# Part I

# Foundations

This first part of the thesis contains a collection of definitions and facts from domain theory and topology. Power domain constructions do not occur in this part although its size is more than one third of the whole thesis. The purpose of this foundations part is to serve as a data base for all fundamental facts and technical proofs that are used in the two remaining parts of the thesis when discussing power domain constructions. Experienced readers should skip this part, immediately start reading Part II, and only occasionally look up concepts, definitions, and results from the foundations part.

The material of the first four chapters is standard except the notion of strong compactness introduced in section 4.7. Many of the concepts and ideas are mentioned in [Law88] together with sketches of their proofs. Fully experienced readers will thus not find much interesting in here. Many computer scientists however do not know much of topology, and conversely, many topologists do not know domain theory. Thus, these four chapters serve as a consistent attempt for an introduction of the mathematical aspects of both theories.

The first four chapters handle the more basic aspects of the theory. After introducing some notations in chapter 1, we present the theory of partially ordered sets and monotonic functions in chapter 2. In chapter 3, the theory of directed complete partial orders and continuous functions follows. To be concise, we call the directed complete partial orders *domains* in this thesis. Thus, our domains neither have to possess a least element nor must they be algebraic. In chapter 4, we then introduce the theory of topological spaces and (topologically) continuous functions. In contrast to classical introductions to topology, we neglect the well separated Hausdorff spaces. Instead, we concentrate on the relations between order theory and topology. Several of the lemmas in chapters 3 and 4 will be heavily used in order-theoretic or topological proofs in the remaining two parts of the thesis.

Chapters 5 through 8 introduce classifications of domains. Among these, there are trivial ones such as finiteness, well known ones such as bounded completeness and algebraicity, almost unknown ones such as sobriety and multi-continuity, and novel classes such as SC and S-RD. Chapter 5 is devoted to completeness properties from bounded completeness in section 5.2 to SC in section 5.5. Chapter 6 deals with algebraicity and continuity. Besides all the standard theory, it contains purely topological characterizations of algebraicity (Th. 6.2.5) and continuity (Th. 6.7.9). These characterizations will be used in Part III of the thesis. Chapter 7 treats finitely algebraic (bifinite) and finitely continuous domains, retracts, and the preservation of domain classes by functors. These results directly apply to power domain constructions because they are special functors as shown in chapter 10. In chapter 8, we study multi-algebraicity and multi-continuity (8.1 – 8.4), sobriety (8.5 – 8.8), and the 'Rudin classes' K-RD and S-RD (8.9 – 8.11). The last section 8.12 of Part I contains tables and diagrams summarizing the definitions and relative inclusions of the introduced domain classes.

As the end of this introduction, we include a table of all the sections of particular interest in the foundations part.

| | |
|---|---|
| • Rudin's Lemma | Section 3.9 |
| • Classes SC and KC | Section 5.5 |
| • Strong compactness | Section 4.7 |
| • Topological criteria for ALG and CONT | Sections 6.2 and 6.7 |
| • Multi-continuity | Sections 8.1 – 8.4 |
| • Sobriety | Sections 8.5 – 8.8 |
| • Rudin classes S-RD and K-RD | Sections 8.9 and 8.10 |

# Chapter 1

# Basic notations

This chapter introduces some basic mathematical notations that we shall use throughout the thesis. In the first section, we present our notations for finite subsets, images, inverse images, etc. In the second section, closure formation is considered in more detail.

## 1.1 Some notations from set theory

Here, we present only those notations that are not standard. For two sets $A$ and $B$, the notation $A \subseteq_f B$ means $A$ is a finite subset of $B$. If $A$ is a subset of some universe $X$, then $\mathsf{co}\, A$ denotes the *complement* of $A$ in $X$, i.e. $\mathsf{co}\, A = X \setminus A$. The *power set* of a set $X$ is denoted by $2^X$.

Let $f : X \to Y$ be a function. We denote the image of a point $x$ in $X$ under $f$ by $fx$, i.e. we mostly omit parentheses in applications. Thus, the multiplication symbol '$\cdot$' will never be omitted, i.e. products are always written $a \cdot b$. Functions are often denoted by $\lambda$-expressions. For instance, the function $f$ defined by $fy = x \cdot y$ where $x$ is a constant, may be concisely referred to by $\lambda y.\, x \cdot y$. We assume the reader to be familiar with the elementary rules of $\alpha$-, $\beta$-, and $\eta$-conversion for equational reasoning in the $\lambda$-calculus.

Usually, applications will group left, i.e. $Gfx$ means $(Gf)\, x$. In rare circumstances only where the type of the involved terms obviously excludes the meaning $(gf)\, x$, we abbreviate $g\,(fx)$ by $gfx$. As the function occurs to the left of its argument in applications, composition operates from right to left. The composition of $f : X \to Y$ and $g : Y \to Z$ is hence denoted by $g \circ f : X \to Z$, i.e. $(g \circ f)\, x$ equals $g\,(fx)$.

Let $f : X \to Y$ be again a function. The image of a subset $A$ of $X$ under $f$ is denoted by $f[A] = \{fa \mid a \in A\}$. Square brackets are used here to let the reader more safely distinguish between applications to single points $x$ and pseudo-applications to whole sets. Accordingly, the inverse image of a subset $B$ of $Y$ is denoted by $f^{-1}[B] = \{x \in X \mid fx \in B\}$.

## 1.2 Closure operations

Closure operations are a special form of set operations. Because of their fundamentality, they occur in various different contexts. In the thesis, we shall introduce several distinct closure operations. In this section, we study the generic properties common to all closure operators.

**Definition 1.2.1**

Let $\mathbf{X}$ be a given non-empty set, and let $\mathcal{C}$ be a set of subsets of $\mathbf{X}$ with

(C1) $\mathbf{X}$ is in $\mathcal{C}$.

(C2) Arbitrary intersections of members of $\mathcal{C}$ are again in $\mathcal{C}$.

Then, for every $A \subseteq \mathbf{X}$, the set $\mathsf{cl}\, A = \bigcap\{B \in \mathcal{C} \mid B \supseteq A\}$ is the *closure* of $A$ w.r.t. $\mathcal{C}$.

Because of the axioms (C1) and (C2), the intersection leading to $\mathsf{cl}\, A$ is not empty and always produces a member of $\mathcal{C}$.

The closure of $A$ is the smallest set in $\mathcal{C}$ that still contains $A$ as a subset. This is formalized in the first three claims of the proposition below. The remaining claims are simple consequences of the first three.

**Proposition 1.2.2**

(1) $A \subseteq \mathsf{cl}\, A$

(2) $\mathsf{cl}\, A \in \mathcal{C}$

(3) If $A \subseteq B$ and $B \in \mathcal{C}$, then $\mathsf{cl}\, A \subseteq B$

(4) $A \subseteq \mathsf{cl}\, B$ is equivalent to $\mathsf{cl}\, A \subseteq \mathsf{cl}\, B$

(5) $A \subseteq B$ implies $\mathsf{cl}\, A \subseteq \mathsf{cl}\, B$

(6) $\mathsf{cl}\, \mathbf{X} = \mathbf{X}$

(7) $A$ is in $\mathcal{C}$ iff $\mathsf{cl}\, A = A$

(8) $\mathsf{cl}\,(\mathsf{cl}\, A) = \mathsf{cl}\, A$

**Proof:**

(1) $A$ is a subset of all those sets whose intersection is $\mathsf{cl}\, A$.

(2) Due to axiom (C2).

(3) If $A \subseteq B$ and $B \in \mathcal{C}$, then $B$ is among those sets whose intersection is $\mathsf{cl}\, A$.

(4) By (2), $\mathsf{cl}\, B$ is in $\mathcal{C}$. Using (3), then $A \subseteq \mathsf{cl}\, B$ implies $\mathsf{cl}\, A \subseteq \mathsf{cl}\, B$. For the opposite direction, (1) is used: $A \subseteq \mathsf{cl}\, A \subseteq \mathsf{cl}\, B$.

(5) By (1), $A \subseteq B \subseteq \mathsf{cl}\, B$ holds. Using (4), $\mathsf{cl}\, A \subseteq \mathsf{cl}\, B$ follows.

(6) By (1), $\mathbf{X} \subseteq \mathsf{cl}\, \mathbf{X}$ holds.

(7) If $A$ is in $\mathcal{C}$, then $A \subseteq A$ implies $\mathsf{cl}\, A \subseteq A$ by (3). The opposite inclusion holds by (1). If $\mathsf{cl}\, A$ equals $A$, then $A$ is in $\mathcal{C}$ by (2).

(8) Follows from (7) and (2). □

The closure operation is uniquely determined by the properties (1) through (3) given above:

**Proposition 1.2.3**

If $A \mapsto \mathsf{cl}'\, A$ is a set operation with

(1) $A \subseteq \mathsf{cl}'\, A$

(2) $\mathsf{cl}' A \in \mathcal{C}$

(3) If $A \subseteq B$ and $B \in \mathcal{C}$, then $\mathsf{cl}' A \subseteq B$

for all $A, B \subseteq \mathbf{X}$, then $\mathsf{cl}\, A = \mathsf{cl}' A$ holds for all $A \subseteq \mathbf{X}$.

**Proof:**
From $A \subseteq \mathsf{cl}' A$ and $\mathsf{cl}' A \in \mathcal{C}$, the inclusion $\mathsf{cl}\, A \subseteq \mathsf{cl}' A$ follows by using Prop. 1.2.2 (3). The inverse inclusion is derived analogously by exchanging the roles of the two operations. □

# Chapter 2

# Partially ordered sets

Semantic domains are partially ordered sets with additional axioms. Their order is to be understood as given by informational content or computational progress. The relation '$x \leq y$' for two semantic values $x$ and $y$ means that $x$ contains less information than $y$ from a static point of view. Considered dynamically, $x$ might be the outcome of a computation at an early state, and might be refined to $y$ as the computation proceeds.

In this chapter, we consider the elementary properties of partially ordered sets. Notions of convergence and other topological properties will be treated later. After having defined partially ordered sets, we investigate their upper, lower, and convex subsets and related closure operations. Finally, we treat upper bounds, monotonic functions, minimal elements, and finitary sets.

Readers who are experienced in poset theory will not find many new results in this chapter. The only non-standard notion is the notion of *finitary sets* introduced in section 2.7. A set is finitary iff its upper closure may be obtained as the upper closure of a finite set. Equivalently, a set is finitary iff there is a minimal element under each of its elements, and the number of minimal elements is finite.

## 2.1 Definition and examples

The following definition is standard:

**Definition 2.1.1** A binary relation '$\leq$' on a set $P$ that satisfies the axioms

| | | |
|---|---|---|
| (1) | $x \leq x$ for all $x \in P$ | (Reflexivity) |
| (2) | $x \leq y$ and $y \leq x$ implies $x = y$ | (Antisymmetry) |
| (3) | $x \leq y$ and $y \leq z$ implies $x \leq z$ | (Transitivity) |

is called a *partial order* or shortly an *order*. If it satisfies axioms (1) and (3) only, it is called a *pre-order*.

A *poset* (partially ordered set) $(P, \leq)$ is a non-empty set $P$ together with a partial order '$\leq$'.

A point $x$ is called *least element* of a poset $(P, \leq)$ iff $x \leq p$ for all $p$ in $P$, and *greatest element* iff $p \leq x$.

We often identify a poset $\mathbf{P} = (P, \leq)$ with its carrier $P$, writing $\mathbf{P}$ for the carrier. Sometimes, the order is denoted by $\leq_{\mathbf{P}}$ for better distinction.

Least and greatest element are, if they exist, uniquely determined because of antisymmetry. They are often denoted by $\perp$ and $\top$ respectively.

Every pre-order '$\lesssim$' on $P$ induces an equivalence relation '$\approx$' on $P$ by the definition $x \approx y$ iff $x \lesssim y$ and $y \lesssim x$. Then, the pre-order defines a partial order '$\leq$' on the set of equivalence classes of $P$ modulo '$\approx$' by $[x] \leq [y]$ iff $x \lesssim y$ where $[x]$ denotes the equivalence class of the point $x$.

For a poset $\mathbf{P} = (P, \leq)$, the *dual poset* $\mathbf{P}^d = (P, \geq)$ has the same carrier set $P$, but the reverse order $x \geq y$ iff $y \leq x$. Many theorems and propositions about posets have a dual version formulated for the dual poset. This dual version need not be explicitly proved.

Examples for posets:

- The smallest poset has just one point $\diamond$. We denote this poset by $\mathbf{1} = \{\diamond\}$.
- Any set $P$ becomes a poset by defining $x \leq y$ iff $x = y$. This poset $P_= = (P, =)$ is called the discrete poset over $P$.
- For any set $S$, there is the so-called flat poset $S_\perp$:

  $$S_\perp = (S \cup \{\perp\}, \leq) \quad \text{where} \quad x \leq y \quad \text{iff} \quad x = \perp \quad \text{or} \quad x = y$$

  $\perp$ is the least element of a flat poset. Examples of flat posets include

  - The two-point poset $\mathbf{2} = \{\top\}_\perp$ i.e. $\mathbf{2} = (\{\perp, \top\}, \leq)$ where $\perp \leq \top$. We shortly write in a symbolic notation $\mathbf{2} = \{\perp < \top\}$.
  - The Boolean poset $\mathbf{B} = \{\mathsf{F}, \mathsf{T}\}_\perp$ i.e. $\mathbf{B} = (\{\perp, \mathsf{F}, \mathsf{T}\}, \leq)$ where $\perp \leq \mathsf{F}$ and $\perp \leq \mathsf{T}$. Symbolically, we write $\mathbf{B} = \{\mathsf{F} > \perp < \mathsf{T}\}$ or $\mathbf{B} = \{\perp < [\mathsf{F}, \mathsf{T}]\}$ where the square brackets group a level of uncomparable points.
  - The flat poset of natural numbers $\mathbf{N}_\perp = \{0, 1, 2, \ldots\}_\perp = \{\perp < [0, 1, 2, \ldots]\}$.
- For any set $S$, the power set $2^S$ forms a poset with the inclusion order '$\subseteq$'. $\emptyset$ is the least element.
- The set of natural numbers $\mathbf{N}_0 = \{0, 1, 2, \ldots\}$ forms a poset with the usual arithmetic order $0 \leq 1 \leq 2 \leq \cdots$. We symbolically denote it by $\mathbf{N}_0 = \{0 < 1 < 2 < \cdots\}$.
- The poset $\mathbf{N}_0^\infty$ consists of the natural numbers together with a greatest element $\infty$. The order is $0 \leq 1 \leq 2 \leq \cdots \leq \infty$. We write $\mathbf{N}_0^\infty = \{0 < 1 < 2 < \cdots < \infty\}$.
- Analogously, the poset $\mathbf{N}_0^{\infty+1} = \{0 < 1 < 2 < \cdots < \infty < \infty + 1\}$ consists of the natural numbers together with $\infty$ and $\infty + 1$.
- Let $\mathbf{N}' = \{0, 1, 2, \ldots, \infty, 0', 1', 2', \ldots\}$ ordered by $0 < 1 < 2 < \cdots < \infty$ and $n < m'$ iff $n \leq m$. Graphically, $\mathbf{N}'$ looks like

  $$\begin{array}{ccccccc} 0' & & 1' & & 2' & & \\ \uparrow & & \uparrow & & \uparrow & & \\ 0 & \rightarrow & 1 & \rightarrow & 2 & \ldots & \infty \end{array}$$

  This is the poset of the so-called 'lazy naturals'. Its points may be understood as constructor terms involving successor, zero, and bottom: the points $n$ correspond to $succ^n \perp$, and the points $n'$ to $succ^n 0$.

Finally, we consider one (of many) methods to construct new posets from given ones.

**Definition 2.1.2** Let $(\mathbf{P}_i)_{i\in I}$ be a family of posets where $\mathbf{P}_i = (P_i, \leq_i)$. Then we define their *(Cartesian) product* $\prod_{i\in I} \mathbf{P}_i = (\prod_{i\in I} P_i, \leq)$ where $(x_i)_{i\in I} \leq (y_i)_{i\in I}$ iff $x_i \leq_i y_i$ for all $i$ in $I$. For $k$ in $I$, the *projection* to dimension $k$ is the function $\pi_k : \prod_{i\in I} \mathbf{P}_i \to \mathbf{P}_k$ with $\pi_k((x_i)_{i\in I}) = x_k$.

The product $\prod_{i\in I} \mathbf{P}_i$ is again a poset. A special instance is the product of just two posets $\mathbf{P}_1$ and $\mathbf{P}_2$. We write $\mathbf{P}_1 \times \mathbf{P}_2$ instead of $\prod_{i\in\{1,2\}} \mathbf{P}_i$. By definition, $(x_1, x_2) \leq (y_1, y_2)$ holds iff $x_1 \leq_1 y_1$ and $x_2 \leq_2 y_2$.

## 2.2 Upper and lower sets

In this section, we consider some special classes of subsets of a general poset and their properties. The theory introduced here is not complex but is used a lot in the sequel. Throughout this section, $\mathbf{P}$ always denotes a fixed poset unless explicitly stated.

**Definition 2.2.1**
A set $A \subseteq \mathbf{P}$ is a *lower set* iff for all $x, y \in P : x \in A$ and $y \leq x$ implies $y \in A$.
Dually, a set $A \subseteq \mathbf{P}$ is an *upper set* iff for all $x, y \in P : x \in A$ and $x \leq y$ implies $y \in A$.

**Proposition 2.2.2**

(1) $\emptyset$ and the whole poset $\mathbf{P}$ are as well lower as upper sets.

(2) If $(A_i)_{i\in I}$ is a family of lower (upper) sets, then $\bigcup_{i\in I} A_i$ and $\bigcap_{i\in I} A_i$ are lower (upper) sets.

(3) If $A_i$ is a lower (upper) set in $\mathbf{P}_i$ for all $i$ in $I$, then $\prod_{i\in I} A_i$ is a lower (upper) set in $\prod_{i\in I} \mathbf{P}_i$.

The proof is left to the interested reader.

Lower and upper set are not only dual notions, but also complementary ones:

**Proposition 2.2.3** The complement of a lower set is an upper set and vice versa.

**Proof:**
Let $A$ be a lower set, and let $B$ be its complement. If $x$ is in $B$ and $x \leq y$, then $y$ must be in $B$ since otherwise, it would be in $A$ and so would $x$. Thus, $B$ is an upper set.
The 'vice versa' follows from duality. □

**Definition 2.2.4** For $x \in \mathbf{P}$, let $\downarrow x = \{y \in \mathbf{P} \mid y \leq x\}$ the *lower cone* of $x$, and $\uparrow x = \{y \in \mathbf{P} \mid y \geq x\}$ the *upper cone*.

Next, we give a list of facts about cones that are easy to verify.

