power construction in terms of closed sets

In this section, we show that the common lower power construction in terms of closed sets is isomorphic to $\mathcal{L}_\Gamma \mathbf{X}$.

Proposition 18.3.1

$\mathcal{L}_\Gamma \mathbf{X}$ is isomorphic to the poset $\mathcal{L}_C \mathbf{X}$ of closed sets of $\mathbf{X}$ ordered by inclusion.

Proof: Given a closed set C, let $\mathcal{G}(C)$ be the set of all open sets of $\mathbf{X}$ that meet C. $\mathcal{G}(C)$ obviously does not contain $\emptyset$. If an open set O meets C, then every superset of O does so; hence, $\mathcal{G}(C)$ is an upper set of open sets. If a union of open sets meets C, then at least one of its constituents meets C. Thus, $\mathcal{G}(C)$ is in particular open and satisfies the implication involving $O \cup O'$. Summarizing, $\mathcal{G}(C)$ is an open grill for all closed sets C of $\mathbf{X}$.

The larger a closed set is, the more open sets meet it. Hence, $C \subseteq C'$ implies $\mathcal{G}(C) \subseteq \mathcal{G}(C')$. Conversely, assume $\mathcal{G}(C) \subseteq \mathcal{G}(C')$ holds, and let c be a point in C. Then every open environment of c is in $\mathcal{G}(C')$, i.e. meets C'. Thus, c is in $\mathsf{cl}\, C' = C'$ by Prop. 4.2.2. Summarizing, $C \subseteq C'$ holds iff $\mathcal{G}(C) \subseteq \mathcal{G}(C')$.

Finally, we have to show that the mapping $\mathcal{G}(.)$ is surjective. Let $\mathcal{G}$ be an open grill. Let U be the union of all open sets of $\mathbf{X}$ that are *not* in $\mathcal{G}$, and let C be its complement. U is open as union of open sets, whence C is closed. We claim $\mathcal{G} = \mathcal{G}(C)$.

The set $\mathcal{S} = \{O \text{ open} \mid O \notin \mathcal{G}\}$ is directed: It is not empty since $\emptyset$ is in it, and $O, O' \notin \mathcal{G}$ implies $O \cup O' \notin \mathcal{G}$. If U, the union of the directed set $\mathcal{S}$, were in $\mathcal{G}$, then one of the members of $\mathcal{S}$ would be in $\mathcal{G}$ as $\mathcal{G}$ is open. Thus, U is not in $\mathcal{G}$.

If an open set O meets C, then O is not a subset of U. Thus, $O \cup U$ is a proper superset of U. Hence, it is in $\mathcal{G}$ since U is the union of all open sets not in $\mathcal{G}$. $O \cup U \in \mathcal{G}$ and $U \notin \mathcal{G}$ imply $O \in \mathcal{G}$.

If O does not meet C, then O is a subset of U. If O were in $\mathcal{G}$, then U were in $\mathcal{G}$, too, as $\mathcal{G}$ is an upper set. The last two paragraphs together show $\mathcal{G}(C) = \mathcal{G}$. □

After establishing this isomorphism, we translate the power operations into terms of closed sets.

Theorem 18.3.2

The final lower power construction $\mathcal{L}_f$ is isomorphic to the power construction $\mathcal{L}_C$ given by

(1) $\mathcal{L}_C \mathbf{X} = \{C \subseteq \mathbf{X} \mid C \text{ is Scott closed}\}$ ordered by inclusion '$\subseteq$',

(2) $\bigsqcup_{i\in I} A_i = \mathsf{cl} \bigcup_{i \in I} A_i$ where 'cl' denotes Scott closure,

(3) $\theta = \emptyset$,

(4) $A \uplus B = A \cup B$,

(5) $\{\!|x|\!\} = {\downarrow} x$,

(6) $\mathit{ext}\, f\, A = \bigsqcup f[A] = \mathsf{cl} \bigcup f[A] = \mathsf{cl} \bigcup_{a\in A} fa$.

Proof:

(1) The isomorphism is already known (Prop. 18.3.1).

(2) Because $\mathsf{cl}\,\bigcup_{i\in I} A_i$ is the least closed superset of $\bigcup_{i\in I} A_i$.

(3) $\mathcal{G}(\emptyset) = \{O \mid O \cap \emptyset \neq \emptyset\} = \emptyset = \theta_\Gamma$.

(4) An open set meets $A \cup B$ iff it meets A or meets B. Hence, $\mathcal{G}(A \cup B) = \mathcal{G}(A) \cup \mathcal{G}(B) = \mathcal{G}(A) \uplus_\Gamma \mathcal{G}(B)$.

(5) An open set meets $\downarrow x$ iff it contains x, since it is upper. Hence, $\mathcal{G}(\downarrow x) = \{O \mid x \in O\} = \{\!|x|\!\}_\Gamma$.

(6) By Prop. 4.2.3, an open set meets $\mathsf{cl}\,S$ iff it meets S. Hence,

$$\begin{aligned}
\mathcal{G}(\mathsf{cl}\bigcup f[A]) &= \{O \mid O \cap \bigcup f[A] \neq \emptyset\}\\
&= \{O \mid \exists a \in A : O \cap fa \neq \emptyset\}\\
&= \{O \mid \exists a \in A : O \in \mathcal{G}(fa)\}\\
&= \{O \mid A \cap \{a \mid O \in \mathcal{G}(fa)\} \neq \emptyset\}\\
&= \{O \mid \{a \mid O \in \mathcal{G}(fa)\} \in \mathcal{G}(A)\}\\
&= \mathit{ext}_\Gamma\ (\mathcal{G}(.) \circ f)\ (\mathcal{G}(A))
\end{aligned}$$

Here, we have to make sure that $\{a \mid O \in \mathcal{G}(fa)\}$ is open. It is the inverse image by f of the open set $\{C' \in \mathcal{L}_C\mathbf{Y} \mid O \in \mathcal{G}(C')\} = \{C' \mid O \cap C' \neq \emptyset\}$.

These equations show that $\mathcal{G}(.)$ becomes a power isomorphism if the operations for closed sets are chosen as in the theorem. □

18.4 Derived operations

In this section, explicit formulae in terms of closed sets are presented for the derived power operations. These formulae all look quite natural.

- $\theta = \emptyset$,
- $A \uplus B = A \cup B$,
- $\{\!|x|\!\} = \downarrow x$,
- Combining union and singleton, one obtains $\{\!|x_1, \ldots, x_n|\!\} = \downarrow\{x_1, \ldots, x_n\}$.
- $\mathit{ext}\ f\ A = \mathsf{cl}\,\bigcup_{a\in A} fa$.
- $$\begin{aligned}
\mathit{map}\ f\ A &= \mathit{ext}\ (\lambda x.\ \{\!|fx|\!\})\ A\\
&= \mathsf{cl}\,\textstyle\bigcup_{a\in A} \downarrow(fa)\\
&= \mathsf{cl}\downarrow f[A] && \text{by Prop. 2.2.8}\\
&= \mathsf{cl}\, f[A] && \text{by Prop. 3.6.3 (1)}
\end{aligned}$$
- The big union is given by
  $UA = \mathit{ext}\ (\lambda x.\ x)\ A = \mathsf{cl}\,\bigcup_{a\in A} a = \mathsf{cl}\,\bigcup A$
- The formal Cartesian product coincides with the mathematical one:
  $$\begin{aligned}
A \overrightarrow{\times} B &= \mathit{ext}\ (\lambda a.\ \mathit{ext}\ (\lambda b.\ \{\!|(a, b)|\!\})\ B)\ A\\
&= \textstyle\bigsqcup_{a\in A} \bigsqcup_{b\in B} \downarrow\{(a, b)\}\\
&= \textstyle\bigsqcup_{c\in A\times B} \downarrow\{c\}\\
&= \mathsf{cl}\,\textstyle\bigcup_{c\in A\times B} \downarrow c\\
&= \mathsf{cl}\downarrow(A \times B)\ =\ A \times B
\end{aligned}$$

The very last equation holds because the product of closed sets is closed again by Prop. 4.8.1. $A \overleftarrow{\times} B$ also results in $A \times B$, whence $\mathcal{L}$ is symmetric.

- External left and right product coincide and are given by $0 \cdot A = \theta$ and $1 \cdot A = A$.
- For the filter operation, let $p : [\mathbf{X} \to \mathbf{L}]$ be a predicate.

$$\begin{aligned} \mathit{filter}\, p\, A &= \mathit{ext}\, (\lambda a.\, p\, a \cdot \{\!|a|\!\})\, A \;=\; \mathsf{cl} \textstyle\bigcup_{a \in A} (p\, a \cdot {\downarrow} a) \\ &= \mathsf{cl} \bigcup \{{\downarrow} a \mid a \in A,\, p\, a = 1\} \;=\; \mathsf{cl}\, \{a \in A \mid p\, a = 1\} \;=\; \mathsf{cl}\, (A \cap p^{-1}[1]) \end{aligned}$$

By continuity of p, the set $p^{-1}[1]$ is open. Hence, the closure operator cannot be omitted.

- Let $p : [\mathbf{X} \to \mathbf{L}]$ be a predicate. Then

$$\mathcal{E}\, A\, p \;=\; \mathit{ext}\, p\, A \;=\; \mathsf{cl} \bigcup_{a \in A} p\, a$$

Identifying $\emptyset$ with 0 and $\{\diamond\}$ with 1, the set $\bigcup_{a \in A} p\, a$ is empty iff $p\, a = 0$ for all a in A, and it is $\{\diamond\}$ iff there is a in A with $p\, a = 1$. The closure operator is not needed since both $\emptyset$ and $\{\diamond\}$ are closed. Hence,

$$\mathcal{E}\, A\, p = \begin{cases} 1 & \text{if } \exists a \in A : p\, a = 1 \\ 0 & \text{otherwise} \end{cases}$$

This result again justifies the view of $\mathcal{E}$ as existential quantification.

- The non-empty predicate also is named properly in this case:

$$\mathit{ne}\ A = \mathcal{E}\, A\, (\lambda x.\, 1) = \begin{cases} 1 & \text{if } A \neq \emptyset \\ 0 & \text{otherwise} \end{cases}$$

18.5 Initiality of $\mathcal{L}$

Our next goal is to prove that the final construction for semiring $\mathbf{L}$ is also initial. To this end, we prove that the lower power domains in terms of closed sets are free $\mathbf{L}$-$\mathbf{X}$-modules.

Theorem 18.5.1 The final power construction for semiring $\mathbf{L}$ is also initial.

Proof: Let $\mathcal{L}\mathbf{X}$ be the lower power domain over $\mathbf{X}$ in terms of closed sets. As a power domain, it is an $\mathbf{L}$-module.

Let $f : [\mathbf{X} \to M]$ be a morphism from $\mathbf{X}$ to some $\mathbf{L}$-module M. Because M is a complete domain, one may define $\hat{f} A = \bigsqcup f[A]$ for every subset A of $\mathbf{X}$. We show three main properties of $\hat{f}$.

(1) $\hat{f}({\downarrow} x) = f x$ follows from the monotonicity of f.

(2) $\hat{f}(\bigcup_{i \in I} A_i) = \bigsqcup_{i \in I} \hat{f} A_i$
Proof: $\hat{f}(\bigcup_{i \in I} A_i) \;=\; \bigsqcup f[\bigcup_{i \in I} A_i] \;=\; \bigsqcup \bigcup_{i \in I} f[A_i] \;=\; \bigsqcup_{i \in I} \bigsqcup f[A_i] \;=\; \bigsqcup_{i \in I} \hat{f} A_i$
Hence, $\hat{f}$ is in particular linear and monotonic w.r.t. '$\subseteq$'.

(3) $\hat{f}(\mathsf{cl}\, A) = \hat{f} A$ for all $A \subseteq \mathbf{X}$.

'$\geq$' follows from $\mathsf{cl}\, A \supseteq A$. For the opposite direction, let $y = \hat{f} A$, and assume there is $x \in \mathsf{cl}\, A$ with $f x \not\leq y$. Then $f x$ is in the open set $O = M \setminus {\downarrow} y$. By continuity of f, $O' = f^{-1}[O]$ is an open environment of x. Since x is in the closure of A, O' meets A, i.e.

there is a in A with $fa \in O$, i.e. $fa \not\leq y = \bigsqcup f[A]$. This contradiction implies $fx \leq y$ for all $x \in \mathsf{cl}\, A$, whence $\hat{f}(\mathsf{cl}\, A) \leq y$.

Now, we restrict $\hat{f}$ to $\bar{f} : \mathcal{L}\mathbf{X} \to M$. $\bar{f}$ is still linear and satisfies $\bar{f}\{\!|x|\!\} = fx$ by (1). It is continuous because

$$\bar{f}\,(\bigsqcup A_i) = \hat{f}(\mathsf{cl} \bigcup A_i) \overset{(3)}{=} \hat{f}(\bigcup A_i) \overset{(2)}{=} \bigsqcup \hat{f} A_i$$

Let $F_1, F_2 : [\mathcal{L}\mathbf{X} \to M]$ be two linear extensions of a morphism $f : [\mathbf{X} \to M]$. Then F_1 and F_2 coincide for cones $\downarrow x$. By linearity, they thus also coincide for all sets $\downarrow F$ where F is finite. If C is an arbitrary closed set, then $\mathcal{D} = \{\downarrow F \mid F \text{ is a finite subset of } C\}$ is directed and has union (and thus limit) C. Hence, $F_1 = F_2$ holds by continuity. $\square$

18.6 Domain-theoretic properties

We already know that $\mathcal{L}\mathbf{X}$ is a complete domain no matter what $\mathbf{X}$ is, and that $\mathcal{L}$ preserves finiteness, algebraicity, and continuity. Hence, it also preserves all classes between algebraic and complete algebraic domains, e.g. the class of bounded complete algebraic domains and the class of finitely algebraic domains. In section 20.5, we show that $\mathcal{L}$ also preserves multi-algebraicity and multi-continuity.

Although we know that $\mathcal{L}\mathbf{X}$ is algebraic whenever $\mathbf{X}$ is, we do not know the base. It is provided by the following theorem:

Theorem 18.6.1 Let $\mathbf{X}$ be an algebraic domain with base $\mathbf{X}^0$. Then $\mathcal{L}\mathbf{X}$ is algebraic with base $\{\downarrow F \mid F \subseteq_f \mathbf{X}^0\}$.

Proof: First, we show that the sets $\downarrow F$ with $F \subseteq_f \mathbf{X}^0$ are isolated. They are closed by Prop. 3.6.3 (2).

Let F be a finite subset of $\mathbf{X}^0$, and let $\downarrow F \subseteq \bigsqcup_{i \in I} C_i = \mathsf{cl} \bigcup_{i \in I} C_i$. Since F consists of isolated points, $F \subseteq \bigcup_{i \in I} C_i$ follows by Prop. 6.1.3. Since F is finite, there is k in I with $F \subseteq C_k$ by Prop. 6.3.3. C_k being lower implies $\downarrow F \subseteq C_k$.

Now, we show that every closed set is the lub of a directed set of isolated closed sets. For given closed set C, let

$$\mathcal{D} = \{\downarrow F \mid F \subseteq_f C \cap \mathbf{X}^0\}$$

We claim $C = \mathsf{cl} \bigcup \mathcal{D}$.

The set $\mathcal{D}$ is directed because of $\downarrow F_1 \cup \downarrow F_2 = \downarrow (F_1 \cup F_2)$. The inclusion '$\supseteq$' certainly holds, because $F \subseteq C$ and C is closed. For the opposite inclusion, let x be a point of C. Since $\mathbf{X}$ is algebraic, there is a directed set $D' \subseteq \mathbf{X}$ with $x = \bigsqcup D'$. For every d in D', $\{d\} \subseteq_f C \cap \mathbf{X}^0$ holds, whence $D' \subseteq \bigcup \mathcal{D} \subseteq \mathsf{cl} \bigcup \mathcal{D}$. Because $\mathsf{cl} \bigcup \mathcal{D}$ is closed, x is a point of it. $\square$

Chapter 19

Upper power constructions

This chapter is concerned with the upper power constructions, i.e. the constructions with semiring $\mathbf{U} = \{1 < 0\}$. Although this semiring looks as simple as the lower semiring $\mathbf{L} = \{0 < 1\}$, the situation here is much more complex. The theory is considerably harder than in the lower case, and nevertheless produces weaker results. For instance, we do not know whether initial and final upper power construction coincide.[1] There is some evidence that they do not; we shall provide hints at the appropriate places.

Again, we start by considering the final $\mathbf{U}$-construction. Its original form in terms of second order predicates is translated into terms of open filters in section 19.1. If the ground domain $\mathbf{X}$ is sober, then $\mathcal{U}\mathbf{X}$ may be translated further into a representation by compact upper sets (section 19.2). The domain-theoretic properties of these power domains are studied in section 19.3. In section 19.4, we attempt to prove the initiality of the final construction $\mathcal{U}$. In doing so, we point out why we did not succeed in finding a general proof such that a proof for continuous ground domains is the only result.

19.1 The upper construction $\mathcal{U}_f$ in terms of open filters

According to Th. 15.1.1, the final construction for semiring $\mathbf{U}$ is given in predicative form by $\mathcal{U}_f\mathbf{X} = [[\mathbf{X} \to \mathbf{U}] \stackrel{rlin}{\to} \mathbf{U}]$. By additivity of $\mathbf{U}$, every additive map between two $\mathbf{U}$-modules is linear, and we obtain $\quad \mathcal{U}_f\mathbf{X} = [[\mathbf{X} \to \mathbf{U}] \stackrel{add}{\to} \mathbf{U}]$.

The carrier domain of $\mathbf{U}$ is $\mathbf{2} = \{\bot < \top\}$. It coincides with that of $\mathbf{L}$. In contrast to $\mathbf{L}$, addition in $\mathbf{U}$ is '$\sqcap$' and 0 is $\top$. Hence, one may also write $\mathcal{U}_f\mathbf{X} = [[\mathbf{X} \to \mathbf{2}] \stackrel{\top\sqcap}{\to} \mathbf{2}]$ indicating at the arrow what operations have to be preserved by the second order predicates.

Appealing again to the correspondence between predicates and open sets, $\mathcal{U}_f\mathbf{X}$ is isomorphic to a subset of $\Omega(\Omega\mathbf{X})$. $P\,(\lambda x.\,\top) = \top$ in $[[\mathbf{X} \to \mathbf{U}] \to \mathbf{U}]$ corresponds to $\mathbf{X} \in \mathcal{F}$ in $\Omega(\Omega\mathbf{X})$ where $\mathcal{F}$ is the set of open sets corresponding to P. In addition, $P\,(p \sqcap q) = P\,p \sqcap P\,q$ has to hold, or equivalently $P\,(p \sqcap q) = \top$ iff $P\,p = \top$ and $P\,q = \top$. This translates into $O \cap O' \in \mathcal{F}$ iff $O \in \mathcal{F}$ and $O' \in \mathcal{F}$. The implication from left to right always holds since $\mathcal{F}$ is an upper set because it is open. Hence, only the opposite implication matters. Comparing

[1] Meanwhile, they are shown to be different.

these two properties of $\mathcal{F}$ with Def. 8.6.1, one sees that $\mathcal{U}_f\mathbf{X}$ corresponds to the set of *open filters* of $\mathbf{X}$.

Open filters are open $\geq$-directed sets in the domain $(\Omega\mathbf{X}, \subseteq)$ of open sets of $\mathbf{X}$. The poset of all open filters of $\mathbf{X}$ ordered by inclusion '$\subseteq$' is denoted by $\mathcal{U}_\Phi\mathbf{X}$.

Arbitrary unions of open sets are open, and directed unions of directed sets are directed by Prop. 3.1.5. Hence, the union of a directed set of open filters is an open filter. This makes $\mathcal{U}_\Phi\mathbf{X}$ a domain with limits given by union. The isomorphism between $\mathcal{U}_f\mathbf{X}$ and $\mathcal{U}_\Phi\mathbf{X}$ is given by

$$\mathcal{E}\mathcal{O} = \lambda p.\begin{cases} \top & \text{if } p^{-1}[\top] \in \mathcal{O} \\ \bot & \text{otherwise} \end{cases} \qquad \text{and its inverse} \qquad \varphi P = \{O \mid P\, p_O = \top\}.$$

as pointed out in section 18.1.

Theorem 19.1.1 $\mathcal{U}_f\mathbf{X}$ and $\mathcal{U}_\Phi\mathbf{X}$ are isomorphic for all ground domains $\mathbf{X}$. The power operations for $\mathcal{U}_\Phi\mathbf{X}$ are given by the following table:

$\mathcal{U}\mathbf{X}$	$[[\mathbf{X} \to \mathbf{U}] \stackrel{add}{\to} \mathbf{U}]$	$\mathcal{U}_\Phi\mathbf{X}$
$A \leq B$	$\forall p : A\, p \leq B\, p$	$A \subseteq B$
$\bigsqcup \mathcal{D}$	$\lambda p.\, \bigsqcup_{D\in\mathcal{D}} Dp$	$\bigcup \mathcal{D}$
θ	$\lambda p.\, 0$	$\Omega\mathbf{X}$
$A \uplus B$	$\lambda p.\, Ap + Bp$	$A \cap B$
$\{\!\|x\|\!\}$	$\lambda p.\, p\, x$	$\mathcal{O}(x) = \{O \mid x \in O\}$
$ext\ f\ A$	$\lambda p.\, A\,(\lambda x.\, fxp)$	$\{O \mid \{x \mid O \in fx\} \in A\}$

Proof: The facts about order and limits are already known.

$\theta = \varphi\,(\lambda p.\, 0) = \{O \mid (\lambda p.\, \top)\, p_O = \top\} = \Omega\mathbf{X}$.

$A \uplus B = \varphi\,(\lambda p. \mathcal{E}A\, p + \mathcal{E}B\, p) = \{O \mid \mathcal{E}A\, p_O \sqcap \mathcal{E}B\, p_O = \top\} = \{O \mid O \in A \text{ and } O \in B\} = A \cap B$

The formulae for the operations $\{\!\|.\|\!\}$ and *ext* and their proofs look exactly as those in Th. 18.2.1. □

Next, we derive formulae for some of the derived operation in $\mathcal{U}_\Phi$.

$\mathcal{U}\mathbf{X}$	$[[\mathbf{X} \to \mathbf{U}] \stackrel{add}{\to} \mathbf{U}]$	$\mathcal{U}_\Phi\mathbf{X}$
$map\ f\ A$	$\lambda p.\, A\,(p \circ f)$	$\{O \mid f^{-1}[O] \in A\}$
$\mathcal{E}A\, p$	$A\, p$	$\mathcal{E}A\, p = 0$ iff $p^{-1}[0] \in A$
$ne\ A$	$A\,(\lambda x.\, 1)$	$ne\ A = 0$ iff $\emptyset \in A$ iff $A = \theta$
$filter\ q\ A$	$\lambda p.\, A\,(q \cdot p)$	$\{O \mid q^{-1}[0] \cup O \in A\}$
$U\,A$	$\lambda p.\, A\,(\lambda a.\, a\, p)$	$\{O \mid \{x \in \mathcal{U}_\Phi\mathbf{X} \mid O \in x\} \in A\}$
$A \overrightarrow{\times} B$	$\lambda p.\, A\,(\lambda a.\, B\,(\lambda b.\, p\,(a, b)))$	$\{O \mid \{a \mid \{b \mid (a, b) \in O\} \in B\} \in A\}$
$A \overleftarrow{\times} B$	$\lambda p.\, B\,(\lambda b.\, A\,(\lambda a.\, p\,(a, b)))$	$\{O \mid \{b \mid \{a \mid (a, b) \in O\} \in A\} \in B\}$

Proof: The formulae for *map*, U, '$\overrightarrow{\times}$', and '$\overleftarrow{\times}$' equal those in $\mathcal{L}_\Gamma$. Their proofs are similar to those in section 18.2.

The statement about $\mathcal{E}$ holds because of the correspondence between $\Omega\mathbf{X}$ and $[\mathbf{X} \to \mathbf{U}]$. For *ne*, note that $(\lambda x.\, 1)^{-1}[0] = \emptyset$. Obviously, $\emptyset$ is contained in $\theta = \Omega\mathbf{X}$. Conversely, if $\emptyset$ is in A, then all open sets O with $O \supseteq \emptyset$ are in A, and these are all.

$$\begin{aligned} \mathit{filter}\; q\, A &= \mathit{ext}\,(\lambda x.\, qx \cdot \{\!|x|\!\})\, A \\ &= \{O \mid \{x \mid O \in (qx \cdot \mathcal{O}(x))\} \in A\} \end{aligned}$$

Note that $qx \cdot \mathcal{O} = \Omega\mathbf{X}$ if $qx = 0$, and $= \mathcal{O}$ if $qx = 1$.

$$\begin{aligned} &= \{O \mid \{x \mid qx = 0 \vee O \in \mathcal{O}(x)\} \in A\} \\ &= \{O \mid \{x \mid qx = 0 \vee x \in O\} \in A\} \\ &= \{O \mid q^{-1}[0] \cup O \in A\} \qquad \Box \end{aligned}$$

Looking at the formulae for $A \overrightarrow{\times} B$ and $A \overleftarrow{\times} B$, it is not obvious whether they always equal or are sometimes different.

Problem: Is the power construction $\mathcal{U}_\Phi = \mathcal{U}_f$ symmetric?

Summarizing, we see that the upper power domain $\mathcal{U}_\Phi\mathbf{X}$ is quite unhandy, and the realization of the power operations, in particular of extension, is quite complex. In the next sections, we demonstrate that one can do better for the large class of sober domains. The final upper power domain may be represented in terms of compact sets for this class.

19.2 The upper construction $\mathcal{U}_K$ in terms of compact upper sets

Following [Smy83], we define the upper power domain in terms of compact upper sets in this section. Unfortunately, this approach does not work out for all domains. The class of allowed domains however is quite large; it contains all sober domains and hence all continuous ones.

For an arbitrary domain $\mathbf{X}$, let $\mathcal{U}_K\mathbf{X}$ be the set of all compact upper sets of $\mathbf{X}$. For every compact set K, the set of open environments of K is an open filter by Prop. 8.6.3. Thus, there is a mapping $\mathcal{O}(.) : \mathcal{U}_K\mathbf{X} \to \mathcal{U}_\Phi\mathbf{X}$. By Prop. 4.4.4, for every two compact upper sets K and K', $K \supseteq K'$ is equivalent to $\mathcal{O}(K) \subseteq \mathcal{O}(K')$. Since we ordered $\mathcal{U}_\Phi\mathbf{X}$ by '$\subseteq$', we have to order $\mathcal{U}_K\mathbf{X}$ by '$\supseteq$'. Then we obtain that $\mathcal{O}(.) : \mathcal{U}_K\mathbf{X} \to \mathcal{U}_\Phi\mathbf{X}$ has the property $K \leq K'$ iff $\mathcal{O}(K) \leq \mathcal{O}(K')$. Unfortunately, $\mathcal{O}(.)$ is not always surjective.

Proposition 19.2.1 $\mathcal{U}_K\mathbf{X}$ and $\mathcal{U}_\Phi\mathbf{X}$ are isomorphic by $\mathcal{O}(.)$ iff $\mathbf{X}$ is sober.

Proof: By Th. 8.7.1, $\mathbf{X}$ is sober iff every open filter is comprising.

If $\mathcal{U}_K\mathbf{X}$ and $\mathcal{U}_\Phi\mathbf{X}$ are isomorphic by $\mathcal{O}(.)$, then every open filter F equals $\mathcal{O}(K)$ for some compact upper set K. Hence, it is comprising.

Conversely, if every open filter is comprising, then every open filter $\mathcal{O}$ equals $\mathcal{O}(\bigcap \mathcal{O})$, and $\bigcap \mathcal{O}$ is a compact upper set by Prop. 8.6.3. $\Box$

Since there are non-sober domains, $\mathcal{U}_K\mathbf{X}$ and $\mathcal{U}_\Phi\mathbf{X}$ are not isomorphic for all domains $\mathbf{X}$.

To analyze the topology of $\mathcal{U}_K\mathbf{X}$, we employ the following Lemma.

Lemma 19.2.2 Let $\mathbf{X}$ be sober and O an open set of $\mathbf{X}$. Then $\mathcal{K}(O) = \{K \in \mathcal{U}_K\mathbf{X} \mid K \subseteq O\}$ is open in $\mathcal{U}_K\mathbf{X}$.

Proof: $\mathcal{K}(O)$ is an upper set since $K \geq K'$ implies $K \subseteq K' \subseteq O$.

By Th. 8.9.2, all sober domains have property K-RD. If $\mathcal{D}$ is a $\supseteq$-directed set of compact upper sets with $\bigsqcup \mathcal{D}$ in $\mathcal{K}(O)$, i.e. $\bigsqcup \mathcal{D} \subseteq O$, then $\bigcap \mathcal{D} \subseteq O$ follows since $K \supseteq \bigsqcup \mathcal{D}$ for all K in $\mathcal{D}$. By property K-RD, there is some K in $\mathcal{D}$ such that $K \subseteq O$, i.e. $K \in \mathcal{K}(O)$. □

In the case of sober ground domains, we may translate the power operations of $\mathcal{U}_\Phi$ into the language of compact sets.

Theorem 19.2.3 Let $\mathbf{X}$ and $\mathbf{Y}$ be two sober ground domains. Then the power domain $\mathcal{U}_K\mathbf{X}$ described below is isomorphic to the final $\mathbf{U}$-powerdomain over $\mathbf{X}$.

(1) $\mathcal{U}_K\mathbf{X}$ is the set of all compact upper sets of $\mathbf{X}$ ordered by $K \leq K'$ iff $K \supseteq K'$

(2) $\bigsqcup \mathcal{K} = \bigcap \mathcal{K}$ for directed sets $\mathcal{K}$ in $\mathcal{U}_K\mathbf{X}$.

(3) $\theta = \emptyset$

(4) $A \uplus B = A \cup B$

(5) $\{\!|x|\!\} = \uparrow x$ for all $x \in \mathbf{X}$.

(6) If $f : [\mathbf{X} \to \mathcal{U}_K\mathbf{Y}]$ is continuous and A is in $\mathcal{U}_K\mathbf{X}$, then $ext\, f\, A = \bigcup_{a \in A} fa = \bigcup f[A]$.

All these operations are well defined and continuous.

Proof:

(1) follows from the discussion at the beginning of this section.

(2) Every sober domain has property K-RD by Th. 8.9.2. In domains with property K-RD, intersections of $\supseteq$-directed sets of compact upper sets are compact upper sets by Prop. 8.9.4. Hence, $\bigcap \mathcal{K}$ is in $\mathcal{U}_K\mathbf{X}$. It is the supremum of $\mathcal{K}$ by set-theoretic arguments.

(3) The empty set is a compact upper set. $\mathcal{O}(\emptyset) = \Omega\mathbf{X} = \theta_\Phi$ holds.

(4) $A \cup B$ is compact by Prop. 4.6.3. If O is an open set in $\mathcal{O}(A \cup B)$, then $O \supseteq A \cup B \supseteq A, B$ holds, whence O is in $\mathcal{O}(A) \cap \mathcal{O}(B)$. Conversely, if O is in the intersection, then $O \supseteq A$ and $O \supseteq B$ implies $O \supseteq A \cup B$. Thus, we get $\mathcal{O}(A \cup B) = \mathcal{O}(A) \cap \mathcal{O}(B) = \mathcal{O}(A) \uplus \mathcal{O}(B)$ by Th. 19.1.1.

(5) $\uparrow x$ is a compact upper set, and $\mathcal{O}(\uparrow x) = \mathcal{O}(x) = \{\!|x|\!\}_\Phi$ holds.

(6) Let $f : [\mathbf{X} \to \mathcal{U}_K\mathbf{Y}]$ be continuous and A in $\mathcal{U}_K\mathbf{X}$.

$$\begin{aligned}
\mathcal{O}(\textstyle\bigcup f[A]) &= \{O \in \Omega\mathbf{Y} \mid \textstyle\bigcup_{a \in A} fa \subseteq O\} \\
&= \{O \in \Omega\mathbf{Y} \mid \forall a \in A : fa \subseteq O\} \\
&= \{O \in \Omega\mathbf{Y} \mid A \subseteq \{x \mid fx \subseteq O\}\} \\
&= \{O \in \Omega\mathbf{Y} \mid \{x \in \mathbf{X} \mid O \in \mathcal{O}(fx)\} \in \mathcal{O}(A)\}
\end{aligned}$$

The set $\{x \mid fx \subseteq O\}$ is open because it is the inverse image of the open set $\mathcal{K}(O)$ by the continuous function f.