**Proposition 2.2.5**

| | | |
|---|---|---|
| (1) | $x \leq y$ iff $x \in \downarrow y$ iff $\downarrow x \subseteq \downarrow y$ iff $\uparrow x \ni y$ iff $\uparrow x \supseteq \uparrow y$ | |
| (2) | $x = y$ iff $\downarrow x = \downarrow y$ iff $\uparrow x = \uparrow y$ | |
| (3) | $A$ is lower iff $\downarrow a \subseteq A$ for all $a \in A$ | $A$ is upper iff $\uparrow a \subseteq A$ for all $a \in A$ |
| (4) | $\downarrow x$ is lower | $\uparrow x$ is upper |
| (5) | $x \in \downarrow x$ | $x \in \uparrow x$ |
| (6) | $\downarrow x \cap \uparrow x = \{x\}$ | |

By Prop. 2.2.2, the set of all lower sets and the set of all upper sets both satisfy the axioms (C1) and (C2) of section 1.2 that are needed to define corresponding closure operations. They are simple enough to be explicitly constructed:

**Definition 2.2.6** For $A \subseteq \mathbf{P}$, let $\downarrow A = \{x \in \mathbf{P} \mid \exists a \in A : x \leq a\}$ be the *lower closure* of $A$, and $\uparrow A = \{x \in \mathbf{P} \mid \exists a \in A : a \leq x\}$ be the *upper closure*.

The operations defined above are really the closure operations for the set of lower sets and the set of upper sets. This may be shown by applying Prop. 1.2.3; the proofs of the three properties required there are straightforward. Hence, all properties of general closure operations as enumerated in Prop. 1.2.2 hold. We repeat them here (in a different order) for later reference.

**Proposition 2.2.7**

| | | |
|---|---|---|
| (1) | $\downarrow\emptyset = \emptyset$ and $\downarrow\mathbf{P} = \mathbf{P}$ | $\uparrow\emptyset = \emptyset$ and $\uparrow\mathbf{P} = \mathbf{P}$ |
| (2) | $\downarrow A$ is a lower set | $\uparrow A$ is an upper set |
| (3) | $A \subseteq \downarrow A$ | $A \subseteq \uparrow A$ |
| (4) | $A \subseteq B$ and $B$ is lower implies $\downarrow A \subseteq B$ | $A \subseteq B$ and $B$ is upper implies $\uparrow A \subseteq B$ |
| (5) | $A$ is lower iff $\downarrow A = A$ | $A$ is upper iff $\uparrow A = A$ |
| (6) | $A \subseteq B$ implies $\downarrow A \subseteq \downarrow B$ | $A \subseteq B$ implies $\uparrow A \subseteq \uparrow B$ |
| (7) | $A \subseteq \downarrow B$ iff $\downarrow A \subseteq \downarrow B$ | $A \subseteq \uparrow B$ iff $\uparrow A \subseteq \uparrow B$ |
| (8) | $\downarrow\downarrow A = \downarrow A$ | $\uparrow\uparrow A = \uparrow A$ |

Next, we consider how the closure operators interfere with set-theoretic operations.

**Proposition 2.2.8** Let $(A_i)_{i \in I}$ be a family of subsets of $\mathbf{P}$. Then

$\downarrow(\bigcup_{i\in I} A_i) = \bigcup_{i\in I} \downarrow A_i$ $\qquad \uparrow(\bigcup_{i\in I} A_i) = \bigcup_{i\in I} \uparrow A_i$

$\downarrow(\bigcap_{i\in I} A_i) \subseteq \bigcap_{i\in I} \downarrow A_i$ $\qquad \uparrow(\bigcap_{i\in I} A_i) \subseteq \bigcap_{i\in I} \uparrow A_i$

With intersection, '=' does not generally hold.

**Proof:**

Union, '$\supseteq$': By monotonicity of '$\downarrow$' (Prop. 2.2.7 (6)), $\downarrow(\bigcup_{i\in I} A_i) \supseteq \downarrow A_i$ holds for each $i$.

Union, '$\subseteq$': $\bigcup_{i\in I} \downarrow A_i$ is a lower superset of $\bigcup_{i\in I} A_i$. Prop. 2.2.7 (4) above implies the claim.

Intersection, '$\subseteq$': By monotonicity of '$\downarrow$', $\downarrow(\bigcap_{i\in I} A_i) \subseteq \downarrow A_i$ holds for all $i$. □

We shall give three counterexamples for inequality with intersection: one involving just two sets, and two with a descending sequence.

**Example 1:** Let $\mathbf{P} = \{\bot < [a, b] < \top\}$ ordered such that $\bot$ is least, $\top$ is greatest, and $a$ and $b$ are incomparable. Then

$$\downarrow(\{a\} \cap \{b\}) = \downarrow\emptyset = \emptyset \qquad \downarrow\{a\} \cap \downarrow\{b\} = \{\bot, a\} \cap \{\bot, b\} = \{\bot\}$$
$$\uparrow(\{a\} \cap \{b\}) = \uparrow\emptyset = \emptyset \qquad \uparrow\{a\} \cap \uparrow\{b\} = \{a, \top\} \cap \{b, \top\} = \{\top\}$$

**Example 2:** Let $\mathbf{P} = \mathbf{N}_0^\infty = \{0 < 1 < \cdots < \infty\}$ be the poset of natural numbers, and $A_n = \{x \mid n \le x < \infty\}$. Then

$$\uparrow(\bigcap_{n=1}^{\infty} A_n) = \uparrow\emptyset = \emptyset \qquad \bigcap_{n=1}^{\infty} \uparrow A_n = \bigcap_{n=1}^{\infty} (A_n \cup \{\infty\}) = \{\infty\}$$

**Example 3:** Let $\mathbf{P} = \mathbf{N}_\bot$ be the flat poset of natural numbers, and $A_n = \{n, n+1, \ldots\}$. Then

$$\downarrow(\bigcap_{n=1}^{\infty} A_n) = \downarrow\emptyset = \emptyset \qquad \bigcap_{n=1}^{\infty} \downarrow A_n = \bigcap_{n=1}^{\infty} (A_n \cup \{\bot\}) = \{\bot\}$$

There is a positive fact about closure and intersection:

**Lemma 2.2.9** If $L$ is a lower set, then $\uparrow(L \cap \uparrow A) = \uparrow(L \cap A)$ holds for all sets $A$.

**Proof:** The inclusion '$\supseteq$' holds since all involved operations are monotonic w.r.t. set inclusion and $\uparrow A \supseteq A$ holds.

For the opposite inclusion, we have to show $L \cap \uparrow A \subseteq \uparrow(L \cap A)$. If $x$ is a point of $L \cap \uparrow A$, then $x$ is in $\uparrow A$ whence there is $a$ in $A$ below $x$. $a$ is in $L$ because $L$ is a lower set. Thus, $x \ge a \in L \cap A$ whence $x \in \uparrow(L \cap A)$. $\square$

Finally, we consider Cartesian product.

**Proposition 2.2.10** Let $(\mathbf{P}_i)_{i\in I}$ be a family of posets, and let $A_i$ be a subset of $\mathbf{P}_i$ for all $i$ in $I$. Then $\downarrow(\prod_{i\in I} A_i) = \prod_{i\in I} \downarrow A_i$ and $\uparrow(\prod_{i\in I} A_i) = \prod_{i\in I} \uparrow A_i$ hold.

For the special case of binary products, one obtains $\downarrow(A \times B) = \downarrow A \times \downarrow B$ and $\uparrow(A \times B) = \uparrow A \times \uparrow B$.

**Proof:** $(x_i)_{i\in I} \in \uparrow(\prod_{i\in I} A_i)$ iff $\exists\, (a_i)_{i\in I} \in \prod_{i\in I} A_i : (a_i)_{i\in I} \le (x_i)_{i\in I}$ iff $\forall i \in I\ \exists\, a_i \in A_i : a_i \le x_i$ iff $\forall i \in I\ x_i \in \uparrow A_i$ iff $(x_i)_{i\in I} \in \prod_{i\in I} \uparrow A_i$ $\square$

## 2.3 Convexity

Convexity is a potential property of subsets of posets. Similar to the properties of being a lower or an upper set, there is an associated closure operation, the convex hull.

A set is convex iff with every two points, all points in between belong to it.

**Definition 2.3.1** A set $A \subseteq \mathbf{P}$ is *convex* iff, whenever $a$ and $c$ are in $A$ and $a \le b \le c$ holds, then $b$ is also in $A$.

**Proposition 2.3.2**

(1) $\emptyset$ and the whole poset $\mathbf{P}$ are convex sets.

(2) If $(A_i)_{i\in I}$ is a family of convex sets, then $\bigcap_{i\in I} A_i$ is convex.

(3) If $A_i$ is a convex set in $\mathbf{P}_i$ for all $i$ in $I$, then $\prod_{i\in I} A_i$ is a convex set in $\prod_{i\in I} \mathbf{P}_i$.

(4) Every lower set and every upper set is convex.

The proof is left to the interested reader.

By part (1) and (2) of this proposition and the results of section 1.2, every set has a least convex superset, the so-called convex hull. It may be constructed by means of lower and upper closure, as indicated by the following definition.

**Definition 2.3.3** For $A \subseteq \mathbf{P}$, let $\updownarrow A = \downarrow A \cap \uparrow A$ be the *convex hull* of $A$.

The name 'convex hull' for the $\updownarrow$-operator is appropriate, as parts (2) through (4) of the following proposition show.

**Proposition 2.3.4**

(1) $\updownarrow\emptyset = \emptyset$ and $\updownarrow\mathbf{P} = \mathbf{P}$

(2) $\updownarrow A$ is convex

(3) $A \subseteq \updownarrow A$

(4) If $A \subseteq B$ and $B$ is convex, then $\updownarrow A \subseteq B$

(5) $A$ is convex iff $\updownarrow A = A$

(6) $A \subseteq B$ implies $\updownarrow A \subseteq \updownarrow B$

(7) $A \subseteq \updownarrow B$ iff $\updownarrow A \subseteq \updownarrow B$

(8) $\updownarrow\updownarrow A = \updownarrow A$

(9) $\downarrow\updownarrow A = \downarrow A$ and $\uparrow\updownarrow A = \uparrow A$

(10) $\updownarrow A = \updownarrow B$ iff $\downarrow A = \downarrow B$ and $\uparrow A = \uparrow B$

**Proof:**

(1) $\updownarrow\emptyset = \downarrow\emptyset \cap \uparrow\emptyset = \emptyset \cap \emptyset = \emptyset$, same with $\mathbf{P}$.

(2) $\downarrow A$ and $\uparrow A$ are convex by Prop. 2.3.2 (4), whence is their intersection $\updownarrow A$ by part (2) of the same proposition.

(3) $A \subseteq \downarrow A$ and $A \subseteq \uparrow A$ by Prop. 2.2.7 (3).

(4) Let $y$ be a point of $\updownarrow A = \downarrow A \cap \uparrow A$. Since $y \in \downarrow A$, there is $z \in A$ with $y \leq z$. Since $y \in \uparrow A$, there is $x \in A$ with $x \leq y$. $A \subseteq B$ implies $x, z \in B$, and convexity of $B$ implies $y \in B$.

Statements (2) through (4) imply that '$\updownarrow$' is indeed the closure operation corresponding to convexity due to Prop. 1.2.3. Statements (5) through (8) are reformulations of parts of Prop. 1.2.2.

(9) $\downarrow\updownarrow A \supseteq \downarrow A$ by (3) and Prop. 2.2.7 (6). $\downarrow\updownarrow A = \downarrow(\downarrow A \cap \uparrow A) \subseteq \downarrow\downarrow A = \downarrow A$ by Prop. 2.2.7 (6) and (8).

(10) $\updownarrow A = \updownarrow B$ implies $\downarrow A = \downarrow B$ and $\uparrow A = \uparrow B$ by (9). The opposite direction follows from the definition of $\updownarrow$. □

## 2.4 Upper bounds

**Definition 2.4.1** Let $\mathbf{P} = (P, \leq)$ be a poset, $S \subseteq P$ and $x \in P$. $x$ is an *upper bound* of $S$ if $x \geq s$ for all $s \in S$. $S$ is called (upwards) *bounded* if it has at least one upper bound.

The *least upper bound (lub)* or *supremum* of $S$ — if it exists — is denoted by $\bigsqcup S$. Hence, the defining properties of the supremum are

(1) $s \le \bigsqcup S$ for all $s \in S$

(2) $\forall u \in \mathbf{P}:$ if $s \le u$ for all $s \in S$, then $\bigsqcup S \le u$

If $\bigsqcup\{a, b\}$ exists, it is also written $a \sqcup b$ using a binary infix notation.

Dual notions are lower bound, greatest lower bound (glb) or infimum, and $a \sqcap b$.

Often, we apply these notions to an indexed family $(x_i)_{i \in I}$ of points instead of a set $S$. An upper bound of the family is an upper bound of the corresponding set $\{x_i \mid i \in I\}$. We abbreviate the supremum of this set by $\bigsqcup_{i \in I} x_i$, or even by $\bigsqcup x_i$ if the index set is clear from context.

The next three propositions are easy to prove:

**Proposition 2.4.2** Each point of $\mathbf{P}$ is an upper bound of the empty set. Thus, $\bigsqcup \emptyset$ exists iff $\mathbf{P}$ has a least element $\perp$, and then, $\bigsqcup \emptyset = \perp$ holds.

**Proposition 2.4.3** If a point $x$ is both an upper bound and a member of a set $A$, then $\bigsqcup A$ exists and equals $x$.

**Proposition 2.4.4** Let $A$ and $B$ be two sets having the same upper bounds. Then $\bigsqcup A$ exists iff $\bigsqcup B$ does, and in this case, both are equal.

The next lemma is not difficult either, but sometimes needed later when one set is replaced by another.

**Lemma 2.4.5** Let $A$ and $B$ be two sets with the same lower closure, i.e. $\downarrow A = \downarrow B$ or equivalently $A \subseteq \downarrow B$ and $B \subseteq \downarrow A$. Then $A$ and $B$ have the same upper bounds.

**Proof:** Let $u$ be an upper bound of $B$. For each $a$ in $A$, there is $b$ in $B$ with $a \le b \le u$ because of $A \subseteq \downarrow B$. Thus, $u$ is an upper bound of $A$. The opposite direction holds by symmetry. □

The intersection of a family of upper cones is again an upper cone, if the vertices of the cones have a least upper bound.

**Lemma 2.4.6** Let $(x_i)_{i \in I}$ be a family of points in a poset with a least upper bound $\bigsqcup_{i \in I} x_i$. Then $\bigcap_{i \in I} \uparrow x_i = \uparrow \bigsqcup_{i \in I} x_i$.

**Proof:** $y \in \bigcap_{i \in I} \uparrow x_i$ iff $y \in \uparrow x_i$ for all $i \in I$ iff $y \ge x_i$ for all $i \in I$ iff
$y$ is an upper bound of the points $x_i$ iff $y \ge \bigsqcup_{i \in I} x_i$ iff $y \in \uparrow \bigsqcup_{i \in I} x_i$ □

The lub operation is monotonic w.r.t. set inclusion:

**Proposition 2.4.7** Let $A \subseteq B$. If $\bigsqcup A$ and $\bigsqcup B$ exist, then $\bigsqcup A \le \bigsqcup B$ holds.

**Proof:** $\bigsqcup B$ is an upper bound of $B$, and hence of $A$. Thus, the least upper bound $\bigsqcup A$ is below $\bigsqcup B$. □

The next lemma deals with the lub of a set of lubs.

**Lemma 2.4.8** Let $(A_i)_{i\in I}$ be a family of sets whose lubs exist. Then the sets $\{\bigsqcup A_i \mid i \in I\}$ and $\bigcup_{i\in I} A_i$ have the same upper bounds. Hence, the lub of one set exists iff the lub of the other one exists, and in this case

$$\bigsqcup\{\bigsqcup A_i \mid i \in I\} = \bigsqcup(\bigcup_{i\in I} A_i)$$

**Proof:** If $u$ is an upper bound of the set of lubs, then $u$ is an upper bound of all sets $A_i$, whence it is an upper bound of $\bigcup_{i\in I} A_i$.

Conversely, if $u$ is an upper bound of the union, it is an upper bound of each set $A_i$ and hence of each lub $\bigsqcup A_i$. □

Lub operations may be exchanged:

**Proposition 2.4.9** Let $(x_{ij})_{i\in I, j\in J}$ be a doubly indexed family of points in a poset such that $\bigsqcup_{j\in J} x_{ij}$ exist for all $i$ in $I$, and $\bigsqcup_{i\in I} x_{ij}$ exist for all $j$ in $J$. Then $\bigsqcup_{i\in I}\bigsqcup_{j\in J} x_{ij}$ exists, iff $\bigsqcup_{j\in J}\bigsqcup_{i\in I} x_{ij}$ exists, iff $\bigsqcup_{(i,j)\in I\times J} x_{ij}$ exists, and all three are equal.

**Proof:** By comparing the upper bounds of the three sets $\{\bigsqcup_{j\in J} x_{ij} \mid i \in I\}$, $\{\bigsqcup_{i\in I} x_{ij} \mid j \in J\}$, and $\{x_{ij} \mid i \in I,\, j \in J\}$. □

Next we consider upper bounds in the product of posets as introduced in section 2.1.

**Proposition 2.4.10** Let $(\mathbf{P}_i)_{i\in I}$ be a family of posets. Let $A$ be a subset of $\prod_{i\in I} \mathbf{P}_i$, and let $A_i = \pi_i[A]$ for all $i$ in $I$.

(1) $A \subseteq \prod_{i\in I} A_i$

(2) $A$ and $\prod_{i\in I} A_i$ have the same upper bounds.

(3) $\bigsqcup A$ exists iff $\bigsqcup A_i$ exists for all $i$ in $I$. In this case, $\bigsqcup A = (\bigsqcup A_i)_{i\in I}$ holds.

**Proof:**

(1) If $(a_i)_{i\in I}$ is in $A$, then $a_i$ is in $A_i$ for all $i$ in $I$.

(2) Any upper bound of $\prod_{i\in I} A_i$ is an upper bound of $A$ because of (1).

Let $u = (u_i)_{i\in I}$ be an upper bound of $A$, and let $(x_i)_{i\in I}$ be a point of $\prod_{i\in I} A_i$. Then for all $k$ in $I$, $x_k \in A_k$ implies there is a point $a$ in $A$ such that $\pi_k a = x_k$. Hence, $a \le u$ follows whence in particular $x_k = \pi_k a \le u_k$. Thus, $x_k \le u_k$ holds for all $k$ in $I$, whence $x \le u$.

(3) By (2), $\bigsqcup A$ exists iff $\bigsqcup(\prod_{i\in I} A_i)$ exists, and in this case, both are equal. The remainder of the statement is trivial. □

## 2.5 Monotonic functions

**Definition 2.5.1** Let $\mathbf{P}$ and $\mathbf{Q}$ be two posets. A function $f$ from $\mathbf{P}$ to $\mathbf{Q}$ is *monotonic* iff $x \le_{\mathbf{P}} y$ implies $fx \le_{\mathbf{Q}} fy$ for all $x, y \in \mathbf{P}$.

A function $f : \mathbf{P} \to \mathbf{Q}$ is a *poset embedding* iff for all $x, x' \in \mathbf{P}$ $x \le_{\mathbf{P}} x'$ and $fx \le_{\mathbf{Q}} fx'$ are equivalent.

A *poset isomorphism* is a bijective function that is monotonic in both directions.

**Proposition 2.5.2**

(1) A poset embedding is always injective.

(2) If $\mathbf{P}$ is a subset of a poset $\mathbf{Q}$ with the inherited order, then the natural inclusion map $i : \mathbf{P} \to \mathbf{Q}$ is a poset embedding.

(3) A surjective poset embedding is a poset isomorphism.

**Proof:**

(1) $fx = fy$ implies $fx \leq fy$ and $fx \geq fy$, whence $x \leq y$ and $y \leq x$, whence $x = y$ by anti-symmetry.

(2) Because $\mathbf{P}$ inherits the order of $\mathbf{Q}$, $x \leq_{\mathbf{P}} y$ is equivalent to $ix \leq_{\mathbf{Q}} iy$.

(3) By (1), $f$ is bijective. By the definition of embedding, both $f$ and its inverse are monotonic. □

**Proposition 2.5.3** A poset isomorphism also preserves all infima and suprema.

With the definition of continuity in the next chapter, a poset isomorphism between two domains is in particular continuous in both directions by this proposition.

**Proof:** Let $f$ be a poset isomorphism between $\mathbf{P}$ and $\mathbf{Q}$. Let $g$ be its inverse mapping. Then both $f$ and $g$ are monotonic.

Let $A$ be a subset of $\mathbf{P}$ with a supremum $\bigsqcup A$. For all $y$ in $f[A]$, $y = fa$ holds for some $a$ in $A$, whence $y = fa \leq f(\bigsqcup A)$. Thus, $f(\bigsqcup A)$ is an upper bound of $f[A]$. Let $u$ be another upper bound of $f[A]$. Then for all $a$ in $A$, $fa \leq u$ holds, whence $a \leq gu$. Thus, we get $\bigsqcup A \leq gu$, whence $f(\bigsqcup A) \leq u$. Together, this proves that $f(\bigsqcup A)$ equals $\bigsqcup(f[A])$.

The proof for infima and for $g$ is analogous. □

**Proposition 2.5.4** If $f : \mathbf{X} \to \mathbf{Y}$ is monotonic, then for all $A \subseteq \mathbf{X}$, the inclusions $f[\downarrow A] \subseteq \downarrow f[A]$ and $f[\uparrow A] \subseteq \uparrow f[A]$ hold.

**Proof:** If $y$ is in $f[\uparrow A]$, then there is $x \in \uparrow A$ with $y = fx$. Further, there is $a$ in $A$ with $a \leq x$, whence $fa \leq fx = y$, i.e. $y$ is in $\uparrow f[A]$.

The proof for '$\downarrow$' is analogous. □

Monotonic functions may also be characterized by means of lower sets.

**Proposition 2.5.5** For a function $f : \mathbf{P} \to \mathbf{Q}$ between two posets, the following statements are equivalent:

(1) $f$ is monotonic.

(2) The inverse image of every lower set of $\mathbf{Q}$ is a lower set of $\mathbf{P}$.

(3) The inverse image of every lower cone in $\mathbf{Q}$ is a lower set in $\mathbf{P}$.

**Proof:**

(1) $\Rightarrow$ (2) Let $B$ be a lower set in $\mathbf{Q}$ and $A = f^{-1}[B]$ its inverse image in $\mathbf{P}$. Let $a$ be a point of $A$ and $a' \leq a$. We have to show $a' \in A$. $a \in A$ implies $fa \in B$. $a' \leq a$ implies $fa' \leq fa$ by monotonicity. Since $B$ is a lower set, $fa'$ is in $B$ whence $a'$ is in $A$.

(2) $\Rightarrow$ (3) is trivial since lower cones are special lower sets.

(3) $\Rightarrow$ (1) Let $x$ and $x'$ be two points in $\mathbf{P}$ with $x \leq x'$. Let $B = \downarrow fx'$ be the lower cone of $fx'$ in $\mathbf{Q}$, and let $A$ be the inverse image of $B$ by $f$. $fx' \in B$ implies $x' \in A$ whence $x \in A$ since $A$ is a lower set. Thus, $fx \in B$ holds whence $fx \leq fx'$. □

Now we consider the relations between product and monotonicity. The next proposition is an immediate consequence of the definition of the order in the product.

**Proposition 2.5.6**
Let $(\mathbf{P}_i)_{i \in I}$ be a family of posets. Then all projections $\pi_k : \prod_{i \in I} \mathbf{P}_i \to \mathbf{P}_k$ are monotonic. A function $f : \mathbf{Q} \to \prod_{i \in I} \mathbf{P}_i$ is monotonic iff all functions $\pi_k \circ f : \mathbf{Q} \to \mathbf{P}_k$ are monotonic.

For functions over pairs, monotonicity may be checked component by component:

**Proposition 2.5.7** Let $f : \mathbf{P}_1 \times \mathbf{P}_2 \to \mathbf{Q}$. For $x \in \mathbf{P}_1$ and $y \in \mathbf{P}_2$, let $f^y : \mathbf{P}_1 \to \mathbf{Q}$ and $f_x : \mathbf{P}_2 \to \mathbf{Q}$ be defined by $f^y(x) = f_x(y) = f(x, y)$. Then $f$ is monotonic iff $f^y$ and $f_x$ are monotonic for all $x \in \mathbf{P}_1$ and $y \in \mathbf{P}_2$.

**Proof:**

'$\Rightarrow$' $x \leq x' \Rightarrow (x, y) \leq (x', y) \Rightarrow f(x, y) \leq f(x', y) \Rightarrow f^y(x) \leq f^y(x')$
Analogously for $f_x$.

'$\Leftarrow$' Let $(x, y) \leq (x', y')$. Then $x \leq x'$ and $y \leq y'$. Hence $f(x, y) = f^y(x) \leq f^y(x') = f(x', y) = f_{x'}(y) \leq f_{x'}(y') = f(x', y')$. □

## 2.6 Minimal elements

In this section, we consider the notion of minimal elements of a subset of a poset. The dual notion of maximal elements has analogous properties, but is not treated explicitly.

An element of a set is minimal if there is no smaller element. On the contrary, a least element of a set is smaller than all other elements.

**Definition 2.6.1** Let $\mathbf{P} = (P, \leq)$ be a poset and $S \subseteq P$. An element $x \in S$ is called *minimal* in $S$ if for all $s \in S$, $s \leq x$ implies $s = x$. It is a *least element* of $S$, if $x \leq s$ for all $s \in S$.

The set of all minimal elements of $S$ is denoted by $min\, S$.

Dual notions are *maximal* and *greatest element* and the set $max\, S$.

A set may have zero, one, or many minimal elements, but there is at most one least element by anti-symmetry. If the set $S$ has a least element $x$, then $x$ is minimal in $S$ and there are no other minimal elements, i.e. $min\, S = \{x\}$.

**Proof:** $x$ is minimal since $s \leq x$ for $s$ in $S$ implies $s = x$ due to antisymmetry because $x \leq s$ holds since $x$ is the least element of $S$. Conversely, if $m$ is minimal in $S$, then $x \leq m$ implies $x = m$. □

The '*min*' operator for sets has simple properties collected in the following proposition.

**Proposition 2.6.2**

(1) $min\, \emptyset = \emptyset$ and $min\, \{x\} = \{x\}$ for all $x \in \mathbf{P}$.

(2) $min\, A \subseteq A$

(3) If $A \subseteq B$ and $min\, B \subseteq A$, then $min\, B \subseteq min\, A$

(4) $min\, A = min\,(\uparrow A)$

(5) If $F$ is finite, then $F \subseteq \uparrow(min\, F)$ holds.

(6) If $f : \mathbf{P} \to \mathbf{Q}$ is monotonic and $B \subseteq A \subseteq \uparrow B$ holds for two sets $A$ and $B$ of $\mathbf{P}$, then $min\, f[A] \subseteq f[B]$ holds in $\mathbf{Q}$.

(7) If $f : \mathbf{P} \to \mathbf{Q}$ is monotonic and $A \subseteq \uparrow(min\, A)$ holds for a set $A$ of $\mathbf{P}$, then $min\, f[A] \subseteq f[min\, A]$ holds in $\mathbf{Q}$.

(8) $B \subseteq A \subseteq \uparrow B$ implies $min\, A \subseteq B$.

**Proof:**

(1) and (2) are obvious.

(3) Let $m \in min\, B$. Then $m$ is in $A$. To show the minimality of $m$ in $A$, let $a \in A$ with $a \leq m$. Then $a$ is also in $B$, whence $a = m$ since $m$ is minimal in $B$.

(4) First we show $min\, A \subseteq min\,(\uparrow A)$. Let $m$ be a point in $min\, A$. Then $m$ is in $A$ and hence in $\uparrow A$. Let $r \in \uparrow A$ with $r \leq m$. By definition of $\uparrow A$, there is $a \in A$ with $a \leq r \leq m$. Since $m$ is minimal in $A$, $a$ equals $m$, whence $r$ equals $m$.

For the opposite direction, we have to show $min\,(\uparrow A) \subseteq A$; then statement (3) can be applied since $A \subseteq \uparrow A$. Let $m$ be a point in $min\,(\uparrow A)$. Then $m$ is in $\uparrow A$, whence there is a point $a$ in $A$ below $m$. $a$ is in $\uparrow A$ because of $A \subseteq \uparrow A$. Minimality of $m$ in $\uparrow A$ implies $m = a \in A$.

(5) Let $x$ be a point of $F$. We inductively define a descending chain $x_1 > x_2 > \cdots$.

Let $x_1 = x$. Given $x_i$, there are two cases: either $x_i$ is in $min\, F$, then we cannot proceed; or $x_i$ is not in $min\, F$, then there is some $x_{i+1}$ in $F$ such that $x_i > x_{i+1}$. Since $F$ is finite, the descending chain cannot go on infinitely. Hence, the first case eventually occurs, i.e. there is some $x_k$ in $min\, F$. $x \geq x_k$ holds, whence $x \in \uparrow(min\, F)$.

(6) Let $y$ be a point of $min\, f[A]$. Then $y = fa$ holds for some $a$ in $A$. By $A \subseteq \uparrow B$, there is $b$ in $B \subseteq A$ such that $a \geq b$. $y = fa \geq fb$ follows. Since $fb$ is in $f[A]$, $y = fb$ follows by minimality of $y$.

(7) From (6) by $B = min\, A$.

(8) From (6) by $f = id$. □

## 2.7 Finitary sets

Finally, we consider sets having a finite number of minimal elements where a minimal element can be found below any arbitrary element. Such sets are called *finitary*. They are needed to characterize M-domains, and also to define stronger forms of compactness.

**Proposition 2.7.1** For a set $A \subseteq \mathbf{P}$, the following statements are equivalent:

(1) $A \subseteq \uparrow(min\, A)$ holds and $min\, A$ is finite.

(2) There is a finite subset $F$ of $A$ with $A \subseteq \uparrow F$.

(3) There is a finite set $F$ with $\uparrow A = \uparrow F$.

In case (2) and (3), $min\, A = min\, F$ holds.

Sets with these properties are called *finitary*.

**Proof:**

(1) $\Rightarrow$ (2) : Let $F = min\, A$.

(2) $\Rightarrow$ (3) : $A \subseteq \uparrow F$ implies $\uparrow A \subseteq \uparrow F$ by Prop. 2.2.7 (7), and $\uparrow A \supseteq \uparrow F$ follows from $A \supseteq F$ by Prop. 2.2.7 (6).

(3) $\Rightarrow$ (1) : $min\, A = min\,(\uparrow A) = min\,(\uparrow F) = min\, F$ holds by Prop. 2.6.2 (4). Hence, $min\, A \subseteq F$ is finite.

Applying Prop. 2.6.2 (5), we obtain $A \subseteq \uparrow A = \uparrow F \subseteq \uparrow(min\, F) = \uparrow(min\, A)$. □

The basic properties of finitary sets are given in the following proposition:

**Proposition 2.7.2**

(1) $A$ is finitary iff $\uparrow A$ is.

(2) Every finite set is finitary.

(3) The union of two finitary sets is again so.

(4) The intersection of a lower set and a finitary set is finitary.

(5) Monotonic images of finitary sets are finitary. More precisely: If $f : \mathbf{P} \to \mathbf{Q}$ is monotonic and $A$ is finitary in $\mathbf{P}$, then $f[A]$ is finitary in $\mathbf{Q}$.

**Proof:**

In the proofs, we use the condition given in part (3) of the proposition above. Only for part (5), condition (2) is used.

(1) $\uparrow\uparrow A = \uparrow A$ holds. Hence, $\uparrow A = \uparrow F$ and $\uparrow\uparrow A = \uparrow F$ are equivalent.

(2) Let $F = A$. Then $F$ is finite with $\uparrow A = \uparrow F$.

(3) Given two finitary sets $A_1$ and $A_2$, let $F_i$ be finite sets with $\uparrow A_i = \uparrow F_i$. Then $F_1 \cup F_2$ is a finite set with $\uparrow(A_1 \cup A_2) = \uparrow A_1 \cup \uparrow A_2 = \uparrow F_1 \cup \uparrow F_2 = \uparrow(F_1 \cup F_2)$.

(4) Let $L$ be the lower set and $A$ the finitary set. Let $F$ be a finite set with $\uparrow A = \uparrow F$. By Lemma 2.2.9, $\uparrow(L \cap A) = \uparrow(L \cap \uparrow A) = \uparrow(L \cap \uparrow F) = \uparrow(L \cap F)$. $L \cap F$ is finite.

(5) Since $A$ is finitary, there is some finite set $F$ with $A \subseteq \uparrow F$. Then $f[A] \subseteq f[\uparrow F] \subseteq \uparrow f[F]$ where Prop. 2.5.4 was used for the second inclusion. The set $f[F]$ is finite. □

# Chapter 3

# Directed sets and domains

*Directed sets* are a useful generalization of ascending sequences. By their nature, sequences are always countable whereas directed sets may have any cardinality. Thus, Zorn's Lemma may be applied if every directed set has an upper bound.

The chapter at hand is organized as follows: In section 3.1, directed sets and their basic properties are introduced. Section 3.2 handles directed lubs and the notion of a domain. Domains are posets where every directed set has a supremum. We neither require a least element nor algebraicity.

Next, we define continuous functions (section 3.3) by means of directed sets. In section 3.4, the domain of continuous functions between two domains is introduced. Next, we introduce directed closed sets (section 3.5) and the Scott topology (section 3.6) whose notion of continuity coincides with that of directed sets. The domain of Scott open sets of $\mathbf{X}$ and the function domain from $\mathbf{X}$ to $\mathbf{2} = \{\bot < \top\}$ are isomorphic as pointed out in section 3.7. In section 3.8, we present a Lemma telling that Scott open sets remain open if some of their minimal points are removed. The final section 3.9 presents Rudin's Lemma.

Readers who are experienced in domain theory will not find much new in this chapter. It however collects a host of statements that are heavily used throughout the thesis.

## 3.1 Directed sets

**Definition 3.1.1** A subset $D$ of a poset is (upwards) *directed* iff all finite subsets of $D$ have an upper bound in $D$.

The notion of directed set is biased because it involves upper bounds, not lower bounds. However, the dual notion of downwards directed sets will not occur much in this paper. In case of doubt, we shall explicitly indicate the order. Directed sets as defined above are $\leq$-directed sets, and downwards directed sets would be $\geq$-directed. Sometimes, $\subseteq$- or $\supseteq$-directed sets of sets will occur.

We often also speak of *directed families* of points. A family $(d_i)_{i \in I}$ of points is directed iff the set $\{d_i \mid i \in I\}$ is directed.

The following proposition is an immediate consequence of the definition:

**Proposition 3.1.2** Every finite directed set contains a greatest element which is also the supremum of the set.

There is a characterization of directed sets involving only two points instead of an arbitrary finite number.

**Proposition 3.1.3**
A set $D$ is directed iff it is not empty and for all $x$, $y$ in $D$ there is a point $z$ in $D$ such that $z \geq x$ and $z \geq y$, i.e. any two points of $D$ have a common upper bound in $D$.

**Proof:** A directed set contains upper bounds for all its finite subsets. Hence, it contains at least an upper bound of the empty set, i.e. it is not empty. If $x$ and $y$ are in $D$, then $\{x, y\}$ is a finite subset of $D$, whence there is an upper bound $z$ of $x$ and $y$ in $D$.

For the opposite direction, let $F$ be a finite subset of $D$. If $F$ is empty, any arbitrary point is an upper bound of $F$. Since $D$ is not empty, there is such a point in $D$. If $F$ is not empty, let $F = \{x_1, \ldots, x_n\}$. Then we recursively construct points $y_1, \ldots, y_n$ in $D$: Let $y_1 = x_1$, and $y_{i+1}$ be an upper bound of $y_i$ and $x_{i+1}$ in $D$. Finally, point $y_n$ is an upper bound of the whole set $F$ and contained in $D$. □

Monotonic images of directed sets are directed.

**Proposition 3.1.4** If $f : \mathbf{P} \to \mathbf{Q}$ is a monotonic function between two posets and $D$ is a directed set of $\mathbf{P}$, then $f[D]$ is a directed set of $\mathbf{Q}$.

**Proof:** We apply criterion 3.1.3. $f[D]$ is not empty since $D$ is not empty. For $y_1$ and $y_2$ in $f[D]$, there are $x_1$ and $x_2$ in $D$ with $y_i = f x_i$. $x_1$ and $x_2$ have a common upper bound $x$ in $D$. Then $fx$ is a common upper bound of $y_1$ and $y_2$ in $f[D]$. □

A directed union of directed sets is directed.

**Lemma 3.1.5** Let $\mathcal{D}$ be a $\subseteq$-directed set of directed sets of a fixed poset $\mathbf{P}$. Then $\bigcup \mathcal{D}$ is again a directed set in $\mathbf{P}$.

**Proof:** Let $D = \bigcup \mathcal{D}$. $D$ is not empty, because $\mathcal{D}$ is not empty, and thus contains at least one non-empty set.

Let $x_1$ and $x_2$ be two points of $D$. Then there are two members $D_1$ and $D_2$ of $\mathcal{D}$ with $x_i \in D_i$. Since $\mathcal{D}$ is $\subseteq$-directed, there is a set $D'$ in $\mathcal{D}$ with $D_i \subseteq D'$, whence $x_i \in D'$. Because $D'$ is directed, there is a point $x' \in D' \subseteq D$ with $x_i \leq x$. □

The property of being directed carries over among sets with the same lower closure:

**Proposition 3.1.6** If $\downarrow A = \downarrow B$ holds, then $A$ is directed iff $B$ is.

**Corollary**: $A$ is directed iff $\downarrow A$ is.

**Proof:** The corollary is easily proved, because $\downarrow A = \downarrow\downarrow A$ holds.

Let $\downarrow A = \downarrow B$ and assume $A$ is directed. We apply Prop. 3.1.3 to show that $B$ is directed. $B$ is non-empty, since otherwise $A \subseteq \downarrow A = \downarrow B = \downarrow \emptyset = \emptyset$ would imply that $A$ is empty.

Let $b_1$ and $b_2$ be two points of $B$. By $B \subseteq \downarrow B = \downarrow A$, there are points $a_1$ and $a_2$ in $A$ with $b_i \leq a_i$. Since $A$ is directed, there is $a' \in A$ with $a_i \leq a'$. By $A \subseteq \downarrow A \subseteq \downarrow B$, there is a point $b' \in B$ with $a' \leq b'$. Thus, we have $b_i \leq a_i \leq a' \leq b'$, i.e. $b'$ is an upper bound of $b_1$ and $b_2$ in $B$. □

The proposition and its corollary indicate that instead of considering general directed sets, one could restrict oneself to directed lower sets.