The set $\mathcal{O}(\bigcup f[A])$ is an open filter since we just demonstrated that it may be represented as $ext_\Phi\, (\mathcal{O}(.) \circ f)\, (\mathcal{O}(A))$. Hence, $\bigcup f[A]$ is a compact upper set by Prop. 8.6.3. □

A direct topological proof of the compactness of $\bigcup f[A]$ is also possible, but would be much more tedious. The same remark is valid for a direct proof of the continuity of $ext\, f : \mathcal{U}_K\mathbf{X} \to \mathcal{U}_K\mathbf{Y}$. Both proofs are unnecessary because one may use that $ext\, f : \mathcal{U}_\Phi\mathbf{X} \to \mathcal{U}_\Phi\mathbf{Y}$ is well-defined and continuous. These facts are in turn inherited from the well-definedness and continuity of the operations in the final power construction defined in terms of functions of higher order.

The advantage of $\mathcal{U}_K$ over $\mathcal{U}_\Phi$ is that the operations look simpler. The main reason is that no longer sets of sets are considered, but simply sets of ground domain points.

Hence, the derived operations also become simpler:

- If $\mathbf{X}$ is sober such that $\mathcal{U}_K\mathbf{X}$ is sober again, then $U\,A = ext\; id\; A = \bigcup id[A] = \bigcup A$.

  Thus, $U\,A = \bigcup A$.

- Let $f : [\mathbf{X} \to \mathbf{Y}]$ where both $\mathbf{X}$ and $\mathbf{Y}$ are sober. Then $map\, f\, A = ext\,(\iota \circ f)\, A = \bigcup_{a \in A} \{\!|f a|\!\} = \bigcup_{a\in A} \uparrow f a = \uparrow \bigcup_{a \in A}\{f a\} = \uparrow f[A]$

  Thus, $map\, f\, A = \uparrow f[A]$.

- For $p : [\mathbf{X} \to \mathbf{U}]$ where $\mathbf{X}$ is sober, $ex\, p\, A = 1$ iff $\bigcup_{a\in A} f a = \{\!|\diamond|\!\} = \{\diamond\}$ iff $\exists a \in A : f a = \{\diamond\}$.

  Thus, $ex\, p\, A = 1$ iff there is a in A with $p\, a = 1$.

- Hence, $ne\, A = 1$ iff there is a in A with $(\lambda x.\, 1)\, a = 1$. This is equivalent to $A \neq \emptyset$.

- Let $p : [\mathbf{X} \to \mathbf{U}]$ where $\mathbf{X}$ is sober. Then *filter* $p\, A = ext\,(\lambda a.\, p\, a \cdot \{\!|a|\!\})\, A = \bigcup_{a\in A} (p\, a \cdot \uparrow a) = \bigcup_{a \in A,\, p\, a = 1} \uparrow a = \uparrow\{a \in A \mid p\, a = 1\} = \uparrow(A \cap p^{-1}[1])$

  Thus, *filter* $p\, A = \uparrow\{a \in A \mid p\, a = 1\}$.

- Let $\star : [\mathbf{X} \times \mathbf{Y} \to \mathcal{U}_K\mathbf{Z}]$, where all three of $\mathbf{X}$, $\mathbf{Y}$, and $\mathbf{Z}$ are sober. Then

$$\begin{aligned} A \overrightarrow{\star} B &= ext\,(\lambda a.\, ext\,(\lambda b.\, a \star b)\, B)\, A \\ &= \textstyle\bigcup_{a\in A} ext\,(\lambda b.\, a \star b)\, B \\ &= \textstyle\bigcup_{a \in A} \bigcup_{b \in B} a \star b \\ &= \textstyle\bigcup_{(a,\, b) \in A \times B} a \star b \\ &= \textstyle\bigcup \star[A \times B] \end{aligned}$$

  $A \overleftarrow{\star} B$ gives the same result, whence the final $\mathcal{U}$-construction becomes symmetric when restricted to the class of sober domains.

- If $\mathbf{X}$ and $\mathbf{Y}$ are sober such that $\mathbf{X} \times \mathbf{Y}$ is sober too, then $A \overrightarrow{\times} B = A \overleftarrow{\times} B = \bigcup_{(a,\, b) \in A \times B} \{\!|(a,\, b)|\!\} = \uparrow(A \times B) = A \times B$ since the product of upper sets is an upper set.

The upper power construction in terms of compact upper sets $\mathcal{U}_K$ admits considerably simpler definitions of the basic and derived power operations than the final power construction $\mathcal{U}_\Phi$. This simplicity allows to prove the symmetry of $\mathcal{U}_K$ in contrast to $\mathcal{U}_\Phi$ where we did not succeed in proving or disproving symmetry.

Concerning sobriety and $\mathcal{U}_K$, we were not able to answer the following question:

Problem 14 If $\mathbf{X}$ is sober, is then $\mathcal{U}_K\mathbf{X}$ sober again?

19.3 Domain-theoretic properties of $\mathcal{U}_f$ and $\mathcal{U}_K$

In this section, we investigate how the final **U**-construction $\mathcal{U}$ copes with algebraicity and continuity.

Theorem 19.3.1 If $\mathbf{X}$ is multi-algebraic, then $\mathcal{U}_K\mathbf{X}$ is algebraic. Its base is the set of all finitary open sets of $\mathbf{X}$.

Proof: Finitary open sets A of $\mathbf{X}$ are isolated points in $\mathcal{U}_K\mathbf{X}$ since their upper cone $\{K \mid K \geq A\} = \mathcal{K}(A)$ is open in $\mathcal{U}_K\mathbf{X}$ by Lemma 19.2.2.

Let K be a compact upper set and let $\mathcal{D} = \mathcal{FO}(K)$ be the set of all finitary open supersets of K. By Th. 8.3.5, $\mathcal{D}$ is directed in $\mathcal{U}_K\mathbf{X}$ and its intersection (limit) is K. □

From the theorem, we may derive:

Corollary 19.3.2 If $\mathbf{X}$ is algebraic, then $\mathcal{U}_K\mathbf{X}$ is algebraic with base $\{\uparrow F \mid F \subseteq_f \mathbf{X}^0\}$.

Proof: By Th. 6.2.4, the open sets in an algebraic domain $\mathbf{X}$ are just the sets $\uparrow S$ where $S \subseteq \mathbf{X}^0$. Hence, the finitary open sets are just the sets $\uparrow F$ where F is a finite subset of the base. □

For continuity, a similar theorem holds.

Theorem 19.3.3 If $\mathbf{X}$ is multi-continuous, then $\mathcal{U}_K\mathbf{X}$ is reduced and continuous.

Proof: Let $K \in \mathcal{O}$ where $\mathcal{O}$ is open in $\mathcal{U}_K\mathbf{X}$. If $\mathcal{F}(K)$ is the set of finitary upper environments of K, then we know from Th. 8.3.2 that $\mathcal{F}(K)$ is $\supseteq$-directed with intersection (limit) K. Since $\mathcal{O}$ is open, there is some F in $\mathcal{F}(K)$ such that $F \in \mathcal{O}$. F in $\mathcal{F}(K)$ implies $K \subseteq O \subseteq F$ for some open set O of $\mathbf{X}$. By Lemma 19.2.2, $\mathcal{K}(O)$ is open in $\mathcal{U}_K\mathbf{X}$.

First, K is in $\mathcal{K}(O)$. Second, $B \in \mathcal{K}(O)$ implies $B \subseteq O \subseteq F$, i.e. $B \geq F$. Thus, $\mathcal{K}(O) \subseteq \uparrow\{F\}$ holds. Third, $F \in \mathcal{O}$ implies $\uparrow\{F\} \subseteq \mathcal{O}$. Summarizing, we obtain $K \in \mathcal{K}(O) \subseteq \uparrow\{F\} \subseteq \mathcal{O}$ as required for continuity.

As mentioned above, every K in $\mathcal{U}_K\mathbf{X}$ is the limit of a directed set of finitary upper sets. A finitary upper set $\uparrow\{e_1, \ldots, e_n\}$ is in the core of $\mathcal{U}_K\mathbf{X}$ since it equals $\{\!|e_1, \ldots, e_n|\!\}$. Thus, every K in $\mathcal{U}_K\mathbf{X}$ is in the core. □

The theorem leaves open the question:

Problem 15 Is the final **U**-construction reduced for all ground domains (or at least for all sober ground domains)?

19.4 The initial upper power construction $\mathcal{U}_i$

In this section, we try to prove the initiality of the final **U**-construction. Unfortunately, we are not able to perform the proof for a class of ground domains larger than CONT. On the other hand, we do not know of any counterexample such that the situation is quite unsatisfactory.[2]

[2] Meanwhile, such a counterexample was found.

In order to point out clearly where continuity is required in our proof, we start with an arbitrary domain and restrict it to a more special class when needed.

Let $\mathbf{X}$ be an arbitrary ground domain. We try to prove that $\mathcal{U}\mathbf{X}$ is initial, i.e. that it is a free $\mathbf{U}$-$\mathbf{X}$-module. To this end, let $(M, \sqcap, \top)$ be an arbitrary $\mathbf{U}$-module, and let $f : [\mathbf{X} \to M]$ be a morphism. We have to provide a linear map $\bar{f} : [\mathcal{U}\mathbf{X} \to M]$ with $\bar{f} \circ \iota = f$ and then show its uniqueness.

At first, we define a function f' on finite subsets of $\mathbf{X}$:

- For finite sets E, let $f'\{e_1, \ldots, e_n\} = fe_1 \sqcap \cdots \sqcap fe_n$.

This function is well defined because '$\sqcap$' is commutative and associative, and has a neutral element 0 in M that can be taken as the result of applying f' to $\emptyset$. f' has the following properties:

(1) $f'(\emptyset) = 0$

(2) $f'(A \cup B) = f'A \sqcap f'B$ for all finite sets A and B.

(3) $f'\{x\} = fx$ for all x in $\mathbf{X}$.

(4) If $\uparrow A \supseteq \uparrow B$, then $f'A \leq f'B$ for all finite sets A and B.

Properties (1) and (3) are obvious. For (2), idempotence of '$\sqcap$' is needed since some operands occurring once in $f'(A \cup B)$ may occur twice in $f'A \sqcap f'B$. The precondition of (4) means $B \subseteq \uparrow A$, whence for all b in B, there is $a(b)$ in A with $b \geq a(b)$. By monotonicity of f, $f'B = \sqcap_{b \in B} fb \geq \sqcap_{b \in B} f(a(b))$ holds. By idempotence of '$\sqcap$', the last term equals $\sqcap_{a \in a[B]} fa$. By adding the remaining elements of A, the last term can only become smaller. Hence, it is above $\sqcap_{a \in A} fa = f'A$.

Because of property (4), $\uparrow A = \uparrow B$ implies $f'A = f'B$. Hence, f' implies a function $\widehat{f}$ mapping all finitary upper sets of $\mathbf{X}$ into M. $\widehat{f}$ is defined by $\widehat{f}(\uparrow E) = f'E$ for finite sets E. Properties (1) through (4) of f' directly translate into corresponding properties of $\widehat{f}$:

(1) $\widehat{f}(\emptyset) = 0$

(2) $\widehat{f}(A \cup B) = \widehat{f}A \sqcap \widehat{f}B$ for all finitary upper sets A and B.

(3) $\widehat{f}(\uparrow x) = fx$

(4) If $A \supseteq B$, then $\widehat{f}A \leq \widehat{f}B$ for all finitary upper sets A and B.

The next step should be the extension of $\widehat{f}$ to all of $\mathcal{U}\mathbf{X}$. The problem with this is that we do not know much about the structure of $\mathcal{U}\mathbf{X}$ in general. If $\mathbf{X}$ is sober however, $\mathcal{U}\mathbf{X}$ is isomorphic to $\mathcal{U}_K\mathbf{X}$.

Let K be a compact upper set in $\mathbf{X}$. There are several completely different approaches to define $\bar{f}K$. First, one could consider the set $\overline{F}K$ of all values $\widehat{f}A$ where A is a finitary upper *subset* of K. $\overline{F}$ is downward directed since $A, B \subseteq K$ implies $A \cup B \subseteq K$, and $\widehat{f}(A \cup B)$ is the greatest lower bound of $\widehat{f}A$ and $\widehat{f}B$. Since $A \subseteq K$ means $A \geq K$ in $\mathcal{U}\mathbf{X}$, all members of $\overline{F}K$ are above the value $\bar{f}K$ to be defined. All this suggests to define $\bar{f}K$ to be the greatest lower bound of $\overline{F}K$. Unfortunately, we do not know whether such a greatest lower bound exists, and if so, whether $\bar{f}$ is continuous.

A second approach considers the set $\underline{F}K$ of all values $\widehat{f}A$ where A is a finitary upper *superset* of K. Since $A \supseteq K$ means $A \leq K$ in $\mathcal{U}\mathbf{X}$, all members of $\underline{F}K$ are below the value

$\bar{f}K$ to be defined. Unfortunately, we do not know in general whether $\underline{F}K$ contains any point at all.

If we assume $\mathbf{X}$ to be multi-continuous, then K is strongly compact by Prop. 8.3.3, and there is a finitary upper superset A of K in every environment of K. Hence, the set $\underline{F}K$ is not empty, and one may guess that it well approximates $\bar{f}K$. All this suggests to define $\bar{f}K$ as the least upper bound of $\underline{F}K$. However, we cannot generally prove the existence of such a least upper bound, because we cannot prove $\underline{F}K$ to be directed.

In the third approach, we consider the set $\underline{\underline{F}}K$ of all values $\widehat{f}A$ where A is a finitary upper *environment* of K. To be more precise, let $\mathcal{F}(K)$ be the set of all finitary upper sets A such that there is an open set O with $K \subseteq O \subseteq A$. We first investigate this set more closely and then turn to $\underline{\underline{F}}K = \widehat{f}[\mathcal{F}(K)] \subseteq M$.

Proposition 19.4.1 Let $\mathbf{X}$ be a sober domain. For all compact upper sets K of $\mathbf{X}$, let $\mathcal{F}(K)$ be the set of all finitary upper environments of K. Then

(1) $\emptyset$ is in $\mathcal{F}(\emptyset)$.

(2) $\mathcal{F}(K_1 \cup K_2) = \{A_1 \cup A_2 \mid A_i \in \mathcal{F}(K_i)\}$.

(3) $K \supseteq K'$ implies $\mathcal{F}(K) \subseteq \mathcal{F}(K')$.

(4) Let $\mathcal{K}$ be a directed set in $\mathcal{U}_K\mathbf{X}$. Then $\mathcal{F}(\bigcap \mathcal{K}) = \bigcup_{K \in \mathcal{K}} \mathcal{F}(K)$.

(5) If $\mathbf{X}$ is multi-continuous, then for all K in $\mathcal{U}_K\mathbf{X}$, the set $\mathcal{F}(K)$ is directed in $\mathcal{U}_K\mathbf{X}$ and $\bigcap \mathcal{F}(K) = K$ holds.

Proof:

(1) $\emptyset$ is a finitary upper open superset of $\emptyset$.

(2) Let A be a member of the left hand side. Then there is some open set O with $K_1 \cup K_2 \subseteq O \subseteq A$. This implies $K_i \subseteq O \subseteq A$, whence A is in both $\mathcal{F}(K_1)$ and $\mathcal{F}(K_2)$. Because of $A = A \cup A$, A is a member of the right hand side.

Conversely, if A is in the set on the right, then there are finitary upper sets A_i and open sets O_i such that $K_i \subseteq O_i \subseteq A_i$. This implies $K_1 \cup K_2 \subseteq O_1 \cup O_2 \subseteq A_1 \cup A_2 = A$.

(3) If $K \subseteq O \subseteq A$ for some open O, then $K' \subseteq K \subseteq O \subseteq A$.

(4) The union is a subset of $\mathcal{F}(\bigcap \mathcal{K})$ because of (3) and $K \supseteq \bigcap \mathcal{K}$ for all K in $\mathcal{K}$.

Assume $\bigcap \mathcal{K} \subseteq O \subseteq A$ for some open set O. Because sober domains are in K-RD, there is some K in $\mathcal{K}$ with $K \subseteq O$. Hence, A is in $\mathcal{F}(K)$.

(5) By Th. 8.3.2. □

We now define $\underline{\underline{F}}K = \widehat{f}[\mathcal{F}(K)]$. If the ground domain is multi-continuous, $\mathcal{F}(K)$ is directed, whence $\underline{\underline{F}}K$ is directed, too, by monotonicity of $\widehat{f}$. Hence, the set $\underline{\underline{F}}K$ has a least upper bound in M. Thus, we can define $\bar{f}K = \bigsqcup \underline{\underline{F}}K$. We have to show that this function is continuous, additive, and $\bar{f}(\uparrow x) = fx$ holds.

By property (1) of $\mathcal{F}(K)$, the value $\top = \widehat{f}\,\emptyset$ is in $\underline{\underline{F}}(\emptyset)$, whence $\bar{f}(\emptyset) = \top$. Using property (2) and continuity of '$\sqcap$' in M, one obtains

$$\begin{aligned}
\bar{f}(K_1 \cup K_2) &= \bigsqcup \widehat{f}\,[\mathcal{F}(K_1 \cup K_2)] \\
&= \bigsqcup \{\widehat{f}(A_1 \cup A_2) \mid A_i \in \mathcal{F}(K_i)\} \\
&= \bigsqcup \{\widehat{f}A_1 \sqcap \widehat{f}A_2 \mid A_i \in \mathcal{F}(K_i)\} \\
&= \bigsqcup \{\widehat{f}A_1 \mid A_1 \in \mathcal{F}(K_1)\} \sqcap \bigsqcup \{\widehat{f}A_2 \mid A_2 \in \mathcal{F}(K_2)\} \\
&= \bar{f}K_1 \sqcap \bar{f}K_2
\end{aligned}$$

By property (3) of the Lemma, $K \leq K'$ implies $\mathcal{F}(K) \subseteq \mathcal{F}(K')$, whence $\widehat{f}\,[\mathcal{F}(K)] \subseteq \widehat{f}\,[\mathcal{F}(K')]$. By Prop. 2.4.7, $\bar{f}K \leq \bar{f}K'$ follows. Hence, $\bar{f}$ is monotonic.

Continuity of $\bar{f}$ is shown using part (4) of the Lemma. Let $\mathcal{K}$ be a directed set in $\mathcal{U}_K\mathbf{X}$.

$$\begin{aligned}
\bar{f}(\bigcap \mathcal{K}) &= \bigsqcup \widehat{f}\,[\mathcal{F}(\bigcap \mathcal{K})] \;=\; \bigsqcup \widehat{f}\,[\bigcup_{K \in \mathcal{K}} \mathcal{F}(K)] \\
&= \bigsqcup \bigcup_{K \in \mathcal{K}} \widehat{f}\,[\mathcal{F}(K)] \;=\; \bigsqcup_{K \in \mathcal{K}} \bigsqcup \widehat{f}\,[\mathcal{F}(K)] \;=\; \bigsqcup_{K \in \mathcal{K}} \bar{f}K
\end{aligned}$$

Finally, we have to show $\bar{f}(\uparrow x) = fx$ for all x in $\mathbf{X}$. For all A in $\mathcal{F}(\uparrow x)$, $A \supseteq \uparrow x$ holds, whence $\widehat{f}A \subseteq \widehat{f}(\uparrow x) = fx$. Thus, $\bar{f}(\uparrow x) = \bigsqcup \widehat{f}\,[\mathcal{F}(\uparrow x)] \leq fx$ follows.

For the opposite direction, we assume $fx \in O$ for some open set O, and have to show $\bar{f}(\uparrow x) \in O$ by Lemma 4.3.1. $fx \in O$ implies $x \in O' = f^{-1}[O]$. By continuity of f, O' is an open environment of x. By multi-continuity, there are a finite set E and an open set O'' with $x \in O'' \subseteq \uparrow E \subseteq O'$. Thus, $\uparrow E$ is in $\mathcal{F}(\uparrow x)$, and $fe \in O$ holds for all e in E. Unfortunately, this does not generally imply $\widehat{f}(\uparrow E) = \sqcap_{e \in E} fe \in O$. Thus, the proof cannot be completed.

If $\mathbf{X}$ is even continuous, then the set E may be assumed to be a singleton. Thus, $\widehat{f}(\uparrow E) = fe$ is in O, whence $\bar{f}(\uparrow x) = \bigsqcup \widehat{f}\,[\mathcal{F}(\uparrow x)]$ is in O, and the proof is completed.

Summarizing, we obtain

Theorem 19.4.2 For continuous ground domain $\mathbf{X}$, initial and final upper power domain over $\mathbf{X}$ (and thus all reduced upper power domains over $\mathbf{X}$) coincide.

As indicated at the beginning of this section, this theorem is not as general as one might wish. We saw that the corresponding proof for multi-continuous ground domain only fails in proving $\bar{f}(\uparrow x) \geq fx$.

Chapter 20

More about L- and U-constructions

In this chapter, we present some more results about **L**- and **U**-constructions. According to the (non-)results of the previous chapter, one has to carefully distinguish between the initial **U**-construction $\mathcal{U}_i$ and the final **U**-construction $\mathcal{U}_f$. In particular, it is possible that $\mathcal{U}_f$ is not reduced. Indeed, we were able to identify a sub-construction $\mathcal{U}_s$ of $\mathcal{U}_f$ (section 20.1). For sober ground domains, its power domains may be represented in terms of strongly compact sets $\mathcal{U}_S\mathbf{X}$. Unfortunately, we were not able to prove that $\mathcal{U}_s$ is a proper sub-construction of $\mathcal{U}_f$.

For sober ground domains, $\mathcal{U}_f\mathbf{X}$ and $\mathcal{U}_K\mathbf{X}$, and also $\mathcal{U}_s\mathbf{X}$ and $\mathcal{U}_S\mathbf{X}$ are isomorphic. In section 20.2, we identify the maximal domain of definition of $\mathcal{U}_K$ and $\mathcal{U}_S$ and show that they are partial power constructions of their own independent from whether they coincide with $\mathcal{U}_f$ and $\mathcal{U}_s$ respectively. The properties of $\mathcal{U}_S$ are investigated in sections 20.3 and 20.4. The analysis of the topology of $\mathcal{U}_S\mathbf{X}$ leads to the class of $\mathcal{U}_S$-conform domains that contains all multi-continuous domains.

In section 20.5, we show by topological reasoning that $\mathcal{L}\mathcal{U}_S\mathbf{X}$ and $\mathcal{U}_S\mathcal{L}\mathbf{X}$ are isomorphic for all $\mathcal{U}_S$-conform ground domains $\mathbf{X}$. In the final section 20.6, we then show by algebraic reasoning that $\mathcal{L}\mathcal{U}_i\mathbf{X}$ and $\mathcal{U}_i\mathcal{L}\mathbf{X}$ are isomorphic for *all* ground domains $\mathbf{X}$. Both results largely generalize the paper [FM90] where the isomorphism was shown for bounded complete algebraic domains $\mathbf{X}$ using information systems.

20.1 The U-construction $\mathcal{U}_s$ in terms of strong filters

Since we were not able to show the final **U**-construction $\mathcal{U}_f$ to be reduced, there might be some sub-constructions of $\mathcal{U}_f$. Indeed, we found a sub-construction $\mathcal{U}_s$ defined in terms of *strong filters*. It is open whether $\mathcal{U}_f$ and $\mathcal{U}_s$ coincide for all ground domains.

For sober ground domains, open filters correspond one-to-one to compact upper sets. The *strong* filters are defined such that they correspond to *strongly* compact upper sets.

Definition 20.1.1 An open filter $\mathcal{F}$ is *strong* iff for all O in $\mathcal{F}$, there is a finite set $E \subseteq O$ such that $\mathcal{O}(E) \subseteq \mathcal{F}$.

In the sequel, we verify that the power operations of $\mathcal{U}_f \cong \mathcal{U}_\Phi$ preserve the 'strength' of filters, i.e. that the strong filters induce a sub-construction $\mathcal{U}_s$ of $\mathcal{U}_f$. The power operations of $\mathcal{U}_f$ are given by Th. 19.1.1 in terms of filters. Remember that open filters are open sets of open sets of $\mathbf{X}$ that contain $\mathbf{X}$ and are closed w.r.t. intersection. They are ordered by set inclusion.

- If $\mathcal{D}$ is a directed set of open filters, then $\bigsqcup \mathcal{D}$ is $\bigcup \mathcal{D}$. Assume all members of $\mathcal{D}$ are strong. If O is in $\bigcup \mathcal{D}$, then O is in $\mathcal{F}$ for some $\mathcal{F}$ in $\mathcal{D}$. Since $\mathcal{F}$ is strong, there is a finite set $E \subseteq O$ such that $\mathcal{O}(E) \subseteq \mathcal{F} \subseteq \bigcup \mathcal{D}$.
- θ is given by $\Omega\mathbf{X} = \mathcal{O}(\emptyset)$ and $\{|x|\}$ by $\mathcal{O}(\{x\})$. We generalize both cases to $\mathcal{O}(E)$ for finite set E.

  If O is in $\mathcal{O}(E)$, then $E \subseteq O$ and $\mathcal{O}(E) \subseteq \mathcal{O}(E)$ hold. Hence, all open filters $\mathcal{O}(E)$ for finite E are strong.
- For two filters $\mathcal{F}_1$ and $\mathcal{F}_2$, $\mathcal{F}_1 \uplus \mathcal{F}_2$ is given by $\mathcal{F}_1 \cap \mathcal{F}_2$. If O is a member of this intersection, then $O \in \mathcal{F}_i$ holds for $i = 1, 2$. If the filters $\mathcal{F}_i$ are strong, then there are finite sets $E_i \subseteq O$ such that $\mathcal{O}(E_i) \subseteq \mathcal{F}_i$. $E = E_1 \cup E_2$ is a finite subset of O. Let O' be a member of $\mathcal{O}(E)$. Then $O' \supseteq E \supseteq E_i$ follows for $i = 1, 2$, whence $O' \in \mathcal{O}(E_i) \subseteq \mathcal{F}_i$, i.e. $O' \in \mathcal{F}_1 \cap \mathcal{F}_2$.
- For $f : [\mathbf{X} \to \mathcal{U}_s\mathbf{Y}]$ and A in $\mathcal{U}_s\mathbf{X}$, *ext* $f\, A$ is given by $B = \{O \mid \{x \mid O \in fx\} \in A\}$. If O is in B, then $O' = \{x \mid O \in fx\}$ is in A. Since A is strong, there is a finite subset E' of O' such that $\mathcal{O}(E') \subseteq A$. By $E' \subseteq O'$, set O is in fe for all $e \in E'$. Since fe is strong, there are finite sets $F_e \subseteq O$ such that $\mathcal{O}(F_e) \subseteq fe$. The union $F = \bigcup_{e \in E'} F_e$ of these sets is a finite subset of O.

  We have to show $\mathcal{O}(F) \subseteq B$. Let P be a member of $\mathcal{O}(F)$. $P \supseteq F \supseteq F_e$ for all e in E' implies $P \in \mathcal{O}(F_e) \subseteq fe$. Hence, $P' = \{x \mid P \in fx\}$ is a superset of E', whence $P' \in \mathcal{O}(E') \subseteq A$. Thus, P is in B.

The arguments above prove the following theorem:

Theorem 20.1.2 The strong filters form a sub-construction $\mathcal{U}_s$ of $\mathcal{U}_f$.

The next proposition shows that strong filters are to open filters as strongly compact sets are to compact sets.

Proposition 20.1.3
Let K be a compact upper set. Then $\mathcal{O}(K)$ is a strong filter iff K is strongly compact.

Proof: $\mathcal{O}(K)$ is strong iff $\forall O \in \mathcal{O}(K)\, \exists E \subseteq_f O : \mathcal{O}(E) \subseteq \mathcal{O}(K)$.
By definition of $\mathcal{O}(.)$, $O \in \mathcal{O}(K)$ is equivalent to $K \subseteq O$. By Lemma 4.4.4, $\mathcal{O}(E) \subseteq \mathcal{O}(K)$ is equivalent to $\uparrow E \supseteq K$. Hence, we get the equivalent formula

$$\forall O \supseteq K\ \exists E \text{ finite} : K \subseteq \uparrow E \subseteq O$$

This is the definition of strong compactness. □

By Prop. 19.2.1, we know that $\mathcal{U}_f\mathbf{X}$ and $\mathcal{U}_K\mathbf{X}$ are isomorphic iff $\mathbf{X}$ is sober. Here, $\mathcal{U}_K\mathbf{X}$ is the set of all compact upper sets of $\mathbf{X}$.

Proposition 20.1.4 If $\mathbf{X}$ is sober, then $\mathcal{U}_s\mathbf{X}$ is isomorphic to the set $\mathcal{U}_S\mathbf{X}$ of all strongly compact upper sets of $\mathbf{X}$ ordered by inverse inclusion '$\supseteq$'.

Proof: We have to show surjectivity of $\mathcal{O}(.) : \mathcal{U}_S\mathbf{X} \to \mathcal{U}_s\mathbf{X}$. If $\mathcal{F}$ is a strong filter, then by sobriety $\mathcal{F} = \mathcal{O}(K)$ holds for some compact set K. By Prop. 20.1.3, K is strongly compact. □

In contrast to Prop. 19.2.1, we cannot show the equivalence to sobriety here. Thus, the following question remains open:

Problem 16 Is sobriety equivalent to $\mathcal{U}_s\mathbf{X} \cong \mathcal{U}_S\mathbf{X}$?

The two constructions $\mathcal{U}_f$ and $\mathcal{U}_s$ coincide for multi-continuous ground domains by Prop. 8.3.3. We do however not know whether they always coincide.

Problem 17 For which domains are all open filters strong?

The construction $\mathcal{U}_s$ is intuitively 'more reduced' than the larger construction $\mathcal{U}_f$. From $\mathcal{U}_f$, we know reducedness for multi-continuous ground domains only; by Th. 8.3.2, $K = \bigcap \mathcal{F}(K)$ holds in this case. $\mathcal{U}_s$ is known to be reduced for domains in FC & SOB in addition since every strongly compact set is a directed intersection of finitary upper sets by Prop. 5.4.4.

Problem 18 For what ground domains $\mathbf{X}$ is $\mathcal{U}_s\mathbf{X}$ reduced?

20.2 The topological U-constructions $\mathcal{U}_K$ and $\mathcal{U}_S$

In case of a sober ground domain $\mathbf{X}$, the final **U**-powerdomain $\mathcal{U}_f\mathbf{X}$ is isomorphic to the **U**-powerdomain $\mathcal{U}_K\mathbf{X}$ in terms of compact upper sets. For non-sober $\mathbf{X}$, $\mathcal{U}_f\mathbf{X}$ and $\mathcal{U}_K\mathbf{X}$ are not isomorphic but $\mathcal{U}_K\mathbf{X}$ might still define a **U**-construction. We show however that $\mathcal{U}_K\mathbf{X}$ is not useful for all domains $\mathbf{X}$. Instead, we show that it forms a sub-powerdomain of $\mathcal{U}_f\mathbf{X}$ just for the class K-RD of domains with the K-Rudin property.

Analogously, one may consider the set $\mathcal{U}_S\mathbf{X}$ of strongly compact upper sets. It forms a sub-powerdomain of $\mathcal{U}_s\mathbf{X}$ for domains in S-RD.

We know that the mapping $\mathcal{O}(.) : \mathcal{U}_K\mathbf{X} \to \mathcal{U}_f\mathbf{X}$ cuts down to $\mathcal{O}(.) : \mathcal{U}_S\mathbf{X} \to \mathcal{U}_s\mathbf{X}$ by Prop. 20.1.3. $K \le K'$ is equivalent to $\mathcal{O}(K) \subseteq \mathcal{O}(K')$. Hence, $\mathcal{U}_K\mathbf{X}$ is isomorphic to a subdomain of $\mathcal{U}_f\mathbf{X}$ by $\mathcal{O}(.)$ iff $\mathcal{U}_K\mathbf{X}$ is a domain and $\mathcal{O}(.)$ is continuous.

Definition 20.2.1

A domain $\mathbf{X}$ is $\mathcal{U}_K$*-admitting* iff $\mathcal{U}_K\mathbf{X}$ is a domain and $\mathcal{O}(.) : \mathcal{U}_K\mathbf{X} \to \mathcal{U}_f\mathbf{X}$ is continuous.