**Definition 3.1.7**

Directed lower sets are called *ideals*. Two sets with the same lower closure are *cofinal*. If $A$ is a subset of $B$ and $A$ and $B$ are cofinal, one says $A$ is cofinal *in* $B$.

$A$ is cofinal in $B$ iff $A \subseteq B \subseteq \downarrow A$ holds. Remember that cofinal sets have the same upper bounds by Lemma 2.4.5. By Prop. 3.1.6, two cofinal sets are either both directed or both not directed. Summarizing, one obtains

**Proposition 3.1.8** If $D$ is directed with supremum $\bigsqcup D$ and $A$ is cofinal in $D$, then $A$ is also directed and $\bigsqcup A = \bigsqcup D$ holds.

In some sense, a directed set cannot be partitioned into two smaller sets.

**Lemma 3.1.9** If $D$ is directed and $D = A \cup B$ holds, then at least one of the two sets $A$ and $B$ is cofinal in $D$ (and hence directed).

**Proof:** Assume $A$ is not cofinal in $D$. Since $A$ is a subset of $D$, $D$ cannot be a subset of $\downarrow A$. Thus, there is a point $d_0$ in $D$ with $d_0 \notin \downarrow A$.

Since $B \subseteq D$ holds anyway, we only have to show $D \subseteq \downarrow B$. Let $d$ be any point of $D$. Since $D$ is directed, there is a common upper bound $d'$ of $d$ and $d_0$ in $D$. If $d'$ were in $A$, $d_0$ would be in $\downarrow A$. Thus, $d'$ is in $B$, whence $d$ is in $\downarrow B$. □

Finally, we consider directed sets in a product of two posets.

**Proposition 3.1.10** Let $\mathbf{P}$ and $\mathbf{Q}$ be two posets, and let $D$ be a directed set in their product $\mathbf{P} \times \mathbf{Q}$. Then $D_P = \{x \mid (x, y) \in D\}$ and $D_Q = \{y \mid (x, y) \in D\}$ are both directed, their product $D_P \times D_Q$ is directed, and $D$ is cofinal in $D_P \times D_Q$.

The cofinality statement cannot be shown in case of infinite products.

**Proof:** Since projections are monotonic, the sets $D_P = \pi_1[D]$ and $D_Q = \pi_2[D]$ are directed by Prop. 3.1.4.

Once we have shown the cofinality of $D$ in $D_P \times D_Q$, the latter set is directed by Prop. 3.1.8.

If $(x, y)$ is in $D$, then $x$ is in $D_P$ and $y$ is in $D_Q$. Thus, $D$ is a subset of $D_P \times D_Q$. Conversely, if $(x, y)$ is a point in $D_P \times D_Q$, then there are points $x'$ and $y'$ such that $(x, y')$ and $(x', y)$ are in $D$. Since $D$ is directed, there is a point $(x'', y'')$ in $D$ above both $(x, y')$ and $(x', y)$. Hence, $x'' \geq x$ and $y'' \geq y$ hold, i.e. $(x'', y'')$ is above $(x, y)$, whence the latter point is in $\downarrow D$. □

## 3.2 Directed lubs and directed completeness

Posets where all directed sets have a lub are particularly important.

**Proposition 3.2.1** For a poset $\mathbf{P}$, the following statements are equivalent:

(1) Every directed set of $\mathbf{P}$ has a lub.

(2) Every ideal of $\mathbf{P}$ has a lub.

Posets with these properties are called *directed complete* posets or *domains*. Directed lubs are also called *(directed) limits*.

Note that for domains, the existence of a least element is not required.

**Proof:** (1) implies (2) since every ideal is a directed set. (2) implies (1), since by Prop. 3.1.6, every directed set $D$ is cofinal in the ideal $\downarrow D$. By Lemma 2.4.5, the lub of $\downarrow D$ is also the lub of $D$. □

The notion of directed completeness is not self-dual: the poset $\mathbf{N}_0$ is itself directed, but has no upper bound; thus it is not directed complete. But it is downwards directed complete since every non-empty subset has a least element.

Directed completeness admits the application of the non-constructive Lemma of Zorn since every totally ordered subset ('chain') of a poset is trivially directed. Hence, we obtain:

**Lemma 3.2.2 (Zorn's Lemma for domains)**
In a domain, there is a maximal point above any given point. Formally:

$$\forall x \in \mathbf{P}\ \exists m \in \mathbf{P} \text{ such that } m \geq x \text{ and } \forall z \in \mathbf{P} : (z \geq m \text{ implies } z = m)$$

**Proposition 3.2.3** The product of a family of domains is a domain again.

**Proof:** If $D$ is a directed set in the product, then the projections $D_i = \pi_i[D]$ are directed by Prop. 3.1.4. Prop. 2.4.10 then states that $\bigsqcup D$ exists and equals $(\bigsqcup D_i)_{i \in I}$. □

Examples for domains:

- Every finite poset is a domain by Prop. 3.1.2.
- Discrete and flat posets are directed complete.
- Power sets are domains.
- $\mathbf{N}_0$ with the arithmetic order is not directed complete as mentioned above.
- $\mathbf{N}_0^\infty$ is directed complete. $\mathbf{N}'$ is also directed complete.
- For a poset $\mathbf{P}$, the set of directed subsets of $\mathbf{P}$ ordered by '$\subseteq$' is a domain by Lemma 3.1.5. Hence, there is a maximal directed superset for every directed set by Lemma 3.2.2.

## 3.3 Continuous functions

Whereas monotonic functions may be defined between two arbitrary posets, we define continuous functions for domains only.

**Definition 3.3.1** Let $\mathbf{X}$ and $\mathbf{Y}$ be domains. $f : \mathbf{X} \to \mathbf{Y}$ is (order) *continuous* if for all directed sets $D$ in $\mathbf{X}$, $\bigsqcup(f[D])$ exists and equals $f(\bigsqcup D)$. Continuous functions are sometimes called *morphisms*.

Later, there will be other definitions of continuity. The notion of continuity introduced here is then coined 'order continuity'.

Often, we shall apply the following criterion for continuity.

**Proposition 3.3.2** Let $\mathbf{X}$ and $\mathbf{Y}$ be domains.

(1) Every continuous $f : \mathbf{X} \to \mathbf{Y}$ is monotonic.

(2) A monotonic $f : \mathbf{X} \to \mathbf{Y}$ is continuous iff for all directed sets $D$ of $\mathbf{X}$, $\bigsqcup(f[D]) \geq f(\bigsqcup D)$ holds.

$\bigsqcup(f[D])$ exists for all monotonic $f$ and directed $D$.

**Proof:**

(1) Let $x \leq y$ in $\mathbf{X}$. Then $\{x, y\}$ is a directed set of $\mathbf{X}$ with supremum $y$. By continuity, $\bigsqcup\{fx, fy\}$ exists and equals $fy$. Hence, $fx \leq fy$ holds.

(2) A continuous function obviously satisfies the criterion since it weakens '=' to '$\geq$'.

For the opposite direction, we have to show $\bigsqcup(f[D]) \leq f(\bigsqcup D)$. For all $d$ in $D$, $d \leq \bigsqcup D$ holds, thus $fd \leq f(\bigsqcup D)$ by monotonicity, whence $\bigsqcup(f[D]) \leq f(\bigsqcup D)$.

Monotonic functions $f$ map directed sets $D$ to directed sets $f[D]$. Hence, $\bigsqcup(f[D])$ exists since $\mathbf{Y}$ is a domain. □

Even a bijective continuous map may have a non-monotonic inverse which is then more than ever non-continuous. An example is given by $f : \{1, 2\} \to \{\bot, \top\}$ with $f1 = \bot$ and $f2 = \top$ where 1 and 2 are incomparable, but $\bot \leq \top$ holds.

A monotonic function from a finite poset is always continuous:

**Proposition 3.3.3** Let $\mathbf{P}$ be a finite poset and $\mathbf{Y}$ a domain. A function $f : \mathbf{P} \to \mathbf{Y}$ is monotonic iff it is continuous.

**Proof:** Let $D$ be a directed set in $\mathbf{P}$. By Prop. 3.1.2, $D$ has a greatest element $x$ which is the lub of $D$. By monotonicity, $fx$ is the greatest element of $f[D]$, whence it is the lub of $f[D]$. Thus, $f(\bigsqcup D) = fx = \bigsqcup(f[D])$ holds. □

Continuous functions may be used as the arrows of a category:

**Proposition 3.3.4** The identity $id : \mathbf{X} \to \mathbf{X}$ is continuous for every domain $\mathbf{X}$. If $\mathbf{X}$, $\mathbf{Y}$, and $\mathbf{Z}$ are domains and $f : \mathbf{X} \to \mathbf{Y}$ and $g : \mathbf{Y} \to \mathbf{Z}$ are continuous, then the composition $g \circ f : \mathbf{X} \to \mathbf{Z}$ is continuous.

**Proof:** Obviously, $\bigsqcup(id[D]) = \bigsqcup D = id(\bigsqcup D)$ holds. Because continuous functions are monotonic, the set $f[D]$ is directed in $\mathbf{Y}$. Hence, $\bigsqcup((g \circ f)[D]) = \bigsqcup g[f[D]] = g(\bigsqcup f[D]) = g(f(\bigsqcup D)) = (g \circ f)(\bigsqcup D)$ holds. □

Next, we consider the relations between continuity and product.

**Proposition 3.3.5** Let $(\mathbf{Y}_i)_{i \in I}$ be a family of domains and $\mathbf{Y} = \prod_{i \in I} \mathbf{Y}_i$ their product. The projections $\pi_k : \mathbf{Y} \to \mathbf{Y}_k$ are continuous for all $k$ in $I$. A function $f : \mathbf{X} \to \mathbf{Y}$ is continuous iff $f_k = \pi_k \circ f : \mathbf{X} \to \mathbf{Y}_k$ is continuous for all $k$ in $I$.

**Proof:** Let $D$ be a directed set of $\mathbf{Y}$. By Prop. 2.4.10, we know $\bigsqcup D = (\bigsqcup \pi_i[D])_{i \in I}$, whence $\pi_k(\bigsqcup D) = \bigsqcup \pi_k[D]$ follows for all $k$ in $I$.

If $f$ is continuous, then all functions $\pi_k \circ f$ are continuous by composition. Conversely, assume all functions $f_k$ are continuous. Then for all $k$ in $I$, $\pi_k(\bigsqcup f[D]) = \bigsqcup(\pi_k[f[D]]) = \bigsqcup(f_k[D]) = f_k(\bigsqcup D) = \pi_k(f(\bigsqcup D))$ holds where the first equality is due to the continuity of $\pi_k$ and the third one due to the continuity of $f_k$. Thus, all components of $\bigsqcup f[D]$ and $f(\bigsqcup D)$ are equal. □

Similar to monotonicity, continuity may be checked component by component. This would however not be possible for infinite products.

**Proposition 3.3.6** Let $f : \mathbf{X} \times \mathbf{Y} \to \mathbf{Z}$ where all posets are domains. Let $f^y$ and $f_x$ be defined as in Prop. 2.5.7. $f$ is continuous iff $f^y$ and $f_x$ are continuous for all $x \in \mathbf{X}$ and $y \in \mathbf{Y}$.

**Proof:** By Prop. 2.5.7, $f$ is monotonic iff all $f^y$ and $f_x$ are monotonic. Hence, all the suprema in the following proof exist since they are suprema of directed sets.

Assume $f : \mathbf{X} \times \mathbf{Y} \to \mathbf{Z}$ is continuous. Fix $x$ in $\mathbf{X}$ and let $D$ be a directed set of $\mathbf{Y}$. Then

$$\begin{array}{rcll} \bigsqcup(f_x[D]) & = & \bigsqcup(f\,[\{x\} \times D]) & \\ & = & f\,(\bigsqcup(\{x\} \times D)) & \text{by continuity of } f \text{ since } \{x\} \times D \text{ is directed} \\ & = & f\,(\bigsqcup\{x\}, \bigsqcup D) & \text{by Prop. 2.4.10} \\ & = & f\,(x, \bigsqcup D) \ = \ f_x(\bigsqcup D) & \end{array}$$

Conversely, let $f_x$ and $f^y$ all be continuous. For a directed set $D \subseteq \mathbf{X} \times \mathbf{Y}$ let $D_X$ and $D_Y$ be the projections to $\mathbf{X}$ and $\mathbf{Y}$ as in Prop. 3.1.10. By this Proposition, $D$ is cofinal in $D_X \times D_Y$, i.e. $D \subseteq D_X \times D_Y \subseteq \downarrow D$. Then $f[D] \subseteq f[D_X \times D_Y] \subseteq f[\downarrow D] \subseteq \downarrow f[D]$ where the last inclusion is due to Prop. 2.5.4, i.e. $f[D]$ is cofinal in $f[D_X \times D_Y]$. Thus, we get

$$\begin{array}{rcll} \bigsqcup(f[D]) & = & \bigsqcup(f\,[D_X \times D_Y]) & \text{by cofinality} \\ & = & \bigsqcup(f\,[\bigcup_{x \in D_X}\{x\} \times D_Y]) & \\ & = & \bigsqcup \bigcup_{x \in D_X}(f[\{x\} \times D_Y]) & \\ & = & \bigsqcup_{x \in D_X} \bigsqcup(f_x[D_Y]) & \text{by Prop. 2.4.8} \\ & = & \bigsqcup_{x \in D_X} f_x(\bigsqcup D_Y) & \text{by continuity of } f_x \\ & = & \bigsqcup\{f(x, \bigsqcup D_Y) \mid x \in D_X\} & \\ & = & \bigsqcup f^{\bigsqcup D_Y}[D_X] & \\ & = & f^{\bigsqcup D_Y}(\bigsqcup D_X) & \text{by continuity of } f^{\bigsqcup D_Y} \\ & = & f(\bigsqcup D_X, \bigsqcup D_Y) & \\ & = & f(\bigsqcup D) & \text{by Prop. 3.1.10} \quad \square \end{array}$$

In section 2.5, we defined poset embeddings to be functions $f : \mathbf{P} \to \mathbf{Q}$ with $x \leq_P x'$ iff $fx \leq_Q fx'$. Such embeddings need not be continuous. Consider as an example the function $f : \mathrm{N}_0^\infty \to \mathrm{N}_0^{\infty+1}$ with $fn = n$ for all $n \in \mathrm{N}_0$ and $f(\infty) = \infty + 1$. Here, the order in the domains is given by $\mathrm{N}_0^\infty = \{0 < 1 < \cdots < \infty\}$ and $\mathrm{N}_0^{\infty+1} = \{0 < 1 < \cdots < \infty < \infty + 1\}$.

In contrast to embeddings, poset isomorphisms are always continuous in both directions because of Prop. 2.5.3.

## 3.4 The function domain

Since we are mainly interested in continuous functions, we define

**Definition 3.4.1** For two domains $\mathbf{X}$ and $\mathbf{Y}$, let $[\mathbf{X} \to \mathbf{Y}]$ be the set of all continuous functions from $\mathbf{X}$ to $\mathbf{Y}$. $[\mathbf{X} \to \mathbf{Y}]$ is ordered pointwise, i.e. $f \leq g$ iff $fx \leq gx$ holds for all $x$ in $\mathbf{X}$.

By this definition, $[\mathbf{X} \to \mathbf{Y}]$ becomes a poset. We shall soon see that it is moreover a domain. First, we however establish some conventions. Whereas the notation $f : \mathbf{X} \to \mathbf{Y}$ simply means, $f$ is a function from $\mathbf{X}$ to $\mathbf{Y}$, the notation $f : [\mathbf{X} \to \mathbf{Y}]$ will immediately imply continuity of $f$ such that this continuity need not be explicitly stated.

**Lemma 3.4.2** For two domains $\mathbf{X}$ and $\mathbf{Y}$, let $\mathcal{F} \subseteq [\mathbf{X} \to \mathbf{Y}]$ be a set of continuous functions such that for every $x$ in $\mathbf{X}$, the set $\{g\,x \mid g \in \mathcal{F}\}$ has a supremum. Then the function $f$ defined by $f x = \bigsqcup_{g \in \mathcal{F}} g\,x$ is continuous and represents the supremum of $\mathcal{F}$.

**Proof:** For continuity of $f$, we have to show $\bigsqcup f[D] = f(\bigsqcup D)$ for all directed sets $D$.

$$
\begin{aligned}
\textstyle\bigsqcup f[D] &= \textstyle\bigsqcup_{d \in D} f d \\
&= \textstyle\bigsqcup_{d \in D} \bigsqcup_{g \in \mathcal{F}} g\,d \\
&= \textstyle\bigsqcup_{g \in \mathcal{F}} \bigsqcup_{d \in D} g\,d && \text{by Prop. 2.4.9} \\
&= \textstyle\bigsqcup_{g \in \mathcal{F}} \bigsqcup g[D] \\
&= \textstyle\bigsqcup_{g \in \mathcal{F}} g\,(\bigsqcup D) && \text{by continuity of } g \\
&= f\,(\textstyle\bigsqcup D)
\end{aligned}
$$

Once $f$ is shown to be continuous, it is the supremum of $\mathcal{F}$ because of the pointwise order of functions. □

From this Lemma, one concludes:

**Proposition 3.4.3** If $\mathbf{X}$ and $\mathbf{Y}$ are domains, then $[\mathbf{X} \to \mathbf{Y}]$ is a domain.

**Proof:** Let $\mathcal{F}$ be a directed set of functions. Then for all $x$ in $\mathbf{X}$, the set $\{f x \mid f \in \mathcal{F}\}$ is directed. Thus, the function $\lambda x.\, \bigsqcup_{f \in \mathcal{F}} f x$ exists, is continuous, and represents the supremum of $\mathcal{F}$. □

Next, we establish some continuous functions around the function domain.

**Proposition 3.4.4**
Application $A : [\mathbf{X} \to \mathbf{Y}] \times \mathbf{X} \to \mathbf{Y}$ defined by $A\,(f,\, x) = f x$ is continuous.

**Proof:** By Prop. 3.3.6, we have to show the continuity in each component separately. For fixed $f$, $A_f = \lambda x.\, A\,(f,\, x) = \lambda x.\, f x = f$ holds, whence it is continuous.
To show the continuity of $A^x$ for fixed $x$, let $\mathcal{F}$ be a directed set of morphisms. Then $\bigsqcup A^x[\mathcal{F}] = \bigsqcup_{f \in \mathcal{F}}(f x) = (\bigsqcup \mathcal{F})\, x = A^x(\bigsqcup \mathcal{F})$ holds because of Lemma 3.4.2. □

**Proposition 3.4.5** For every $f : [\mathbf{X} \times \mathbf{Y} \to \mathbf{Z}]$, there is a curried version
$f' = \lambda x.\, (\lambda y.\, f\,(x,\, y)) :\ [\mathbf{X} \to [\mathbf{Y} \to \mathbf{Z}]]$.

**Proof:** We have to show the continuity of $f'x$ for every $x$ in $\mathbf{X}$ (the inner square brackets), and the continuity of $f'$ itself (the outer square brackets).
For every $x$ in $\mathbf{X}$, $f'x = \lambda y.\, f\,(x,\, y) = f_x$ is continuous by Prop. 3.3.6. For continuity of $f'$, let $D$ be a directed set of $\mathbf{X}$. Then for every $y$ in $\mathbf{Y}$

$$
\begin{aligned}
(\textstyle\bigsqcup f'[D])\, y &= (\textstyle\bigsqcup_{d \in D} f_d)\, y \\
&= \textstyle\bigsqcup_{d \in D} (f_d\, y) && \text{by Prop. 3.4.2} \\
&= \textstyle\bigsqcup_{d \in D} f\,(d,\, y) \\
&= \textstyle\bigsqcup f^y[D] \;=\; f^y\,(\bigsqcup D) && \text{by Prop. 3.3.6} \\
&= f\,(\textstyle\bigsqcup D,\, y) \;=\; (f'\,(\bigsqcup D))\, y
\end{aligned}
$$

By extensionality, $\bigsqcup f'[D] = f'(\bigsqcup D)$ follows. $\square$

Because of the propositions 3.2.3 (product of domains is a domain), 3.3.5, 3.4.3 (function domain), 3.4.4 (application), and 3.4.5 (currying), domains and continuous functions form a Cartesian closed category.

Besides application and currying, there are some more useful functional combinators.

**Proposition 3.4.6** If $\mathbf{X}$, $\mathbf{Y}$, and $\mathbf{Z}$ are domains, then functional composition $\circ : [[\mathbf{Y} \to \mathbf{Z}] \times [\mathbf{X} \to \mathbf{Y}] \to [\mathbf{X} \to \mathbf{Z}]]$ is continuous.

The proof is a straightforward application of Prop. 3.3.6, Prop. 3.4.2, and continuity of $g$.

**Proposition 3.4.7** If $\mathbf{X}_1$, $\mathbf{X}_2$, $\mathbf{Y}_1$, and $\mathbf{Y}_2$ are domains, then the Cartesian combinator $\times : [[\mathbf{X}_1 \to \mathbf{Y}_1] \times [\mathbf{X}_2 \to \mathbf{Y}_2] \to [\mathbf{X}_1 \times \mathbf{X}_2 \to \mathbf{Y}_1 \times \mathbf{Y}_2]]$ defined by $(f_1 \times f_2)(x_1, x_2) = (f_1 x_1, f_2 x_2)$ is continuous.

The proof is also straightforward by using the same auxiliary facts as in the proof of the previous proposition.

## 3.5 Directed closed sets

In this section, we investigate those sets which are closed w.r.t. suprema of directed subsets. In the next section, they are restricted to the more important Scott closed sets.

**Definition 3.5.1** Let $\mathbf{P}$ be a domain. A subset $A$ of $\mathbf{P}$ is *directed closed* or shortly *d-closed* iff the suprema w.r.t. $\mathbf{P}$ of all directed subsets of $A$ are again in $A$.

Directed closed sets $A$ of $\mathbf{P}$ are also called *sub-domains* of $\mathbf{P}$ because they are domains in the order inherited from $\mathbf{P}$.

In the sequel, we study the basic properties of d-closed sets.

**Proposition 3.5.2**

(1) $\emptyset$ and $\mathbf{P}$ are d-closed.

(2) The intersection of a family of d-closed sets is d-closed.

(3) The union of two d-closed sets is d-closed.

**Proof:**

(1) $\emptyset$ has no directed subset since directed sets are not empty. Hence, the condition for being d-closed is vacuously satisfied.

The whole domain $\mathbf{P}$ contains the lubs of all its subsets.

(2) Let $(A_i)_{i \in I}$ be a family of d-closed sets, and $A$ its intersection. If $D$ is a directed subset of $A$, then it is a subset of all sets $A_i$ whence its lub is in all sets $A_i$ whence it is in $A$.

(3) Let $A$ and $B$ be two d-closed sets, and let $D \subseteq A \cup B$. Then $D = D \cap (A \cup B) = (D \cap A) \cup (D \cap B)$ holds. By Lemma 3.1.9, one of these two sets — say $D \cap A$ — is cofinal in $D$. By Prop. 3.1.8, $D \cap A$ is directed and $\bigsqcup (D \cap A) = \bigsqcup D$ holds. Since $A$ is d-closed and $D \cap A$ is a directed subset of $A$, this implies $\bigsqcup D \in A \subseteq A \cup B$ as required. $\square$

Next, we establish some important classes of d-closed sets.

**Proposition 3.5.3**

(1) All upper sets are d-closed.

(2) All finite sets are d-closed.

(3) If $F$ is finite, then $\downarrow F$ is d-closed.

**Proof:**

(1) Let $A$ be an upper set, $D \subseteq A$ a directed set and $x$ the lub of $D$. Since $D$ is not empty, there is some member $d \in D \subseteq A$. Then $x$ is in $A$, too, since $A$ is an upper set and $d \leq x$ holds.

(2) We show that singletons are d-closed and apply Prop. 3.5.2 (3).
If $A = \{x\}$ is a singleton, then the only directed subset of $A$ is $A$ itself, whose lub $x$ is contained in $A$.

(3) We show that cones are d-closed and apply Prop. 3.5.2 (3). If $A = \downarrow x$ is a cone, and $D$ is a directed subset of $A$, then all elements of $D$ are below $x$, whence their lub is so. □

Considering part (2) and (3) of the definition above, one might believe that for all sets $A$, $A$ is d-closed iff $\downarrow A$ is. This belief is strengthened by the intuition that being d-closed is a property of the upper margin of a set only. It is however completely wrong as two examples show:

**Example 1:** Let $\mathbf{N}'$ be the poset of 'lazy natural' numbers as introduced in chapter 2. Let $A = \{0, s0, s^2 0, \ldots\}$. This is an upper set, and thus d-closed. $\downarrow A = \mathbf{N}' - \{s^\infty \bot\}$ however is not d-closed.

**Example 2:** Let $\mathbf{P} = \mathbf{N}_0^{\infty+1}$ and $A = \mathbf{P} - \{\infty\}$. Then $A$ is not d-closed, but $\downarrow A = \mathbf{P}$ is.

For convex sets, at least one implication holds:

**Proposition 3.5.4** If $A$ is convex and $\downarrow A$ is d-closed, then $A$ is d-closed.

**Proof:** Let $D$ be a directed subset of $A$. Then $D \subseteq A \subseteq \downarrow A$ holds, whence $\bigsqcup D \in \downarrow A$ since $\downarrow A$ is d-closed. This implies there is $a \in A$ such that $\bigsqcup D \leq a$. Since $D$ is not empty, there is $d \in D \subseteq A$. $d \leq \bigsqcup D \leq a$ implies $\bigsqcup D \in A$ by convexity of $A$. □

Now, we consider the complementary notion of d-open sets.

**Proposition 3.5.5**

For a subset $A$ of a domain $\mathbf{P}$, the following statements are equivalent:

(1) For all directed sets $D \subseteq \mathbf{P}$ with $\bigsqcup D \in A$, $D \cap A$ is cofinal in $D$.

(2) For all directed sets $D \subseteq \mathbf{P}$ with $\bigsqcup D \in A$, $D \cap A$ is not empty.

(3) $A$ is *d-open*, i.e. the complement of a d-closed set.

**Proof:**

(1) ⇒ (2) By Prop. 3.1.8, $D \cap A$ is directed since it is cofinal in the directed set $D$. Directed sets are not empty.

(2) ⇒ (3) Let $B$ be the complement of $A$, and $D$ be a directed subset of $B$. If $\bigsqcup D$ were not in $B$, it would be in $A$, whence some element of $D$ would be in $A$ by (2) in contradiction to $D \subseteq B$.

(3) ⇒ (1) Let $B$ be again the complement of $A$. By (3), $B$ is d-closed. $D = (D \cap A) \cup (D \cap B)$ holds, whence one of these two sets is cofinal in $D$ by Prop. 3.1.9. Assume $D' = D \cap B$ were cofinal in $D$. By Prop. 3.1.8, $D'$ is directed and $\bigsqcup D' = \bigsqcup D$ holds. Then $D' \subseteq B$ would follow, but $\bigsqcup D'$ is in $A$ in contradiction to $B$ being d-closed. □

Now, we investigate the relationship between continuous functions and d-closed sets.

**Proposition 3.5.6** Let $f : \mathbf{P} \to \mathbf{Q}$ be a continuous map between two domains. Then the inverse image of every d-closed set in $\mathbf{Q}$ is d-closed in $\mathbf{P}$.

**Proof:** Let $B$ be a d-closed subset of $\mathbf{Q}$ and let $A = f^{-1}[B]$ be the inverse image of $B$. Let $D$ be a directed subset of $A$, and let $x$ be the lub of $D$. Then $fx$ is the lub of the directed set $f[D]$ by continuity of $f$. Since $f[D]$ is a subset of $B$ and $B$ is d-closed, $fx$ is in $B$, whence $x$ is in $A$. □

The converse implication does not hold. The mapping $f : \{\bot, \top\} \to \{\bot, \top\}$ with $f\bot = \top$ and $f\top = \bot$ for instance is not continuous since it is not even monotonic. However, it backwards maps d-closed sets to d-closed sets since every subset of $\{\bot, \top\}$ is d-closed because of Prop. 3.5.3 (2).

Towards the end of the next section, we shall see that the converse implication however holds if the function is monotonic.

Because of Prop. 3.5.2 (1) and (2), there is a least d-closed superset to every set in a domain.

**Definition 3.5.7**
Let $\mathbf{X}$ be a domain and $A \subseteq \mathbf{X}$. The least d-closed superset of $A$ is denoted by $\overline{A}$.

A continuous function on $\overline{A}$ is uniquely determined by its values on $A$.

**Proposition 3.5.8**
Let $\mathbf{X}$ and $\mathbf{Y}$ be domains, $A \subseteq \mathbf{X}$, and $f, g : \overline{A} \to \mathbf{Y}$ be two continuous functions. If $fa = ga$ holds for all $a$ in $A$, then $fx = gx$ also holds for all $x$ in $\overline{A}$.

**Proof:** Let $B = \{x \in \overline{A} \mid fx = gx\}$. By the pre-condition, $A \subseteq B$ holds. Let $D$ be a directed subset of $B$ with lub $x$. Then $fx = f(\bigsqcup D) = \bigsqcup f[D] = \bigsqcup g[D] = g(\bigsqcup D) = gx$ holds by continuity of $f$ and $g$ and since both functions coincide on $B$. Thus, $B$ is d-closed. From $A \subseteq B$, $\overline{A} \subseteq B$ follows. □

We now provide an upper bound for the cardinality of the d-closure of a set.

**Proposition 3.5.9** Let $A$ be a subset of a domain $\mathbf{X}$. Then $|\overline{A}| \leq 2^{|A|}$ holds.

**Proof:** Let $B = \{\bigsqcup S \mid S \subseteq A, \bigsqcup S \text{ exists}\}$. Then $A \subseteq B$ holds since $a = \bigsqcup\{a\}$ holds for all $a \in A$. We show that $B$ is d-closed. If $D$ is a directed subset of $B$, then for all $d$ in $D$ there is a subset $S_d$ of $A$ such that $d = \bigsqcup S_d$. Then $\bigsqcup D = \bigsqcup_{d \in D}(\bigsqcup S_d) = \bigsqcup(\bigcup_{d \in D} S_d)$ by Prop. 2.4.8. Since $\bigcup_{d \in D} S_d$ is a subset of $A$, $\bigsqcup D$ is a member of $B$.
Since $B$ is a d-closed superset of $A$, $\overline{A} \subseteq B$ follows, whence $|\overline{A}| \leq |B| \leq |2^A| = 2^{|A|}$. □

Note that the set $B$ in this proof contained the lubs of *all* subsets of $A$ that exist, not only the lubs of the directed subsets of $A$. One might believe that the set $\widehat{A}$ of all lubs of *directed* subsets of $A$ equals $\overline{A}$. This belief is however wrong; in general, $\widehat{A}$ does not contain the lubs of directed sets of lubs of directed sets of $A$.

## 3.6 Scott closed sets

In this section, we define the important notions of Scott closed and Scott open sets. Scott closed sets may be used to characterize continuous functions.

**Definition 3.6.1** Let $\mathbf{P}$ be a domain. A subset $A$ of $\mathbf{P}$ is called *Scott closed* iff it is a d-closed lower set. The complements of Scott closed sets are called *Scott open*.

By Prop. 2.2.3, Scott open sets are upper d-open sets. For the properties of d-open sets see Prop. 3.5.5.

**Proposition 3.6.2**

(1) $\emptyset$ and $\mathbf{P}$ are Scott closed.

(2) Arbitrary intersections of Scott closed sets are Scott closed.

(3) Finite unions of Scott closed sets are Scott closed.

**Proof:** Just apply Prop. 2.2.2 for lower sets and Prop. 3.5.2 for d-closed sets. □

By part (1) and (2) of this proposition, every set $A$ has a least Scott closed superset $\mathsf{cl}_S\, A$.

**Proposition 3.6.3** The Scott closure operator '$\mathsf{cl}_S$' has the following properties:

(1) $\mathsf{cl}_S\, A \supseteq {\downarrow}A$ for all $A \subseteq \mathbf{P}$.

(2) $\mathsf{cl}_S\, A = {\downarrow}A$ for all finite sets $A \subseteq \mathbf{P}$.

**Proof:**

(1) ${\downarrow}A$ is the least lower superset of $A$. $\mathsf{cl}_S\, A$ is also a lower superset of $A$, since it is the least Scott closed one.

(2) '$\supseteq$' holds by (1). ${\downarrow}A$ is a Scott closed superset of $A$, since it is d-closed by Prop. 3.5.3 (3). Hence, $\mathsf{cl}_S\, A \subseteq {\downarrow}A$ holds since $\mathsf{cl}_S\, A$ is the least Scott closed superset of $A$. □

In contrast to the d-closure, the cardinality of the Scott closure of a set $A$ is not bounded by the cardinality of $A$. Let for instance $S$ be a set, and let the domain $\mathbf{X}$ be the powerset of $S$ ordered by inclusion. Since $S$ is the greatest element of $\mathbf{X}$, the Scott closure of the singleton set $\{S\}$ is the whole domain $\mathbf{X}$. There are also examples where a continuous function on $\mathsf{cl}_S\, A$ is not uniquely determined by its values on $A$.

The order continuous functions of section 3.3 may be characterized in terms of Scott closed sets:

**Proposition 3.6.4** For a function $f : \mathbf{P} \to \mathbf{Q}$ between two domains, the following statements are equivalent:

(1) $f$ is order continuous.

(2) $f$ is monotonic, and the inverse image of every d-closed set of $\mathbf{Q}$ is a d-closed set of $\mathbf{P}$.

(3) The inverse image of every Scott closed set of $\mathbf{Q}$ is a Scott closed set of $\mathbf{P}$.

(4) The inverse image of every lower cone in $\mathbf{Q}$ is a Scott closed set in $\mathbf{P}$.

(5) The inverse image of every Scott open set of $\mathbf{Q}$ is Scott open in $\mathbf{P}$.

**Proof:**

(1) $\Rightarrow$ (2) by Prop. 3.3.2 and Prop. 3.5.6.

(2) $\Rightarrow$ (3) Let $B$ be a Scott closed set in $\mathbf{Q}$ and $A$ its inverse image by $f$. By Prop. 2.5.5, $A$ is a lower set by monotonicity of $f$, and it is d-closed by (2).

(3) $\Rightarrow$ (4) is trivial since every lower cone is Scott closed by Prop. 3.6.3 (2).

(3) $\Leftrightarrow$ (5) since the complement of an inverse image of a set $A$ is the inverse image of $\mathsf{co}\, A$.

(4) $\Rightarrow$ (1) The inverse image of every lower cone is a lower set whence $f$ is monotonic by Prop. 2.5.5. Let $D$ be a directed set in $\mathbf{P}$ with lub $x$. By monotonicity, $f[D]$ is a directed set in $\mathbf{Q}$ with lub $y$, and $fx \geq y$ holds. We have to show $fx \leq y$. Let $B = \downarrow y$ be the lower cone of $y$ in $\mathbf{Q}$, and let $A$ be its inverse image in $\mathbf{P}$, which is Scott closed by (4). Since $y$ is an upper bound of $f[D]$, $f[D] \subseteq B$ holds, whence $D \subseteq A$. Since $A$ is d-closed, it contains $x$, whence $fx \in B$, i.e. $fx \leq y$. □

## 3.7 Open sets and continuous functions

Let $\mathbf{X}$ be a domain. In this section, we compare the domain $\Omega\mathbf{X}$ of (Scott) open sets of $\mathbf{X}$ with the domain of continuous functions from $\mathbf{X}$ to $\mathbf{2} = \{\bot < \top\}$. The open sets form a domain when ordered by inclusion '$\subseteq$' because arbitrary unions of open sets are open.

If $f : [\mathbf{X} \to \mathbf{2}]$ is a morphism, then $f^{-1}[\{\top\}]$ is open in $\mathbf{X}$ because $\{\top\}$ is open in $\mathbf{2}$. Conversely, if an open set $O$ of $\mathbf{X}$ is given, define a function $f : \mathbf{X} \to \mathbf{2}$ by $fx = \top$ for $x$ in $O$ and $fx = \bot$ otherwise. Continuity of this function is asserted by the following Lemma that deals with a more general situation.

**Lemma 3.7.1** Let $\mathbf{X}$ and $\mathbf{Y}$ be two domains, $O$ an open set of $\mathbf{X}$, and $u$ and $v$ two points of $\mathbf{Y}$ with $u \leq v$. Then the function $f = \lambda x. \begin{cases} v & \text{if } x \in O \\ u & \text{otherwise} \end{cases}$ is continuous.

**Proof:** Applying Prop. 3.6.4, we have to show that the inverse image of every open set of $\mathbf{Y}$ is open in $\mathbf{X}$. Let $O'$ be an open set of $\mathbf{Y}$. Concerning $u$ and $v$, there are three cases:

(1) Both $u$ and $v$ are in $O'$. Then $f^{-1}[O'] = \mathbf{X}$.

(2) Neither $u$ nor $v$ is in $O'$. Then $f^{-1}[O'] = \emptyset$.

(3) $v$ is in $O'$, but $u$ is not. Then $f^{-1}[O'] = O$.

In any case, $f^{-1}[O']$ is open. The case where $u$ is in $O'$ but $v$ is not, is impossible since $u \leq v$ and $O'$ is upper. □

Thus, we get a one-to-one correspondence between $\Omega\mathbf{X}$ and $[\mathbf{X} \to \mathbf{2}]$ by means of $O \mapsto f_O = \lambda x. \begin{cases} \top & \text{if } x \in O \\ \bot & \text{otherwise} \end{cases}$ and $f \mapsto f^{-1}[\{\top\}]$. This correspondence is an order isomorphism since

$$\begin{array}{lll} f \leq g & \text{iff} & fx \leq gx \text{ for all } x \in \mathbf{X} \\ & \text{iff} & fx = \top \text{ implies } gx = \top \text{ for all } x \in \mathbf{X} \\ & \text{iff} & x \in f^{-1}[\{\top\}] \text{ implies } x \in g^{-1}[\{\top\}] \text{ for all } x \in \mathbf{X} \\ & \text{iff} & f^{-1}[\{\top\}] \subseteq g^{-1}[\{\top\}] \end{array}$$

Order isomorphisms preserve all suprema and infima. Hence, union in $\Omega\mathbf{X}$ corresponds to pointwise lub in $[\mathbf{X} \to \mathbf{2}]$, and binary intersection in $\Omega\mathbf{X}$ corresponds to binary infimum in $[\mathbf{X} \to \mathbf{2}]$. Furthermore, $\emptyset$ and $\mathbf{X}$ in $\Omega\mathbf{X}$ are associated with $\lambda x.\bot$ and $\lambda x.\top$ in $[\mathbf{X} \to \mathbf{2}]$ respectively.