A domain $\mathbf{X}$ is $\mathcal{U}_S$*-admitting* iff $\mathcal{U}_S\mathbf{X}$ is a domain and $\mathcal{O}(.) : \mathcal{U}_S\mathbf{X} \to \mathcal{U}_f\mathbf{X}$ is continuous.

There are some equivalent formulations of the $\mathcal{U}_K$-admitting property:

Theorem 20.2.2 For a domain $\mathbf{X}$, the following statements are equivalent:

(1) $\mathbf{X}$ is $\mathcal{U}_K$-admitting.

(2) $\mathcal{U}_K\mathbf{X}$ is a domain, and for all open sets O in $\mathbf{X}$, the set $\mathcal{K}(O) = \{K \in \mathcal{U}_K\mathbf{X} \mid K \subseteq O\}$ is open in $\mathcal{U}_K\mathbf{X}$.

(3) For every open set O in $\mathbf{X}$ and every directed set $\mathcal{K}$ in $\mathcal{U}_K\mathbf{X}$, $O \supseteq \bigcap \mathcal{K}$ implies there is some K in $\mathcal{K}$ with $O \supseteq K$. This is property K-RD.

In this case, the limit of a directed set $\mathcal{K}$ in $\mathcal{U}_K\mathbf{X}$ is its intersection $\bigcap\mathcal{K}$.

An analogous statement holds for $\mathcal{U}_S$-admitting, S-RD, and strongly compact sets.

In the sequel, part (3) will be the main instrument to cope with when investigating $\mathcal{U}_K\mathbf{X}$.

Proof: We start by deriving (3) from (1). Assume $\mathbf{X}$ is $\mathcal{U}_K$-admitting, and let $\mathcal{K}$ be a directed set in it. Then $\mathcal{K}$ has a limit K, and $\bigcap\mathcal{K} \supseteq K$ holds because '$\supseteq$' is the order in $\mathcal{U}_K\mathbf{X}$. By continuity of $\mathcal{O}(.)$, $\mathcal{O}(K) = \bigcup_{A\in\mathcal{K}}\mathcal{O}(A)$ holds. Let O be an open set in $\mathbf{X}$ with $O \supseteq \bigcap\mathcal{K}$. Then $K \subseteq \bigcap\mathcal{K} \subseteq O$ whence $O \in \mathcal{O}(K)$. Thus, there is A in $\mathcal{K}$ with $O \in \mathcal{O}(A)$ i.e. $O \supseteq A$.

The fact $\bigsqcup\mathcal{K} = \bigcap\mathcal{K}$ follows from (3) because of Prop. 8.9.4.

Now, we show the equivalence of (2) and (3). The set $\mathcal{K}(O)$ occurring in (2) is always an upper set in $\mathcal{U}_K$ since $K \le K'$ and $K \in \mathcal{K}(O)$ imply $K' \subseteq K \subseteq O$, whence $K' \in \mathcal{K}(O)$. Hence, $\mathcal{K}(O)$ is open iff $\bigsqcup\mathcal{K} \subseteq O$ implies $A \subseteq O$ for some A in $\mathcal{K}$. (3) directly implies this, since $\bigsqcup\mathcal{K} = \bigcap\mathcal{K}$. Conversely, if (2) holds, then $\bigcap\mathcal{K} \subseteq O$ implies $\bigsqcup\mathcal{K} \subseteq O$ and further $A \subseteq O$ for some A in $\mathcal{K}$.

Finally, we show that (3) implies (1). (3) implies the addition, i.e. $\bigsqcup\mathcal{K}$ exists and is $\bigcap\mathcal{K}$. Thus, $\mathcal{U}_K\mathbf{X}$ is a domain. For continuity of $\mathcal{O}(.)$, we have to show $\mathcal{O}(\bigcap\mathcal{K}) = \bigcup_{A\in\mathcal{K}}\mathcal{O}(A)$. '$\supseteq$' holds since $A \subseteq O$ implies $\bigcap\mathcal{K} \subseteq A \subseteq O$. '$\subseteq$' holds by (3): if $\bigcap\mathcal{K} \subseteq O$, then $A \subseteq O$ for some $A \in \mathcal{K}$.

The proof for strong compactness is completely analogous. We did not use the specific properties of compact sets in the proof above. □

If $\mathbf{X}$ is $\mathcal{U}_K$-admitting, then $\mathcal{U}_K\mathbf{X}$ is isomorphic to a sub-domain of $\mathcal{U}_\Phi\mathbf{X}$ by $\mathcal{O}(.)$. This sub-domain is closed w.r.t. the power operations in $\mathcal{U}_\Phi\mathbf{X}$. Hence, $\mathcal{U}_K\mathbf{X}$ may also be considered an upper power domain over $\mathbf{X}$ if the operations in $\mathcal{O}[\mathcal{U}_K\mathbf{X}] \subseteq \mathcal{U}_\Phi\mathbf{X}$ are translated back into operations in $\mathcal{U}_K\mathbf{X}$.

Theorem 20.2.3 $\mathcal{U}_K$ is a partial upper power construction defined for the class K-RD of all $\mathcal{U}_K$-admitting domains. The power domains and operations are given by

(1) $\mathcal{U}_K\mathbf{X}$ is the set of all compact upper sets of $\mathbf{X}$.

(2) $K \le K'$ iff $K \supseteq K'$

(3) $\bigsqcup\mathcal{K} = \bigcap\mathcal{K}$ for directed sets $\mathcal{K}$ in $\mathcal{U}_K\mathbf{X}$.

(4) $\theta = \emptyset$

(5) $A \uplus B = A \cup B$

(6) $\{\!|x|\!\} = \uparrow x$ for all $x \in \mathbf{X}$.

(7) If $f : [\mathbf{X} \to \mathcal{U}_K\mathbf{Y}]$ is continuous and A is in $\mathcal{U}_K\mathbf{X}$, then $\mathit{ext}\, f\, A = \bigcup_{a\in A} fa = \bigcup f[A]$.

All these operations are well defined and continuous.

$\mathcal{U}_S$ is also a partial power construction. It is defined for the class S-RD of all $\mathcal{U}_S$-admitting domains. The power operations are like those above.

Proof: The proof of Th. 19.2.3 did not make specific use of sobriety or compactness. Hence, it may also be taken to prove the theorem at hand. □

Now, we turn to the completeness properties of $\mathcal{U}_K\mathbf{X}$ and $\mathcal{U}_S\mathbf{X}$.

Theorem 20.2.4 If $\mathbf{X}$ is in KC & K-RD, then $\mathcal{U}_K\mathbf{X}$ is in CC. If $\mathbf{X}$ is in SC & S-RD, then $\mathcal{U}_S\mathbf{X}$ is in CC.

Proof: We show the statement for $\mathcal{U}_K$. The statement for $\mathcal{U}_S$ is shown analogously.

Property KC means that the whole of $\mathbf{X}$ is compact and the intersection of two upper cones is compact. We have to show that $\mathcal{U}\mathbf{X}$ has a least element and lubs of every two points. Obviously, $\mathbf{X}$ itself is the least element.

For every two points a and b of $\mathbf{X}$, let $a * b = \uparrow a \cap \uparrow b$. By KC, '$*$' is a map from $\mathbf{X} \times \mathbf{X}$ to $\mathcal{U}\mathbf{X}$. It is continuous because $\uparrow a \cap \uparrow \bigsqcup D = \bigcap_{d \in D}(\uparrow a \cap \uparrow d)$ holds. By double extension, we get $A \Cap B = \mathit{ext}\,(\lambda a.\, \mathit{ext}\,(\lambda b.\, a * b)\, B)\, A$. Well-definedness of the power operations in $\mathcal{U}_K\mathbf{X}$ yields compactness of $A \Cap B$. We claim $A \Cap B = A \cap B$, i.e. $A \Cap B$ is the least upper bound of A and B.

By Th. 19.2.3, $A \Cap B = \bigcup_{a \in A} \bigcup_{b \in B} \uparrow a \cap \uparrow b$ holds. If x is a member of this set, then x is above some point a of A and also above some b of B. Hence, x is in $A \cap B$. Conversely, if y is a point of the intersection, then y is in both A and B, and y in $\uparrow y \cap \uparrow y$ holds. □

By this proposition, initial and final **U**-construction preserve the classes BC & ALG and BC & CONT since they coincide with $\mathcal{U}_K$ and $\mathcal{U}_S$ for continuous ground domains.

We were not able to decide whether the constructions $\mathcal{U}_K$ and $\mathcal{U}_S$ may be iterated:

Problem 19 If $\mathbf{X}$ is $\mathcal{U}_K$-admitting, is then $\mathcal{U}_K\mathbf{X}$ $\mathcal{U}_K$-admitting again? If $\mathbf{X}$ is $\mathcal{U}_S$-admitting, is then $\mathcal{U}_S\mathbf{X}$ $\mathcal{U}_S$-admitting again?

20.3 Open sets and additive functions in $\mathcal{U}_S$

In this section, we state and prove some properties of the **U**-construction $\mathcal{U}_S$ in terms of strongly compact upper sets. It is defined for the class S-RD of $\mathcal{U}_S$-admitting domains.

By Th. 20.2.2, the sets $\mathcal{K}_S(O) = \{S \in \mathcal{U}_S\mathbf{X} \mid S \subseteq O\}$ are open in $\mathcal{U}_S\mathbf{X}$ if O is open in $\mathbf{X}$. In the next Lemma, we characterize all open sets of $\mathcal{U}_S\mathbf{X}$ that are obtained by $\mathcal{K}_S(.)$.

Lemma 20.3.1 Let $\mathbf{X}$ be $\mathcal{U}_S$-admitting. For a subset $\mathcal{O}$ of $\mathcal{U}_S\mathbf{X}$, the following two statements are equivalent:

(1) There is an open set O of $\mathbf{X}$ such that $\mathcal{O} = \mathcal{K}_S(O)$.

(2) $\mathcal{O}$ is open and closed w.r.t. binary union.

Proof: The sets $\mathcal{K}_S(O)$ are open by Th. 20.2.2. They are closed w.r.t. union since $S_1, S_2 \subseteq O$ implies $S_1 \cup S_2 \subseteq O$.

Let conversely $\mathcal{O}$ be an open set of $\mathcal{U}_S\mathbf{X}$ that is closed w.r.t. union. Let $O = \iota^{-1}[\mathcal{O}] = \{x \in \mathbf{X} \mid \uparrow x \in \mathcal{O}\}$. O is open as inverse image of $\mathcal{O}$ by ι. We claim $\mathcal{O} = \mathcal{K}_S(O)$.

Let S be in $\mathcal{O}$. x in S implies $\uparrow x \subseteq S$, i.e. $S \leq \uparrow x$. Hence, $\uparrow x$ is in $\mathcal{O}$, whence x is in O. Thus, $S \subseteq O$ holds. Conversely, $S \subseteq O$ implies $S \subseteq \uparrow E \subseteq O$ for some finite set E by strong compactness. By $\uparrow E \subseteq O$, every e in E is in O, whence $\uparrow e \in \mathcal{O}$ for all e in E. Thus, $S \geq \uparrow E = \bigcup_{e \in E} \uparrow e \in \mathcal{O}$ because $\mathcal{O}$ is closed w.r.t. finite union. □

Since we used strong compactness explicitly in this proof, an analogous proof for $\mathcal{U}_K$ is impossible. We may however prove a similar criterion where 'binary union' is replaced by 'arbitrary union'. This criterion would not be quite useful however.

The criterion for $\mathcal{U}_S$ may be used to show that $\mathcal{U}_S$ has unique additive extensions.

Lemma 20.3.2 Let $\mathbf{X}$ and $\mathbf{Y}$ be two $\mathcal{U}_S$-admitting domains. If $F : [\mathcal{U}_S\mathbf{X} \to \mathcal{U}_S\mathbf{Y}]$ is additive, then $FS = \bigcap\{FA \mid A \in \mathcal{F}'(S)\}$ holds for all S in $\mathcal{U}_S\mathbf{X}$ where $\mathcal{F}'(S)$ is the set of all finitary upper supersets of S.

Proof: If $A \supseteq S$, then $FA \supseteq FS$. Thus, FS is a subset of the intersection.

Conversely, we have to show $T = \bigcap\{FA \mid A \in \mathcal{F}'(S)\} \subseteq FS$. Applying Lemma 4.4.4, we have to show that $FS \subseteq O$ implies $T \subseteq O$ for all open sets O of $\mathbf{Y}$. Let $FS \subseteq O$. Then FS is in $\mathcal{K}_S(O)$, whence S is in the open set $\mathcal{O} = F^{-1}[\mathcal{K}_S(O)]$. $\mathcal{O}$ is closed w.r.t. binary union since F is additive: $A, B \in \mathcal{O}$ implies $FA, FB \in \mathcal{K}_S(O)$, whence $F(A \cup B) = FA \cup FB \in \mathcal{K}_S(O)$, whence $A \cup B \in \mathcal{O}$. By Lemma 20.3.1, there is some open set O' of $\mathbf{X}$ such that $\mathcal{O} = \mathcal{K}_S(O')$. $S \subseteq O'$ implies $S \subseteq A \subseteq O'$ for some finitary upper set A by strong compactness of S. $A \subseteq O'$ means $FA \subseteq O$, whence $T \subseteq O$. □

The Lemma shows that additive morphisms from $\mathcal{U}_S\mathbf{X}$ to $\mathcal{U}_S\mathbf{Y}$ only depend on their values on finitary upper sets. Thus, we get

Theorem 20.3.3 Let $\mathbf{X}$ and $\mathbf{Y}$ be two $\mathcal{U}_S$-admitting domains. If $F, G : [\mathcal{U}_S\mathbf{X} \to \mathcal{U}_S\mathbf{Y}]$ are two additive functions with $F \circ \iota = G \circ \iota$, then $F = G$ follows.

Proof: The precondition means $F(\uparrow x) = G(\uparrow x)$ for all x in $\mathbf{X}$. By additivity, $FA = GA$ holds for all finitary upper sets A. By Lemma 20.3.2, $FS = GS$ follows for all S in $\mathcal{U}_S\mathbf{X}$. □

An analogous theorem for $\mathcal{U}_K$ could not be proved.

20.4 $\mathcal{U}_S$ and domain classes

In this section, we consider the relations between $\mathcal{U}_S$ and the domain classes of (multi-)algebraicity and (multi-)continuity. We also introduce a novel class, the class of $\mathcal{U}_S$-conform domains.

Theorem 20.4.1 Let $\mathbf{X}$ be $\mathcal{U}_S$-admitting.

(1) $\mathcal{U}_S\mathbf{X}$ is algebraic iff $\mathbf{X}$ is multi-algebraic.

(2) $\mathcal{U}_S\mathbf{X}$ is continuous iff $\mathbf{X}$ is multi-continuous.

Hence, $\mathcal{U}_S$ preserves ALG, M-ALG, CONT, and M-CONT.

Proof: The implications from right to left are already provided by Th. 19.3.1 and Th. 19.3.3. These two theorems were formulated for $\mathcal{U}_K$. For multi-continuous $\mathbf{X}$, $\mathcal{U}_K\mathbf{X}$ and $\mathcal{U}_S\mathbf{X}$ coincide by Prop. 8.3.3.

Let now $\mathcal{U}_S\mathbf{X}$ be continuous. Let x be a point of an open set O of $\mathbf{X}$. Then $\uparrow x \in \mathcal{K}_S(O)$ whence by continuity of $\mathcal{U}_S\mathbf{X}$ there are an open set $\mathcal{O}$ of $\mathcal{U}_S\mathbf{X}$ and a member A of $\mathcal{U}_S\mathbf{X}$ such that $\uparrow x \in \mathcal{O} \subseteq \uparrow\{A\} \subseteq \mathcal{K}_S(O)$. Let $O' = \iota^{-1}[\mathcal{O}]$. By continuity of ι, O' is an open set of $\mathbf{X}$.

First, $\iota x = \uparrow x \in \mathcal{O}$ implies $x \in O'$. Second, $y \in O'$ means $\uparrow y \in \mathcal{O} \subseteq \uparrow\{A\}$, whence $\uparrow y \geq A$, i.e. $\uparrow y \subseteq A$. Thus, we obtain $O' \subseteq A$. Third, $\uparrow\{A\} \subseteq \mathcal{K}_S(O)$ implies $A \in \mathcal{K}_S(O)$, whence $A \subseteq O$. Summarizing, we obtain $x \in O' \subseteq A \subseteq O$. A being strongly compact implies the existence of a finitary set F such that we finally get $x \in O' \subseteq A \subseteq F \subseteq O$.

Finally, let $\mathcal{U}_S\mathbf{X}$ be algebraic. Let x be a point of an open set O of $\mathbf{X}$. Then $\uparrow x \in \mathcal{K}_S(O)$ whence by algebraicity of $\mathcal{U}_S\mathbf{X}$ there is a member A of $\mathcal{U}_S\mathbf{X}$ such that $\uparrow\{A\}$ is open in $\mathcal{U}_S\mathbf{X}$ and $\uparrow x \in \uparrow\{A\} \subseteq \mathcal{K}_S(O)$. A point a is in A iff $\uparrow a \subseteq A$ iff $\uparrow a \geq A$ iff $\uparrow a \in \uparrow\{A\}$. Thus, $A = \iota^{-1}[\uparrow\{A\}]$ is open in $\mathbf{X}$. Since it is also strongly compact, it is finitary. $\uparrow x \in \uparrow\{A\} \subseteq \mathcal{K}_S(O)$ implies $x \in A \subseteq O$ as required. $\square$

$\mathcal{U}_S$-admission allows to map open sets of the ground domain to open sets of the power domain by means of $\mathcal{K}_S(.)$ (see Th. 20.2.2). If even the contrary is possible we speak of $\mathcal{U}_S$-conformity.

Definition 20.4.2 A domain $\mathbf{X}$ is *$\mathcal{U}_S$-conform* iff it is $\mathcal{U}_S$-admitting, and for all $S \in \mathcal{O}$ where $\mathcal{O}$ is an open set of $\mathcal{U}_S\mathbf{X}$, there is an open set O in $\mathbf{X}$ such that $S \in \mathcal{K}_S(O) \subseteq \mathcal{O}$.

Topologically speaking, $\mathcal{U}_S$-conformity means that the Scott topology of $\mathcal{U}_S\mathbf{X}$ has as a base the sets $\mathcal{K}_S(O)$ where O is open in $\mathbf{X}$.

Theorem 20.4.3 Every multi-continuous domain is $\mathcal{U}_S$-conform.

Proof:
Every multi-continuous domain is in S-RD, i.e. $\mathcal{U}_S$-admitting. For $\mathcal{U}_S$-conformity, let $A \in \mathcal{O}$ where $\mathcal{O}$ is open in $\mathcal{U}_S\mathbf{X}$. From Th. 8.3.2, we know that A is the supremum of the directed set $\mathcal{F}(A)$ of finitary upper environments of A. Since $\mathcal{O}$ is open, there is some $F \in \mathcal{F}(A)$ with $F \in \mathcal{O}$. From $F \in \mathcal{F}(A)$, we get an open set O with $A \subseteq O \subseteq F$. Then $A \in \mathcal{K}_S(O)$ holds, and $B \in \mathcal{K}_S(O)$ implies $B \subseteq O \subseteq F$, i.e. $B \geq F$, whence $B \in \mathcal{O}$. Thus, we obtain $A \in \mathcal{K}(O) \subseteq \mathcal{O}$. $\square$

Problem 20 If $\mathbf{X}$ is $\mathcal{U}_S$-conform, is then $\mathcal{U}_S\mathbf{X}$ again $\mathcal{U}_S$-conform?

Problem 21 What are the relations between $\mathcal{U}_S$-conformity and sobriety?

20.5 $\mathcal{L}$ and $\mathcal{U}_S$ may be exchanged

In [FM90], it was shown that $\mathcal{U}(\mathcal{L}\mathbf{X})$ and $\mathcal{L}(\mathcal{U}\mathbf{X})$ are isomorphic for bounded complete algebraic ground domains $\mathbf{X}$. The longish proof was done by means of the theory of information systems. We were able to largely generalize this statement up to the class of $\mathcal{U}_S$-conform domains using $\mathcal{U}_S$ for $\mathcal{U}$. Our proof is still longish, but completely differs from the proof in [FM90] in that it is done by topological means. In section 20.6, we shall show by algebraic means that $\mathcal{L}$ and the initial **U**-construction $\mathcal{U}_i$ may also be exchanged.

Theorem 20.5.1 If $\mathbf{X}$ is $\mathcal{U}_S$-conform, then $\mathcal{U}_S\mathcal{L}\mathbf{X}$ and $\mathcal{L}\mathcal{U}_S\mathbf{X}$ are isomorphic.

Before we are going to prove this theorem, we note that it allows to elegantly prove the preservation of M-ALG and M-CONT under $\mathcal{L}$.

Theorem 20.5.2 $\mathcal{L}$ preserves M-ALG and M-CONT.

Proof: If $\mathbf{X}$ is multi-algebraic, then it is $\mathcal{U}_S$-conform by Th. 20.4.3, and thus Th. 20.5.1 applies. $\mathcal{U}_S\mathbf{X}$ is algebraic by Th. 20.4.1, whence $\mathcal{L}\mathcal{U}_S\mathbf{X}$ is algebraic by Th. 18.6.1. By isomorphism, $\mathcal{U}_S\mathcal{L}\mathbf{X}$ is algebraic, too, and by Th. 20.4.1, $\mathcal{L}\mathbf{X}$ is multi-algebraic. The proof for M-CONT is analogous. □

The proof of Th. 20.5.1 proceeds in several steps. The precondition of $\mathcal{U}$-conformity[1] is not needed until the very last step. Hence, we start by assuming that $\mathbf{X}$ is only $\mathcal{U}$-admitting. Then, $\mathcal{L}\mathcal{U}\mathbf{X}$ is a domain. In contrast, $\mathcal{U}\mathcal{L}\mathbf{X}$ is a domain even for arbitrary $\mathbf{X}$ since $\mathcal{L}\mathbf{X}$ is complete and complete domains are $\mathcal{U}$-admitting by Th. 8.9.3.

Step 1: A monotonic function $\varphi : \mathcal{U}\mathcal{L}\mathbf{X} \to \mathcal{L}\mathcal{U}\mathbf{X}$

For a strongly compact upper set $\mathcal{K}$ of $\mathcal{L}\mathbf{X}$, we define $\varphi\mathcal{K} = \{K \in \mathcal{U}\mathbf{X} \mid C \cap K \neq \emptyset \; \forall C \in \mathcal{K}\}$. We have to show that $\varphi\mathcal{K}$ is a closed subset of $\mathcal{U}\mathbf{X}$. It is a lower set since $K' \leq K$ means $K' \supseteq K$, and thus, $C \cap K \neq \emptyset$ implies $C \cap K' \neq \emptyset$.

Let $(K_i)_{i\in I}$ be a directed family in $\varphi\mathcal{K}$. If we assume $\bigcap_{i\in I} K_i \notin \varphi\mathcal{K}$, then there is a member C of $\mathcal{K}$ such that $C \cap \bigcap_{i\in I} K_i = \emptyset$. Hence, $\bigcap_{i\in I} K_i \subseteq \mathsf{co}\, C$ holds, whence $K_k \subseteq \mathsf{co}\, C$ for some $k \in I$ by $\mathcal{U}$-admission. Here, the closedness of C was used. $K_k \subseteq \mathsf{co}\, C$ means $K_k \cap C = \emptyset$, whence K_k is not in $\varphi\mathcal{K}$. Because of this contradiction, $\bigcap_{i\in I} K_i$ is in $\varphi\mathcal{K}$.

$\mathcal{K} \leq \mathcal{K}'$ means $\mathcal{K} \supseteq \mathcal{K}'$. If K is in $\varphi\mathcal{K}$, then K meets all members of $\mathcal{K}$, and thus it also meets all members of $\mathcal{K}'$. Thus, $\mathcal{K} \leq \mathcal{K}'$ implies $\varphi\mathcal{K} \leq \varphi\mathcal{K}'$.

Step 2: A monotonic function $\psi : \mathcal{L}\mathcal{U}\mathbf{X} \to \mathcal{U}\mathcal{L}\mathbf{X}$

For a closed set $\mathcal{C}$ of $\mathcal{U}\mathbf{X}$, we define $\psi\mathcal{C} = \{C \in \mathcal{L}\mathbf{X} \mid K \cap C \neq \emptyset \; \forall K \in \mathcal{C}\}$.[2] We have to show that $\psi\mathcal{C}$ is a strongly compact upper set of $\mathcal{L}\mathbf{X}$.

For a strongly compact upper set K, we define $\psi'(K) = \{C \in \mathcal{L}\mathbf{X} \mid K \cap C \neq \emptyset\}$. We first show $\psi'(K) \in \mathcal{U}\mathcal{L}\mathbf{X}$ and later conclude $\psi\mathcal{C} \in \mathcal{U}\mathcal{L}\mathbf{X}$.

We claim $\psi' = map\, \iota_{\mathcal{L}}$.[3] Since the latter maps from $\mathcal{U}\mathbf{X}$ to $\mathcal{U}\mathcal{L}\mathbf{X}$, we get $\psi'(K) \in \mathcal{U}\mathcal{L}\mathbf{X}$. $map\, \iota_{\mathcal{L}}\, K = \uparrow(\iota_{\mathcal{L}}[K]) = \uparrow\{\downarrow x \mid x \in K\}$ holds. Let C be a member of this set. Then $C \geq \downarrow x$, i.e. $C \supseteq \downarrow x$, for some $x \in K$, whence $x \in C \cap K$. Conversely, assume there is some x in $C \cap K$. Then $\downarrow x \subseteq C$ since C is closed, whence $C \geq \downarrow x$ where $x \in K$.

Now, $\psi\mathcal{C} = \bigcap_{K\in\mathcal{C}} \psi'(K)$ holds. Since $\mathcal{L}\mathbf{X}$ is complete, $\mathcal{U}\mathcal{L}\mathbf{X}$ is also complete by Th. 20.2.4, and all suprema are given by intersection. Hence, arbitrary intersections of members of $\mathcal{U}\mathcal{L}\mathbf{X}$ are back in $\mathcal{U}\mathcal{L}\mathbf{X}$ again.

$\mathcal{C} \leq \mathcal{C}'$ means $\mathcal{C} \subseteq \mathcal{C}'$. Thus, whenever a set A meets all members of $\mathcal{C}'$, then it meets all members of $\mathcal{C}$. Hence, $\psi\mathcal{C}' \subseteq \psi\mathcal{C}$ holds, i.e. $\psi\mathcal{C} \leq \psi\mathcal{C}'$. Thus, ψ is monotonic.

Step 3: Relations between φ and ψ

Let $\mathcal{K}$ be a member of $\mathcal{U}\mathcal{L}\mathbf{X}$, and let C in turn be a member of $\mathcal{K}$. By definition of φ, C meets all members of $\varphi\mathcal{K}$. By definition of ψ, C is then in $\psi(\varphi\mathcal{K})$. Thus, we get $\mathcal{K} \subseteq \psi(\varphi\mathcal{K})$, or in terms of order $\mathcal{K} \geq \psi(\varphi\mathcal{K})$, or by abstraction $\psi \circ \varphi \leq id$.

[1] We drop the index 'S' of $\mathcal{U}_S$ in this proof.

[2] Interestingly, the definitions φ and ψ only differ in the domain and co-domain.

[3] This was the way I found the mapping ψ. The mapping φ may be analogously derived.

Analogously, $\mathcal{C} \subseteq \varphi(\psi\mathcal{C})$ holds for all $\mathcal{C}$ in $\mathcal{LU}\mathbf{X}$. This means $\mathcal{C} \leq \varphi(\psi\mathcal{C})$, whence $\varphi \circ \psi \geq id$. As an aside, this means that φ is an upper adjoint and preserves all infima, whereas ψ is a lower adjoint and preserves all suprema, i.e. it is in particular continuous.

The two relations $\psi \circ \varphi \leq id$ and $\varphi \circ \psi \geq id$ imply $\varphi \circ \psi \circ \varphi = (\varphi \circ \psi) \circ \varphi \geq \varphi$ and $\varphi \circ \psi \circ \varphi = \varphi \circ (\psi \circ \varphi) \leq \varphi$, whence $\varphi \circ \psi \circ \varphi = \varphi$. Similarly, $\psi \circ \varphi \circ \psi = \psi$ holds. In Step 4, we show that φ is injective, whence $\varphi \circ \psi \circ \varphi = \varphi$ implies $\psi \circ \varphi = id$. In the final Step 5, we show that ψ is injective, too, whence also $\varphi \circ \psi = id$.

Step 4: φ is injective: $\varphi\mathcal{K} \leq \varphi\mathcal{K}'$ implies $\mathcal{K} \leq \mathcal{K}'$

Let $\mathcal{K}$ and $\mathcal{K}'$ be two strongly compact upper sets of $\mathcal{L}\mathbf{X}$. $\varphi\mathcal{K} \leq \varphi\mathcal{K}'$ means $\varphi\mathcal{K} \subseteq \varphi\mathcal{K}'$. We have to conclude $\mathcal{K} \leq \mathcal{K}'$, i.e. $\mathcal{K} \supseteq \mathcal{K}'$. Applying Lemma 4.4.4, we assume $\mathcal{K} \subseteq \mathcal{O}$ where $\mathcal{O}$ is an open set of $\mathcal{L}\mathbf{X}$, and have to show $\mathcal{K}' \subseteq \mathcal{O}$.

Since $\mathcal{K}$ is strongly compact, $\mathcal{K} \subseteq \mathcal{O}$ implies there is a finitary set $\mathcal{F}$ in $\mathcal{L}\mathbf{X}$ such that $\mathcal{K} \subseteq \mathcal{F} \subseteq \mathcal{O}$. As $\mathcal{F}$ is finitary, there is a finite set $\mathcal{E}$ of closed sets $E_1, \ldots, E_n$ such that $\mathcal{F} = \uparrow\mathcal{E}$. We show $\mathcal{K}' \subseteq \mathcal{F}$, whence $\mathcal{K}' \subseteq \mathcal{O}$ immediately follows by $\mathcal{F} \subseteq \mathcal{O}$.

Let C be a member of $\mathcal{K}'$. C is closed, whence its complement $O = \mathsf{co}\, C$ is open. The sets $O_i = \mathsf{co}\, E_i$ are also open. With these notions, we first show an auxiliary property: $\mathcal{K}(O) \subseteq \mathcal{K}(O_1) \cup \cdots \cup \mathcal{K}(O_n)$. This property will later be used to show that there is some k such that $O \subseteq O_k$. By complementing, this implies $C \supseteq E_k$, i.e. $C \geq E_k$ for some k, whence $C \in \uparrow\mathcal{E} = \mathcal{F}$.

Proof of $\mathcal{K}(O) \subseteq \mathcal{K}(O_1) \cup \cdots \cup \mathcal{K}(O_n)$:
Let K be a member of $\mathcal{K}(O)$, i.e. a strongly compact upper subset of O. Then K does not meet O's complement C, whence K is not in $\varphi\mathcal{K}'$. Thus, K is neither in $\varphi\mathcal{K}$ because of the precondition $\varphi\mathcal{K} \subseteq \varphi\mathcal{K}'$. Hence, there is a member A of $\mathcal{K}$ such that $A \cap K = \emptyset$. Because of $\mathcal{K} \subseteq \uparrow\mathcal{E}$, there is some E_i in $\mathcal{E}$ such that $A \geq E_i$, i.e. $A \supseteq E_i$. Hence, $E_i \cap K = \emptyset$ also holds, whence $K \subseteq O_i$, i.e. $K \in \mathcal{K}(O_i)$.

Proof of $O \subseteq O_k$ for some k:
Assume $O \not\subseteq O_k$ for all k. Then for all k there is some x_k in O with $x_k \notin O_k$. Let $K = \uparrow\{x_1, \ldots, x_n\} \subseteq O$. By finiteness, this is a strongly compact set. Hence, $K \in \mathcal{K}(O)$, whence there is some i such that $K \subseteq O_i$, i.e. in particular $x_i \in K \subseteq O_i$. Thus, we derived a contradiction.

Step 4 is now concluded. Notice that we did not use $\mathcal{U}$-conformity so far, i.e. all results proved till now are valid for general $\mathcal{U}$-admitting ground domains $\mathbf{X}$.

Step 5: ψ is injective: $\psi\mathcal{C} \leq \psi\mathcal{C}'$ implies $\mathcal{C} \leq \mathcal{C}'$

Let $\mathcal{C}$ and $\mathcal{C}'$ be two closed sets of $\mathcal{U}\mathbf{X}$. $\psi\mathcal{C} \leq \psi\mathcal{C}'$ means $\psi\mathcal{C} \supseteq \psi\mathcal{C}'$. We have to conclude $\mathcal{C} \leq \mathcal{C}'$, i.e. $\mathcal{C} \subseteq \mathcal{C}'$.

Let K be in $\mathcal{C}$, and assume K is not in $\mathcal{C}'$. Then K is in the complement $\mathcal{O}$ of $\mathcal{C}'$. Since $\mathcal{O}$ is open in $\mathcal{U}\mathbf{X}$, there is an open set O of $\mathbf{X}$ such that $K \in \mathcal{K}(O) \subseteq \mathcal{O}$ by $\mathcal{U}$-conformity. $K \subseteq O$ implies $K \cap \mathsf{co}\, O = \emptyset$. Hence, $\mathsf{co}\, O$ is not in $\psi\mathcal{C}$, whence it is in $\psi\mathcal{C}'$ neither. This means there is some $K' \in \mathcal{C}'$ such that $K' \cap \mathsf{co}\, O = \emptyset$. Thus, $K' \subseteq O$, whence $K' \in \mathcal{K}(O) \subseteq \mathcal{O}$ contradicting $K' \in \mathcal{C}'$. □

20.6 $\mathcal{L}$ and $\mathcal{U}_i$ may be exchanged

In this section, we investigate algebraic structures that are both **L**- and **U**-modules. The theory of these structures helps to prove that the initial **L**-construction $\mathcal{L}$ and the initial **U**-construction $\mathcal{U} = \mathcal{U}_i$ commute for *every* ground domain **X**. In contrast to the proof for $\mathcal{L}$ and $\mathcal{U}_S$ in the previous section, this proof will be mainly performed by algebraic methods. Thus, it has a completely different structure. The proved theorem is also unrelated to Th. 20.5.1 unless somebody succeeds in proving $\mathcal{U}_S$ to be initial.

The proof of $\mathcal{L}\mathcal{U}\mathbf{X} \cong \mathcal{U}\mathcal{L}\mathbf{X}$ is done by introducing the algebraic theory of **L**-**U**-modules and showing that both $\mathcal{L}\mathcal{U}\mathbf{X}$ and $\mathcal{U}\mathcal{L}\mathbf{X}$ are free **L**-**U**-modules over **X**. **L**-**U**-modules are domains that are **L**-modules as well as **U**-modules such that the respective additions distribute over each other.

Definition 20.6.1 A domain **M** is an **L**-**U**-*module* iff

(1) There are a least element $\perp$ and a least upper bound $a \sqcup b$ for every two points a and b.

(2) There are a greatest element $\top$ and a greatest lower bound $a \sqcap b$ for every two points a and b. The operation '$\sqcap$' is continuous.

(3) $a \sqcup (b_1 \sqcap b_2) = (a \sqcup b_1) \sqcap (a \sqcup b_2)$ and $a \sqcap (b_1 \sqcup b_2) = (a \sqcap b_1) \sqcup (a \sqcap b_2)$ hold for all a, b_1, b_2 in **M**.

An **L**-**U**-**X**-*module* is a pair $(\mathbf{M}, \eta)$ of an **L**-**U**-module **M** and a morphism $\eta : [\mathbf{X} \to \mathbf{M}]$.

A function $f : [\mathbf{M} \to \mathbf{M}']$ between two **L**-**U**-modules is **L**-**U**-*linear* iff it is both **L**-linear and **U**-linear, i.e. $f(\perp) = \perp$, $f(\top) = \top$, $f(a \sqcup b) = fa \sqcup fb$, and $f(a \sqcap b) = fa \sqcap fb$ hold.

A function f between two **L**-**U**-**X**-modules $(\mathbf{M}, \eta)$ and $(\mathbf{M}', \eta')$ is **L**-**U**-**X**-*linear* iff in addition $f \circ \eta = \eta'$ holds.

By Prop. 17.2.2, part (1) of the definition states that **M** be an **L**-module, whereas part (2) requires it to be a **U**-module. Continuity of '$\sqcup$' need not be explicitly required because of Prop. 5.3.5. The two distributivities in (3) are equivalent to each other as known from lattice theory. Thus, one need only check one of them in every case. The definitions involving **X** are analogues of the definition of R-**X**-modules.

We start the development of the **L**-**U**-theory by two theorems about the generation of **L**-**U**-modules.

Theorem 20.6.2 If **X** is a **U**-module, then $\mathcal{L}\mathbf{X}$ is an **L**-**U**-module and the singleton map is **U**-linear, i.e. $\{\!|\top|\!\} = \top$ and $\{\!|a \sqcap b|\!\} = \{\!|a|\!\} \sqcap \{\!|b|\!\}$ hold.

Proof: We use the explicit representation of $\mathcal{L}\mathbf{X}$ in terms of closed sets. $\{\!|\top|\!\} = {\downarrow}\top = \mathbf{X}$ is the greatest element of $\mathcal{L}\mathbf{X}$.

From $\sqcap : [\mathbf{X} \times \mathbf{X} \to \mathbf{X}]$, we get the operation '$\Cap$' by double extension:

$$\begin{aligned} A \Cap B &= ext\,(\lambda a.\, ext\,(\lambda b.\, \{\!|a \sqcap b|\!\})\, B)\, A \\ &= \mathsf{cl} \textstyle\bigcup_{a \in A} \mathsf{cl} \bigcup_{b \in B} {\downarrow}(a \sqcap b) \\ &= \mathsf{cl} \textstyle\bigcup_{a \in A} \bigcup_{b \in B} \{a \sqcap b\} \\ &= \mathsf{cl}\, \{a \sqcap b \mid a \in A,\, b \in B\} \end{aligned}$$

We claim that the latter set equals $A \cap B$. If a is in A and b in B, then $a \sqcap b$ is in $A \cap B$ since A and B are lower. Conversely, if x is in $A \cap B$, then x is in both A and B, and $x = x \sqcap x$ holds.

Thus, we derived $A \Cap B = A \cap B = A \sqcap B$. The reason for this derivation was to show continuity of '$\sqcap$' in $\mathcal{L}\mathbf{X}$. The operators '$\sqcap$' = '$\cap$' and '$\sqcup$' = '$\cup$' distribute by set theory. $\{\!|a|\!\} \Cap \{\!|b|\!\} = \{\!|a \sqcap b|\!\}$ holds because '$\Cap$' is the double extension of '$\sqcap$'. □

The next theorem is the dual of Th. 20.6.2. Its proof is more complex since the initial **U**-powerdomains $\mathcal{U}\mathbf{X}$ are not known explicitly in general.

Theorem 20.6.3 If $\mathbf{X}$ is an **L**-module, then $\mathcal{U}\mathbf{X}$ is an **L**-**U**-module and the singleton map is **L**-linear, i.e. $\{\!|\bot|\!\} = \bot$ and $\{\!|a \sqcup b|\!\} = \{\!|a|\!\} \sqcup \{\!|b|\!\}$ hold.

Proof: By Prop. 14.3.3 (4), $\mathcal{U}\mathbf{X}$ has a least element, namely $\bot_{\mathbf{U}} \cdot \{\!|\bot_{\mathbf{X}}|\!\} = 1 \cdot \{\!|\bot|\!\} = \{\!|\bot|\!\}$. By double extension, we raise the operation '$\sqcup$' of $\mathbf{X}$ to

$$A \Cap B = ext\,(\lambda a.\, ext\,(\lambda b.\, \{\!|a \sqcup b|\!\})\, B)\, A$$

Then $\{\!|a|\!\} \Cap \{\!|b|\!\} = \{\!|a \sqcup b|\!\}$ holds, and '$\Cap$' is additive in both arguments. Thus, we are done when $A \Cap B = A \sqcup B$ is proved. Additivity of '$\Cap$' implies the distributive laws in connection with '$\uplus$' = '$\sqcap$'.

We show $A \Cap B = A \sqcup B$ in four steps.

Step 1: $A \Cap B \geq B$

$$\begin{array}{lcl} A \Cap B & \geq & ext\,(\lambda a.\, ext\,(\lambda b.\, \{\!|b|\!\})\, B)\, A \\ & \overset{(S3)}{=} & ext\,(\lambda a.\, B)\, A \\ & \geq & ext\,(\lambda a.\, B)\, \{\!|\bot|\!\} \\ & \overset{(P3)}{=} & (\lambda a.\, B)\, \bot \;=\; B \end{array}$$

Step 2: $A \Cap B \geq A$

$$\begin{array}{lcl} A \Cap B & \geq & ext\,(\lambda a.\, ext\,(\lambda b.\, \{\!|a|\!\})\, B)\, A \\ & \geq & ext\,(\lambda a.\, ext\,(\lambda b.\, \{\!|a|\!\})\, \{\!|\bot|\!\})\, A \\ & \overset{(P3)}{=} & ext\,(\lambda a.\, \{\!|a|\!\})\, A \\ & \overset{(S3)}{=} & A \end{array}$$

Step 3: $A \Cap A = A$

We show that the set $S = \{A \in \mathcal{U}\mathbf{X} \mid A \Cap A = A\}$ admits all **U**-**X**-module operations, whence $S = \mathcal{U}\mathbf{X}$ follows by reducedness of $\mathcal{U}\mathbf{X}$.

- θ is in S, since $\theta \Cap \theta = ext\,(\ldots)\, \theta = \theta$ holds by (P1).
- All singletons are in S because of $\{\!|c|\!\} \Cap \{\!|c|\!\} = \{\!|c \sqcup c|\!\} = \{\!|c|\!\}$.
- Limits of directed subsets of S are in S by continuity of '$\Cap$'.
- Let A and B be two members of S. By additivity of '$\Cap$', we obtain

$$\begin{array}{rcl} (A \uplus B) \Cap (A \uplus B) & = & (A \Cap A) \uplus (A \Cap B) \uplus (B \Cap A) \uplus (B \Cap B) \\ & = & A \uplus (A \Cap B) \uplus (B \Cap A) \uplus B \end{array}$$

  Since '$\uplus$' is greatest lower bound[4] and we already showed $A \leq A \Cap B$ and $B \leq B \Cap A$, the result simplifies to $A \uplus B$. Thus, $A \uplus B$ is back in S again.

[4]It is this argument that makes the proof specific for **L**- and **U**-constructions.

Step 4: $A \Cap B = A \sqcup B$.

By $A \Cap B \geq A, B$, the formal set $A \Cap B$ is an upper bound of A and B. If both A and B are below some C, then $A \Cap B \leq C \Cap C = C$ follows from monotonicity of '$\Cap$' and step 3.□

At next, we show that extension preserves linearity of the opposite kind.

Theorem 20.6.4

(1) Let $\mathbf{X}$ be a $\mathbf{U}$-module, $\mathbf{M}$ an $\mathbf{L}$-$\mathbf{U}$-module, and $f : [\mathbf{X} \to \mathbf{M}]$ a $\mathbf{U}$-linear map. Then its (unique) extension $\overline{f} : [\mathcal{L}\mathbf{X} \to \mathbf{M}]$ is $\mathbf{L}$-$\mathbf{U}$-linear.

(2) Let $\mathbf{X}$ be an $\mathbf{L}$-module, $\mathbf{M}$ an $\mathbf{L}$-$\mathbf{U}$-module, and $f : [\mathbf{X} \to \mathbf{M}]$ an $\mathbf{L}$-linear map. Then its (unique) extension $\overline{f} : [\mathcal{U}\mathbf{X} \to \mathbf{M}]$ is $\mathbf{L}$-$\mathbf{U}$-linear.

Proof: We start by proving (1). As an extension generated by $\mathcal{L}$, $\overline{f}$ is $\mathbf{L}$-linear, i.e. $\overline{f}(\bot) = \bot$ and $\overline{f}(A \sqcup B) = \overline{f}(A) \sqcup \overline{f}(B)$ hold.

By $\mathbf{U}$-linearity of f, we get $\overline{f}\{\!|\top|\!\} = f(\top) = \top$. The last equation to show is $\overline{f}(A \sqcap B) = \overline{f}A \sqcap \overline{f}B$. Let F be defined by $F(A, B) = \overline{f}(A \sqcap B)$ and G by $G(A, B) = \overline{f}A \sqcap \overline{f}B$. By Th. 20.6.2, $\mathcal{L}\mathbf{X}$ is an $\mathbf{L}$-$\mathbf{U}$-module as $\mathbf{M}$ is. By the distributive law in part (3) of the definition of $\mathbf{L}$-$\mathbf{U}$-modules, '$\sqcap$' is $\mathbf{L}$-linear in both arguments, whence both F and G are $\mathbf{L}$-linear in both arguments. They coincide for singletons:

$$
\begin{aligned}
\overline{f}(\{\!|a|\!\} \sqcap \{\!|b|\!\}) &= \overline{f}\{\!|a \sqcap b|\!\} && \text{by } \mathbf{U}\text{-linearity of } \{\!|.|\!\} \\
&= f(a \sqcap b) && \text{since } \overline{f} \text{ extends } f \\
&= fa \sqcap fb && \text{by } \mathbf{U}\text{-linearity of } f \\
&= \overline{f}\{\!|a|\!\} \sqcap \overline{f}\{\!|b|\!\}
\end{aligned}
$$

Thus, F and G coincide by initiality of $\mathcal{L}$.

Now, part (1) is completely proved. Part (2) may be proved analogously since the proof of part (1) is completely algebraic, i.e. it does not involve any order-theoretic or topological arguments. □

After these preliminaries, we now come to the main theorem.

Theorem 20.6.5

(1) For every ground domain $\mathbf{X}$, $(\mathcal{L}\mathcal{U}\mathbf{X},\ \iota_{\mathcal{L}} \circ \iota_{\mathcal{U}})$ is an initial $\mathbf{L}$-$\mathbf{U}$-$\mathbf{X}$-module, i.e. there is exactly one $\mathbf{L}$-$\mathbf{U}$-$\mathbf{X}$-linear map to every $\mathbf{L}$-$\mathbf{U}$-$\mathbf{X}$-module.

(2) For every ground domain $\mathbf{X}$, $(\mathcal{U}\mathcal{L}\mathbf{X},\ \iota_{\mathcal{U}} \circ \iota_{\mathcal{L}})$ is also an initial $\mathbf{L}$-$\mathbf{U}$-$\mathbf{X}$-module.

(3) For every ground domain $\mathbf{X}$, the iterated power domains $\mathcal{L}\mathcal{U}\mathbf{X}$ and $\mathcal{U}\mathcal{L}\mathbf{X}$ are isomorphic by an $\mathbf{L}$-$\mathbf{U}$-linear morphism that maps $\{\!|\{\!|x|\!\}_{\mathcal{U}}|\!\}_{\mathcal{L}}$ to $\{\!|\{\!|x|\!\}_{\mathcal{L}}|\!\}_{\mathcal{U}}$.

Proof: We first show (1). Let $(\mathbf{M}, f)$ be a given $\mathbf{L}$-$\mathbf{U}$-$\mathbf{X}$-module. Then $f : [\mathbf{X} \to \mathbf{M}]$ holds. Since $\mathbf{M}$ is a $\mathbf{U}$-module, there is a $\mathbf{U}$-linear extension $\widehat{f} : [\mathcal{U}\mathbf{X} \to \mathbf{M}]$. Since $\mathbf{M}$ is also an $\mathbf{L}$-module, there is an extension $\overline{f} : [\mathcal{L}\mathcal{U}\mathbf{X} \to \mathbf{M}]$ of $\widehat{f}$. By Th. 20.6.4, $\overline{f}$ is $\mathbf{L}$-$\mathbf{U}$-linear. Because of its generation by extending twice, $\overline{f} \circ \iota_{\mathcal{L}} \circ \iota_{\mathcal{U}} = \widehat{f} \circ \iota_{\mathcal{U}} = f$ holds. Hence, $\overline{f}$ is $\mathbf{L}$-$\mathbf{U}$-$\mathbf{X}$-linear from $(\mathcal{L}\mathcal{U}\mathbf{X},\ \iota_{\mathcal{L}} \circ \iota_{\mathcal{U}})$ to $(\mathbf{M}, f)$.

Let F_1 and F_2 be two such $\mathbf{L}$-$\mathbf{U}$-$\mathbf{X}$-linear maps, i.e. $F_i \circ \iota_{\mathcal{L}} \circ \iota_{\mathcal{U}} = f$ holds. We have to show $F_1 = F_2$. Let $F_i' = F_i \circ \iota_{\mathcal{L}}$. The mapping $\iota_{\mathcal{L}}$ is $\mathbf{U}$-linear by Th. 20.6.2. Thus, F_i' are two $\mathbf{U}$-linear maps from $\mathcal{U}\mathbf{X}$ to $\mathbf{M}$ with $F_i' \circ \iota_{\mathcal{U}} = f$. By initiality of $\mathcal{U}$, $F_1' = F_2'$ follows. Thus,

F_1 and F_2 are two **L**-linear maps from $\mathcal{L}\mathcal{U}\mathbf{X}$ to $\mathbf{M}$ with $F_1 \circ \iota_{\mathcal{L}} = F_2 \circ \iota_{\mathcal{L}}$. By initiality of $\mathcal{L}$, $F_1 = F_2$ follows.

Part (2) is proved analogously. Parts (1) and (2) together imply part (3). $\square$

Chapter 21

Subconstructions of the final D-construction

In this chapter, we investigate the sub-constructions of the final **D**-construction $\mathcal{D}$. Existential restriction to the three sub-semirings **B**, $\overline{\mathbf{B}}$, and **C** of **D** gives us three sub-constructions, which we call $\mathcal{S}$, $\overline{\mathcal{S}}$, and $\mathcal{T}$ respectively. The name $\mathcal{S}$ was chosen since $\mathcal{S}$ generalizes the sandwich power construction of Buneman [BDW88] that is defined for algebraic ground domain only. $\overline{\mathcal{S}}$ belongs to $\overline{\mathbf{B}}$ as $\mathcal{S}$ belongs to **B**. The name $\mathcal{T}$ has no deeper meaning except that the letter '$\mathcal{T}$' is the successor of '$\mathcal{S}$'.

The topics of the chapter are prepared by studying the logical properties of predicates with values in **D** in section 21.1. Then, the final **D**-construction $\mathcal{D}$ and its operations are translated into topological terms in section 21.2 for sober ground domains.

$\mathcal{S}$, the existential restriction of $\mathcal{D}$ to **B** is introduced algebraically in section 21.3. In the case of sober ground domains, it is translated into topological terms in section 21.4. Its defining condition, the sandwich condition, can be drastically simplified provided the ground domain is in the class KC & M-CONT (see section 21.5). The sandwich power domains for algebraic ground domains are studied in section 21.6.

In section 21.7, the dual sandwich construction $\overline{\mathcal{S}}$ is introduced as the existential restriction of $\mathcal{D}$ to the semiring $\overline{\mathbf{B}}$. In case of sober ground domains, its defining condition is translated into the language of topology and largely simplified. The restriction $\mathcal{T}$ to semiring **C** and intersections of all these sub-constructions of $\mathcal{D}$ are dealt with in the final section 21.8.

21.1 D-predicates

By Th. 16.3.1, the final **D**-construction $\mathcal{D}$ is isomorphic to the product of the final **L**-construction $\mathcal{L}$ and the final **U**-construction $\mathcal{U}$:

$$\mathcal{D}\mathbf{X} = [[\mathbf{X} \to \mathbf{D}] \stackrel{add}{\to} \mathbf{D}] = [[\mathbf{X} \to \mathbf{L}] \stackrel{add}{\to} \mathbf{L}] \times [[\mathbf{X} \to \mathbf{U}] \stackrel{add}{\to} \mathbf{U}]$$

Although the second equality is only an isomorphism, we do not write down the isomorphisms explicitly for simplification. Instead, we directly apply pairs of functions to pairs of predicates subsuming an equality $(P^L,\, P^U)(p^L,\, p^U) = (P^L p^L,\, P^U p^U)$.

Due to the equation $\mathbf{D} = \mathbf{L} \times \mathbf{U}$, the predicate domains $[\mathbf{X} \to \mathbf{D}]$ and $[\mathbf{X} \to \mathbf{L}] \times [\mathbf{X} \to \mathbf{U}]$ are also isomorphic. Again, we assume equality here, i.e. a pair (p^L, p^U) in $[\mathbf{X} \to \mathbf{L}] \times [\mathbf{X} \to \mathbf{U}]$ may be directly applied to a point x resulting in $(p^L, p^U)\, x = (p^L x, p^U x)$.

In section 17.1, we mentioned negation $\neg : [\mathbf{D} \to \mathbf{D}]$ that maps F to T, T to F, and $\perp$ and W to themselves. In terms of $\mathbf{L} \times \mathbf{U}$, $(0, 0)$ is mapped to $(1, 1)$ and vice versa, whereas $(0, 1)$ and $(1, 0)$ are mapped to themselves. Assuming two operations $\neg_L : [\mathbf{L} \to \mathbf{U}]$ and $\neg_U : [\mathbf{U} \to \mathbf{L}]$ which both exchange 0 and 1, one obtains $\neg(l, u) = (\neg_U u, \neg_L l)$. Note that $\neg_L$ and $\neg_U$ are inverse order isomorphisms between $\mathbf{L} = \{0 < 1\}$ and $\mathbf{U} = \{1 < 0\}$.

The operations $\neg_L$ and $\neg_U$ may also be used to convert $\mathbf{L}$-predicates into $\mathbf{U}$-predicates and vice versa. For a given $\mathbf{L}$-predicate $p : [\mathbf{X} \to \mathbf{L}]$, $\widetilde{p} = \neg_L \circ p : [\mathbf{X} \to \mathbf{U}]$ provides a $\mathbf{U}$-predicate. Conversely, a $\mathbf{U}$-predicate $q : [\mathbf{X} \to \mathbf{U}]$ is transformed into the $\mathbf{L}$-predicate $\widetilde{q} = \neg_U \circ q : [\mathbf{X} \to \mathbf{L}]$. De Morgan's formulae $\widetilde{p_1 + p_2} = \widetilde{p_1} \cdot \widetilde{p_2}$ and $\widetilde{p_1 \cdot p_2} = \widetilde{p_1} + \widetilde{p_2}$ are easily verified for two $\mathbf{L}$-predicates p_1 and p_2 as well as for two $\mathbf{U}$-predicates. The negation of a $\mathbf{D}$-predicate $(p, q) : [\mathbf{X} \to \mathbf{D}] = [\mathbf{X} \to \mathbf{L}] \times [\mathbf{X} \to \mathbf{U}]$ then becomes $\neg(p, q) = (\widetilde{q}, \widetilde{p})$.

21.2 $\mathcal{D}$ in topological terms

By Th. 16.3.1, the final $\mathbf{D}$-construction $\mathcal{D}$ is the product of $\mathcal{L}$ and $\mathcal{U}$. For sober ground domain, we know topological representations of $\mathcal{L}$ and $\mathcal{U}$. Hence, $\mathcal{D}$ may also be represented in topological terms:

- $\mathcal{D}\mathbf{X}$ is the set of all pairs (C, K) of a closed subset C of $\mathbf{X}$ with a compact upper set K of $\mathbf{X}$.
- $(C, K) \le (C', K')$ iff $C \subseteq C'$ and $K \supseteq K'$.
- The limit of a directed family $(C_i, K_i)_{i \in I}$ is $(\mathsf{cl} \bigcup_{i \in I} C_i, \bigcap_{i \in I} K_i)$.
- $\theta = (\emptyset, \emptyset)$
- $(C, K) \uplus (C', K') = (C \cup C', K \cup K')$
- $\{\!|x|\!\} = (\downarrow x, \uparrow x)$
- For $f : [\mathbf{X} \to \mathcal{D}\mathbf{Y}]$, $\mathit{ext}\, f\, A = (\mathsf{cl} \bigcup f^L[A^L], \bigcup f^U[A^U])$.

For extension and the derived operations below, we adopt the convention to denote the lower component of a pair A by A^L and the upper component by A^U. Similarly, a function $f : [\mathbf{X} \to \mathcal{D}\mathbf{Y}]$ is identified with a pair $(f^L, f^U) : [\mathbf{X} \to \mathcal{L}\mathbf{Y}] \times [\mathbf{X} \to \mathcal{U}\mathbf{Y}]$.

- $\mathit{map}\, f\, A = (\mathsf{cl}\, f^L[A^L], \uparrow f^U[A^U])$
- $A \overrightarrow{\times} B = A \overleftarrow{\times} B = (A^L \times B^L, A^U \times B^U)$
  This equation shows that $\mathcal{D}$ becomes symmetric when restricted to a class of domains closed w.r.t. Cartesian product and contained in the class of sober domains. The largest such class we know of is the class of multi-continuous domains by Prop. 8.4.2 and Th. 8.8.1.

- $r \cdot (C, K) = \begin{cases} (C, K) & \text{if } r = \mathsf{T} \\ (\emptyset, \emptyset) & \text{if } r = \mathsf{F} \\ (\emptyset, K) & \text{if } r = \bot \\ (C, \emptyset) & \text{if } r = \mathsf{W} \end{cases}$

  For the last two cases, remember $\bot = (1, 0)$ and $\mathsf{W} = (0, 1)$.

- $$\begin{aligned} \textit{filter}\ p\ A &= (\mathsf{cl}\,\{a \in A^L \mid p^L\, a = 1\},\ \uparrow\{a \in A^U \mid p^U\, a = 1\}) \\ &= (\mathsf{cl}\,\{a \in A^L \mid p\, a \geq \mathsf{T}\},\ \uparrow\{a \in A^U \mid p\, a \leq \mathsf{T}\}) \end{aligned}$$

21.3 $\mathcal{S}$ — the existential restriction of $\mathcal{D}$ to B

Since **B** is a sub-semiring of **D**, we know from Th. 14.6.2 that the construction $\mathcal{S}$ defined by

- $\mathcal{S}\mathbf{X} = \{A \in \mathcal{D}\mathbf{X} \mid \forall p \in [\mathbf{X} \to \mathbf{B}] : Ap \in \mathbf{B}\}$

is a power construction with characteristic semiring **B**. In Th. 21.6.4 below, we shall see that $\mathcal{S}$ is a generalization of the *sandwich power construction* defined in [BDW88] for algebraic ground domains and investigated further in [Gun89b, Gun90, Hec90c, Hec90a]. In anticipation of the theorem, we chose the abbreviation $\mathcal{S}$ and call the domain $\mathcal{S}\mathbf{X}$ *sandwich power domain* and its elements *sandwiches*. Consequently, the condition restricting $\mathcal{D}\mathbf{X}$ to $\mathcal{S}\mathbf{X}$ is called *sandwich condition* or shorter *condition S*. Since $\mathcal{D}\mathbf{X}$ is isomorphic to the product $\mathcal{L}\mathbf{X} \times \mathcal{U}\mathbf{X}$, we sometimes call its elements *pairs*.

We start the investigation of $\mathcal{S}$ by illustrating the location of $\mathcal{S}\mathbf{X}$ inside of $\mathcal{D}\mathbf{X}$:

Proposition 21.3.1
$\mathcal{S}\mathbf{X}$ is a closed subset of $\mathcal{D}\mathbf{X}$. Hence, every pair below a sandwich is a sandwich.

Proof: The sandwich power domain is characterized by the condition $Ap \in \mathbf{B}$ for all $p : [\mathbf{X} \to \mathbf{B}]$. Let $A' \leq A$. Then for all $p : [\mathbf{X} \to \mathbf{B}]$, $A'p \leq Ap$ holds. Since **B** is a lower set in **D**, $Ap \in \mathbf{B}$ implies $A'p \in \mathbf{B}$. Thus, $\mathcal{S}\mathbf{X}$ is a lower set in $\mathcal{D}\mathbf{X}$. It is closed since it is a sub-domain. Or more directly: since $(\bigsqcup_{i\in I} A_i)\, p = \bigsqcup_{i\in I}(A_i p)$ and **B** is closed in **D**. □

In the sequel, we shall transform the sandwich condition gradually. We start by turning the universal quantification into an existential one.

$$\begin{aligned} \forall p \in [\mathbf{X} \to \mathbf{B}] : A\, p \in \mathbf{B} \quad &\text{iff} \quad \forall p \in [\mathbf{X} \to \mathbf{D}] : ((\forall x \in \mathbf{X} : p\, x \in \mathbf{B}) \Rightarrow A\, p \in \mathbf{B}) \\ &\text{iff} \quad \forall p \in [\mathbf{X} \to \mathbf{D}] : (A\, p = \mathsf{W} \Rightarrow \exists x \in \mathbf{X} : p\, x = \mathsf{W}) \end{aligned}$$

This formula may be interpreted such that $\mathcal{S}\mathbf{X}$ consists of all consistent second order predicates in $\mathcal{D}\mathbf{X}$. A consistent second order predicate does not create inconsistencies by itself. If it results in an inconsistency ($Ap = \mathsf{W}$), then its argument already was inconsistent ($p\, x = \mathsf{W}$ for some x).

By splitting the pairs into components, we obtain further by $\mathsf{W} = (1, 0)$:

$$\begin{aligned} \text{iff} \quad & \forall p^L \in [\mathbf{X} \to \mathbf{L}],\ p^U \in [\mathbf{X} \to \mathbf{U}] : \\ & \qquad (A^L\, p^L = 1 \text{ and } A^U\, p^U = 0 \Rightarrow \exists x \in \mathbf{X} : p^L\, x = 1 \text{ and } p^U\, x = 0) \end{aligned}$$

Renaming p^L into p and p^U into q, the conclusion looks like $p\,x = 1$ and $q\,x = 0$ for some x. It may be more elegantly formulated. Negating q, one gets $p\,x = 1$ and $\widetilde{q}\,x = 1$ for some x. Both p and $\widetilde{q}$ are **L**-predicates and thus may be multiplied. We equivalently obtain $(p \cdot \widetilde{q})\,x = 1$ for some x, or $p \cdot \widetilde{q} \neq \underline{0}$ where $\underline{0}$ is the constant **L**-predicate $\lambda x.\,0$. Thus, we finally get for (A^L, A^U) in $\mathcal{D}\mathbf{X}$:

Proposition 21.3.2

$$(A^L, A^U) \in \mathcal{S}\mathbf{X} \ \text{ iff } \ \forall p : [\mathbf{X} \to \mathbf{L}],\, q : [\mathbf{X} \to \mathbf{U}] : (A^L\,p = 1 \text{ and } A^U\,q = 0 \text{ implies } p \cdot \widetilde{q} \neq \underline{0})$$

21.4 $\mathcal{S}$ in topological terms

In this section, we first want to represent $\mathcal{S}$ in terms of open grills and filters. In case of a sober ground domain, we then translate the grills into closed sets and the filters into compact upper sets.

For the translation into grills and filters, we use the condition in the form

$$(A^L\,p^L = 1 \text{ and } A^U\,p^U = 0 \Rightarrow \exists x \in \mathbf{X} : p^L\,x = 1 \text{ and } p^U\,x = 0)$$

Translating 0 and 1 into $\bot$ and $\top$, we obtain

$$(A^L\,p^L = \top \text{ and } A^U\,p^U = \top \Rightarrow \exists x \in \mathbf{X} : p^L\,x = \top \text{ and } p^U\,x = \top)$$

since $\top_{\mathbf{L}} = 1$ and $\top_{\mathbf{U}} = 0$.

Let $\mathcal{G}$ be the open grill belonging to A^L and $\mathcal{O}$ the open filter belonging to A^U. To complete the translation to set notation, we represent the predicates p^L and p^U by open sets O^L and O^U. Then $p^L\,x = \top$ means $x \in O^L$, and same for O^U. Similarly, $A^L p^L = \top$ means $O^L \in \mathcal{G}$, and $A^U p^U = \top$ becomes $O^U \in \mathcal{O}$. Hence, we obtain

$$\begin{aligned}(\mathcal{G}, \mathcal{O}) \in \mathcal{S}\mathbf{X} \quad &\text{iff} \quad \forall O^L, O^U \in \Omega\mathbf{X} : \\ &\qquad\qquad (O^L \in \mathcal{G} \text{ and } O^U \in \mathcal{O} \Rightarrow \exists x \in \mathbf{X} : x \in O^L \text{ and } x \in O^U) \\ &\text{iff} \quad \forall O^L \in \mathcal{G},\, O^U \in \mathcal{O} : O^L \cap O^U \neq \emptyset\end{aligned}$$

Formulating this in English words, one obtains

Proposition 21.4.1 The sandwich power domain $\mathcal{S}\mathbf{X}$ over some ground domain $\mathbf{X}$ is isomorphic to the set of all pairs $(\mathcal{G}, \mathcal{O})$ of an open grill and an open filter of $\mathbf{X}$ where every member of $\mathcal{G}$ meets all members of $\mathcal{O}$.