## 3.8 Removing minimal points of open sets

In this section, we state and prove a technical Lemma that informally reads as follows: if one removes a part of the minimal points of a Scott open set $O$, then the set remains open.

**Lemma 3.8.1** Let $O$ be a Scott open set in a domain $\mathbf{X}$, and let $M$ be a subset of $min\, O$. Then $O \setminus M$ is Scott open.

**Proof:**
Let $O' = O \setminus M$ and $O'' = O \setminus min\, O$. Because of $M \subseteq min\, O$, $O'' \subseteq O' \subseteq O$ holds.

<u>$O'$ is upper:</u> Let $x$ in $O'$ and $x \leq y$. Since $O$ is upper, $x \in O$ implies $y \in O$. If $y$ is not in $min\, O$, then $y \in O'' \subseteq O'$ follows. If $y$ is in $min\, O$, then $x \leq y$ implies $y = x \in O'$.

<u>$O'$ is d-open:</u> Let $D$ be a directed set with $\bigsqcup D$ in $O'$. We have to show that $D$ meets $O'$. Assume it does not. Since $\bigsqcup D \in O' \subseteq O$, there is some $x$ in $D \cap O$. Let $d$ be an arbitrary point of $D$. The two points $d$ and $x$ have a common upper bound $y$ in $D$. $x$ in $O$ and $O$ being upper implies $y$ in $O$. By assumption, $y$ is not in $O'$, whence it is in $M \subseteq min\, O$. $x \leq y$ in $O$ implies $x = y \geq d$. Thus, $x$ is an upper bound of $D$. $\bigsqcup D \leq x$ follows, whence $x$ is in $O'$ as $\bigsqcup D$ is. This contradicts the assumption that $D$ does not meet $O'$. □

## 3.9 Rudin's Lemma and its consequences

In [GLS83] and [Jun88], an interesting Lemma is stated and proved that in its original weaker form dates back to [Rud80]. Although the Lemma is concerned with arbitrary posets, we present it in this chapter since we derive some consequences from it that only hold in domains.

In the version of [Jun88], the Lemma reads as follows:

**Lemma 3.9.1** Let $\mathbf{X}$ be a poset and $\mathcal{E}$ a set of *non-empty finite* subsets of $\mathbf{X}$ such that $\uparrow[\mathcal{E}] = \{\uparrow E \mid E \in \mathcal{E}\}$ is $\supseteq$-*directed*. Then there is a *directed* subset $D$ of $\bigcup \mathcal{E}$ that *meets* all members of $\mathcal{E}$.

For the proof of this Lemma, we refer to [Jun88], or to [GLS83] for a proof of a slightly different version. In the sequel, we derive some consequences of the Lemma.

**Lemma 3.9.2** Let $\mathbf{X}$ be a *domain* and $\mathcal{E}$ a set of *non-empty finite* subsets of $\mathbf{X}$ such that $\uparrow[\mathcal{E}] = \{\uparrow E \mid E \in \mathcal{E}\}$ is $\supseteq$*-directed*. Then there is a point $x$ that is both in $\overline{\bigcup \mathcal{E}}$ and in $\bigcap \uparrow[\mathcal{E}]$.

**Proof:** Lemma 3.9.1 gives us a directed subset $D$ of $\bigcup \mathcal{E}$ that meets all sets $E$ in $\mathcal{E}$, i.e. there is a point $d_E$ in $D \cap E$ for all $E$ in $\mathcal{E}$. Since $\mathbf{X}$ is a domain, the supremum $x$ of $D$ exists. For every $E$ in $\mathcal{E}$, $x \geq d_E \in E$ implies $x \in \uparrow E$. $D \subseteq \bigcup \mathcal{E}$ implies $x \in \overline{\bigcup \mathcal{E}}$. □

The next Lemma is concerned with a (Scott) closed set.

**Lemma 3.9.3** Let $\mathbf{X}$ be a *domain* and $\mathcal{F}$ a $\supseteq$*-directed* set of *finitary upper* sets of $\mathbf{X}$. If all members of $\mathcal{F}$ meet a closed set $C$, then $\bigcap \mathcal{F}$ meets $C$.

**Proof:** By Prop. 2.7.1, for every $F$ in $\mathcal{F}$, $min\, F$ is finite and $F = \uparrow(min\, F)$ holds. Let $E_F = C \cap min\, F$. We show some properties of the sets $E_F$.

(1) The sets $E_F$ are finite since $E_F \subseteq min\, F$.

(2) The sets $E_F$ are not empty.
Proof: $F$ meets $C$, whence there is a point $u$ in $C \cap F$. By $F = \uparrow(min\, F)$, there is a point $v \in min\, F$ below $u$. Since $C$ is lower, $v$ also is in $C$.

(3) $F \supseteq F'$ implies $\uparrow E_F \supseteq \uparrow E_{F'}$.
Proof: Let $u$ be a point in $E_{F'} = C \cap min\, F'$. Then $u$ is in $min\, F' \subseteq F' \subseteq F = \uparrow(min\, F)$, whence there is a point $v \in min\, F$ below $u$. Since $C$ is lower, $v$ is also in $C$. Thus, $v \in C \cap min\, F = E_F$, whence $u \in \uparrow E_F$.

Now let $\mathcal{E} = \{E_F \mid F \in \mathcal{F}\}$. By (1) and (2), $\mathcal{E}$ is a set of non-empty finite sets. By (3), $\uparrow[\mathcal{E}]$ is directed since $\mathcal{F}$ is directed. Hence, Lemma 3.9.2 applies, and there is a point $x$ in both $\overline{\bigcup \mathcal{E}}$ and $\bigcap \uparrow[\mathcal{E}]$. By the first containment, $x \in \overline{\bigcup \mathcal{E}} \subseteq \overline{C} = C$ holds since all members of $\mathcal{E}$ are subsets of $C$ by definition. By the second containment, $x \in \uparrow E_F \subseteq \uparrow(min\, F) = F$ holds for all $F$ in $\mathcal{F}$. □

The last consequence involves a (Scott) open set.

**Lemma 3.9.4** Let $\mathbf{X}$ be a *domain*, $\mathcal{F}$ a $\supseteq$*-directed* set of *finitary upper* subsets of $\mathbf{X}$, and $O$ an open set of $\mathbf{X}$ such that $\bigcap \mathcal{F} \subseteq O$ holds. Then $F \subseteq O$ holds for some $F \in \mathcal{F}$.

**Proof:** Assume $F \not\subseteq O$ for all $F \in \mathcal{F}$. Then $F \cap \mathsf{co}\, O \neq \emptyset$ follows for all $F \in \mathcal{F}$. By Lemma 3.9.3, $\bigcap \mathcal{F} \cap \mathsf{co}\, O \neq \emptyset$ follows, whence $\bigcap \mathcal{F} \not\subseteq O$ — a contradiction. □

# Chapter 4

# Set-theoretic topology

This chapter presents general notions from topology such as closed and open sets, closure, compact sets, and continuous functions. Besides these well known notions, the novel concept of *strong compactness* is introduced in section 4.7. As the name suggests, strongly compact sets have properties analogous to those of compact sets.

In contrast to usual introductions to topology, we are especially concerned with the connections between order theory and topology. The previous chapter showed that every domain is equipped with a standard topology, the Scott topology. In this chapter, we show that every topological space induces a pre-order on its carrier that becomes an order under some mild assumption.

In section 4.1, we define topological spaces and their closed and open sets. Section 4.2 handles topological closure. The specialization pre-order is introduced in section 4.3. The separation properties (T0) and (T1) are defined in terms of this pre-order. In section 4.4, we investigate the relations between order-theoretic and topological notions.

In section 4.5, topological continuity and its properties are studied. Section 4.6 is devoted to the theory of compact sets, and the subsequent section 4.7 introduces strongly compact sets. Section 4.8 investigates product domains topologically, and the final section 4.9 deals with strongly compact sets in products.

Readers who are experienced in domain-oriented topology will not find much new in this chapter. An exception are the strongly compact sets of section 4.7. Strong compactness seems to be a novel concept. It is used in the theory of the upper power domain in chapter 20.

Prop. 4.3.1 and Lemma 4.4.4 (2) deserve particular attention. They allow to prove inequalities and inclusions respectively by topological means and will be used heavily throughout the thesis.

## 4.1 Closed and open sets

The Propositions 2.2.2, 3.5.2, and 3.6.2 show some common properties of lower sets, d-closed sets, and Scott closed sets respectively. By abstraction, we obtain the following definition:

**Definition 4.1.1 (Topological space)**

A topological space $\mathbf{X}$ is a non-empty set $X$ together with a set $\Gamma$ of subsets of $X$ such that

(C1) $\emptyset$ and $X$ are in $\Gamma$.

(C2) Arbitrary intersections of members of $\Gamma$ are again in $\Gamma$.

(C3) Finite unions of members of $\Gamma$ are again in $\Gamma$.

The sets in $\Gamma$ are *closed sets*, their complements are called *open*.

A set $R$ is an *environment* of a point $x$ or a set $S$ if there is an open set $O$ such that $x \in O \subseteq R$ or $S \subseteq O \subseteq R$ respectively hold.

The set of all open environments of a set $S$ is denoted by $\mathcal{O}(S)$. $\mathcal{O}(\{x\})$ is often abbreviated to $\mathcal{O}(x)$.

As with posets, we often identify the topological space $\mathbf{X}$ and its carrier $X$.

From the definition, one may immediately deduce that $\emptyset$ and $\mathbf{X}$ are open, and arbitrary union and finite intersection of open sets are open. Usually, topological spaces are defined by their open sets.

As indicated in the beginning, every domain implies (at least) three topological spaces: the *Alexandroff space* where the closed sets are the lower sets, the *d-space* with the d-closed sets, and the *Scott space* with the Scott closed sets. From these, the Scott space is the most important. The Scott space of a domain $\mathbf{X}$ is denoted by $\Sigma\mathbf{X}$, and the Alexandroff space by $\mathrm{A}\mathbf{X}$. Mostly, we shall identify $\mathbf{X}$ and $\Sigma\mathbf{X}$.

## 4.2 Topological closure

The set $\Gamma$ of closed sets oversupplies the axioms of section 1.2 needed to establish closure operations. Hence, there is a topological closure $\mathsf{cl}\, A = \bigcap\{B \in \Gamma \mid B \supseteq A\}$. The closure of a singleton set $\mathsf{cl}\,\{x\}$ will often be abbreviated by $\mathsf{cl}\, x$. The topological closure enjoys all properties given in Prop. 1.2.2. In addition, it has some further properties:

**Proposition 4.2.1**

(1) $\mathsf{cl}\,\emptyset = \emptyset$

(2) $\mathsf{cl}\,(A \cup B) = \mathsf{cl}\, A \cup \mathsf{cl}\, B$

**Proof:**

(1) By Prop. 1.2.2 (8) and $\emptyset \in \Gamma$.

(2) $A \cup B \subseteq \mathsf{cl}\, A \cup \mathsf{cl}\, B$ holds by Prop. 1.2.2 (1). Since the latter set is closed by axiom (C3), Prop. 1.2.2 (4) implies $\mathsf{cl}\,(A \cup B) \subseteq \mathsf{cl}\, A \cup \mathsf{cl}\, B$.

The opposite inclusion follows from $\mathsf{cl}\, A \subseteq \mathsf{cl}\,(A \cup B)$ and $\mathsf{cl}\, B \subseteq \mathsf{cl}\,(A \cup B)$, which in turn follow from $A \subseteq A \cup B$ and $B \subseteq A \cup B$ by Prop. 1.2.2 (5). $\square$

As introduced above, we denote the generic topological closure by '$\mathsf{cl}$'. The Scott closure is denoted for the moment by '$\mathsf{cl}_S$', whereas the d-closure is denoted by overlining. The Alexandroff closure is already known as lower closure '$\downarrow$'.

The topological closure may also be characterized in terms of environments:

**Proposition 4.2.2** For all $x \in \mathbf{X}$ and $S \subseteq \mathbf{X}$, $x$ is in $\mathsf{cl}\, S$, iff all environments of $x$ meet $S$, iff all *open* environments of $x$ meet $S$.

**Proof:** Let the three statements be numbered by (1) through (3).

$(1) \Rightarrow (2)$ : Let $x$ be in $\mathsf{cl}\, S$ and let $R$ be an environment of $x$ by the open set $O$ with $x \in O \subseteq R$. Then the complement of $O$ is a closed set $C$ not containing $x$. If $S$ were a subset of $C$, then $\mathsf{cl}\, S \subseteq C$ would hold, whence $x$ would be in $C$. Hence $S$ is not a subset of $C$, i.e. $S \cap O \neq \emptyset$, whence more than ever $S \cap R \neq \emptyset$.

$(2) \Rightarrow (3)$ : Open environments are special environments.

$(3) \Rightarrow (1)$ : Let $x$ be a point such that all open environments of $x$ meet $S$. If $x$ were not in $\mathsf{cl}\, S$, then the complement of $\mathsf{cl}\, S$ would be an open environment $O$ of $x$, whence $S \cap O \neq \emptyset$ would hold, i.e. $S$ would not be a subset of $\mathsf{cl}\, S$. □

From this proposition, we may deduce the next one:

**Proposition 4.2.3**
Let $A$ be an arbitrary set. If an open set $O$ meets $\mathsf{cl}\, A$, then it also meets $A$.

**Proof:** Let $x$ be a point of $O \cap \mathsf{cl}\, A$. Then $x$ is in $\mathsf{cl}\, A$, and $O$ is an environment of $x$. By Prop. 4.2.2, $O$ meets $A$. □

## 4.3 The specialization pre-order

A domain may be turned into a topological space by the Alexandroff, Scott, or d-topology. In this section, we consider the opposite direction. Given a topological space $\mathbf{X}$, we define a pre-order on the carrier of $\mathbf{X}$. Later, we establish conditions when this pre-ordered set is a poset or a domain.

**Proposition 4.3.1** Let $\mathbf{X}$ be a topological space. For points $x$ and $y$ of $\mathbf{X}$, the following statements are equivalent:

(1) $x \in \mathsf{cl}\, y$

(2) $\mathsf{cl}\, x \subseteq \mathsf{cl}\, y$

(3) $\mathcal{O}(x) \subseteq \mathcal{O}(y)$

We denote this situation by $x \lesssim y$.

**Proof:** (1) and (2) are equivalent because of Prop. 1.2.2 (4).

By Prop. 4.2.2, $x \in \mathsf{cl}\, y$ is equivalent to the fact that every open environment of $x$ meets $\{y\}$. This in turn is equivalent to the fact that every member of $\mathcal{O}(x)$ contains $y$, i.e. is in $\mathcal{O}(y)$.□

**Proposition 4.3.2** For every space $\mathbf{X}$, the relation '$\lesssim$' is reflexive and transitive.

**Proof:** Since $x \lesssim y$ iff $\mathsf{cl}\, x \subseteq \mathsf{cl}\, y$. □

Applying Prop. 4.3.1, it is not difficult to characterize those spaces whose pre-order is an order.

**Proposition 4.3.3 ((T0)-spaces)**

For a space $\mathbf{X}$, the following three statements are equivalent:

(1) '$\lesssim$' is an order, i.e. $x \lesssim y$ and $y \lesssim x$ implies $x = y$.

(2) For all $x, y \in \mathbf{X}$, $\mathsf{cl}\, x = \mathsf{cl}\, y$ implies $x = y$.

(3) For all $x, y \in \mathbf{X}$, $\mathcal{O}(x) = \mathcal{O}(y)$ implies $x = y$.

A topological space with these properties is called *(T0)-space*.

In the sequel, we shall mostly consider (T0)-spaces. The poset belonging to a (T0)-space $\mathbf{X}$ is denoted by $\Delta\mathbf{X}$. As we always did in chapter 2, we denote the order by '$\leq$'.

If we start with a domain $\mathbf{X}$, then form the corresponding Alexandroff space $\mathrm{A}\mathbf{X}$ or Scott space $\Sigma\mathbf{X}$, and finally form the pre-order of these spaces, then we get back the original domain $\mathbf{X}$.

**Proposition 4.3.4** For all domains $\mathbf{X}$, $\Delta(\mathrm{A}\mathbf{X})$ and $\Delta(\Sigma\mathbf{X})$ are isomorphic to $\mathbf{X}$.

**Proof:** Let '$\leq$' be the order in $\mathbf{X}$. $x \leq y$ holds iff $x \in {\downarrow} y$. By Prop. 3.6.3, ${\downarrow} y = \mathsf{cl}_S\, y$ holds. Hence, '$\leq$' is also the order of $\Delta(\mathrm{A}\mathbf{X})$ and $\Delta(\Sigma\mathbf{X})$. $\square$

The proposition does not hold for the d-space. By Prop. 3.5.3 (2), all singleton sets are d-closed, whence the order induced by the d-space of a domain degenerates to equality (cf. Prop. 4.3.5 below).

The converse of the proposition is not true. From the poset belonging to a (T0)-space, the space, i.e. the closed sets, cannot be recovered. It is possible that different (T0)-spaces induce the same poset.

The poset belonging to a (T0)-space becomes uninteresting if the order degenerates to equality. Such spaces are known as (T1)-spaces.

**Proposition 4.3.5 ((T1)-spaces)**

For a space $\mathbf{X}$, the following seven statements are equivalent:

(1) For all $x, y \in \mathbf{X}$, $x \lesssim y$ iff $x = y$.

(2) For all $x, y \in \mathbf{X}$, $\mathcal{O}(x) \subseteq \mathcal{O}(y)$ implies $x = y$.

(3) For all $x, y \in \mathbf{X}$, $\mathsf{cl}\, x \subseteq \mathsf{cl}\, y$ implies $x = y$.

(4) For all $x$ in $\mathbf{X}$, $\mathsf{cl}\, x = \{x\}$ holds.

(5) Every singleton set is closed.

(6) Every finite set is closed.

(7) For all $x, y \in \mathbf{X}$ with $x \neq y$, there is an open set $O$ such that $x \in O$ and $y \notin O$.

Topological spaces with these properties are called (T1)-spaces.

**Proof:**

(1) through (3) are equivalent by Prop. 4.3.1. (3) implies (4) since $a \in \mathsf{cl}\, x$ implies $\mathsf{cl}\, a \subseteq \mathsf{cl}\, x$ whence $a = x$. (4) implies (5) since all closures are closed. (5) implies (6) since the empty set is closed (C1), and finite unions of closed sets are closed (C3).

(6) $\Rightarrow$ (7): Let $O$ be the complement of $\{y\}$. Then $x$ is in $O$, but $y$ is not. $O$ is open as complement of the finite set $\{y\}$.