For sober ground domain $\mathbf{X}$, one can go one step further and translate the open filters into compact upper sets K. The translation of open grills into closed sets C is always possible. $O^L \in \mathcal{G}$ becomes $C \cap O^L \neq \emptyset$, and $O^U \in \mathcal{O}$ becomes $K \subseteq O^U$.

Hence, the restriction translates into: for all open sets O^L and O^U, if C meets O^L and $K \subseteq O^U$ then O^L meets O^U. For fixed C and O^U, the following holds:

Every open set meeting C meets O^U
  iff every open environment of every point of C meets O^U
  iff every point of C is in the closure of O^U by Prop. 4.2.2
  iff $C \subseteq \mathsf{cl}\, O^U$

Hence, one obtains

Theorem 21.4.2 The sandwich power domain $\mathcal{S}\mathbf{X}$ over a sober ground domain $\mathbf{X}$ is isomorphic to the set of all pairs (C, K) of a closed set C and a compact upper set K such that for all open sets O with $K \subseteq O$ the inclusion $C \subseteq \mathsf{cl}\, O$ holds. Order and power operations are inherited from $\mathcal{D}$ (see section 21.2).

Two remarks seem to be appropriate. First, the condition '$K \subseteq O$ implies $C \subseteq \mathsf{cl}\, O$' looks quite strange, and it is not obvious how it could have been found without considering the second order predicates. Thus, this strange topological condition is motivated by a natural condition on second order predicates. Second, if we had defined a power domain construction directly as in the theorem above, we would have been forced to verify that each power operation respects the topological criterion. This would have been a non-trivial task, in particular for the extension functional.

Before we investigate the restriction '$K \subseteq O$ implies $C \subseteq \mathsf{cl}\, O$' more closely in the next section, we briefly consider existential quantification in $\mathcal{S}\mathbf{X}$.

- $ex\, p\, A = (ex_L\; p^L A^L,\; ex_U\, p^U A^U)$ for $p : [\mathbf{X} \to \mathbf{B}]$.
  The pair on the right hand side denotes a Boolean. We want to translate this expression into ordinary Boolean notation.

  The result is $\mathsf{T} = (1, 1)$ iff $ex_L\; p^L A^L = 1$. The second component does not matter because $(1, 0)$ is not a legal Boolean. The equation $ex_L\; p^L A^L = 1$ in turn is equivalent to the existence of $x \in A^L$ such that $p^L x = 1$. For this x, $p^U x$ cannot be 0 since 0 is not a legal Boolean. Thus, $p^L x = 1$ is equivalent to $p\, x = \mathsf{T}$.

  Similarly, the result is $\mathsf{F} = (0, 0)$ iff $ex_U\, p^U A^U = 0$. This in turn is equivalent to the non-existence of $x \in A^L$ such that $p^U x = 1$, i.e. $p^U x$ is 0 for all $x \in A^U$. Since $(1, 0)$ is not a legal Boolean, this is equivalent to $p\, x = \mathsf{F}$.

  Thus, $ex\; p\; A = \begin{cases} \mathsf{T} & \text{if } \exists\, u \in A^L : p\, u = \mathsf{T} \\ \mathsf{F} & \text{if } \forall\, v \in A^U : p\, v = \mathsf{F} \\ \bot & \text{otherwise} \end{cases}$

  The sandwich condition guarantees that the first two cases exclude each other.

- The formula for ne is derived from that for ex:

  $$ne\; A = ex\; (\lambda x.\, \mathsf{T})\; A = \begin{cases} \mathsf{T} & \text{if } A^L \neq \emptyset \\ \mathsf{F} & \text{if } A^U = \emptyset \\ \bot & \text{otherwise} \end{cases}$$

21.5 Simplification of the sandwich condition

In this section, we try to simplify the restriction '$K \subseteq O$ open implies $C \subseteq \mathsf{cl}\, O$' for pairs (C, K) of a closed and a compact upper set. A drastic simplification of this condition is achieved for a special class of ground domains. An example is presented showing that general simplification is impossible however.

Theorem 21.5.1 Let $\mathbf{X}$ be a multi-continuous domain in class KC. Then for all closed $C \subseteq \mathbf{X}$ and all compact upper sets $K \subseteq \mathbf{X}$, the following two statements are equivalent:

(1) The sandwich condition: If $K \subseteq O$ for some open set O, then $C \subseteq \mathsf{cl}\, O$.

(2) $C \subseteq {\downarrow} K$.

Property KC was introduced by Prop. 5.5.2 and means that the intersection of two upper cones is compact. All finitely continuous domains are continuous and satisfy this condition (Prop. 7.2.2), i.e. they are covered by the theorem above.

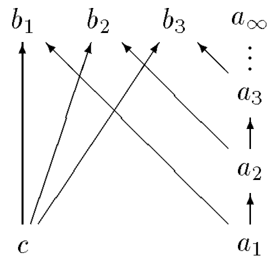
Proof: If $C \subseteq {\downarrow} K$ holds, then $K \subseteq O$ implies $C \subseteq {\downarrow} K \subseteq {\downarrow} O \subseteq \mathsf{cl}\, O$.

The opposite direction is much more difficult. A multi-continuous domain is sober, whence in K-RD. Thus, the pre-condition $\mathbf{X}$ in KC & K-RD of the lemmas 8.10.1 and 8.10.2 is satisfied.

Let C be a closed set and K be a compact upper set satisfying the sandwich condition. Let $\mathcal{F}$ be the set of all finitary upper sets F such that there is an open set O with $K \subseteq O \subseteq F$. From Th. 8.3.2, we know that $\mathcal{F}$ is a $\supseteq$-directed set whose intersection is K.

For all sets $F \in \mathcal{F}$, $K \subseteq O \subseteq F$ holds, whence $C \subseteq \mathsf{cl}\, O \subseteq \mathsf{cl}\, F$ follows from the sandwich condition. Lemma 8.10.1 implies $\mathsf{cl}\, F = {\downarrow} F$ if $\mathbf{X}$ is chosen as the closed set in the Lemma. Hence, $C \subseteq \bigcap_{F \in \mathcal{F}} {\downarrow} F = {\downarrow} \bigcap_{F \in \mathcal{F}} F = {\downarrow} K$ applying Lemma 8.10.2. □

Property KC is important for the validity of the theorem. Without it, the theorem does not hold even for algebraic ground domains. Consider the following example. Let $\mathbf{X} = \{a_1, a_2, a_3, \ldots, a_\infty, b_1, b_2, b_3, \ldots, c\}$. There is no point b_∞. The a-points form an ascending sequence: $a_1 < a_2 < \cdots < a_\infty$, whereas the b-points are incomparable. Every a-point is below the corresponding b-point: $a_n < b_n$. The remaining point c is below all b-points, but not below any a-point, not even below a_∞. The domain is visualized by the following picture:



This domain is algebraic. Property KC is not satisfied since ${\uparrow} c \cap {\uparrow} a_1 = \{b_1, b_2, \ldots\}$ is an infinite discrete open set covered by the infinite number of open sets $\{b_i\}$, whence it is not compact. (Interestingly, the additional inequality $c \leq a_\infty$ would establish property KC, but would destroy algebraicity on the other hand.) Let $C = {\downarrow} c = \{c\}$ and let $K = {\uparrow} a_\infty = \{a_\infty\}$. C and K satisfy the sandwich condition although $C \subseteq {\downarrow} K$, i.e. $c \leq a_\infty$ does not hold. Every open set with $K \subseteq O$, i.e. $a_\infty \in O$, contains some point a_n with $n < \infty$. Hence, it contains b_n because it is an upper set, whence c is in ${\downarrow} O$ by $c \leq b_n$. Thus, c is more than ever in $\mathsf{cl}\, O$. □

21.6 $\mathcal{S}$ for algebraic ground domains

For algebraic ground domain, the sandwich power domain is shown to be algebraic again. The sandwich condition looks particularly simple for its isolated points. This allows to show that our sandwich power domain generalizes the sandwich power domain of [BDW88].

If $\mathbf{X}$ is algebraic, then both $\mathcal{L}\mathbf{X}$ and $\mathcal{U}\mathbf{X}$ are algebraic. Their bases are given by the sets of all $\downarrow F$ and $\uparrow F$ respectively for finite subsets of $\mathbf{X}^0$. By Prop. 8.4.2, $\mathcal{D}\mathbf{X}$ is algebraic, and by Prop. 6.1.6, its base is the set of all pairs $(\downarrow E, \uparrow F)$ where E and F are finite subsets of $\mathbf{X}^0$. These pairs are also isolated in $\mathcal{S}\mathbf{X}$ provided they satisfy the sandwich condition because $\mathcal{S}\mathbf{X}$ is a sub-domain of $\mathcal{D}\mathbf{X}$. Every point in $\mathcal{D}\mathbf{X}$ is the directed limit of such pairs. Since all pairs below a sandwich are sandwiches again by Prop. 21.3.1, every point of $\mathcal{S}\mathbf{X}$ is the limit of a directed set of isolated sandwiches. Thus, we obtain

Proposition 21.6.1
The sandwich power domain over an algebraic ground domain is algebraic. Its base is the set of all sandwiches $(\downarrow E, \uparrow F)$ where E and F are finite subsets of $\mathbf{X}^0$.

The sandwich criterion simplifies drastically for such isolated pairs:

Lemma 21.6.2 Let $\mathbf{X}$ be a domain. If E and F are finite sets of isolated points of $\mathbf{X}$, then $(\downarrow E, \uparrow F)$ satisfies the sandwich condition iff $E \subseteq \downarrow\uparrow F$.

Proof: Assume $E \subseteq \downarrow\uparrow F$. Then for every $O \supseteq \uparrow F$, $\downarrow E \subseteq \downarrow\uparrow F \subseteq \downarrow O \subseteq \mathsf{cl}\, O$ holds.

For the opposite, note that $\uparrow F$ is open since F consists of isolated points. Thus, the sandwich condition implies $E \subseteq \downarrow E \subseteq \mathsf{cl}\uparrow F$. Since E consists of isolated points, Prop. 6.1.3 yields $E \subseteq \downarrow\uparrow F$. □

The representation of the base of $\mathcal{S}\mathbf{X}$ may even be further simplified choosing suitable sets E and F.

Lemma 21.6.3 Let $\mathbf{X}$ be algebraic and let E and F be finite subsets of $\mathbf{X}^0$ with $E \subseteq \downarrow\uparrow F$. Then there is a finite subset F' of $\mathbf{X}^0$ with $\uparrow F = \uparrow F'$ and $E \subseteq \downarrow F'$.

Proof: Since $E \subseteq \downarrow\uparrow F$, for every $e \in E$ there is some point $x_e \in \mathbf{X}$ and some point $f_e \in F$ such that $e \leq x_e \geq f_e$. By Prop. 6.4.1, the points x_e may be assumed to be in the base $\mathbf{X}^0$. With $E' = \{x_e \mid e \in E\}$, we define $F' = E' \cup F$. E' is a finite subset of $\mathbf{X}^0$, whence F' also is. All points e in E are below x_e in F', whence $E \subseteq \downarrow F'$ follows. $\uparrow F \subseteq \uparrow F'$ immediately follows from $F \subseteq F'$. For the opposite inclusion, x_e is above f_e for all e in E, whence $E' \subseteq \uparrow F$ whence $F' \subseteq \uparrow F$. □

Summarizing, we obtain the following theorem:

Theorem 21.6.4 For algebraic ground domain $\mathbf{X}$, our sandwich power domain over $\mathbf{X}$ is algebraic and coincides with the sandwich power domain of [BDW88] and [Gun89b, Gun90]. Its base is the set of all pairs $(\downarrow E, \uparrow F)$ with $E \subseteq \downarrow\uparrow F$, or equivalently the set of all pairs $(\downarrow E, \uparrow F)$ with $E \subseteq \downarrow F$, where in both cases E and F are finite subsets of $\mathbf{X}^0$.

Proof: For the comparison with the sandwich power domain in [BDW88, Gun89b, Gun90] notice that the authors of these papers write the sandwiches the other way round, i.e. the lower set to the right. Correcting this and translating notation, the paper [Gun89b] defines the sandwich power domain to be the ideal completion of all pairs (E, F) of finite subsets of $\mathbf{X}^0$ such that there is a finite subset G of $\mathbf{X}^0$ with $E \subseteq {\downarrow}G$ and $G \subseteq {\uparrow}F$. This directly implies $E \subseteq {\downarrow}{\uparrow}F$, and conversely, G may be chosen as the set E' in the proof of Lemma 21.6.3. These pairs are pre-ordered by $(E, F) \preceq (E', F')$ iff ${\downarrow}E \subseteq {\downarrow}E'$ and ${\uparrow}F \supseteq {\uparrow}F'$. Hence, the poset of equivalence classes of this pre-ordered set is just our base as presented in the theorem. □

Since the existential restriction to $\mathbf{B}$ of the final double power construction properly generalizes the sandwich power construction to all domains, we are allowed to denote it by $\mathcal{S}$ as we have already done throughout the preceding sections.

21.7 $\overline{\mathcal{S}}$ — the existential restriction to $\overline{\mathbf{B}}$

Instead of restricting $\mathcal{D}$ to the semiring $\mathbf{B} = \{\bot, \mathsf{F}, \mathsf{T}\}$, we may also restrict it to the dual semiring $\overline{\mathbf{B}} = \{\mathsf{F}, \mathsf{T}, \mathsf{W}\}$. The result is denoted by $\overline{\mathcal{S}}$. By Th. 14.6.2, it is defined by

- $\overline{\mathcal{S}}\mathbf{X} = \{A \in \mathcal{D}\mathbf{X} \mid \forall p \in [\mathbf{X} \to \overline{\mathbf{B}}] : A\,p \in \overline{\mathbf{B}}\}$

It is a power construction with characteristic semiring $\overline{\mathbf{B}}$. In contrast to $\mathcal{S}$, this construction was never proposed in the literature. Hence, there is no established name for its members. We sometimes call them *dual sandwiches*. The condition restricting $\mathcal{D}$ to $\overline{\mathcal{S}}$ is called *condition* $\overline{S}$.

We start the investigation of $\overline{\mathcal{S}}$ by illustrating the location of $\overline{\mathcal{S}}\mathbf{X}$ inside of $\mathcal{D}\mathbf{X}$:

Proposition 21.7.1 $\overline{\mathcal{S}}\mathbf{X}$ is an upper subset of $\mathcal{D}\mathbf{X}$. Hence, every pair above a dual sandwich is a dual sandwich.

Proof: Let $A \le A'$ where A is in $\overline{\mathcal{S}}\mathbf{X}$. Then for all $p : [\mathbf{X} \to \overline{\mathbf{B}}]$, $Ap \le A'p$ holds. Since $\overline{\mathbf{B}}$ is an upper set in $\mathbf{D}$, $Ap \in \overline{\mathbf{B}}$ implies $A'p \in \overline{\mathbf{B}}$. Thus, $\overline{\mathcal{S}}\mathbf{X}$ is an upper set in $\mathcal{D}\mathbf{X}$. □

Condition $\overline{S}$ for $A = (A^L, A^U)$ may be transformed as follows:

$$\forall p \in [\mathbf{X} \to \overline{\mathbf{B}}] : A\,p \in \overline{\mathbf{B}} \quad \text{iff} \quad \forall p \in [\mathbf{X} \to \mathbf{D}] : (A\,p = \bot \Rightarrow \exists x \in \mathbf{X} : p\,x = \bot)$$

This formula may be interpreted such that $\overline{\mathcal{S}}\mathbf{X}$ consists of all omniscient second order predicates of $\mathcal{D}\mathbf{X}$. An omniscient second order predicate does not create ignorance by itself. If it results in a state of ignorance, $(Ap = \bot)$, then its argument already did not know everything ($p\,x = \bot$ for some x).

By splitting the pairs into components, we obtain further by $\bot = (0, 1)$:

$$\begin{aligned}\text{iff} \quad & \forall p^L \in [\mathbf{X} \to \mathbf{L}], p^U \in [\mathbf{X} \to \mathbf{U}] : \\ & (A^L\,p^L = 0 \text{ and } A^U\,p^U = 1 \Rightarrow \exists x \in \mathbf{X} : p^L\,x = 0 \text{ and } p^U\,x = 1)\end{aligned}$$

In transforming further, we employ the same trick as we did for condition S. Renaming p^L into p and p^U into q, the conclusion $p\,x = 0$ and $q\,x = 1$ for some x is equivalent to $\widetilde{p} \cdot q \ne \underline{0}$ in terms of $\mathbf{U}$-predicates. Thus, we finally get for (A^L, A^U) in $\mathcal{D}\mathbf{X}$:

Proposition 21.7.2

$$(A^L, A^U) \in \overline{\mathcal{S}}\mathbf{X} \text{ iff } \forall p : [\mathbf{X} \to \mathbf{L}],\ q : [\mathbf{X} \to \mathbf{U}] : (A^L\, p = 0 \text{ and } A^U\, q = 1 \text{ implies } \widetilde{p} \cdot q \neq \underline{0})$$

Next, we go on to open grills $\mathcal{G}$ and open filters $\mathcal{O}$. We use the condition

$$(A^L\, p^L = 0 \text{ and } A^U\, p^U = 1 \Rightarrow \exists x \in \mathbf{X} : p^L\, x = 0 \text{ and } p^U\, x = 1)$$

The predicates p^L and p^U are replaced by open sets O^L and O^U. All atomic components of the formula above are just negations of the corresponding components in case of the condition S. Hence

$$\begin{array}{lll}(\mathcal{G}, \mathcal{O}) \in \overline{\mathcal{S}}\mathbf{X} & \text{iff} & \forall O^L, O^U \in \Omega\mathbf{X} : \\ & & \qquad (O^L \notin \mathcal{G} \text{ and } O^U \notin \mathcal{O} \Rightarrow \exists x \in \mathbf{X} : x \notin O^L \text{ and } x \notin O^U) \\ & \text{iff} & \forall O^L, O^U \in \Omega\mathbf{X} : \\ & & \qquad (O^L \cup O^U = \mathbf{X} \Rightarrow O^L \in \mathcal{G} \text{ or } O^U \in \mathcal{O})\end{array}$$

For sober ground domain $\mathbf{X}$, we translate the open filters $\mathcal{O}$ into compact upper sets K, and the open grills $\mathcal{G}$ into closed sets C. For pairs (C, K) of a closed set and a compact upper set, condition $\overline{S}$ then becomes

$$O^L \cup O^U = \mathbf{X} \Rightarrow C \cap O^L \neq \emptyset \text{ or } K \subseteq O^U$$

We claim that the last implication is equivalent to '$C \subseteq O^U \Rightarrow K \subseteq O^U$'. Let $C \subseteq O^U$, and let O^L be the complement of C. Then $C \cap O^L = \emptyset$ holds, and $C \subseteq O^U$ implies $O^L \cup O^U = \mathbf{X}$. By condition $\overline{S}$, one may conclude $K \subseteq O^U$.

Conversely, assume $O^L \cup O^U = \mathbf{X}$ and $C \cap O^L = \emptyset$. Then $C = C \cap \mathbf{X} = C \cap (O^L \cup O^U) = C \cap O^U$, whence $C \subseteq O^U$. By the new condition, $K \subseteq O^U$ follows.

We now simplified $\overline{S}$ to the condition that $C \subseteq O$ implies $K \subseteq O$ for all open sets O. By Lemma 4.4.4, this is equivalent to $K \subseteq \uparrow C$.

Theorem 21.7.3 The sandwich power domain $\overline{\mathcal{S}}\mathbf{X}$ over a sober ground domain $\mathbf{X}$ is isomorphic to the set of all pairs (C, K) of a closed set C and a compact upper set K such that $K \subseteq \uparrow C$ holds. Order and power operations are inherited from $\mathcal{D}$ (see section 21.2).

The inherent asymmetry of domains effects that condition $\overline{S}$ may always be simplified to $K \subseteq \uparrow C$. Condition S however can be simplified to the comparable form $C \subseteq \downarrow K$ in special cases only.

An isolated pair $(\downarrow E, \uparrow F)$ where E and F are finite subsets of $\mathbf{X}^0$ is a dual sandwich iff $F \subseteq \uparrow\downarrow E$. Later, we shall see that $\overline{\mathcal{S}}\mathbf{X}$ is algebraic whenever $\mathbf{X}$ is algebraic. The proof of this claim is postponed to section 22.10 where it is handled together with some other algebraicity proofs.

21.8 $\mathcal{T}$ and some intersections

The third sub-semiring $\mathbf{C}$ of $\mathbf{D}$ also gives rise to a sub-construction of $\mathcal{D}$. We call it $\mathcal{T}$; this name has no deeper meaning. $\mathcal{T}$ is defined by

- $\mathcal{T}\mathbf{X} = \{P \in \mathcal{D}\mathbf{X} \mid \forall p \in [\mathbf{X} \to \mathbf{C}] : Pp \in \mathbf{C}\}$

From Th. 14.6.2, we know that $\mathcal{T}$ is a power construction with characteristic semiring $\mathbf{C}$. We do not transform the defining condition of $\mathcal{T}$ further since the outcome would not be very interesting.

Since the intersection of sub-constructions is a sub-construction again, we obtain some more sub-constructions of $\mathcal{D}$ by intersecting $\mathcal{S}$, $\overline{\mathcal{S}}$, and $\mathcal{T}$. We denote them by $\mathcal{S}\overline{\mathcal{S}} = \mathcal{S} \cap \overline{\mathcal{S}}$, $\mathcal{S}\mathcal{T} = \mathcal{S} \cap \mathcal{T}$, and $\mathcal{T}\overline{\mathcal{S}} = \mathcal{T} \cap \overline{\mathcal{S}}$.The intersection of all three is $\mathcal{S}\overline{\mathcal{S}}$ because of the following inclusion statement:

Proposition 21.8.1 For all ground domains $\mathbf{X}$, $\mathcal{S}\overline{\mathcal{S}}\mathbf{X} \subseteq \mathcal{T}\mathbf{X}$ holds.

Proof: Let A be in $\mathcal{S}\overline{\mathcal{S}}\mathbf{X}$, and let $p : [\mathbf{X} \to \mathbf{C}]$. Then p is also in $[\mathbf{X} \to \mathbf{B}]$ and $[\mathbf{X} \to \overline{\mathbf{B}}]$, whence $A\,p$ is in both $\mathbf{B}$ and $\overline{\mathbf{B}}$ as A is in both $\mathcal{S}\mathbf{X}$ and $\overline{\mathcal{S}}\mathbf{X}$. Thus, $A\,p \in \mathbf{C}$. □

The mutual inclusions of these power constructions are visualized by the following diagram.

$$\begin{tikzcd}
 & \mathcal{D} & \\
\mathcal{S} \arrow[ur] & & \overline{\mathcal{S}} \arrow[ul] \\
 & \mathcal{T} \arrow[uu] & \\
\mathcal{S}\mathcal{T} \arrow[uu] \arrow[ur] & & \mathcal{T}\overline{\mathcal{S}} \arrow[uu] \arrow[ul] \\
 & \mathcal{S}\overline{\mathcal{S}} \arrow[ul] \arrow[ur] &
\end{tikzcd}$$

The dotted lines separate the constructions with different characteristic semirings. $\mathcal{D}$ has semiring $\mathbf{D}$, $\mathcal{S}$ has $\mathbf{B}$, $\overline{\mathcal{S}}$ has $\overline{\mathbf{B}}$, and the constructions $\mathcal{T}$, $\mathcal{S}\mathcal{T}$, $\mathcal{T}\overline{\mathcal{S}}$, and $\mathcal{S}\overline{\mathcal{S}}$ have semiring $\mathbf{C}$. All these constructions are shown to be different in section 22.8.

Chapter 22

The conditions M and $\overline{\mathrm{M}}$

In the previous chapter, we introduced all sub-constructions of the final **D**-construction $\mathcal{D}$ that are obtained by existential restriction to sub-semirings of **D** and intersection. In this chapter, we present some smaller sub-constructions of $\mathcal{D}$ characterized by additional logical conditions on the second order predicates. The need for smaller constructions is also due to the fact that the sandwich construction and its dual are not reduced (see section 22.1).

The additional logical conditions are prepared by studying the notions of lower and upper implication of $\mathcal{D}$-predicates in section 22.2. In section 22.3, we establish the logical conditions M and $\overline{M}$ on second order predicates and show that they induce two power domain constructions that we call $\mathcal{M}$ and $\overline{\mathcal{M}}$.

The defining condition M of the construction $\mathcal{M}$ is transformed in section 22.4, translated into topological terms in section 22.5 for sober ground domains, and drastically simplified in section 22.6 for ground domains in KC & M-CONT. The dual mixed construction $\overline{\mathcal{M}}$ is considered algebraically and topologically in section 22.7.

By intersecting the sub-constructions of $\mathcal{D}$ introduced so far, we obtain some more sub-constructions. All these 14 sub-constructions are summarized in section 22.8 and shown to be different even for small finite ground domains.

The most interesting sub-construction that is obtained by intersection is $\mathcal{C}$, the intersection of $\mathcal{M}$ and $\overline{\mathcal{M}}$. In section 22.9, we show that every member of $\mathcal{M}\mathbf{X}$ is a union of a member of $\mathcal{C}\mathbf{X}$ and a member of $\mathcal{M}\mathbf{X}$ that is below the empty set.

The case of an algebraic ground domain is considered in section 22.10. In this case, $\mathcal{S}$ is the known sandwich construction of [BDW88], $\mathcal{M}$ the mixed construction of [Gun89b, Gun90], and $\mathcal{C}$ Plotkin's construction of [Plo76].

22.1 Are sandwich power domains reduced?

We now start to investigate the question whether the sandwich power construction is reduced, and if not what its core is. One reason for being not reduced could be the fact that the sandwich power construction is the existential restriction of the *final* **D**-construction that in turn is built using the *final* **U**-construction. If $\mathcal{U}_f$ happens to be not reduced (an open question), then this could make the sandwich power construction non-reduced. The effect would however be small; we know that $\mathcal{U}_f\mathbf{X}$ is reduced for multi-continuous $\mathbf{X}$.

Independently from this uncertain effect, a second reason for 'junk' in the sandwich power domains could be the weakness of the sandwich condition. The condition was derived as the weakest possible condition that lets existential quantification w.r.t. $\mathbf{B}$ be well defined. There might well be stronger conditions that are also respected by all power operations. This is indeed true; we shall soon meet such a stronger condition.

To be precise, sandwich power domains are generally not reduced, not even for the simplest ground domains. As an example, we consider the ground domain $\mathbf{2} = \{\bot < \top\}$. Since this domain is finite, all lower sets are closed, and all upper sets are compact. Hence, both $\mathcal{L}\mathbf{2}$ and $\mathcal{U}\mathbf{2}$ contain 3 elements, and therefore, $\mathcal{D}\mathbf{2}$ has 9 elements. They are listed in the following table. The order is such that the lower left corner contains the least element, and elements increase when going to the right or upward.

	$\emptyset$	$\{\!\vert\,\vert\!\} = \theta$	$-$	$-$
$\uparrow$	$\{\top\}$	$\bot \cdot \{\!\vert\top\vert\!\}$	$*$	$\{\!\vert\top\vert\!\}$
$\mathcal{U}\mathbf{2}$	$\{\bot, \top\}$	$\bot \cdot \{\!\vert\bot\vert\!\}$	$\{\!\vert\bot\vert\!\}$	$\{\!\vert\bot, \top\vert\!\}$
		$\emptyset$	$\{\bot\}$	$\{\bot, \top\}$
		$\mathcal{L}\mathbf{2}$	$\longrightarrow$	

The two places marked by '$-$' do not contain sandwiches. Since $\mathbf{2}$ is finite, Th. 21.5.1 holds, and (C, K) is a sandwich iff $C \subseteq {\downarrow}K$. This is not true if K is empty and C is not empty. Hence, the sandwich power domain $\mathcal{S}\mathbf{2}$ has 7 elements. From these, 6 can be built up using the power operations θ, '$\uplus$', and '$\cdot$', as indicated in the table. The remaining sandwich $S = (\{\bot\}, \{\top\})$ at position '$*$' cannot be built such way. It is neither θ nor $\{\!|x|\!\}$ nor $\bot \cdot \{\!|x|\!\}$ for some x since all these sandwiches occur on other positions in the table. It also cannot be built by union: if $A \uplus B = (\{\bot\}, \{\top\})$, then $A^L \cup B^L = \{\bot\}$ holds. Hence, one of these, say A^L, equals $\{\bot\}$. Because of $A^U \cup B^U = \{\top\}$, $A^U \subseteq \{\top\}$ holds. A^U cannot be empty, because $(\{\bot\}, \emptyset)$ is not a sandwich. Hence $A = (\{\bot\}, \{\top\}) = S$. Because all is finite, S can neither be obtained as limit of a directed set of the remaining 6 sandwiches. Thus, S is not in the core of $\mathcal{S}\mathbf{2}$.

Summarizing, we found that $\mathcal{S}\mathbf{2}$ has 7 elements, whereas its core only has 6. The unreachable sandwich S is characterized by the fact that its lower part is not empty, but does not meet its upper part.

Unfortunately, there is no easy way to see what condition distinguishes the 6 sandwiches in the core from the one not in the core. By studying various sandwich power domains for more complex ground domains, the correct condition was found. We call it *mix condition* or condition M because Th. 22.10.3 will show that it restricts the sandwich power domain to the *mixed power domain* defined in [Gun89b, Gun90] for algebraic ground domains.

22.2 Lower and upper implication for D-predicates

The definition of condition M and its dual, condition $\overline{M}$ is prepared by investigating the logic of $\mathbf{D}$ more closely.

To obtain a concise presentation, we always work with **D**-*predicates* $[\mathbf{X} \to \mathbf{D}]$ in the sequel where $\mathbf{X}$ is an arbitrary domain. All results are valid for $\mathbf{D}$ itself also since $\mathbf{D}$ is isomorphic to $[\mathbf{1} \to \mathbf{D}]$. $[\mathbf{X} \to \mathbf{D}]$ is isomorphic to $[\mathbf{X} \to \mathbf{L}] \times [\mathbf{X} \to \mathbf{U}]$, whence a in $[\mathbf{X} \to \mathbf{D}]$ may be written as pair (a^L, a^U).

In addition to the logical operations of disjunction '$+$', conjunction '$\cdot$', and negation '$\neg$', we introduce a kind of difference: $a - b = a \cdot \neg b$. It is mainly used as a notational abbreviation. The following relations are easily verified:

Proposition 22.2.1 For all **D**-predicates a and b:

(1) $(a+b)^L = a^L + b^L$ and $(a+b)^U = a^U + b^U$

(2) $(a \cdot b)^L = a^L \cdot b^L$ and $(a \cdot b)^U = a^U \cdot b^U$

(3) $(\neg a)^L = \widetilde{a^U}$ and $(\neg a)^U = \widetilde{a^L}$ (cf. section 21.1)

(4) $(a-b)^L = a^L \cdot \widetilde{b^U}$ and $(a-b)^U = a^U \cdot \widetilde{b^L}$.

Statement (4) is an immediate consequence of statements (2) and (3). It shows that difference '$-$' combines the lower and upper components of its arguments. This property explains the usage of difference in condition M and $\overline{M}$ as will be seen below.

The next proposition claims the equivalence of various conditions. They are coined as lower and upper implication.

Proposition 22.2.2 For **D**-predicates, the following equivalences hold:

(1) $a^L \leq b^L$ iff $a + b \leq b$. In this case, we say that a and b are in the relation of *lower implication* '$\overset{L}{\mapsto}$'.

(2) $a^U \geq b^U$ iff $a + b \geq b$. In this case, we say that a and b are in the relation of *upper implication* '$\overset{U}{\mapsto}$'.

Proof:

(1) By part (1) of Prop. 22.2.1, $a + b \leq b$ holds iff $a^L + b^L \leq b^L$ and $a^U + b^U \leq b^U$. Since the inequality involving 'U' is a tautology, it can be dropped. Hence, $a+b \leq b$ iff $a^L + b^L \leq b^L$. This inequality is equivalent to $a^L \leq b^L$.

(2) By part (1) of Prop. 22.2.1, $a + b \geq b$ holds iff $a^L + b^L \geq b^L$ and $a^U + b^U \geq b^U$. Here, the inequality with 'L' is a tautology. $a^U + b^U \geq b^U$ is equivalent to $a^U \geq b^U$. □

Lower and upper implication enjoy some properties that are needed in the next section.

Proposition 22.2.3 Let $X = L$ or U in the following.

(1) The relation '$\overset{X}{\mapsto}$' is reflexive and transitive.

(2) If $a \overset{X}{\mapsto} a'$ and $b \overset{X}{\mapsto} b'$, then $a + b \overset{X}{\mapsto} a' + b'$.

(3) $(a+b) - (a'+b') \overset{X}{\mapsto} (a - a') + (b - b')$.

(4) If P is an additive second order predicate, then $a \overset{X}{\mapsto} b$ implies $Pa \overset{X}{\mapsto} Pb$.

(5) If $(a_i)_{i\in I}$ and $(b_i)_{i\in I}$ are directed families of $\mathbf{D}$-predicates with $a_i \overset{X}{\mapsto} b_i$ for all $i \in I$, then $\bigsqcup_{i\in I} a_i \overset{X}{\mapsto} \bigsqcup_{i\in I} b_i$.

(6) $\perp \cdot a \overset{L}{\mapsto} \perp \cdot b$ no matter what $\mathbf{D}$-predicates a and b are. Similarly, $\mathsf{W} \cdot a \overset{U}{\mapsto} \mathsf{W} \cdot b$.

Proof:

(1) Immediate by definition.

(2) $a^L \le a'^L$ and $b^L \le b'^L$ implies $(a+b)^L = a^L + b^L \le a'^L + b'^L = (a'+b')^L$. The proof for '$\overset{U}{\mapsto}$' is similarly simple.

(3) For '$\overset{L}{\mapsto}$': $((a+b)-(a'+b'))^L = (a^L+b^L)\cdot \widetilde{a'^U + b'^U} = (a^L+b^L)\cdot \widetilde{a'^U}\cdot\widetilde{b'^U} = (a^L\cdot\widetilde{a'^U}\cdot\widetilde{b'^U}) + (b^L\cdot\widetilde{a'^U}\cdot\widetilde{b'^U}) \le (a^L\cdot\widetilde{a'^U}) + (b^L\cdot\widetilde{b'^U}) = (a-a')^L + (b-b')^L = ((a-a')+(b-b'))^L$.

Here, '$\le$' holds since $p\cdot q \le p$ holds for L-predicates p and q. The proof for '$\overset{U}{\mapsto}$' is analogous; here, '$\ge$' holds since $p\cdot q \ge p$ holds for U-predicates.

(4) For '$\overset{L}{\mapsto}$': $a \overset{L}{\mapsto} b$ implies $a+b \le b$, whence $Pa + Pb = P(a+b) \le Pb$, i.e. $Pa \overset{L}{\mapsto} Pb$. The proof for '$\overset{U}{\mapsto}$' is similar.

(5) We use the equivalence $a \overset{L}{\mapsto} b$ iff $a^L \le b^L$. If $(a_i)_{i\in I}$ is directed, then $(a_i^L)_{i\in I}$ is directed, too. $a_i^L \le b_i^L$ implies $(\bigsqcup_{i\in I} a_i)^L = \bigsqcup_{i\in I} a_i^L \le \bigsqcup_{i\in I} b_i^L = (\bigsqcup_{i\in I} b_i)^L$. The proof for '$\overset{U}{\mapsto}$' is analogous.

(6) $(\perp\cdot a)^L = 0\cdot a^L = 0$ and also $(\perp\cdot b)^L = 0$. Similarly, $(\mathsf{W}\cdot a)^U = 0\cdot a^U = 0$. □

22.3 The conditions M and $\overline{M}$

After the preliminaries of the previous section, we are now able to define the conditions M and $\overline{M}$:

Definition 22.3.1

A in $\mathcal{D}\mathbf{X}$ satisfies condition M iff $Ap - Aq \overset{L}{\mapsto} A(p-q)$ for all predicates $p, q : [\mathbf{X} \to \mathbf{D}]$.

A in $\mathcal{D}\mathbf{X}$ satisfies condition $\overline{M}$ iff $Ap - Aq \overset{U}{\mapsto} A(p-q)$ for all predicates $p, q : [\mathbf{X} \to \mathbf{D}]$.

These definitions may seem to be ad hoc and badly motivated. We shall however see in Th. 22.10.3 that condition M restricts the final $\mathbf{D}$-powerdomain $\mathcal{D}\mathbf{X}$ to the mixed power domain $\mathcal{M}\mathbf{X}$ of [Gun89b, Gun90] if the ground domain $\mathbf{X}$ is algebraic, and further restricting $\mathcal{M}\mathbf{X}$ by means of condition $\overline{M}$ results in Plotkin's power construction.

In order to establish the power constructions $\mathcal{M}\mathbf{X}$ and $\overline{\mathcal{M}}\mathbf{X}$, we have to show that the power operations preserve the restrictions. Using the generic Prop. 22.2.3, the proofs for M and $\overline{M}$ are completely analogous. We formulate it for M.

- $\theta = \lambda p.\,0$, whence $\theta p - \theta q = 0 - 0 = 0 = \theta(p-q)$. By reflexivity of lower and upper implication (Prop. 22.2.3 (1)), $\theta \in \mathcal{M}\mathbf{X}$ and $\theta \in \overline{\mathcal{M}}\mathbf{X}$ follow.
- $\{\!|x|\!\} = \lambda p.\, p\,x$, whence $\{\!|x|\!\}\,p - \{\!|x|\!\}\,q = p\,x - q\,x = (p-q)\,x = \{\!|x|\!\}\,(p-q)$.
  $p\,x - q\,x = (p-q)\,x$ holds since all logical operations are defined pointwise on predicates. Reflexivity again yields the desired result.

- For A, B in $\mathcal{M}\mathbf{X}$,

$$\begin{array}{rcll} (A \uplus B)\,p - (A \uplus B)\,q & = & (A\,p + B\,p) - (A\,q + B\,q) & \\ & \overset{L}{\mapsto} & (A\,p - A\,q) + (B\,p - B\,q) & \text{by Prop. 22.2.3 (3)} \\ & \overset{L}{\mapsto} & A\,(p-q) + B\,(p-q) & \\ & & \quad \text{since } A,\, B \text{ in } \mathcal{M}\mathbf{X} \text{ by Prop. 22.2.3 (2)} & \\ & = & (A \uplus B)\,(p-q) & \end{array}$$

- For $f : [\mathbf{X} \to \mathcal{M}\mathbf{Y}]$ and A in $\mathcal{M}\mathbf{X}$,

$$\begin{array}{rcll} (ext\,f\,A)\,p - (ext\,f\,A)\,q & = & A\,(\lambda x.\, fxp) - A\,(\lambda x.\, fxq) & \\ & \overset{L}{\mapsto} & A\,(\lambda x.\, fxp - fxq) & \text{since } A \text{ in } \mathcal{M}\mathbf{X} \\ & \overset{L}{\mapsto} & A\,(\lambda x.\, fx\,(p-q)) & \\ & & \quad \text{since } fx \text{ in } \mathcal{M}\mathbf{X} \text{ by Prop. 22.2.3 (4); } A \text{ is additive} & \\ & = & (ext\,f\,A)\,(p-q) & \end{array}$$

- If $(A_i)_{i\in I}$ is a directed family in $\mathcal{M}\mathbf{X}$, then both $(A_i p - A_i q)_{i\in I}$ and $(A_i\,(p-q))_{i\in I}$ are directed families with $A_i p - A_i q \overset{L}{\mapsto} A\,(p-q)$ for all $i \in I$. By Prop. 22.2.3 (5), $A\,p - A\,q \overset{L}{\mapsto} A\,(p-q)$ follows where $A = \bigsqcup_{i\in I} A_i$.
- Finally, we show two facts about the external product: $\perp \cdot A$ satisfies M for all A in $\mathcal{D}\mathbf{X}$ — even those that do not satisfy M. Analogously, $\mathsf{W} \cdot A$ satisfies $\overline{M}$ for all A.

  The first statement holds since $\perp \cdot Ap - \perp \cdot Aq = \perp \cdot Ap \cdot \neg(\perp \cdot Aq) \overset{L}{\mapsto} \perp \cdot A\,(p-q)$ by Prop. 22.2.3 (6).

22.4 $\mathcal{M}$ — the mixed power domain construction

In this section, we investigate the power domain construction derived from $\mathcal{D}\mathbf{X}$ by means of condition M. We abbreviate this construction by $\mathcal{M}$ since it will turn out to be a generalization of Gunter's mixed power domain construction [Gun89b, Gun90] that was defined for algebraic ground domains only. Hence, we chose the names *condition M* or *mix condition*. The elements of $\mathcal{M}\mathbf{X}$ are called *mixes*.

$$\mathcal{M}\mathbf{X} = \{P : [[\mathbf{X} \to \mathbf{D}] \overset{add}{\to} \mathbf{D}] \mid \forall p,\, q \in [\mathbf{X} \to D] : Pp - Pq \overset{L}{\mapsto} P\,(p-q)\}$$

In the previous section, we have proved that $\mathcal{M}$ is a power domain construction. Its characteristic semiring is $\mathbf{B}$ since it contains all three of $\mathsf{T} = \{\!|\diamond|\!\}$, $\mathsf{F} = \theta$, and $\perp = \perp \cdot \{\!|\diamond|\!\}$, and since it does not contain W because the mix condition implies the sandwich condition as we shall see below.

In the sequel, we want to translate the mix condition into topological terms. This is done in analogy to the sandwich power construction. The first step leads to pairs of open grills and open filters, and the second step to pairs of closed sets and compact upper sets. In the course of this translation, we also prove that condition M implies condition S, i.e. the mixed power domains are subsets of the sandwich power domains.

Let $p = (p^L, p^U)$ and $q = (q^L, q^U)$ be two predicates. For $A = (A^L, A^U)$, the mix condition may then be transformed using the facts collected in Prop. 22.2.1.

$$
\begin{array}{lll}
A\,p - A\,q \overset{L}{\mapsto} A\,(p-q) & \text{iff} & (A\,p - A\,q)^L \leq (A\,(p-q))^L \\
& \text{iff} & (A\,p - A\,q)^L = 1 \Rightarrow (A\,(p-q))^L = 1 \\
& \text{iff} & A^L p^L \cdot \widetilde{A^U q^U} = 1 \Rightarrow A^L (p^L \cdot \widetilde{q^U}) = 1 \\
& \text{iff} & A^L p^L = 1 \text{ and } A^U q^U = 0 \Rightarrow A^L (p^L \cdot \widetilde{q^U}) = 1
\end{array}
$$

Thus, we finally obtain:

Proposition 22.4.1

$(A^L, A^U) \in \mathcal{M}\mathbf{X}$ iff

$\forall p : [\mathbf{X} \to \mathbf{L}],\ q : [\mathbf{X} \to \mathbf{U}] : A^L p = 1$ and $A^U q = 0$ implies $A^L(p \cdot \widetilde{q}) = 1$

From this equivalence, one may easily conclude $\mathcal{M}\mathbf{X} \subseteq \mathcal{S}\mathbf{X}$. The precondition $A^L p = 1$ and $A^U q = 0$ of the sandwich condition in the form of Prop. 21.3.2 equals that of the mix condition. By the mix condition, $A^L(p \cdot \widetilde{q}) = 1$ follows. If $p \cdot \widetilde{q}$ were $\underline{0}$, then $A^L(p \cdot \widetilde{q}) = 0$ would hold because of additivity of A^L. Thus, the mix condition implies $p \cdot \widetilde{q} \neq \underline{0}$.

22.5 $\mathcal{M}$ in topological terms

In this section, we translate condition M into topological terms. At first, we translate the predicates to open sets. p becomes O^L and q becomes O^U. Then $p \cdot \widetilde{q} = p \sqcap \widetilde{q}$ corresponds to $O^L \cap O^U$. The lower second order predicate A^L is translated into an open grill $\mathcal{G}$, and the upper one into an open filter $\mathcal{O}$. We remember $A^L p = 1 = \top$ iff $O^L \in \mathcal{G}$, and $A^U q = 0 = \top$ iff $O^U \in \mathcal{O}$. Thus, we obtain

$$(\mathcal{G}, \mathcal{O}) \in \mathcal{M}\mathbf{X} \text{ iff } \forall O^L \in \mathcal{G},\ O^U \in \mathcal{O} : O^L \cap O^U \in \mathcal{G}$$

For sober ground domain $\mathbf{X}$, one can go one step further and translate the open grills $\mathcal{G}$ into closed sets C, and the open filters $\mathcal{O}$ into compact upper sets K. $O \in \mathcal{G}$ then becomes $C \cap O \neq \emptyset$, and $O' \in \mathcal{O}$ becomes $K \subseteq O'$. Hence, the mix condition translates into:
for all open sets O and O', if C meets O and $K \subseteq O'$ then $C \cap O \cap O' \neq \emptyset$.
For fixed C and O', the following holds:

Every open set meeting C meets $C \cap O'$
- iff every open environment of every point of C meets $C \cap O'$
- iff every point of C is in the closure of $C \cap O'$ by Prop. 4.2.2
- iff $C \subseteq \mathsf{cl}\,(C \cap O')$.

Hence, one obtains

Theorem 22.5.1 The mixed power domain $\mathcal{M}\mathbf{X}$ over a sober ground domain $\mathbf{X}$ is isomorphic to the set of all pairs (C, K) of a closed set C and a compact upper set K such that for all open sets O with $K \subseteq O$ the inclusion $C \subseteq \mathsf{cl}\,(C \cap O)$ holds. The order is given by $(C, K) \leq (C', K')$ iff $C \subseteq C'$ and $K \supseteq K'$. The power operations are defined as in $\mathcal{D}\mathbf{X}$ (see 21.2).

22.6 Simplification of the mix condition

In this section, we investigate the mix restriction '$K \subseteq O$ open implies $C \subseteq \mathsf{cl}\,(C \cap O)$' for pairs (C, K) of a closed and a compact upper set. As with the sandwich condition, we present a drastic simplification for a special class of ground domains.

Theorem 22.6.1 Let $\mathbf{X}$ be a multi-continuous domain in class KC. Then for all closed $C \subseteq \mathbf{X}$ and all compact upper sets $K \subseteq \mathbf{X}$, the following two statements are equivalent:

(1) The mix condition: If $K \subseteq O$ for some open set O, then $C \subseteq \mathsf{cl}\,(C \cap O)$.

(2) $C \subseteq \downarrow(C \cap K)$.

Condition KC was introduced by Prop. 5.5.2 and means that the intersection of two upper cones is compact. All finitely continuous domains are continuous and satisfy this condition (Prop. 7.2.2), i.e. they are covered by the theorem above.

Proof: If $C \subseteq \downarrow(C \cap K)$ holds, then $K \subseteq O$ implies $C \subseteq \downarrow(C \cap K) \subseteq \downarrow(C \cap O) \subseteq \mathsf{cl}\,(C \cap O)$.

As pointed out in the proof of Th. 21.5.1, the preconditions of the lemmas 8.10.1 and 8.10.2 are satisfied.

Let C be a closed set and K a compact upper set satisfying the mix condition. Let $\mathcal{F}$ be the set of all finitary upper sets F such that there is an open set O with $K \subseteq O \subseteq F$. From Th. 8.3.2, we know that $\mathcal{F}$ is a $\supseteq$-directed set whose intersection is K.

For all sets $F \in \mathcal{F}$, $K \subseteq O \subseteq F$ holds, whence $C \subseteq \mathsf{cl}\,(C \cap O) \subseteq \mathsf{cl}\,(C \cap F)$ follows from the mix condition. Lemma 8.10.1 implies $\mathsf{cl}\,(C \cap F) = \downarrow(C \cap F)$. Hence, $C \subseteq \bigcap_{F \in \mathcal{F}} \downarrow(C \cap F) = \downarrow \bigcap_{F \in \mathcal{F}}(C \cap F) = \downarrow(C \cap K)$ applying Lemma 8.10.2. □

In contrast to the situation with the sandwich power domain, we do not know whether condition KC is important for the validity of the theorem. The example following Th. 21.5.1 does not matter here since the sandwich occurring in it does not satisfy the mix condition.

22.7 $\overline{\mathcal{M}}$ — the dual mixed power domain construction

In this section, we investigate the power domain construction $\overline{\mathcal{M}}$ derived from $\mathcal{D}$ by means of condition $\overline{M}$. The elements of $\overline{\mathcal{M}}\mathbf{X}$ are called *dual mixes*. In section 22.3, we have proved that $\overline{\mathcal{M}}$ is a power domain construction. Its characteristic semiring is $\overline{\mathbf{B}}$ since it contains all three of $\mathsf{T} = \{\!|\diamond|\!\}$, $\mathsf{F} = \theta$, and $\mathsf{W} = \mathsf{W} \cdot \{\!|\diamond|\!\}$, and condition $\overline{M}$ implies condition $\overline{S}$ as we shall see below.

Let $p = (p^L, p^U)$ and $q = (q^L, q^U)$ be two predicates. For $A = (A^L, A^U)$, the condition may then be transformed using the facts collected in Prop. 22.2.1.

$$
\begin{array}{lll}
A\,p - A\,q \overset{U}{\mapsto} A\,(p-q) & \text{iff} & (A\,p - A\,q)^U \geq (A\,(p-q))^U \\
 & \text{iff} & (A\,p - A\,q)^U = 1 \Rightarrow (A\,(p-q))^U = 1 \\
 & \text{iff} & A^U p^U \cdot \widetilde{A^L q^L} = 1 \Rightarrow A^U (p^U \cdot \widetilde{q^L}) = 1 \\
 & \text{iff} & A^U p^U = 1 \text{ and } A^L q^L = 0 \Rightarrow A^U (p^U \cdot \widetilde{q^L}) = 1
\end{array}
$$

Thus, we finally obtain:

Proposition 22.7.1

$$(A^L, A^U) \in \overline{\mathcal{M}}\mathbf{X} \text{ iff}$$
$$\forall p : [\mathbf{X} \to \mathbf{U}],\ q : [\mathbf{X} \to \mathbf{L}] : A^U p = 1 \text{ and } A^L q = 0 \text{ implies } A^U(p \cdot \widetilde{q}) = 1$$

From this equivalence, one may easily conclude $\overline{\mathcal{M}}\mathbf{X} \subseteq \overline{\mathcal{S}}\mathbf{X}$. If $p \cdot \widetilde{q}$ were $\underline{0}$, then $A^U(p \cdot \widetilde{q}) = 0$ would hold by additivity.

The transformation of condition $\overline{M}$ proceeds by translating the predicates to open sets. p becomes O^U and q becomes O^L. Then $p \cdot \widetilde{q} = p \sqcup \widetilde{q}$ corresponds to $O^U \cup O^L$. The lower second order predicate A^L is translated into an open grill $\mathcal{G}$, and the upper one into an open filter $\mathcal{O}$. All equations in the formula above are just negations of those in Prop. 22.4.1. Thus, we obtain

$$\begin{array}{lll} (\mathcal{G}, \mathcal{O}) \in \overline{\mathcal{M}}\mathbf{X} & \text{iff} & O^L \notin \mathcal{G} \text{ and } O^U \notin \mathcal{O} \Rightarrow O^L \cup O^U \notin \mathcal{O} \\ & \text{iff} & O^L \cup O^U \in \mathcal{O} \Rightarrow O^L \in \mathcal{G} \text{ or } O^U \in \mathcal{O} \end{array}$$

For sober ground domain $\mathbf{X}$, we translate open grills $\mathcal{G}$ into closed sets C and open filters $\mathcal{O}$ into compact upper sets K.

$$(C, K) \in \overline{\mathcal{M}}\mathbf{X} \quad \text{iff} \quad K \subseteq O^L \cup O^U \Rightarrow C \cap O^L \neq \emptyset \text{ or } K \subseteq O^U$$

Let C' be the complement of C. Then we claim that the condition above is equivalent to '$K \subseteq C' \cup O \Rightarrow K \subseteq O$'. The old condition implies the new one since $C \cap C' = \emptyset$ and C' is open. The new condition implies the old one since $C \cap O^L = \emptyset$ implies $O^L \subseteq C'$ whence $K \subseteq O^L \cup O^U \subseteq C' \cup O^U$.

To simplify further, note that $K \subseteq C' \cup O$ is equivalent to $C \cap K \subseteq O$. Thus, we obtain $(C, K) \in \overline{\mathcal{M}}\mathbf{X}$ iff $C \cap K \subseteq O$ implies $K \subseteq O$ for all open sets O. By Lemma 4.4.4, this is equivalent to $K \subseteq \uparrow(C \cap K)$.

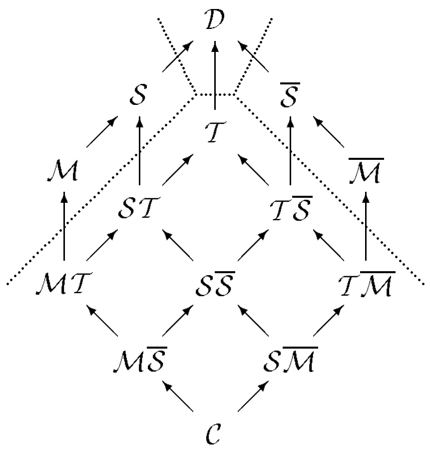
Theorem 22.7.2 The dual mixed power domain $\overline{\mathcal{M}}\mathbf{X}$ over a sober ground domain $\mathbf{X}$ is isomorphic to the set of all pairs (C, K) of a closed set C and a compact upper set K such that $K \subseteq \uparrow(C \cap K)$ holds.

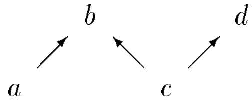
22.8 Other sub-constructions of $\mathcal{D}$

The final $\mathbf{D}$-construction $\mathcal{D}$ has 5 'atomic' sub-constructions obtained by restricting $\mathcal{D}$ using the conditions S, $\overline{S}$, M, $\overline{M}$, and T. The resulting constructions $\mathcal{S}$ and $\mathcal{M}$ have semiring $\mathbf{B}$, whereas the dual constructions $\overline{\mathcal{S}}$ and $\overline{\mathcal{M}}$ have semiring $\overline{\mathbf{B}}$, and $\mathcal{T}$ has semiring $\mathbf{C}$. For all ground domains $\mathbf{X}$, the inclusions $\mathcal{M}\mathbf{X} \subseteq \mathcal{S}\mathbf{X}$, $\overline{\mathcal{M}}\mathbf{X} \subseteq \overline{\mathcal{S}}\mathbf{X}$ and $\mathcal{S}\mathbf{X} \cap \overline{\mathcal{S}}\mathbf{X} \subseteq \mathcal{T}\mathbf{X}$ hold.

Further sub-constructions of $\mathcal{D}$ result from intersections of the constructions above. Because of the subset relations, 8 constructions are obtained. The most interesting one is $\mathcal{C} = \mathcal{M} \cap \overline{\mathcal{M}}$ because it coincides with Plotkin's power construction for algebraic ground domains.

The relative order of the constructions is indicated by the following diagram where the arrows denote subset relations. All constructions below $\mathcal{T}$ have semiring $\mathbf{C}$, $\mathcal{M}$ and $\mathcal{S}$ have semiring $\mathbf{B}$, $\overline{\mathcal{M}}$ and $\overline{\mathcal{S}}$ have $\overline{\mathbf{B}}$, and $\mathcal{D}$ has $\mathbf{D}$ (see the dotted lines).





$\mathbf{X}$

Examples that all these constructions differ are obtained when using the simple ground domain $\mathbf{X} = \{a < b > c < d\}$. For finite ground domains, the simple formulae for conditions S and M apply. A pair (C, K) satisfies S iff $C \subseteq {\downarrow}K$, $\overline{S}$ iff $K \subseteq {\uparrow}C$, M iff $C \subseteq {\downarrow}(C \cap K)$, and $\overline{M}$ iff $K \subseteq {\uparrow}(C \cap K)$. Since $\mathbf{X}$ admits only two predicates $[\mathbf{X} \to \mathbf{C}]$, namely $\lambda x.\, \mathsf{F}$ and $\lambda x.\, \mathsf{T}$, one easily deduces $(C, K) \in \mathcal{T}\mathbf{X}$ iff both C and K are not empty or else $C = K = \emptyset$.

- $(\{a\}, \{a, b\})$ is in $\mathcal{C}\mathbf{X}$.
- $(\{a, b, c\}, \{b, d\})$ is in $\mathcal{M}\overline{\mathcal{S}}\mathbf{X}$, but not in $\overline{\mathcal{M}}\mathbf{X}$.
- $(\{a, c\}, \{b, c, d\})$ is in $\mathcal{S}\overline{\mathcal{M}}\mathbf{X}$, but not in $\mathcal{M}\mathbf{X}$.
- $(\{a\}, \{b\})$ is in $\mathcal{S}\overline{\mathcal{S}}\mathbf{X}$, but neither in $\mathcal{M}\mathbf{X}$ nor in $\overline{\mathcal{M}}\mathbf{X}$.
- $(\{a\}, \{a, b, d\})$ is in $\mathcal{MT}\mathbf{X}$, but not in $\overline{\mathcal{S}}\mathbf{X}$.
- $(\{a, c, d\}, \{d\})$ is in $\mathcal{T}\overline{\mathcal{M}}\mathbf{X}$, but not in $\mathcal{S}\mathbf{X}$.
- $(\emptyset, \{d\})$ is in $\mathcal{M}\mathbf{X}$, but neither in $\mathcal{T}\mathbf{X}$ nor in $\overline{\mathcal{S}}\mathbf{X}$.
- $(\{a\}, \emptyset)$ is in $\overline{\mathcal{M}}\mathbf{X}$, but neither in $\mathcal{T}\mathbf{X}$ nor in $\mathcal{S}\mathbf{X}$.
- $(\{a\}, \{d\})$ is in $\mathcal{T}\mathbf{X}$, but neither in $\mathcal{S}\mathbf{X}$ nor in $\overline{\mathcal{S}}\mathbf{X}$.

22.9 The $\mathcal{M}$-$\mathcal{C}$-Theorem

Our next effort in this chapter is the derivation of a strong relation between the subconstructions $\mathcal{M}$ and $\mathcal{C}$ of the final $\mathbf{D}$-construction $\mathcal{D}$.

Theorem 22.9.1 For every ground domain $\mathbf{X}$ holds: For every m in $\mathcal{M}\mathbf{X}$, there is some c in $\mathcal{C}\mathbf{X}$ such that $m = m? \uplus c$.

Proof: Let $m = (L, U)$ where L is in the final $\mathbf{L}$-powerdomain $\mathcal{L}\mathbf{X}$ and U is in the final $\mathbf{U}$-powerdomain $\mathcal{U}\mathbf{X}$. Then $m? = \bot \cdot m = (0, 1) \cdot (L, U) = (\theta, U)$ holds.

Let S be the set of all $\mathbf{L}$-predicates p with $Lp = 0$. $\underline{0}$ is in S by additivity of L. If p_1 and p_2 are in S, then $p_1 + p_2$ is in S by $L\,(p_1 + p_2) = L\,p_1 + L\,p_2 = 0 + 0 = 0$. Since addition in $\mathbf{L}$ is least upper bound, this shows that S is directed. Let s be its supremum. By continuity, $L\,s = 0$ holds.

The **L**-predicate s induces a **U**-predicate $\widetilde{s}$. Let $U' = \mathit{filter}\,\widetilde{s}\,U$. Since 1 is least in **U**, $\widetilde{s} \geq \underline{1}$ holds, whence $U' \geq \mathit{filter}\,\underline{1}\,U = U$ follows. In **U**-powerdomains, union is greatest lower bound, whence $U \uplus U' = U$. Thus, with $c = (L, U')$, we obtain $m? \uplus c = (\theta, U) \uplus (L, U') = (\theta \uplus L, U \uplus U') = (L, U) = m$ as required.

We still have to show c in $\mathcal{C}\mathbf{X}$. $(\mathit{filter}\,\widetilde{s}\,U)\,q = U\,(\widetilde{s} \cdot q)$ holds as computed in section 15.6. To show condition M for $c = (L, U')$, we have to derive $L\,(p \cdot \widetilde{q}) = 1$ from $L\,p = 1$ and $U'\,q = 0$. The precondition implies $L\,p = 1$ and $U\,(\widetilde{s} \cdot q) = 0$. Since $m = (L, U)$ satisfies condition M, we conclude $1 = L\,(p \cdot \widetilde{\widetilde{s} \cdot q}) = L\,(p \cdot (s + \widetilde{q})) = L\,(p \cdot s) + L\,(p \cdot \widetilde{q}) \leq L\,s + L\,(p \cdot \widetilde{q}) = L\,(p \cdot \widetilde{q})$. The '$\leq$' relation holds since '$\cdot$' is greatest lower bound in **L**. $L\,s$ is 0 as mentioned above.

To show condition $\overline{M}$ for $c = (L, U')$, we have to derive $U'\,(p \cdot \widetilde{q}) = 1$ from $U'\,p = 1$ and $L\,q = 0$. Equivalently, we have to derive $U\,(p \cdot \widetilde{s} \cdot \widetilde{q}) = 1$ from $U\,(p \cdot \widetilde{s}) = 1$ and $L\,q = 0$. $L\,q = 0$ implies $q \leq s$ since s is the supremum of all predicates that are mapped to 0 by L. Since addition in **L** is lub, $q + s = s$ follows, whence $\widetilde{s} = \widetilde{q + s} = \widetilde{q} \cdot \widetilde{s}$. Thus $U\,(p \cdot \widetilde{s}) = 1$ implies $U\,(p \cdot \widetilde{s} \cdot \widetilde{q}) = 1$. $\square$

An analogous $\overline{\mathcal{M}}$-$\mathcal{C}$-theorem cannot be proved analogously since the first step, i.e. the construction of s, does not work out. In the world of **U**-predicates, the set of all p with $U\,p = 0$ is downward directed, and its infimum may not exist. Here, the inherent asymmetry of domains matters.

22.10 The case of an algebraic ground domain

Next, we consider the case of an algebraic ground domain. We claim that for all $\mathcal{R} \in \{\mathcal{S}, \overline{\mathcal{S}}, \mathcal{M}, \overline{\mathcal{M}}, \mathcal{S}\overline{\mathcal{S}}, \mathcal{M}\overline{\mathcal{S}}, \mathcal{S}\overline{\mathcal{M}}, \mathcal{C}\}$, the power domain $\mathcal{R}\mathbf{X}$ is algebraic if the ground domain is, and the base of $\mathcal{R}\mathbf{X}$ equals the part of the base of $\mathcal{D}\mathbf{X}$ which is within $\mathcal{R}\mathbf{X}$, i.e. $(\mathcal{R}\mathbf{X})^0 = (\mathcal{D}\mathbf{X})^0 \cap \mathcal{R}\mathbf{X}$. The power constructions involving $\mathcal{T}$ are excluded from this claim because they are of minor interest. To prove the claim, we show two lemmata.

Lemma 22.10.1 Every pair in $(\mathcal{D}\mathbf{X})^0 \cap \mathcal{R}\mathbf{X}$ is isolated in $\mathcal{R}\mathbf{X}$.

Proof: Let A be a member of $\mathcal{R}\mathbf{X}$ that is isolated in $\mathcal{D}\mathbf{X}$, and let $A \leq \bigsqcup_{\mathcal{R}\mathbf{X}} \mathcal{D}$ where $\mathcal{D}$ is a directed set in $\mathcal{R}\mathbf{X}$. Because $\mathcal{R}\mathbf{X}$ is a sub-domain of $\mathcal{D}\mathbf{X}$, lubs of directed sets w.r.t. $\mathcal{R}\mathbf{X}$ coincide with lubs w.r.t. $\mathcal{D}\mathbf{X}$. Hence $A \leq D$ for some D in $\mathcal{D}$ since A is isolated in $\mathcal{D}\mathbf{X}$. $\square$

The second Lemma is less trivial.