(7) $\Rightarrow$ (3): Let $\mathsf{cl}\, x$ be a subset of $\mathsf{cl}\, y$ and assume $x \neq y$. By (7), there is an open set $O$ with $x \in O$ and $y \notin O$. Let $C$ be the complement of $O$, i.e. $x \notin C$ and $y \in C$. Then $x \in \mathsf{cl}\, x \subseteq \mathsf{cl}\, y \subseteq C$ holds in contradiction to $x \notin C$. □

Statement (7) in the definition of (T1)-spaces may be strengthened. This results in the defining condition of (T2)-spaces.

**Definition 4.3.6 ((T2)-space)** A space $\mathbf{X}$ is a (T2)-space iff for all points $x$ and $y$ with $x \neq y$ there are disjoint open sets $O_x$ and $O_y$ with $x \in O_x$ and $y \in O_y$.

Classical topologists mainly investigate (T2)-spaces. They are usually not interested in (T1)-spaces and do not consider (T0)-spaces at all. The Scott space of a domain however satisfies (T0), but not in general (T1).

## 4.4 The structure of (T0)-spaces

Throughout this section, we state and prove some technical properties concerning the relations between closed and lower sets, and between open and upper sets in (T0)-spaces. The notions of lower and upper sets and the operators '$\downarrow$' and '$\uparrow$' refer to the poset induced by the (T0)-space. The results of this section in particular hold for the Scott spaces of domains.

**Proposition 4.4.1** Let $\mathbf{X}$ be a (T0)-space. All closed sets of $\mathbf{X}$ are lower sets, and all open sets are upper sets.

**Proof:** Let $C$ be a closed set, and let $x \in C$ and $x' \leq x$. Since $C$ is closed, $x \in C$ implies $\mathsf{cl}\, x \subseteq C$, whence by definition of '$\leq$', $x' \in \mathsf{cl}\, x \subseteq C$ follows.

Open sets are upper sets, because closed sets are lower, and complements of lower sets are upper sets (Prop. 2.2.3). □

Some properties that we already proved for Scott spaces also hold in this more general setting. The following Proposition generalizes Prop. 3.6.3.

**Proposition 4.4.2** Let $\mathbf{X}$ be a (T0)-space. For all subsets $A$ of $\mathbf{X}$, $\mathsf{cl}\, A \supseteq \downarrow A$ holds. If $F$ is a finite subset of $\mathbf{X}$, then $\mathsf{cl}\, F = \downarrow F$ holds.

**Proof:** Since $A$ is lower and $A \subseteq \mathsf{cl}\, A$ holds, $\downarrow A \subseteq \mathsf{cl}\, A$ follows.

Because $x \in \mathsf{cl}\,\{y\}$ is equivalent to $x \leq y$, we obtain $\mathsf{cl}\,\{y\} = \downarrow\{y\}$. By means of $\mathsf{cl}\,\emptyset = \downarrow\emptyset = \emptyset$, $\downarrow(A \cup B) = \downarrow A \cup \downarrow B$, and $\mathsf{cl}\,(A \cup B) = \mathsf{cl}\, A \cup \mathsf{cl}\, B$, we may deduce $\mathsf{cl}\, F = \downarrow F$ for all finite sets $F$. □

Lower and upper sets may be obtained as unions of closed sets and intersections of open sets respectively.

**Lemma 4.4.3**

(1) Every lower set $A$ is the union of all its closed subsets:

$$A = \bigcup\{C \mid C \text{ closed, } C \subseteq A\}$$

(2) Every upper set $B$ is the intersection of all its open supersets
$$B = \bigcap\{O \mid O \text{ open}, O \supseteq B\} = \bigcap \mathcal{O}(B)$$

**Proof:**

(1) '$\supseteq$' is trivial. Let $a \in A$. Then $a \in \downarrow a \subseteq A$, since $A$ is a lower set. $\downarrow a$ is a closed subset of $A$ because of Prop. 4.4.2.

(2) follows from (1) by complementing all sets. □

Consequently, upper closures are obtained as the intersection of all open environments, and may be compared by comparing the environments:

**Lemma 4.4.4**

(1) For all sets $A$, $\uparrow A = \bigcap \mathcal{O}(A)$ holds.

(2) For two sets $A$ and $B$, $\uparrow A \supseteq \uparrow B$ is equivalent to $\mathcal{O}(A) \subseteq \mathcal{O}(B)$.

**Proof:**

(1) Since open sets are upper, $\mathcal{O}(A) = \mathcal{O}(\uparrow A)$ holds. By Prop. 4.4.3 (2), $\bigcap \mathcal{O}(\uparrow A)$ is $\uparrow A$.

(2) If $\uparrow A \supseteq \uparrow B$, then $A \subseteq O$ where $O$ is open implies $B \subseteq \uparrow A \subseteq O$.

Conversely assume $\mathcal{O}(A) \subseteq \mathcal{O}(B)$. If $x$ is in $\uparrow B = \bigcap \mathcal{O}(B)$, then it is in all members of $\mathcal{O}(B)$. Thus, it is more than ever in all members of $\mathcal{O}(A)$, whence it is in $\uparrow A = \bigcap \mathcal{O}(A)$.□

Part (2) of the Lemma above will be heavily used to prove inclusions of upper sets. If $A$ and $B$ are upper sets, one may prove $A \subseteq B$ by showing that $B \subseteq O$ for some open set $O$ implies $A \subseteq O$.

## 4.5 Continuity

In section 3.3, we defined the notion of order continuity for functions between domains. By Prop. 3.6.4, a function is order continuous iff the inverse images of all Scott closed sets are Scott closed. Generalizing from Scott spaces to arbitrary spaces, we define a function between topological spaces to be continuous iff it backwards maps closed sets to closed sets. By the above-mentioned Prop. 3.6.4, order continuity and Scott continuity coincide. Furthermore, monotonicity coincides with Alexandroff continuity by Prop. 2.5.5, and all order continuous functions are d-continuous by Prop. 3.5.6.

The definition of continuity comes in several equivalent shapes:

**Theorem 4.5.1 (Continuity)** For a function $f : \mathbf{X} \to \mathbf{Y}$ between two topological spaces, the following statements are equivalent:

(1) The inverse image of every closed set of $\mathbf{Y}$ is closed in $\mathbf{X}$.

(2) The inverse image of every open set of $\mathbf{Y}$ is open in $\mathbf{X}$.

(3) For every $x$ in $\mathbf{X}$ and every environment $R$ of $fx$, there is an environment $R'$ of $x$ such that $f[R'] \subseteq R$.

(4) For all subsets $A$ of $\mathbf{X}$, $f[\mathsf{cl}\, A] \subseteq \mathsf{cl}\, f[A]$ holds.

A function satisfying these properties is called *continuous*.

Statement (3) corresponds to the definition of continuity known from real calculus.

**Proof:**

(1) $\Rightarrow$ (2) : If $O$ is open in $\mathbf{Y}$, then $\mathsf{co}\,O$ is closed, whence $\mathsf{co}\,(f^{-1}[O]) = f^{-1}[\mathsf{co}\,O]$ is closed in $\mathbf{X}$ by (1), whence $f^{-1}[O]$ is open.

(2) $\Rightarrow$ (3) : Let $x$ in $\mathbf{X}$, and let $R$ be an environment of $fx$ by the open set $O$ with $fx \in O \subseteq R$. Then $fx \in O$ holds, whence $x \in f^{-1}[O]$. The latter set is open by (2). Hence, it provides an (open) environment $R' = f^{-1}[O]$ of $x$. $O \subseteq R$ implies $f[R'] \subseteq R$.

(3) $\Rightarrow$ (4) : Let $x$ be a point of $\mathsf{cl}\,A$. We have to show $fx \in \mathsf{cl}\,f[A]$. Applying Prop. 4.2.2, let $R$ be an environment of $fx$. By (3), there is an environment $R'$ of $x$ such that $f[R'] \subseteq R$. Since $x$ is in $\mathsf{cl}\,A$, $R'$ meets $A$, i.e. there is a point $p$ in $R' \cap A$. Then $fp \in f[R'] \cap f[A] \subseteq R \cap f[A]$ follows, i.e. $R$ meets $f[A]$.

(4) $\Rightarrow$ (1) : Let $B$ be closed in $\mathbf{Y}$, and let $A = f^{-1}[B]$. To show that $A$ is closed, we prove $\mathsf{cl}\,A \subseteq A$. Applying (4), $x \in \mathsf{cl}\,A$ implies $fx \in f[\mathsf{cl}\,A] \subseteq \mathsf{cl}\,f[A] \subseteq \mathsf{cl}\,B = B$, whence $x \in f^{-1}[B] = A$. □

Obviously, the identity function and the composition of continuous functions is continuous. The inverse of a bijective continuous function however need not be continuous. Let $\mathbf{X}$ be the topological space with carrier set $\{1, 2\}$ where all sets are closed, and let $\mathbf{Y}$ be the space with the same carrier set where only $\emptyset$ and $\{1, 2\}$ are closed. The function mapping 1 to 1 and 2 to 2 is obviously bijective and continuous, whereas its inverse is not because $\{1\}$ is not closed in $\mathbf{Y}$.

With the definition of topological continuity, the Propositions 2.5.5 and 3.6.4 may be reformulated:

**Proposition 4.5.2**

(1) A function between two posets is monotonic iff it is Alexandroff continuous.

(2) A function between two domains is order continuous, iff it is Scott continuous, iff it is monotonic and d-continuous.

By this proposition and part (4) of Prop. 4.5.1, we obtain

**Proposition 4.5.3** If $f$ is an order continuous function between two domains $\mathbf{X}$ and $\mathbf{Y}$, then for all $A \subseteq \mathbf{X}$, the inclusions $f[\overline{A}] \subseteq \overline{f[A]}$ and $f[\mathsf{cl}_S\,A] \subseteq \mathsf{cl}_S\,f[A]$ hold.

Remember that overlining denotes the d-closure, and '$\mathsf{cl}_S$' is the Scott closure.

Not only the Scott continuous functions are monotonic, but all continuous functions.

**Proposition 4.5.4** Let $\mathbf{X}$ and $\mathbf{Y}$ be (T0)-spaces. Every continuous $f : \mathbf{X} \to \mathbf{Y}$ is monotonic in the induced orders.

**Proof:** By Prop. 2.5.5, a function is monotonic iff the inverse image of every lower cone in $\mathbf{Y}$ is a lower set in $\mathbf{X}$. Now, lower cones in $\mathbf{Y}$ are closed in $\mathbf{Y}$ since $\downarrow\{x\} = \mathsf{cl}\,\{x\}$, whence their inverse image is closed in $\mathbf{X}$ by continuity. Closed sets are lower sets by Prop. 4.4.1. □

## 4.6 Compact sets

Besides open and closed sets, there is another important class of sets in a topological space: the compact sets. They provide a generalization of finite sets in some sense.

There are several equivalent definitions of compact sets where (1) is the standard definition:

**Proposition 4.6.1 (Compact sets)**

For a subset $K$ of a topological space $\mathbf{X}$, the following statements are equivalent:

(1) For every set $\mathcal{O}$ of open sets with $K \subseteq \bigcup \mathcal{O}$, there is a finite subcollection $\mathcal{F}$ of $\mathcal{O}$ with $K \subseteq \bigcup \mathcal{F}$.

(2) For every $\subseteq$-directed set $\mathcal{O}$ of open sets with $K \subseteq \bigcup \mathcal{O}$, there is a member $O$ of $\mathcal{O}$ with $K \subseteq O$.

(3) If $\mathcal{C}$ is a $\supseteq$-directed set of closed sets that all meet $K$, then the intersection $\bigcap \mathcal{C}$ meets $K$.

In this case, the set $K$ is called *compact*.

**Proof:**

$(1) \Rightarrow (2)$ : Let $\mathcal{O}$ be a $\subseteq$-directed set of open sets with $K \subseteq \bigcup \mathcal{O}$. By (1), there is a finite subset $\mathcal{F}$ of $\mathcal{O}$ such that $K \subseteq \bigcup \mathcal{F}$. Since $\mathcal{O}$ is directed, there is an upper bound $O$ of $\mathcal{F}$ in $\mathcal{O}$. $K \subseteq \bigcup \mathcal{F}$ implies $K \subseteq O$.

$(2) \Rightarrow (1)$ : For given $\mathcal{O}$, let $\mathcal{O}' = \{\bigcup \mathcal{F} \mid \mathcal{F} \subseteq_f \mathcal{O}\}$. Then $\bigcup \mathcal{O} = \bigcup \mathcal{O}'$ holds, and $\mathcal{O}'$ is $\subseteq$-directed. By (2), $K \subseteq \bigcup \mathcal{O} = \bigcup \mathcal{O}'$ implies there is $O \in \mathcal{O}'$ such that $K \subseteq O$. Then $K \subseteq \bigcup \mathcal{F}$ for some finite subset $\mathcal{F}$ of $\mathcal{O}$ as required.

$(2) \Rightarrow (3)$ : For given $\supseteq$-directed $\mathcal{C}$, let $\mathcal{O} = \{\mathsf{co}\, C \mid C \in \mathcal{C}\}$. $\mathcal{O}$ is a $\subseteq$-directed set of open sets. If $K \cap \bigcap \mathcal{C}$ were empty, then $K \subseteq \mathsf{co} \bigcap \mathcal{C} = \bigcup \mathcal{O}$ would hold, whence by (2), there would be a member $O$ of $\mathcal{O}$ such that $K \subseteq O$. Thus, there would be a member $C$ of $\mathcal{C}$, namely $\mathsf{co}\, O$, such that $K \cap C = \emptyset$ in contradiction to the precondition.

$(3) \Rightarrow (2)$ : follows the same lines. □

Topologists often define compactness only for the case $K = \mathbf{X}$. Classically, they then require the additional property (T2), and speak of quasi-compactness if (T2) is not satisfied.

The properties of compact sets are analogous to those of finitary sets as expressed in Prop. 2.7.2. Indeed, the finitary sets of a poset are just the compact sets of its Alexandroff space. We now present the properties of compact sets as a sequence of propositions.

**Proposition 4.6.2** A set $A$ is compact iff $\uparrow A$ is compact.

**Proof:** $A$ is a subset of a union of open sets iff $\uparrow A$ is. This holds because open sets are upper sets. □

**Proposition 4.6.3** The union of two compact sets is compact.

**Proof:** Let $K_1$ and $K_2$ be the compact sets, and $K = K_1 \cup K_2$. If $K \subseteq \bigcup \mathcal{O}$ holds, then $K_i \subseteq \bigcup \mathcal{O}$ follows for all $i = 1, 2$. Hence, there are finite collections $\mathcal{F}_i \subseteq \mathcal{O}$ with $K_i \subseteq \bigcup \mathcal{F}_i$. Then, $\mathcal{F} = \mathcal{F}_1 \cup \mathcal{F}_2$ is a finite subset of $\mathcal{O}$ with $K \subseteq \bigcup \mathcal{F}$. □

There is also some relationship to closed sets:

**Proposition 4.6.4** If $C$ is closed and $K$ is compact, then $C \cap K$ is compact.

**Proof:** Let $O$ be the complement of $C$. If $C \cap K \subseteq \bigcup \mathcal{O}$ for some collection $\mathcal{O}$ of open sets, then $K = (C \cap K) \cup (O \cap K) \subseteq (C \cap K) \cup O \subseteq \bigcup(\mathcal{O} \cup \{O\})$. Since $K$ is compact, there is a finite collection $\mathcal{F} \subseteq \mathcal{O} \cup \{O\}$ with $K \subseteq \bigcup \mathcal{F}$. Let $\mathcal{F}' = \mathcal{F} \cap \mathcal{O}$. Then $K \subseteq O \cup \bigcup \mathcal{F}'$ holds. This implies $C \cap K \subseteq (C \cap O) \cup (C \cap \bigcup \mathcal{F}') \subseteq \bigcup \mathcal{F}'$. □

**Proposition 4.6.5**
If $f : \mathbf{X} \to \mathbf{Y}$ is continuous and $K \subseteq \mathbf{X}$ is compact, then $f[K] \subseteq \mathbf{Y}$ is compact.

**Proof:** Let $\mathcal{O}$ be a collection of open sets of $\mathbf{Y}$ with $f[K] \subseteq \bigcup \mathcal{O}$. Then $K \subseteq \bigcup\{f^{-1}[O] \mid O \in \mathcal{O}\}$ follows. By continuity of $f$, this is a collection of open sets whence there is a finite subset $\mathcal{F}$ of $\mathcal{O}$ with $K \subseteq \bigcup\{f^{-1}[O] \mid O \in \mathcal{F}\}$. From this, one finally obtains $f[K] \subseteq \bigcup \mathcal{F}$. □

In a general (T0)-space, compact sets need not be closed. However, they possess a closure property w.r.t. *descending* directed sets.

**Proposition 4.6.6** Let $K$ be a compact set in a (T0)-space $\mathbf{X}$, and let $D$ be a $\geq$-directed set in $K$. Then $D$ has a lower bound in $K$.

**Proof:** Let $\mathcal{C} = \{\downarrow d \mid d \in D\}$. By Prop. 4.4.2, the lower cones $\downarrow d = \mathsf{cl}\, d$ are closed. Since $d' \leq d$ implies $\downarrow d' \subseteq \downarrow d$, the set $\mathcal{C}$ is $\supseteq$-directed. Because of $D \subseteq K$, all members of $\mathcal{C}$ meet $K$. By Prop. 4.6.1 (3), the intersection $\bigcap \mathcal{C}$ meets $K$. Let $x$ be a point of $\bigcap \mathcal{C} \cap K$. Then for all $d$ in $D$, $x$ is in $\downarrow d$, i.e. $x \leq d$ holds. Thus, $x$ is a lower bound of $D$ in $K$. □

Applying Zorn's Lemma (for the dual order), one obtains that there is a minimal element of $K$ below every point of $K$. Thus,

**Proposition 4.6.7** Let $K$ be a compact set in a (T0)-space $\mathbf{X}$. Then $K \subseteq \uparrow(\mathit{min}\, K)$ holds. If $K$ is a compact upper set, then $K = \uparrow(\mathit{min}\, K)$ follows.

Combining these results with Lemma 3.8.1, one obtains

**Proposition 4.6.8**
Let $\mathbf{X}$ be a *domain*. Then every compact open set in the Scott space of $\mathbf{X}$ is finitary.

**Proof:** Let $K$ be the compact open set. By Prop. 4.6.7, $K = \uparrow(\mathit{min}\, K)$ holds. We have to show that $\mathit{min}\, K$ is finite.

For the following, let $K' = K \setminus \mathit{min}\, K$. For every element $m$ of $\mathit{min}\, K$, the set $M = \mathit{min}\, K \setminus \{m\}$ is a subset of $\mathit{min}\, K$, whence $O_m = K \setminus M = K' \cup \{m\}$ is open by Lemma 3.8.1. The sets $O_m$ cover $K$ because of $K = K' \cup \mathit{min}\, K = \bigcup_{m \in \mathit{min}\, K}(K' \cup \{m\}) = \bigcup_{m \in \mathit{min}\, K} O_m$. Since $K$ is compact, there is a finite subset $E$ of $\mathit{min}\, K$ such that $K' \cup \mathit{min}\, K = K = \bigcup_{e \in E} O_e = K' \cup E$. $\mathit{min}\, K = E$ follows. □

The proposition above was shown for Scott spaces only because Lemma 3.8.1 was proved by means of directed sets.