Lemma 22.10.2 Let P be a member of $\mathcal{R}\mathbf{X}$, and let A be an isolated point of $\mathcal{D}\mathbf{X}$ below P. Then there is an isolated point B in $\mathcal{D}\mathbf{X}$ that lies within $\mathcal{R}\mathbf{X}$ and is between A and P.

$$A \in (\mathcal{D}\mathbf{X})^0,\ P \in \mathcal{R}\mathbf{X},\ A \leq P \Rightarrow \exists B \in (\mathcal{D}\mathbf{X})^0 \cap \mathcal{R}\mathbf{X} : A \leq B \leq P.$$

Before we are going to prove this lemma, we show that the two lemmata imply algebraicity. Let P be in $\mathcal{R}\mathbf{X}$. Then let $\mathcal{A} = \{A \in (\mathcal{D}\mathbf{X})^0 \mid A \leq P\}$ and $\mathcal{B} = \{B \in (\mathcal{D}\mathbf{X})^0 \cap \mathcal{R}\mathbf{X} \mid B \leq P\}$. Since $\mathcal{D}\mathbf{X}$ is algebraic, $\mathcal{A}$ is directed with lub P. Obviously, $\mathcal{B} \subseteq \mathcal{A}$ holds, and Lemma 22.10.2 implies $\mathcal{A} \subseteq {\downarrow}\mathcal{B}$. Thus, $\mathcal{A}$ and $\mathcal{B}$ are cofinal, whence $\mathcal{B}$ is also directed with the same lub P by Prop. 3.1.8. Lemma 22.10.1 states that $\mathcal{B}$ is a set of isolated points in $\mathcal{R}\mathbf{X}$.

Proof of the Lemma:
We have to show the claim for each $\mathcal{R}$ separately. Generally, $A = (\downarrow E, \uparrow F)$ holds where E and F are finite subsets of $\mathbf{X}^0$, and $P = (C, K)$ where C is closed, K is a compact upper set, and $E \subseteq C$ and $K \subseteq \uparrow F$ hold because of $A \le P$. Two finite subsets E' and F' of $\mathbf{X}^0$ are to be found that satisfy the conditions of $\mathcal{R}$ and lie between A and P, i.e. $E \subseteq \downarrow E'$, $E' \subseteq C$, and $K \subseteq \uparrow F' \subseteq \uparrow F$ have to hold.

$\mathcal{S}$: A itself is in $\mathcal{S}\mathbf{X}$ since $\mathcal{S}\mathbf{X}$ is lower in $\mathcal{D}\mathbf{X}$. (Indeed, we already proved the algebraicity of $\mathcal{S}\mathbf{X}$ in section 21.5.)

$\overline{\mathcal{S}}$: We know $E \subseteq C$, $K \subseteq \uparrow F$, and $K \subseteq \uparrow C$ because of condition $\overline{S}$.

Let $\mathcal{E} = \{E' \subseteq C \mid E' \subseteq_{fin} \mathbf{X}^0,\ E \subseteq E'\}$. $\mathcal{E}$ contains E and is closed w.r.t. union. Hence, it is $\subseteq$-directed. $\bigcup \mathcal{E} = C$ holds, whence $\uparrow C = \uparrow \bigcup \mathcal{E} = \bigcup \uparrow[\mathcal{E}]$. The set $\uparrow[\mathcal{E}]$ is a $\subseteq$-directed collection of open sets that cover K since $K \subseteq \uparrow C$. By compactness of K, there is E' in $\mathcal{E}$ with $K \subseteq \uparrow E'$. $E \subseteq E' \subseteq C$ holds as required.

$K \subseteq \uparrow E'$ and $K \subseteq \uparrow F$ implies $K \subseteq \uparrow E' \cap \uparrow F$. This set is open as intersection of two open sets. Since K is compact and $\mathbf{X}$ is algebraic, there is a finite set of isolated points F' with $K \subseteq \uparrow F' \subseteq \uparrow E' \cap \uparrow F$. Then $K \subseteq \uparrow F' \subseteq \uparrow F$ holds as required, and $\uparrow F' \subseteq \uparrow E' \subseteq \uparrow\downarrow E'$ shows that $(\downarrow E', \uparrow F')$ satisfies $\overline{S}$.

$\mathcal{S}\overline{\mathcal{S}}$: $A \le P \in \mathcal{S}\overline{\mathcal{S}}\mathbf{X} \subseteq \overline{\mathcal{S}}\mathbf{X}$ implies the existence of B in $(\mathcal{D}\mathbf{X})^0 \cap \overline{\mathcal{S}}\mathbf{X}$ with $A \le B \le P$ as we have seen above. Since $\mathcal{S}\mathbf{X}$ is a lower set in $\mathcal{D}\mathbf{X}$, $B \in \mathcal{S}\mathbf{X}$ also holds.

$\mathcal{M}$: $(\downarrow E, \uparrow F) \le (C, K)$ means $E \subseteq C$ and $K \subseteq \uparrow F$. $\uparrow F$ is open by Prop. 6.2.4, whence we obtain $C \subseteq \mathsf{cl}(C \cap \uparrow F)$ by the mix property of (C, K). This implies $E \subseteq \mathsf{cl}(C \cap \uparrow F)$, whence even $E \subseteq \downarrow(C \cap \uparrow F)$ follows by Prop. 6.1.3.

Hence, for all e in E, there is c_e in C and f_e in F such that $e \le c_e \ge f_e$. As $\mathbf{X}$ is algebraic, there are isolated points g_e with $e, f_e \le g_e \le c_e$. Let $E' = \{g_e \mid e \in E\}$. Then E' is a finite subset of $\mathbf{X}^0$.

$e \le g_e \in E'$ for all e in E implies $E \subseteq \downarrow E'$. $g_e \le c_e \in C$ for all e in E implies $E' \subseteq \downarrow C = C$. $g_e \ge f_e$ for all e in E implies $E' \subseteq \uparrow F$, whence $(\downarrow E', \uparrow F)$ is a mix because it implies $E' = E' \cap \uparrow F \subseteq \mathsf{cl}(\downarrow E' \cap \uparrow F)$.

$\overline{\mathcal{M}}$: $K \subseteq \uparrow F$ and condition $\overline{M}$ imply $K \subseteq \uparrow(C \cap K) \subseteq \uparrow(C \cap \uparrow F) = \uparrow(C \cap F)$. The last equality holds since C is lower. Let $F' = C \cap F$. Then $K \subseteq \uparrow F' \subseteq \uparrow F$ holds as required.

Let $E' = E \cup F'$. Then $E \subseteq E' \subseteq C$ holds since $F' \subseteq C$. Finally, $F' \subseteq E'$ implies $F' = E' \cap F' \subseteq \uparrow(\downarrow E' \cap \uparrow F')$.

$\mathcal{S}\overline{\mathcal{M}}$: Follows from '$\overline{\mathcal{M}}$' as '$\mathcal{S}\overline{\mathcal{S}}$' follows from '$\overline{\mathcal{S}}$' since $\mathcal{S}\mathbf{X}$ is a lower set.

$\mathcal{M}\overline{\mathcal{S}}$: $A \le P \in \mathcal{M}\overline{\mathcal{S}}\mathbf{X} \subseteq \overline{\mathcal{S}}\mathbf{X}$ implies a point B_1 in $(\mathcal{D}\mathbf{X})^0 \cap \overline{\mathcal{S}}\mathbf{X}$ with $A \le B_1 \le P$. Then $P \in \mathcal{M}\mathbf{X}$ implies B in $(\mathcal{D}\mathbf{X})^0 \cap \mathcal{M}\mathbf{X}$ with $B_1 \le B \le P$. Because $\overline{\mathcal{S}}\mathbf{X}$ is an upper set in $\mathcal{D}\mathbf{X}$, B is also in $\overline{\mathcal{S}}\mathbf{X}$.

$\mathcal{C}$: As in '$\overline{\mathcal{M}}$', $K \subseteq \uparrow(C \cap F)$ holds, whence $K \subseteq \uparrow F' \subseteq \uparrow F$ where $F' = C \cap F$. By defining E' as in '$\mathcal{M}$', $E \subseteq \downarrow E' \subseteq C$ holds. $E' \subseteq \uparrow F'$ holds since $c_e \ge g_e \ge f_e$, i.e. $f_e \in F'$.

Now let $G = E' \cup F'$. We claim that $(\downarrow G, \uparrow G)$ is the desired pair. $E \subseteq \downarrow E' \subseteq \downarrow G$ holds, and $G \subseteq C$ since $F' \subseteq C$. In addition, $K \subseteq \uparrow F' \subseteq \uparrow G$ holds and also $G \subseteq \uparrow F' \subseteq \uparrow F$ since $E' \subseteq \uparrow F'$.

$(\downarrow G, \uparrow G)$ is in $\mathcal{C}\mathbf{X}$ since $G \subseteq \downarrow G \cap \uparrow G$ whence conditions M and $\overline{M}$ follow. □

The proof above not only shows the algebraicity of $\mathcal{R}\mathbf{X}$, but also provides nice representations for the bases of these power domains. The most important ones are included in the following theorem.

Theorem 22.10.3 All the constructions $\mathcal{D}$, $\mathcal{S}$, $\mathcal{M}$, $\overline{\mathcal{S}}$, $\overline{\mathcal{M}}$, $\mathcal{S}\overline{\mathcal{S}}$, $\mathcal{S}\overline{\mathcal{M}}$, $\mathcal{M}\overline{\mathcal{S}}$, and $\mathcal{C}$ preserve algebraicity, continuity, finiteness, finite algebraicity, and finite continuity of the ground domain. In case of algebraic ground domain, their bases are the intersections of the base of $\mathcal{D}$ with the respective power domains. The base of $\mathcal{D}\mathbf{X}$ is the set of all pairs $(\downarrow E, \uparrow F)$ where E and F are finite subsets of the base of $\mathbf{X}$. In particular, the bases of the most important constructions are characterized as follows:

$$\begin{array}{llllll} \mathcal{S}: & E \subseteq \downarrow F & \overline{\mathcal{S}}: & F \subseteq \uparrow E & & \\ \mathcal{M}: & E \subseteq F & \overline{\mathcal{M}}: & F \subseteq E & \mathcal{C}: & E = F \end{array}$$

Proof: Preservation of algebraicity was shown above. Preservation of continuity follows by the theory of retracts. All these constructions are sub-constructions of $\mathcal{D}$, whence they preserve finiteness as $\mathcal{D}$ does.

The characterization of the base of $\mathcal{S}$ is taken over from Th. 21.6.4, whereas '$\overline{\mathcal{S}}$', '$\overline{\mathcal{M}}$', and '$\mathcal{C}$' follow from the proof of Lemma 22.10.2. For $\mathcal{M}$, this Lemma only provided $E \subseteq \uparrow F$. Given $(\downarrow E, \uparrow F)$ with $E \subseteq \uparrow F$, let $F' = E \cup F$. Then $\uparrow F' = \uparrow E \cup \uparrow F = \uparrow F$ holds since $E \subseteq \uparrow F$, and $E \subseteq F'$ obviously holds. This implies the form in the theorem. □

We already know that our sandwich power construction $\mathcal{S}$ generalizes the one in [BDW88] and [Gun89b, Gun90] that was defined for algebraic ground domain only. The original characterization $E \subseteq \uparrow F$ — $E \geq^\sharp F$ in Gunter's notation — shows that our mixed power construction generalizes Gunter's [Gun89b, Gun90].

The base of $\mathcal{C}\mathbf{X}$ is the set of all pairs $(\downarrow F, \uparrow F)$ where F is a finite subset of $\mathbf{X}^0$. The intersection of $\downarrow F$ and $\uparrow F$ is the convex hull $\updownarrow F$ of F. It suffices to recover $\downarrow F$ and $\uparrow F$ since $\downarrow F = \downarrow\updownarrow F$ and $\uparrow F = \uparrow\updownarrow F$. The ordering of these convex sets is given by $\updownarrow F \leq \updownarrow F'$ iff $\updownarrow F \subseteq \downarrow\updownarrow F'$ and $\updownarrow F' \subseteq \uparrow\updownarrow F$. This is the Egli-Milner ordering. Hence, $\mathcal{C}\mathbf{X}$ equals Plotkin's power domain for algebraic ground domain.

Chapter 23

Properties of B- and C-constructions

In this chapter, we first consider the preservation of bounded completeness under some **B**- and **C**-constructions. The sandwich construction $\mathcal{S}$ and the non-empty part of $\mathcal{S}\overline{\mathcal{S}}$ are shown to preserve bounded completeness, whereas the smaller constructions $\mathcal{M}$ and $\mathcal{C}$ do not (section 23.1).

In section 23.2, we investigate initiality and reducedness of the constructions $\mathcal{M}$ and $\mathcal{C}$. For algebraic ground domains, they are known to be initial. This property carries over to continuous ground domains by Th. 14.4.3. For multi-continuous ground domains, we furthermore prove the reducedness of $\mathcal{M}$.

Then, we consider the final power constructions for semirings **B** and **C**, and investigate whether they are sub-constructions of the final **D**-construction $\mathcal{D}$. In the case of semiring **B**, the answer is yes: the final **B**-construction is just the existential restriction $\mathcal{S}$ of $\mathcal{D}$ to **B** (section 23.3). In the case of semiring **C**, the answer is no: the final **C**-construction is not faithful (section 23.4). Hence, it cannot be a sub-construction of any **D**-construction since all these constructions are faithful. In section 23.5, we reveal some more awkward properties of the final **C**-construction. If the ground domain is an infinite discrete domain, then linear functions are not uniquely determined by their values on singletons. In section 23.6, we briefly consider the core of the final **C**-construction.

23.1 Notes on bounded completeness

The sandwich power construction $\mathcal{S}$ and the **C**-construction $\mathcal{S}\overline{\mathcal{S}}$ have the advantage to preserve bounded completeness.

Proposition 23.1.1 If $\mathbf{X}$ is bounded complete and sober, then $\mathcal{S}\mathbf{X}$ is bounded complete. Thus, $\mathcal{S}$ preserves BC & ALG and BC & CONT.

Proof: By Prop. 20.2.4, $\mathcal{U}\mathbf{X}$ is complete in this case. The lub of two members K_1 and K_2 is given by $K_1 \cap K_2$. $\mathcal{L}\mathbf{X}$ is complete for all ground domains; the lub of C_1 and C_2 is $C_1 \cup C_2$.

If $\mathbf{X}$ has a least element $\perp$, then $\mathcal{D}\mathbf{X}$ has the least element $\perp\cdot\{\!|\perp|\!\} = (0,\, 1)\cdot(\downarrow\perp,\, \uparrow\perp) = (\emptyset,\, \mathbf{X})$. This pair obviously satisfies the sandwich condition, whence it is also the least element of $\mathcal{S}\mathbf{X}$.

If two sandwiches $S_1 = (C_1,\, K_1)$ and $S_2 = (C_2,\, K_2)$ are bounded by some sandwich S', then the pair $S = (C_1 \cup C_2,\, K_1 \cap K_2)$ is the least upper bound of S_1 and S_2 in $\mathcal{D}\mathbf{X}$. S is a sandwich by Prop. 21.3.1 since it is below S'. Thus, it is the lub of S_1 and S_2 in $\mathcal{S}\mathbf{X}$. □

The **C**-construction $\mathcal{S}\overline{\mathcal{S}}$ given by the intersection of $\mathcal{S}$ and $\overline{\mathcal{S}}$ does not preserve bounded completeness since its power domains have no least element. They fall into two incomparable parts: an empty part mapped to 0 by ne, and a non-empty part $\mathcal{S}\overline{\mathcal{S}}^1\mathbf{X}$ mapped to 1 (cf. section 17.7). We claim that the non-empty part preserves bounded completeness.

Proposition 23.1.2 If $\mathbf{X}$ is bounded complete and sober, then $\mathcal{S}\overline{\mathcal{S}}^1\mathbf{X}$ is bounded complete. Thus, $\mathcal{S}\overline{\mathcal{S}}^1$ preserves BC & ALG and BC & CONT.

Proof: If $\perp$ is the least element of $\mathbf{X}$, then the pair $\{\!|\perp|\!\} = (\downarrow\perp,\, \uparrow\perp) = (\{\perp\},\, \mathbf{X})$ is in $\mathcal{S}\overline{\mathcal{S}}\mathbf{X}$ since $\mathcal{S}\overline{\mathcal{S}}$ is a sub-construction of $\mathcal{D}$. It is in $\mathcal{S}\overline{\mathcal{S}}^1\mathbf{X}$ since all singletons are mapped to 1 by ne. Let $(C,\, K)$ be an arbitrary member of $\mathcal{S}\overline{\mathcal{S}}^1\mathbf{X}$. $(\{\perp\},\, K)$ is below $(C,\, K)$ iff C is not empty. If C were empty, then K would also be empty by the dual sandwich condition $K \subseteq \uparrow C$. $(\emptyset,\, \emptyset) = \theta$ is however mapped to 0 by ne. Thus, $\{\!|\perp|\!\}$ is the least element of $\mathcal{S}\overline{\mathcal{S}}^1\mathbf{X}$.
Let S_1 and S_2 be two members of $\mathcal{S}\overline{\mathcal{S}}^1\mathbf{X}$ that are bounded by some S' in $\mathcal{S}\overline{\mathcal{S}}^1\mathbf{X}$. Then S_1 and S_2 are also bounded in $\mathcal{S}\mathbf{X}$, whence they have a lub S in $\mathcal{S}\mathbf{X}$ by Prop. 23.1.1. Since $S \geq S_1$ and S_1 is in $\mathcal{S}\overline{\mathcal{S}}\mathbf{X}$, S is in $\mathcal{S}\overline{\mathcal{S}}\mathbf{X}$ because $\overline{\mathcal{S}}\mathbf{X}$ is an upper set in $\mathcal{D}\mathbf{X}$ by Prop. 21.7.1. $ne\, S \geq ne\, S_1 = 1$ implies $S \in \mathcal{S}\overline{\mathcal{S}}^1\mathbf{X}$. □

In contrast to the sandwich power domains, the power domains $\mathcal{M}\mathbf{X}$ and $\mathcal{C}\mathbf{X}$ are not necessarily bounded complete, even in the case of finite bounded complete ground domain $\mathbf{X}$. The example is identical with Plotkin's example for his power domain.

Let $\mathbf{X} = \mathbf{B} \times \mathbf{B}$. To be concise, we write the domain members as xy instead of $(x,\, y)$, e.g. $\perp\perp$, TF. Since this domain is finite, all its lower subsets are closed and all subsets are compact. Thus, Th. 22.6.1 applies and condition M reads $(L,\, U) \in \mathcal{M}\mathbf{X}$ iff $L \subseteq \downarrow(L \cap U)$. Condition $\overline{M}$ is $U \subseteq \uparrow(L \cap U)$.

Let $U = \{\mathsf{T}\perp,\, \mathsf{F}\perp\}$ and $V = \{\perp\mathsf{T},\, \perp\mathsf{F}\}$, and let $Y = \{\mathsf{TT},\, \mathsf{FF}\}$ and $Z = \{\mathsf{TF},\, \mathsf{FT}\}$. Then consider the corresponding formal sets $A = \{\!|\mathsf{T}\perp,\, \mathsf{F}\perp|\!\} = (\downarrow U,\, \uparrow U)$, $B = (\downarrow V,\, \uparrow V)$, $C = (\downarrow Y,\, \uparrow Y)$ and $D = (\downarrow Z,\, \uparrow Z)$. A, B, C, and D are members of $\mathcal{C}\mathbf{X}$ since they satisfy the criterion for the base of $\mathcal{C}\mathbf{X}$ of Th. 22.10.3. It is easy to verify that both of A and B are below both of C and D.

A least upper bound X of A and B, and also a greatest lower bound X of C and D must lie in between, i.e. $A,\, B \leq X \leq C,\, D$. This means $A^U,\, B^U \supseteq X^U \supseteq C^U,\, D^U$, whence $A^U \cap B^U \supseteq X^U \supseteq C^U \cup D^U$, or $\uparrow U \cap \uparrow V \supseteq X^U \supseteq \uparrow Y \cup \uparrow Z$. Since $\uparrow U \cap \uparrow V$ equals $\uparrow Y \cup \uparrow Z$, X^U is uniquely determined to be $\{\mathsf{TT},\, \mathsf{TF},\, \mathsf{FT},\, \mathsf{FF}\}$.
Analogously, $A^L \cup B^L \subseteq X^L \subseteq C^L \cap D^L$ holds. This determines X^L since both sides are equal. Thus, we obtain $X = (\{\perp\perp,\, \mathsf{T}\perp,\, \mathsf{F}\perp,\, \perp\mathsf{T},\, \perp\mathsf{F}\},\, \{\mathsf{TT},\, \mathsf{TF},\, \mathsf{FT},\, \mathsf{FF}\})$, i.e. X^L and X^U are disjoint, although they are not empty. Thus, X is neither a member of $\mathcal{M}\mathbf{X}$ nor a member

of $\overline{\mathcal{M}}\mathbf{X}$ (nevertheless, it is in $\mathcal{S}\overline{\mathcal{S}}\mathbf{X}$). This shows that A and B have no lub, and C and D no glb in $\mathcal{M}\mathbf{X}$, $\overline{\mathcal{M}}\mathbf{X}$, and $\mathcal{C}\mathbf{X}$.

Since $\mathcal{S}$ preserves bounded completeness but is not reduced, whereas $\mathcal{M}$ is reduced but does not preserve BC, one might believe that the power domains $\mathcal{S}\mathbf{X}$ are reduced if one restricts attention to bounded complete sub-powerdomains. This belief is wrong since for instance $\mathcal{M}\mathbf{2}$ and $\mathcal{S}\mathbf{2}$ differ as pointed out in section 22.1, but are both bounded complete. Nevertheless, $\mathcal{S}$ might be a minimal $\mathbf{B}$-construction that preserves BC & ALG.

23.2 Notes on initiality and reducedness

In [Gun89b], Gunter proves the initiality of $\mathcal{M}$ in case of an algebraic ground domain $\mathbf{X}$. Hence, $\mathcal{M}\mathbf{X}$ and $\mathcal{P}_i^{\mathbf{B}}\mathbf{X}$ are isomorphic for all algebraic ground domains $\mathbf{X}$. By Th. 14.4.3, this isomorphism extends to all continuous ground domains. Similarly, the initiality of $\mathcal{C}^1\mathbf{X}$ was claimed in [HP79] for algebraic ground domain $\mathbf{X}$ w.r.t. the class of $\mathbf{C}$-modules without 0. One easily verifies that the addition of the empty set to $\mathcal{C}^1\mathbf{X}$ does not matter if only $\mathbf{C}$-modules according to our definition are considered. Applying Th. 14.4.3 once more, we obtain

Theorem 23.2.1 For continuous ground domains, $\mathcal{M}$ and the initial $\mathbf{B}$-construction coincide as well as $\mathcal{C}$ and the initial $\mathbf{C}$-construction.

In the remainder of this section, we show that $\mathcal{M}\mathbf{X}$ is reduced for multi-continuous $\mathbf{X}$. We do not know whether it is not reduced for some strange ground domains. If $\mathcal{M}$ is not reduced, it differs from its core $\mathcal{M}^c$. Then $\mathcal{M}$ and $\mathcal{M}^c$ will be two different generalizations of Gunter's mixed power construction that have to be carefully distinguished. If in addition $\mathcal{M}^c$ and the initial $\mathbf{B}$-construction differ,[1] one gets even a third generalization.

The proof of $\mathcal{M}^c\mathbf{X} = \mathcal{M}\mathbf{X}$ for multi-continuous $\mathbf{X}$ proceeds in several steps. In each step, we prove a larger class of mixes to be contained in the core. The first steps also work in the larger class of sober domains.

Proposition 23.2.2 Let $\mathbf{X}$ be a sober ground domain. If E and F are finite sets of $\mathbf{X}$ such that $E \subseteq \uparrow F$, then $(\downarrow E, \uparrow F)$ is in $\mathcal{M}^c\mathbf{X}$.

Proof: First of all, $(\downarrow E, \uparrow F)$ is a mix since $E \subseteq \uparrow F$ implies $E = E \cap \uparrow F \subseteq \mathsf{cl}\,(\downarrow E \cap O)$ for all supersets O of $\uparrow F$. Let $E = \{e_1, \ldots, e_n\}$ and $F = \{f_1, \ldots, f_m\}$. We claim $(\downarrow E, \uparrow F) = \{\!|e_1, \ldots, e_n, f_1?, \ldots, f_m?|\!\}$.

Since $\{\!|e|\!\} = (\downarrow e, \uparrow e)$ and $\{\!|f?|\!\} = (\emptyset, \uparrow f)$, the right hand side equals $(\downarrow E \cup \emptyset, \uparrow E \cup \uparrow F)$. $\uparrow E \cup \uparrow F$ is $\uparrow F$ since $E \subseteq \uparrow F$.

$\{\!|e_1, \ldots, e_n, f_1?, \ldots, f_m?|\!\}$ is a finite union (by means of '$\uplus$') of singletons and uncertain singletons. In module theory, one would say it is a finite linear combination of singletons. Hence, it is in $\mathcal{M}^c\mathbf{X}$. □

In the next step, we make the lower part arbitrary, whereas the upper part remains finitary.

[1] Meanwhile, they are shown to differ.

Proposition 23.2.3 Let $\mathbf{X}$ be a sober ground domain. If C is a closed set and F a finitary upper set such that $C \subseteq \mathsf{cl}\,(C \cap F)$, then (C, F) is in $\mathcal{M}^c\mathbf{X}$.

Proof: Let $\mathcal{E}$ be the set of all finite subsets of $C \cap F$, and let $\mathcal{D} = \{(\downarrow E, F) \mid E \in \mathcal{E}\}$. Because of $E \subseteq F$, this set is a subset of $\mathcal{M}^c\mathbf{X}$ by Prop. 23.2.2. It is directed, since $E_1, E_2 \subseteq_f C \cap F$ implies $E_1 \cup E_2 \subseteq_f C \cap F$, and $(\downarrow(E_1 \cup E_2), F)$ is the lub of $(\downarrow E_i, F)$.

We claim that (C, F) is the limit of $\mathcal{D}$. We have to show $\mathsf{cl} \bigcup \downarrow[\mathcal{E}] = C$. All members of $\mathcal{E}$ are subsets of C, whence '$\subseteq$' follows. For the opposite, notice that $\mathcal{E}$ contains all singleton subsets of $C \cap F$, whence $\bigcup \mathcal{E} = C \cap F$. Thus, $\mathsf{cl} \bigcup \downarrow[\mathcal{E}] = \mathsf{cl} \bigcup \mathcal{E} = \mathsf{cl}\,(C \cap F)$ follows. $\mathsf{cl}\,(C \cap F) \supseteq C$ holds by the precondition.

(C, F) is in $\mathcal{M}^c\mathbf{X}$ since it is the lub of a directed set of core elements. $\square$

In the final step, we generalize the upper set.

Proposition 23.2.4 Let $\mathbf{X}$ be a multi-continuous ground domain. Then every mix (C, K) belongs to $\mathcal{M}^c\mathbf{X}$.

Proof: Let $\mathcal{F}$ be the set of all finitary upper environments of K. From Th. 8.3.2, we know that $\mathcal{F}$ is directed in $\mathcal{U}_f\mathbf{X}$ with lub K. Let $\mathcal{D} = \{(C, F) \mid F \in \mathcal{F}\}$. The members of this set satisfy the preconditions of Prop. 23.2.3: if F is in $\mathcal{F}$, then there is an open set O with $K \subseteq O \subseteq F$. Because (C, K) is a mix, $C \subseteq \mathsf{cl}\,(C \cap O) \subseteq \mathsf{cl}\,(C \cap F)$ holds.

Thus, $\mathcal{D}$ is a subset of $\mathcal{M}^c\mathbf{X}$. Since $\mathcal{F}$ is directed with lub K, $\mathcal{D}$ is directed, too, with lub (C, K). Thus, (C, K) is in $\mathcal{M}^c\mathbf{X}$. $\square$

Summarizing, we obtain

Theorem 23.2.5 For multi-continuous ground domain $\mathbf{X}$, $\mathcal{M}^c\mathbf{X} = \mathcal{M}\mathbf{X}$ holds.

23.3 The final B-construction

The Boolean semiring $\mathbf{B}$ is a sub-semiring of the double semiring $\mathbf{D}$. Generally, the existential restriction of the final construction for the larger semiring is completely different from the final power construction of the sub-semiring. An example will be presented in section 23.4. In the case of $\mathbf{B}$ and $\mathbf{D}$ however, these two constructions happen to coincide. Hence, we claim

Theorem 23.3.1 The final $\mathbf{B}$-construction is isomorphic to the existential restriction of the final $\mathbf{D}$-construction, i.e. the sandwich power construction: $\mathcal{P}_f^{\mathbf{B}} = \mathcal{S}$.

Proof: We have to establish an isomorphism between $\mathcal{P}_f^{\mathbf{B}}\mathbf{X} = [[\mathbf{X} \to \mathbf{B}] \overset{add}{\to} \mathbf{B}]$ and $\mathcal{S}\mathbf{X} = \{P \in [[\mathbf{X} \to \mathbf{D}] \overset{add}{\to} \mathbf{D}] \mid \forall p : [\mathbf{X} \to \mathbf{B}] : Pp \in \mathbf{B}\}$. An obvious choice is the restriction and co-restriction $\mathcal{R}$ of functions in $\mathcal{S}\mathbf{X}$ to arguments in $[\mathbf{X} \to \mathbf{B}]$ and results in $\mathbf{B}$. Since the power operations in $\mathcal{S}\mathbf{X}$ are inherited from those in $\mathcal{D}_f$, restriction $\mathcal{R}$ coincides with existential quantification in $\mathcal{S}\mathbf{X}$, whence it is a power homomorphism as indicated in chapter 15. Employing Prop. 12.2.1, we only have to show that it is bijective and its inverse is monotonic, then it is a power isomorphism.

From Prop. 16.3.1, we know that $\mathcal{D}_f\mathbf{X}$ and $\mathcal{L}_f\mathbf{X} \times \mathcal{U}_f\mathbf{X}$ are isomorphic. Hence, every $P : [[\mathbf{X} \to \mathbf{D}] \overset{add}{\to} \mathbf{D}]$ may be considered a pair (P^L, P^U) where $P^L : [[\mathbf{X} \to \mathbf{L}] \overset{add}{\to} \mathbf{L}]$ and $P^U : [[\mathbf{X} \to \mathbf{U}] \overset{add}{\to} \mathbf{U}]$. Accordingly, every function from somewhere to $\mathbf{D}$ may be considered a pair of functions to $\mathbf{L}$ and $\mathbf{U}$ respectively.

Let P and Q be two members of $\mathcal{S}\mathbf{X}$ such that $\mathcal{R}P \leq \mathcal{R}Q$. We have to show $P \leq Q$. For every $f : [\mathbf{X} \to \mathbf{D}]$ resp. $(f^L, f^U) : [\mathbf{X} \to \mathbf{L}] \times [\mathbf{X} \to \mathbf{U}]$, $Pf = P(f^L, f^U) = P(f^L, \underline{0}) + P(\underline{0}, f^U)$ where $\underline{0} = \lambda x.\, 0$ holds by additivity of P. The second summand is handled easily: function $(\underline{0}, f^U)$ can produce the values $(0, 0) = 0$ and $(0, 1) = \perp$ only. Both are in $\mathbf{B}$, whence by $\mathcal{R}P \leq \mathcal{R}Q$ the inequality $P(\underline{0}, f^U) \leq Q(\underline{0}, f^U)$ follows. The first summand cannot be handled this way since its argument may produce $(1, 0) = \mathsf{W}$. We consider the argument $(f^L, \underline{1})$ instead. It can produce the values $(0, 1) = \perp$ and $(1, 1) = 1$ which are both in $\mathbf{B}$. Hence, $(P^L f^L, P^U \underline{1}) = P(f^L, \underline{1}) \leq Q(f^L, \underline{1}) = (Q^L f^L, Q^U \underline{1})$ holds, whence $P^L f^L \leq Q^L f^L$ follows. Since $P^U \underline{0} = 0 = Q^U \underline{0}$, one obtains by combination $P(f^L, \underline{0}) \leq Q(f^L, \underline{0})$. Returning to the sum, this proves $P(f^L, f^U) \leq Q(f^L, f^U)$. Thus, we have shown $P \leq Q$.

We now know that $\mathcal{R}$ is injective and that its inverse — if it exists — is monotonic. Hence, only surjectivity of $\mathcal{R}$ remains to be shown. Let $Q : [[\mathbf{X} \to \mathbf{B}] \overset{add}{\to} \mathbf{B}]$ be given. We now consider $\mathbf{B}$ as subset of $\mathbf{D} = \mathbf{L} \times \mathbf{U}$, i.e. consisting of pairs. Hence, one may assume $Q = (Q_1, Q_2)$ where $Q_1 : [[\mathbf{X} \to \mathbf{B}] \overset{add}{\to} \mathbf{L}]$ and $Q_2 : [[\mathbf{X} \to \mathbf{B}] \overset{add}{\to} \mathbf{U}]$. We show that Q_1 does not depend on the second component of its argument, and correspondingly, Q_2 does not depend on the first.

(1) $Q_1(f^L, f^U) = Q_1(f^L, \underline{1})$ for all $(f^L, f^U) : [\mathbf{X} \to \mathbf{B}]$.

This statement makes sense because $(f^L, \underline{1})$ always maps to $\mathbf{B}$ as we saw above. In $\mathbf{U}$, 1 is least, whence $Q_1(f^L, f^U) \geq Q_1(f^L, \underline{1})$ by monotonicity. By additivity of Q_1, $Q_1(f^L, \underline{1}) = Q_1(f^L, f^U) + Q_1(f^L, \underline{1}) \geq Q_1(f^L, f^U)$ since $1 = x + 1$ in $\mathbf{U}$ and $x \geq 0$ in $\mathbf{L}$.

(2) $Q_2(f^L, f^U) = Q_2(\underline{0}, f^U)$ for all $(f^L, f^U) : [\mathbf{X} \to \mathbf{B}]$.

This statement makes sense because $(\underline{0}, f^U)$ always maps to $\mathbf{B}$ as pointed out above. 0 is the least element of $\mathbf{L}$, whence $Q_2(f^L, f^U) \geq Q_2(\underline{0}, f^U)$ by monotonicity. By additivity of Q_2, $Q_2(f^L, f^U) = Q_2(f^L, f^U) + Q_2(\underline{0}, f^U) \leq Q_2(\underline{0}, f^U)$ since $x = x + 0$ in $\mathbf{L}$ and $x \leq 0$ in $\mathbf{U}$.

For $f^U : [\mathbf{X} \to \mathbf{U}]$, $(\underline{0}, f^U)$ maps from $\mathbf{X}$ to $\mathbf{B}$ whence the definition $P^U f^U = Q_2(\underline{0}, f^U)$ makes sense. Similarly, for $f^L : [\mathbf{X} \to \mathbf{L}]$, $(f^L, \underline{1})$ is in $[\mathbf{X} \to \mathbf{B}]$ whence $P^L f^L = Q_1(f^L, \underline{1})$ may be defined. P^L and P^U are maps in $[[\mathbf{X} \to \mathbf{L}] \to \mathbf{L}]$ and $[[\mathbf{X} \to \mathbf{U}] \to \mathbf{U}]$ respectively. In the sequel, we show their additivity. Thus, they may be combined to $P = (P^L, P^U)$ in $[[\mathbf{X} \to \mathbf{D}] \overset{add}{\to} \mathbf{D}]$. We then have to show $\mathcal{R}P = Q$; this automatically implies $P \in \mathcal{S}\mathbf{X}$.

$P^U \underline{0} = Q_2(\underline{0}, \underline{0}) = 0$ and $P^U(f+g) = P^U f + P^U g$ hold by additivity of Q because $(f+g, \underline{0}) = (f, \underline{0}) + (g, \underline{0})$.

$P^L(f + g) = P^L f + P^L g$ holds by additivity of Q because $(f + g, \underline{1}) = (f, \underline{1}) + (g, \underline{1})$. $P^L \underline{0} = Q_1(\underline{0}, \underline{1}) = Q_1(\underline{0}, \underline{0}) = 0$ holds; here statement (1) from above is applied.

Finally, we have to show $\mathcal{R}P = Q$, i.e. for $f : [\mathbf{X} \to \mathbf{B}]$, $Pf = Qf$ holds. Applying statements (1) and (2) from above, we obtain $Pf = (P^L f^L, P^U f^U) = (Q_1(f^L, \underline{1}), Q_2(\underline{0}, f^U)) = (Q_1(f^L, f^U), Q_2(f^L, f^U)) = Q(f^L, f^U) = Qf$. □

The theorem allows a slight simplification of the definition of $\mathcal{M}$. In section 22.3, we defined

- $\mathcal{M}\mathbf{X} = \{P \in [[\mathbf{X} \to \mathbf{D}] \stackrel{add}{\to} \mathbf{D}] \mid Pp - Pq \geq \mathsf{T} \Rightarrow P(p-q) \geq \mathsf{T}\ \forall p, q : [\mathbf{X} \to \mathbf{D}]\}$

Now, we know $\mathcal{M}\mathbf{X} \subseteq \mathcal{S}\mathbf{X} = [[X \to \mathbf{B}] \stackrel{add}{\to} \mathbf{B}]$. Hence, one may replace all occurrences of $\mathbf{D}$ by $\mathbf{B}$ in the formula above. For values b in $\mathbf{B}$, $b \geq \mathsf{T}$ is equivalent to $b = \mathsf{T}$. Thus, one obtains

- $\mathcal{M}\mathbf{X} = \{P \in [[\mathbf{X} \to \mathbf{B}] \stackrel{add}{\to} \mathbf{B}] \mid Pp - Pq = \mathsf{T} \Rightarrow P(p-q) = \mathsf{T}\ \forall p, q : [\mathbf{X} \to \mathbf{B}]\}$

23.4 The final C-construction

In this section, we investigate the final $\mathbf{C}$-construction $\mathcal{P}_f^{\mathbf{C}}$. For the sake of simplicity, we abbreviate it by $\mathcal{P}_f$ in the sequel.

The properties of $\mathcal{P}_f$ are quite awkward.

Proposition 23.4.1

(1) $\mathcal{P}_f\mathbf{X}$ is discrete for all ground domains $\mathbf{X}$.

(2) If $x \leq x'$ holds in $\mathbf{X}$, then $\{\!|x|\!\} = \{\!|x'|\!\}$ holds in $\mathcal{P}_f\mathbf{X}$.

(3) If $\mathbf{X}$ has a least element $\perp$, then $\mathcal{P}_f\mathbf{X}$ has exactly two elements: θ and $\{\!|\perp|\!\}$. They are incomparable.

Proof:

(1) $P \leq Q$ holds iff $Pp \leq Qp$ for all $p : [\mathbf{X} \to \mathbf{C}]$. Since Pp and Qp are in $\mathbf{C}$, and $\mathbf{C}$ is discrete, $Pp = Qp$ follows for all p, whence $P = Q$.

(2) $x \leq x'$ implies $\{\!|x|\!\} \leq \{\!|x'|\!\}$ by monotonicity. By (1), $\{\!|x|\!\} = \{\!|x'|\!\}$ follows.

(3) If $f : [\mathbf{X} \to \mathbf{C}]$, then $fx \geq f\perp$ holds for all x in $\mathbf{X}$. As $\mathbf{C}$ is discrete, $fx = f\perp$ follows. Hence, $[\mathbf{X} \to \mathbf{C}]$ has exactly two elements: $\lambda x.\,0$ and $\lambda x.\,1$. Every additive second order predicate must map $\lambda x.0$ to 0; it only has the choice to map $\lambda x.1$. Thus, $[[\mathbf{X} \to \mathbf{C}] \stackrel{add}{\to} \mathbf{C}]$ has at most two elements. Conversely, it has at least two elements, namely θ and $\{\!|\perp|\!\}$. They are different, even incomparable, because they are distinguished by ne. □

By this proposition, $\mathcal{P}_f$ is not faithful. It maps all ground domains with a least element — no matter how large they might be — to a power domain of just two elements. Probably, this is the reason why this construction never was proposed in the literature.

23.5 C-constructions on discrete ground domains

One might guess that $\mathcal{P}_f\mathbf{X}$ always be quite small. This however is not true: discrete ground domains on the contrary are mapped to huge power domains. Remember we already know two $\mathbf{C}$-constructions defined on the class of discrete domains: the set of arbitrary subsets $\mathcal{P}_{set}$ and the set of finite subsets $\mathcal{P}_{fin}$. The resulting power domains are always ordered discretely. Since functions from a discrete domain are continuous always, continuity is of no concern in this section.

Proposition 23.5.1 For discrete X, $\mathcal{P}_{fin}X$ may be embedded into $\mathcal{P}_{set}X$, and this may in turn be embedded into $\mathcal{P}_f X$. Both embeddings are surjective iff X is finite.

Proof: The statements about $\mathcal{P}_{fin}X$ and $\mathcal{P}_{set}X$ are obvious. Abbreviating the latter by $\mathcal{P}X$, we have to provide an embedding from $\mathcal{P}X$ into $\mathcal{P}_f X$. By section 15.5, existential quantification is the only power homomorphism from $\mathcal{P}$ to $\mathcal{P}_f$. We have to show $\mathcal{E}A = \mathcal{E}B$ implies $A = B$ for A, B in $\mathcal{P}X$.

For a in X, let $f_a : X \to \mathbf{C}$ be defined such that $f_a x = 1$ iff $a = x$, and $= 0$ otherwise. Then $\mathcal{E}Af_a = 1$ iff there is x in A with $a = x$ iff $a \in A$. Hence, $\mathcal{E}A = \mathcal{E}B$ implies $\mathcal{E}Af_a = 1$ iff $\mathcal{E}Bf_a = 1$, whence $a \in A$ iff $a \in B$, i.e. $A = B$.

Finally, we have to show $\mathcal{E}$ is surjective iff X is finite. Let X be finite. For $P : (X \to \mathbf{C}) \xrightarrow{add} \mathbf{C}$, let $A = \{a \in X \mid Pf_a = 1\}$. We claim $\mathcal{E}A = P$. Let $p : X \to \mathbf{C}$, and let $B = \{b \in X \mid p\, b = 1\}$. Then $p = \sum_{b\in B} f_b$ holds; this sum is well-defined since X and more than ever B are finite. By additivity of P, $Pp = \sum_{b\in B} Pf_b$ holds. This value is 1 iff there is b in B with $Pf_b = 1$ iff there is b in B with $b \in A$ iff there is b in A with $p\, b = 1$ iff $\mathcal{E}Ap = 1$.

Now let X be infinite. We have to show that $\mathcal{E}$ is not surjective. Let $P : (X \to \mathbf{C}) \to \mathbf{C}$ be defined such that Pp is 0 iff $p^{-1}[1]$ is finite. This function is additive since $P(\lambda x.\, 0)$ is 0, and $P\,(p + q)$ is 0 iff $(p + q)^{-1}[1]$ is finite iff $p^{-1}[1] \cup q^{-1}[1]$ is finite iff both $p^{-1}[1]$ and $q^{-1}[1]$ are finite iff $Pp = 0$ and $Pq = 0$ iff $Pp + Pq = 0$. Assume there is a set A with $\mathcal{E}A = P$. For all a in A, $\mathcal{E}Af_a = 1$ holds, whereas $Pf_a = 0$ because $f_a^{-1}[1] = \{a\}$. Hence, A is empty. Thus, $\mathcal{E}A\,p = 0$ for all $p : X \to \mathbf{C}$. $P(\lambda x.\, 1)$ however is 1 since $(\lambda x.\, 1)^{-1}[1] = X$ is infinite. □

By this proposition, we see that $\mathcal{P}_f X$ is even larger than the power set of X if X is an infinite discrete domain. In section 15.8, we wondered whether all final power constructions have unique extension, and claimed the answer be no. An example is provided by the final $\mathbf{C}$-construction.

Proposition 23.5.2 For discrete ground domain X, $\mathcal{P}_{fin}X$ is the initial $\mathbf{C}$-powerdomain over X. The cores of $\mathcal{P}_{set}X$ and of $\mathcal{P}_f X$ are both (isomorphic to) $\mathcal{P}_{fin}X$. $\mathcal{P}_{set}X$ and $\mathcal{P}_f X$ have unique extensions iff X is finite.

Proof: Let M be a not necessarily discrete $\mathbf{C}$-module, i.e. a commutative idempotent monoid, and let $f : X \to M$ be a function. Then define $ext\, f\, A = \sum_{a\in A} fa$ for all finite subsets of X. By idempotence, $ext\, f$ is additive. $ext\, f\,\{x\} = fx$ obviously holds. Because of the required additivity, $ext\, f$ is unique with these properties.

Because of Prop. 23.5.1, $\mathcal{P}_{fin}X$ is contained in both of $\mathcal{P}_{set}X$ and $\mathcal{P}_f X$. Hence, it is their core because it is reduced.

For finite X, all three considered power domains are isomorphic by Prop. 23.5.1. Since $\mathcal{P}_{fin}X$ is initial, the remaining two also have unique extensions.

Let X be infinite. We define two different linear functions from $\mathcal{P}_{set}X$ to $\mathbf{C}$ that coincide for singletons. Let FA be 0 for all $A \subseteq X$, and let GA be 0 iff A is finite, and 1 otherwise. F obviously is additive, and G is so since a binary union is finite iff its constituents are. $F\{x\} = G\{x\} = 0$ holds for all x in X, whereas $FX = 0$ and $GX = 1$ since X is infinite.

This example cannot be taken directly for $\mathcal{P}_f X$ because it is not obvious how to define G for all members of $\mathcal{P}_f X$. We call $A \in \mathcal{P}_f X$ *finite* iff there are $x_1, \ldots, x_k \in X$, such that

$A = \{\!|x_1, \ldots, x_k|\!\}$, and we call $B \in \mathcal{P}_f X$ *sub-finite* iff there is an arbitrary C and a finite A such that $B \uplus C = A$. Because of $A \uplus A = A$, every finite set is sub-finite. After these preliminaries, we define

$$FA = 0 \text{ for all } A \in \mathcal{P}_f X$$

$$GA = \begin{cases} 0 & \text{if } A \text{ is sub-finite} \\ 1 & \text{otherwise} \end{cases}$$

F is obviously linear. For G, some more work is needed. $G\theta = 0$ holds since θ is sub-finite. For linearity, we then have to show $G(A \uplus B) = GA + GB$. $G(A \uplus B) = 0$ holds iff $A \uplus B$ is sub-finite, and $GA + GB = 0$ holds iff $GA = 0$ and $GB = 0$ iff A and B are sub-finite. Hence, we have to show $A \uplus B$ is sub-finite iff A and B are sub-finite.

If A and B are sub-finite, there are arbitrary U and V and finite C and D such that $A \uplus U = C$ and $B \uplus V = D$ whence $A \uplus B \uplus U \uplus V = C \uplus D$ whence $A \uplus B$ is sub-finite because the union of finite members of $\mathcal{P}_f X$ is finite again. Conversely, if $A \uplus B$ is sub-finite then there are some U and a finite C with $A \uplus B \uplus U = C$. This equation may be directly interpreted such that A and B are sub-finite.

The linear mappings F and G coincide for singletons; both map all singletons $\{\!|x|\!\}$ to 0 because singletons are sub-finite. Nevertheless, F and G are different. This is shown by presenting an A in $\mathcal{P}_f X$ that is not sub-finite.

Let A be defined by $A\,p = \begin{cases} 0 & \text{if } p\,x = 0 \text{ for all } x \in X \\ 1 & \text{otherwise} \end{cases}$

Obviously, $A\,(\lambda x.\,0) = 0$ holds. A is linear by the following derivation:

$$\begin{array}{lll} A\,(p+q) = 0 & \text{iff} & (p+q)\,x = 0 \text{ for all } x \\ & \text{iff} & px + qx = 0 \text{ for all } x \\ & \text{iff} & (px = 0 \text{ and } qx = 0) \text{ for all } x \\ & \text{iff} & (px = 0 \text{ for all } x) \text{ and } (qx = 0 \text{ for all } x) \\ & \text{iff} & A\,p = 0 \text{ and } A\,q = 0 \\ & \text{iff} & A\,p + A\,q = 0 \end{array}$$

Finally, we show that A is not sub-finite. Assume there are U in $\mathcal{P}_f X$ and $x_1, \ldots, x_k \in X$ such that $A \uplus U = \{\!|x_1, \ldots, x_k|\!\}$. Let x' be a point of X that is not among $x_1, \ldots, x_k$, and let $p : X \to \mathbf{C}$ be defined such that x' is mapped to 1 and all other points are mapped to 0. Then $(A \uplus U)\,p = A\,p + U\,p = 1 + U\,p = 1$ holds, whereas $\{\!|x_1, \ldots, x_k|\!\}\,p = p\,x_1 + \cdots + p\,x_k = 0$ — a contradiction. Hence, $FA = 0$, but $GA = 1$. □

23.6 The core of the final C-construction

For ground domains with least element, $\mathcal{P}_f^{\mathbf{C}}\mathbf{X}$ has only two elements: θ and $\{\!|\bot|\!\}$ by Prop. 23.4.1 (3). Thus, it is reduced and coincides with its core. This provides an example for a reduced power construction that is not faithful. For infinite discrete ground domains however, $\mathcal{P}_f^{\mathbf{C}}\mathbf{X}$ and its core $\mathcal{P}_c^{\mathbf{C}}\mathbf{X}$ are far from being equal. The core of $\mathcal{P}_f^{\mathbf{C}}\mathbf{X}$ is initial in this case.

The lower semiring $\mathbf{L}$ is a commutative idempotent monoid, i.e. a $\mathbf{C}$-module. It is not discrete, whence by Prop. 14.5.3, the initial $\mathbf{C}$-construction $\mathcal{P}_i^{\mathbf{C}}$ is faithful. Thus, $\mathcal{P}_i^{\mathbf{C}}\mathbf{X}$ and $\mathcal{P}_c^{\mathbf{C}}\mathbf{X}$ are not isomorphic if $\mathbf{X}$ is a domain with bottom and at least one further element.

Hence, we have seen that the three autochthonous $\mathbf{C}$-constructions $\mathcal{P}_i^{\mathbf{C}}$, $\mathcal{P}_c^{\mathbf{C}}$, and $\mathcal{P}_f^{\mathbf{C}}$ considerably differ. The difference shows up even for complete algebraic ground domains.

Chapter 24

Sandwich theory

In case of the power constructions $\mathcal{L}$, $\mathcal{U}$, $\mathcal{C}$, and $\mathcal{M}$, the power domain over a continuous ground domain is algebraically characterized as the free (initial) R-module with appropriate semiring R. In Gunter's papers, the freedom of the sandwich power domains is an open question. Our results show that they cannot be characterized as free modules of some semiring since $\mathcal{M}$ and $\mathcal{S}$ share the same semiring, but nevertheless differ.

The chapter at hand shows that the sandwich power domains are also free algebras belonging to an algebraic theory that however cannot be seen as theory of R-modules for some semiring R. This *sandwich theory* involves a partial operation with a strange meaning. The problem is that all reasonable operations on sandwiches seem to preserve the mix condition, whence only 'unreasonable' operations remain.

The chapter at hand consists of only two sections: in section 24.1, the axioms and theorems of the sandwich theory are introduced, and in section 24.2, the sandwich power domains are shown to be free sandwich algebras if the ground domain is algebraic.

24.1 Axioms and theorems of the sandwich theory

To define sandwich algebras, we use an operation that *marries* sandwiches to each other i.e. $(A^L, A^U) \circledcirc (B^L, B^U) = (A^L, B^U)$. A problem is that the partners may not harmonize such that the marriage is not durable. This means that '$\circledcirc$' is a partial operation only. The exact definition for the sandwich power domain is

$$(A^L, A^U) \circledcirc (B^L, B^U) = \begin{cases} (A^L, B^U) & \text{if } (A^L, B^U) \in \mathcal{S}\mathbf{X} \\ \textit{undefined} & \text{otherwise} \end{cases}$$

'$\circledcirc$' becomes a total continuous operation if '*undefined*' is added to the sandwich power domain as an artificial top element. This is done in the following definition which enumerates the axioms of a sandwich algebra.

Definition 24.1.1 (Sandwich algebras)

A *sandwich algebra* $(\mathbf{P}, +, 0, \circledcirc)$ is a commutative idempotent monoid domain $(\mathbf{P}, +, 0)$

with an additional continuous operation $\_ \circledcirc \_ : [\mathbf{P} \times \mathbf{P} \to \mathbf{P} \cup \{\mathit{undefined}\}]$ where $x \leq \mathit{undefined}$ for all $x \in \mathbf{P}$ satisfying the following 4 axioms

$(A1)$ $A \circledcirc A = A$

$(A2)$ $A \circledcirc B \leq (A + A') \circledcirc B$ $\quad$ $(A3)$ $A \circledcirc B \geq A \circledcirc (B + B')$

$(A4)$ If $A \circledcirc B$ and $A' \circledcirc B'$ are defined, then $(A + A') \circledcirc (B + B') = (A \circledcirc B) + (A' \circledcirc B')$

A mapping f between two sandwich algebras is a *sandwich homomorphism* iff it is continuous and additive and satisfies $f(A \circledcirc B) = (fA) \circledcirc (fB)$ whenever $A \circledcirc B$ is defined, i.e. in $\mathbf{P}$.

One may easily verify that sandwich power domains become sandwich algebras by defining '$\circledcirc$' as indicated above.

As for the mix theory, some theorems may be derived from the axioms. In the sequel, idempotence is abbreviated by (I) and neutrality of 0 by (N).

(T1) If $A \circledcirc B$ and $A \circledcirc B'$ are defined, then $A \circledcirc (B + B') = (A \circledcirc B) + (A \circledcirc B')$
Proof: let $A' = A$ in (A4) and apply (I).

(T2) If $A \circledcirc B$ and $A' \circledcirc B$ are defined, then $(A + A') \circledcirc B = (A \circledcirc B) + (A' \circledcirc B)$
Proof: let $B' = B$ in (A4) and apply (I).

(T3) If $A \circledcirc B$ is defined, then $(A + B) \circledcirc B = (A \circledcirc B) + B$ and $A \circledcirc (A + B) = A + (A \circledcirc B)$.
Proof: the first equation by (T2) and (A1), the second one by (T1) and (A1).

(T4) $A \circledcirc B$ and B are both $\leq (A + B) \circledcirc B$.
Proof: $B = B \circledcirc B$ by (A1), then apply (A2) in both cases.

(T5) $A \circledcirc (A + B) \leq$ both of A and $A \circledcirc B$.
Proof: $A = A \circledcirc A$ by (A1), then apply (A3) in both cases.

(T6) $0 \circledcirc B \leq B$ $\quad$ since $0 \circledcirc B \overset{T4}{\leq} (0 + B) \circledcirc B \overset{N}{=} B \circledcirc B \overset{A1}{=} B$

(T7) $0 \circledcirc B \leq 0$ $\quad$ since $0 \circledcirc B \overset{N}{=} 0 \circledcirc (0 + B) \overset{T5}{\leq} 0$

(T8) Every sandwich algebra is a mix algebra (a $\mathbf{B}$-module) by the definition $B? = 0 \circledcirc B$.
Proof: (T6) means $B? \leq B$ and (T7) means $B? \leq 0$. These inequations also show that $B?$ is defined for all B. (T3) implies $B? + B = (0 \circledcirc B) + B = (0 + B) \circledcirc B = B$ by (N) and (A1). (T1) gives $(A + B)? = A? + B?$.

(T9) $A \leq A + B$ iff $0 \circledcirc A \leq 0 \circledcirc B$ iff for all C: $C \circledcirc A \leq C \circledcirc B$.
Proof: '$(1) \Rightarrow (3)$': $C \circledcirc A \overset{1}{\leq} C \circledcirc (A + B) \overset{A3}{\leq} C \circledcirc B$
(3) obviously implies (2), and (2) implies (1) by (T8) and theorem (T8) of the mix theory.

(T10) $A + B \leq B$ iff for all C: $A \circledcirc C \leq B \circledcirc C$.
Proof: '$\Rightarrow$': $A \circledcirc C \overset{A2}{\leq} (A + B) \circledcirc C \overset{lhs}{\leq} B \circledcirc C$
'$\Leftarrow$': $A + B \overset{A1}{=} (A + B) \circledcirc (A + B) \overset{T2}{=} (A \circledcirc (A + B)) + (B \circledcirc (A + B))$. (T2) is applicable since both marriages to the right are defined due to (T5) viz. $A \circledcirc (A + B) \leq A$ and $B \circledcirc (A + B) \leq B$. The precondition with $C = A + B$ may be applied to the last term resulting in $A + B \leq (B \circledcirc (A + B)) + (B \circledcirc (A + B)) \overset{I}{=} B \circledcirc (A + B) \overset{T5}{\leq} B$.

(T11) $A \leq A + B \geq B$ iff for all C: $C \otimes A = C \otimes B$.
This is deduced directly from (T9). We call A and B in this case *upper equivalent* and write $A \approx_U B$. This relation obviously is an equivalence relation. For the sandwich power domain, one can show $A \approx_U B$ iff $A^U = B^U$, hence the name.

(T12) $A \geq A + B \leq B$ iff for all C: $A \otimes C = B \otimes C$.
This is deduced directly from (T10). We call A and B in this case *lower equivalent* and write $A \approx_L B$. For the sandwich power domain, one can show $A \approx_L B$ iff $A^L = B^L$, hence the name.

(T13) If $A \otimes B$ is defined, then it is in the intersection of the lower class of A and the upper class of B, i.e. $A \approx_L A \otimes B \approx_U B$.
Proof: The lower equivalence by (T3) and (T5), and the upper one by (T3) and (T4).

(T14) For given A and B, there is an X such that $A \approx_L X \approx_U B$ iff $A \otimes B$ is defined. In this case, $A \otimes B = X$ holds. $A \otimes B$ is undefined iff there is no such X.
Proof: If $A \otimes B$ is defined, there is such an X by (T13). Conversely, assume such an X. Note that (T11) and (T12) hold no matter whether the married couples involving 'C' are defined or not. Hence, $A \otimes B \overset{T12}{=} X \otimes B \overset{T11}{=} X \otimes X \overset{A1}{=} X$, i.e. $A \otimes B$ is defined and equals X.

Similar to mix algebras, (T14) implies that the operation '$\otimes$' is uniquely determined in a given sandwich algebra, i.e. for given commutative idempotent monoid domain, there is at most one choice for the operation '$\otimes$' to turn it into a sandwich algebra. Another important consequence is the following theorem:

Theorem 24.1.2 An additive mapping between two sandwich algebras is automatically a sandwich homomorphism.

Proof: Let $f : [\mathbf{X} \to \mathbf{Y}]$ be a (continuous) additive map between the two sandwich algebras $\mathbf{X}$ and $\mathbf{Y}$. Let A and B be members of $\mathbf{X}$ such that $A \otimes B$ is defined. Then $A \approx_L A \otimes B \approx_U B$ holds by (T13). Using the equivalence of $S \approx_L T$ with $S \geq S + T \leq T$ and of $S \approx_U T$ with $S \leq S + T \geq T$, one obtains by monotonicity and linearity of f the relations $fA \approx_L f(A \otimes B) \approx_U fB$. (T14) then implies $fA \otimes fB = f(A \otimes B)$. □

24.2 Sandwich power domains as free sandwich algebras

In this section, we show that the sandwich power domains are free sandwich algebras:

Theorem 24.2.1 For every domain $\mathbf{X}$, the sandwich power domain $\mathcal{S}\mathbf{X}$ is a sandwich domain. If $\mathbf{X}$ is algebraic, then for every arbitrary sandwich domain $\mathbf{Y}$ and every continuous map $f : [\mathbf{X} \to \mathbf{Y}]$, there is a unique additive continuous map $\bar{f} : [\mathcal{S}\mathbf{X} \to \mathbf{Y}]$ with $\bar{f}\{\!|x|\!\} = fx$. $\bar{f}$ is a sandwich homomorphism.

Proof: For a finite set of isolated points $E \subseteq \mathbf{X}$, we define $f'E = \sum_{e \in E} fe$. The isolated sandwiches are given by $(\downarrow E, \uparrow F)$ with $E \subseteq \downarrow\uparrow F$. We define for them $\bar{f}(\downarrow E, \uparrow F) = f'E \otimes f'F$.
We first show that the married couple on the right hand side is always defined. If E is empty, $f'E \otimes f'F = 0 \otimes f'F \leq 0$ is defined by (T7). Let e be some member of E. By $E \subseteq \downarrow\uparrow F$, there

are points y and x such that $x \in F$ and $e, x \le y$. Then $fe \otimes fx \le fy \otimes fy = fy$ is defined by (A1). By (A3), $fe \otimes f'F = fe \otimes (fx + f'(F \setminus \{x\})) \le fe \otimes fx$ holds, whence $fe \otimes f'F$ is defined for all e in E. Applying (T2) iteratedly, we obtain $f'E \otimes f'F = (\sum_{e \in E} fe) \otimes f'F = \sum_{e \in E}(fe \otimes f'F)$ is defined.

We also have to show that $\bar{f}$ is well-defined, i.e. independent from the specific choice of E and F. We do this by showing its monotonicity. Well-definedness then follows from antisymmetry in $\mathbf{Y}$.

Assume $(\downarrow E, \uparrow F) \le (\downarrow E', \uparrow F')$ holds, i.e. $E \subseteq \downarrow E'$ and $F' \subseteq \uparrow F$. We have to show $f'E \otimes f'F \le f'E' \otimes f'F'$. $E \subseteq \downarrow E'$ means there is a member $y(x)$ in E' for every x in E such that $y(x) \ge x$. Thus, we obtain

$$\begin{array}{rcl} f'E \otimes f'F & = & \sum_{x \in E} fx \otimes f'F \\ & \le & \sum_{x \in E} f(y(x)) \otimes f'F \\ & \overset{(I)}{=} & \sum_{y \in y[E]} fy \otimes f'F \\ & \overset{(A2)}{\le} & \sum_{y \in E'} fy \otimes f'F \ = \ f'E' \otimes f'F \end{array}$$

Next, $F' \subseteq \uparrow F$ means there is a member $y(x)$ in F for every x in F' such that $y(x) \le x$. Thus, we obtain

$$\begin{array}{rcl} f'E' \otimes f'F' & = & f'E' \otimes \sum_{x \in F'} fx \\ & \ge & f'E' \otimes \sum_{x \in F'} f(y(x)) \\ & \overset{(I)}{=} & f'E' \otimes \sum_{y \in y[F']} fy \\ & \overset{(A3)}{\ge} & f'E' \otimes \sum_{y \in F} fy \ = \ f'E' \otimes f'F \end{array}$$

Combining both inequations results in $f'E \otimes f'F \le f'E' \otimes f'F'$ as required.

Now, we know that $\bar{f}$ is well-defined and monotonic. Next, we show its additivity.

$$\bar{f}\theta \ = \ \bar{f}(\emptyset, \emptyset) \ = \ f'(\emptyset) \otimes f'(\emptyset) \ = \ 0 \otimes 0 \ = \ 0$$

holds using (A1).

$$\begin{array}{rcl} \bar{f}(A \uplus B) & = & \bar{f}(A^L \cup B^L,\, A^U \cup B^U) \\ & = & (f'A^L + f'B^L) \otimes (f'A^U + f'B^U) \\ & \overset{(A4)}{=} & (f'A^L \otimes f'A^U) + (f'B^L \otimes f'B^U) \\ & = & \bar{f}A + \bar{f}B \end{array}$$

shows the additivity of $\bar{f}$.

$\bar{f}$ can be extended from the base of $\mathcal{S}\mathbf{X}$ to the whole of $\mathcal{S}\mathbf{X}$ resulting in a continuous additive map $\overline{f} : [\mathcal{S}\mathbf{X} \to \mathbf{Y}]$.

Uniqueness is shown by using the fact $(\downarrow E, \uparrow F) = (\downarrow E, \uparrow E) \otimes (\downarrow F, \uparrow F) = \{\!|e \mid e \in E|\!\} \otimes \{\!|y \mid y \in F|\!\}$ and theorem 24.1.2. □

Corollary 24.2.2

If $\mathbf{X}$ is algebraic, then for every continuous map $f : [\mathbf{X} \to \mathcal{S}\mathbf{Y}]$, there is a unique additive continuous extension $\bar{f} : [\mathcal{S}\mathbf{X} \to \mathcal{S}\mathbf{Y}]$ with $\bar{f} \circ \{\!|.|\!\} = f$. It is explicitly given by $\bar{f} = \mathit{ext}\, f$ where *ext* is defined as in section 21.2. □

This corollary is particularly nice because it neither mentions sandwich theory nor marrying, i.e. it is independent from this partial and semantically dubious operation. It shows that even power constructions which are far from being reduced may have unique extensions.