## 4.7 Strongly compact sets

In [Smy78], Smyth used finitely generable sets to define the first upper power construction. A set $S$ is *finitely generable* iff it is obtained as set of all labels of the infinite border of a finitarily branching tree with infinite branches only. The nodes of this tree are labeled by domain elements such that the labels of parents are smaller than the labels of their children. Intuitively, $S$ is approximated by the sequence of horizontal tree cuts. All these cuts are finite. This suggests the following definition of the more abstract notion of strong compactness.

**Definition 4.7.1** A set $S$ in a topological space $\mathbf{X}$ is *strongly compact* iff for all open sets $O$ with $S \subseteq O$ there is a finitary set $F$ with $S \subseteq F \subseteq O$.

Without restriction, one may assume the finitary set $F$ to be an upper set, i.e. to be obtained as upper closure of a finite set.

The name 'strong compactness' was chosen since the properties of this notion are analogous to those of compactness.

**Proposition 4.7.2**
Every finite set is finitary, every finitary set is strongly compact, and every strongly compact set is compact.

**Proof:** The first two claims are trivial.

Let $S$ be strongly compact, and $S \subseteq \bigcup \mathcal{O}$ where $\mathcal{O}$ is a set of open sets. Because the union is open, too, there is a finite set $E$ such that $S \subseteq \uparrow E \subseteq \bigcup \mathcal{O}$. Every $e$ in $E$ is in some open set $O_e \in \mathcal{O}$. Hence, $S \subseteq \bigcup_{e \in E} \uparrow e \subseteq \bigcup_{e \in E} O_e$. □

**Remark:** In (T1)-spaces where the pre-order degenerates to equality, a set is strongly compact iff it is finite. Thus, there are compact sets on the real line that are not strongly compact. The topology of the real line however is not the Scott topology of some domain.

**Problem 1** Are there domains where not every Scott compact set is strongly compact?

Next, we present the properties of strongly compact sets. They are analogous to those of compact sets.

**Proposition 4.7.3** $A$ is strongly compact iff $\uparrow A$ is so.

**Proof:** Since open sets are upper sets, and $A \subseteq \uparrow E$ iff $\uparrow A \subseteq \uparrow E$. □

**Proposition 4.7.4** If $A$ and $B$ are strongly compact, then so is $A \cup B$.

**Proof:** $A \cup B \subseteq O$ open implies $A \subseteq \uparrow E \subseteq O$ and $B \subseteq \uparrow F \subseteq O$ for some finite sets $E$ and $F$, whence $A \cup B \subseteq \uparrow(E \cup F) \subseteq O$. □

**Proposition 4.7.5**
If $C$ is closed and $K$ is strongly compact, then $C \cap K$ is strongly compact.

**Proof:** $C \cap K \subseteq O$ where $O$ is open implies $K \subseteq \mathsf{co}\, C \cup O$. By strong compactness, there is a finitary set $F$ such that $K \subseteq F \subseteq \mathsf{co}\, C \cup O$. Intersecting by $C$ yields $C \cap K \subseteq C \cap F \subseteq C \cap O \subseteq O$. The set $C \cap F$ is finitary by Prop. 2.7.2 since $C$ is lower. □

**Proposition 4.7.6** If $A$ is strongly compact in $\mathbf{X}$ and $f : [\mathbf{X} \to \mathbf{Y}]$ is continuous, then $f[A]$ is strongly compact in $\mathbf{Y}$.

**Proof:** $f[A] \subseteq O$ implies $A \subseteq f^{-1}[O]$ whence $A \subseteq F \subseteq f^{-1}[O]$ for some finitary $F$. This implies $f[A] \subseteq f[F] \subseteq O$. The set $f[F]$ is finitary as a monotonic image of the finitary set $F$ by Prop. 2.7.2. $\square$

Prop. 4.6.8 claimed that compact open sets in *Scott spaces* are finitary. With strongly compact sets, we can do better:

**Proposition 4.7.7** Every strongly compact open set in a (T0)-space $\mathbf{X}$ is finitary.

**Proof:** If $S$ is strongly compact and open, then $S \subseteq S$ implies there is a finitary set $F$ such that $S \subseteq F \subseteq S$. $\square$

By Prop. 4.7.2, every finitary set is compact. In Scott spaces, moreover directed intersections of finitary upper sets result in strongly compact sets.

**Proposition 4.7.8** Let $\mathbf{X}$ be a domain, and $\mathcal{F}$ a $\supseteq$-directed set of finitary upper sets. Then $\bigcap \mathcal{F}$ is strongly compact in the Scott topology.

**Proof:** Let $O$ be an open set such that $\bigcap \mathcal{F} \subseteq O$. By Lemma 3.9.4, there is some $F$ in $\mathcal{F}$ such that $F \subseteq O$. $\bigcap \mathcal{F} \subseteq F$ holds anyway. $\square$

Note that there might also be strongly compact sets which are not obtained as $\supseteq$-directed intersections of finite upper sets.

## 4.8 The product domain in topological view

In this section, we assume two given domains $\mathbf{X}$ and $\mathbf{Y}$, and investigate the topological properties of their product domain $\mathbf{X} \times \mathbf{Y}$. In doing so, we are mainly concerned with the Scott topologies of $\mathbf{X}$, $\mathbf{Y}$, and $\mathbf{X} \times \mathbf{Y}$. Thus, we simply write '$\mathsf{cl}$' for the Scott closure instead of '$\mathsf{cl}_S$', say Scott closed instead of closed etc. As usual, the d-closure is denoted by overlining.

First, we collect some facts we already know:

- The projections $\pi_1 : \mathbf{X} \times \mathbf{Y} \to \mathbf{X}$ and $\pi_2 : \mathbf{X} \times \mathbf{Y} \to \mathbf{Y}$ are continuous.
- A function $f : \mathbf{X} \times \mathbf{Y} \to \mathbf{Z}$ is continuous iff for all $x$ in $\mathbf{X}$ and $y$ in $\mathbf{Y}$, the functions $\lambda x.\, f(x, y) : \mathbf{X} \to \mathbf{Z}$ and $\lambda y.\, f(x, y) : \mathbf{Y} \to \mathbf{Z}$ are continuous.
- For every $y$ in $\mathbf{Y}$, the function $\sigma^y = \lambda x.\, (x, y) :\ \mathbf{X} \to \mathbf{X} \times \mathbf{Y}$ is continuous. For every $x$ in $\mathbf{X}$, the function $\sigma_x = \lambda y.\, (x, y) :\ \mathbf{Y} \to \mathbf{X} \times \mathbf{Y}$ is continuous.

The first fact is Prop. 3.3.5, and the second one is Prop. 3.3.6. The third fact is derived from the second one by assuming $f = id :\ \mathbf{X} \times \mathbf{Y} \to \mathbf{X} \times \mathbf{Y}$.

Next, we consider the effect of Cartesian product to various kinds of sets.

**Proposition 4.8.1** If $A$ in $\mathbf{X}$ and $B$ in $\mathbf{Y}$ are both lower / upper / finitary / d-closed / closed / open, then their product $A \times B$ has the same property in $\mathbf{X} \times \mathbf{Y}$.

**Proof:** If $A$ and $B$ are lower sets, then $\downarrow(A \times B) = \downarrow A \times \downarrow B = A \times B$ holds by Prop. 2.2.10, whence $A \times B$ is lower. The proof for upper sets is analogous.

If $A$ and $B$ are finitary, then there are finite subsets $E$ of $A$ and $F$ of $B$ such that $A \subseteq \uparrow E$ and $B \subseteq \uparrow F$. Then $E \times F$ is a finite subset of $A \times B$, and $A \times B \subseteq \uparrow E \times \uparrow F = \uparrow(E \times F)$ holds by Prop. 2.2.10.

Next, we consider closed sets. Since $\pi_1$ is continuous, the inverse image $\pi_1^{-1}[A] = A \times \mathbf{Y}$ is closed as $A$ is. Similarly, $\mathbf{X} \times B$ is closed. Thus, their intersection $A \times \mathbf{Y} \cap \mathbf{X} \times B = A \times B$ is also closed.

The proof for open and d-closed sets is analogous; continuous functions are also d-continuous.□

Despite the topological Theorem of Tychonoff, we cannot prove the compactness of a product of compact sets since the Scott space of the product domain in general is not the topological product of the Scott spaces of the factors (see also Def. 4.8.6 below). In contrast to compactness, we are able to show the strong compactness of a product of two strongly compact sets (Prop. 4.9.1). This is the first in a long sequence of results we could show for strongly compact sets, but not for compact sets.

First however we consider how a property of the product of two sets carries over to the factors.

**Proposition 4.8.2** Let $\mathbf{X}$ and $\mathbf{Y}$ be two domains, and let $A \subseteq \mathbf{X}$ and $B \subseteq \mathbf{Y}$ be two *non-empty* sets. If $A \times B$ is lower / upper / finitary / d-closed / closed / open / compact / strongly compact in $\mathbf{X} \times \mathbf{Y}$, then both factors $A$ and $B$ have the corresponding property.

**Proof:** Let $b$ be a member of $B$. Then $a \in A$ holds iff $\sigma^b a = (a, b)$ is in $A \times B$, whence $A = {\sigma^b}^{-1}[A \times B]$. Since $\sigma^b$ is continuous, this proves the claim for lower, upper, d-closed, closed, and open sets.

For all $a$ in $A$, $a = \pi_1(a, b)$ holds. Hence, $A = \pi_1[A \times B]$ is a continuous image of $A \times B$. This proves the claim for finitary, strongly compact, and compact sets. □

The next propositions concerns Scott and d-closure in the product.

**Proposition 4.8.3** Let $\mathbf{X}$ and $\mathbf{Y}$ be two domains, and $A \subseteq \mathbf{X}$ and $B \subseteq \mathbf{Y}$. Then $\overline{A \times B} = \overline{A} \times \overline{B}$ and $\mathsf{cl}\,(A \times B) = \mathsf{cl}\,A \times \mathsf{cl}\,B$ hold.

**Proof:** We show the statement for the Scott closure. The other proof is fully analogous.

$\mathsf{cl}\,A$ and $\mathsf{cl}\,B$ are closed, whence their product is closed by Prop. 4.8.1. Thus, $A \times B \subseteq \mathsf{cl}\,A \times \mathsf{cl}\,B$ implies $\mathsf{cl}\,(A \times B) \subseteq \mathsf{cl}\,A \times \mathsf{cl}\,B$.

For the opposite direction, we use Prop. 4.5.1: for continuous $f$, $f[\mathsf{cl}\,S] \subseteq \mathsf{cl}\,f[S]$ holds for all $S$. Using $\sigma_u = \lambda y.\,(u, y)$, one obtains for arbitrary sets $U \subseteq \mathbf{X}$ and $V \subseteq \mathbf{Y}$ the inclusion $U \times \mathsf{cl}\,V = \bigcup_{u \in U} \sigma_u[\mathsf{cl}\,V] \subseteq \bigcup_{u \in U} \mathsf{cl}\,(\sigma_u[V]) \subseteq \bigcup_{u \in U} \mathsf{cl}\,(U \times V) = \mathsf{cl}\,(U \times V)$. Analogously, one may show $\mathsf{cl}\,U \times V \subseteq \mathsf{cl}\,(U \times V)$. Combining both inclusions, one finally obtains $\mathsf{cl}\,A \times \mathsf{cl}\,B \subseteq \mathsf{cl}\,(A \times \mathsf{cl}\,B) \subseteq \mathsf{cl}\,(\mathsf{cl}\,(A \times B)) = \mathsf{cl}\,(A \times B)$. □

The Proposition above allows to prove two statements about closure properties of sets w.r.t. continuous operations.

**Proposition 4.8.4** Let $\mathbf{X}$ be a domain and $A$ a subset of $\mathbf{X}$.

(1) If $A$ is closed w.r.t. a continuous unary operation $f : [\mathbf{X} \to \mathbf{X}]$, i.e. $f[A] \subseteq A$, then $\overline{A}$ and $\mathsf{cl}\, A$ are also closed w.r.t. $f$.

(2) If $A$ is closed w.r.t. a continuous binary operation $g : [\mathbf{X} \times \mathbf{X} \to \mathbf{X}]$, i.e. $g[A \times A] \subseteq A$, then $\overline{A}$ and $\mathsf{cl}\, A$ are also closed w.r.t. $g$.

**Proof:** We show the statements for the Scott closure operator '$\mathsf{cl}$'. The statements for the d-closure operator '$^{-}$' are shown analogously; remember that all Scott continuous functions are d-continuous.

(1) By Th. 4.5.1 (4), $f[\mathsf{cl}\, A] \subseteq \mathsf{cl}\, f[A] \subseteq \mathsf{cl}\, A$ holds.

(2) By the same statement and Prop. 4.8.3, $g[\mathsf{cl}\, A \times \mathsf{cl}\, A] = g[\mathsf{cl}\,(A \times A)] \subseteq \mathsf{cl}\, g[A \times A] \subseteq \mathsf{cl}\, A$ holds. □

Next, we investigate the general structure of open sets in the product.

**Lemma 4.8.5** Let $\mathbf{X}$ and $\mathbf{Y}$ be two domains. Let $O$ be an open set of $\mathbf{X} \times \mathbf{Y}$ and $(x, y)$ a point of $O$. Then there are open sets $O_1$ in $\mathbf{X}$ and $O_2$ in $\mathbf{Y}$ such that $(x, y) \in O_1 \times \{y\} \subseteq O$ and $(x, y) \in \{x\} \times O_2 \subseteq O$.

**Proof:** For $\sigma^y = \lambda u.\, (u, y) : [\mathbf{X} \to \mathbf{X} \times \mathbf{Y}]$, let $O_1 = \sigma^{y^{-1}}[O]$. By continuity, this is an open set in $\mathbf{X}$. By definition of $O_1$ and $(x, y) \in O$, $(x, y) \in O_1 \times \{y\} \subseteq O$ follows as required. □

Instead of the statement of this Lemma, one often would like to be able to conclude $(x, y) \in O_1 \times O_2 \subseteq O$. This would characterize the topological product of spaces as topologists define it. An example in the introductory chapter of [Bar80] however shows that such a strong statement does not generally hold. Therefore, we define

**Definition 4.8.6** A pair of domains $\mathbf{X}$ and $\mathbf{Y}$ is $\times$-*conform* iff for every open set $O$ of $\mathbf{X} \times \mathbf{Y}$ and every point $(x, y)$ of $O$, there are open sets $O_1$ of $\mathbf{X}$ and $O_2$ of $\mathbf{Y}$ such that $(x, y) \in O_1 \times O_2 \subseteq O$.

If a pair of domains is $\times$-conform, then the topological Theorem of Tychonoff applies and one may conclude

**Proposition 4.8.7** If $\mathbf{X}$ and $\mathbf{Y}$ form a $\times$-conform pair of domains, then the product of a compact set of $\mathbf{X}$ and a compact set of $\mathbf{Y}$ is compact in $\mathbf{X} \times \mathbf{Y}$.

We do not include the proof here.

**Problem 2** Is the product of compact sets compact again even if the ground domains are not $\times$-conform?

## 4.9 The product of strongly compact sets

In contrast to compact sets, we are able to show that strong compactness is preserved by binary products.

**Theorem 4.9.1** The product of strongly compact sets of $\mathbf{X}$ and $\mathbf{Y}$ is strongly compact in $\mathbf{X} \times \mathbf{Y}$ (no matter whether $\mathbf{X}$ and $\mathbf{Y}$ are $\times$-conform or not).

The theorem is proved by means of some auxiliary lemmas. The first Lemma generalizes Lemma 4.8.5 from single points $y$ to finitary sets.

**Lemma 4.9.2** Let $x$ be a point of $\mathbf{X}$, $F$ a finitary set of $\mathbf{Y}$, and $O$ an open set of $\mathbf{X} \times \mathbf{Y}$ with $\{x\} \times F \subseteq O$. Then there is an open set $O'$ of $\mathbf{X}$ such that $x \in O'$ and $O' \times F \subseteq O$.

**Proof:** As $F$ is finitary, there is a finite subset $E$ of $F$ such that $F \subseteq \uparrow E$. For all $e$ in $E$, $(x, e) \in O$ holds. By Lemma 4.8.5, there is an open set $O_e$ for every $e$ in $E$ such that $(x, e) \in O_e \times \{e\} \subseteq O$. Let $O' = \bigcap_{e \in E} O_e$ (and $O' = \mathbf{X}$ if $E$ is empty). $O'$ is open as *finite* intersection of open sets. $x$ is in $O'$ since it is in all sets $O_e$.

Let $(u, v)$ be a point of $O' \times F$. $v \in F \subseteq \uparrow E$ implies there is $e$ in $E$ such that $v \geq e$. $u \in O' \subseteq O_e$ implies $(u, e) \in O$, whence also $(u, v) \in O$ because $(u, v) \geq (u, e)$ and $O$ is upper. □

In the next step, the finitary set is generalized to a strongly compact one.

**Lemma 4.9.3** Let $x$ be a point of $\mathbf{X}$, $S$ a strongly compact set of $\mathbf{Y}$, and $O$ an open set of $\mathbf{X} \times \mathbf{Y}$ with $\{x\} \times S \subseteq O$. Then there are an open set $O'$ of $\mathbf{X}$ and a finitary set $F$ of $\mathbf{Y}$ such that $\{x\} \times S \subseteq O' \times F \subseteq O$.

**Proof:** Using the continuous function $\sigma_x = \lambda y. (x, y)$, the inclusion $\{x\} \times S \subseteq O$ implies $S \subseteq \sigma_x^{-1}[O]$ where the latter set is open. By strong compactness, there is a finitary set $F$ such that $S \subseteq F \subseteq \sigma_x^{-1}[O]$. Applying $\sigma_x$, one obtains $\{x\} \times S \subseteq \{x\} \times F \subseteq O$. Lemma 4.9.2 yields an open set $O'$ of $\mathbf{X}$ such that $x \in O'$ and $O' \times F \subseteq O$ whence the claim follows. □

Now we are able to prove the theorem itself.

**Proof:** Let $A$ and $B$ be the two strongly compact sets, and assume $A \times B \subseteq O$ where $O$ is open. Then for every $a$ in $A$, $\{a\} \times B \subseteq O$ holds, whence by Lemma 4.9.3 there are open sets $O_a$ of $\mathbf{X}$ and finitary sets $F_a$ of $\mathbf{Y}$ such that $\{a\} \times B \subseteq O_a \times F_a \subseteq O$.

Thus, $A \subseteq \bigcup_{a \in A} O_a$ holds whence by strong compactness, there is a finitary set $G$ such that $A \subseteq G \subseteq \bigcup_{a \in A} O_a$. Let $E$ be a finite subset of $G$ with $G \subseteq \uparrow E$. Because of $E \subseteq G \subseteq \bigcup_{a \in A} O_a$, for every $e$ in $E$ there is $a(e)$ in $A$ such that $e \in O_{a(e)}$. Let $H = \bigcup_{e \in E} (\uparrow e \times F_{a(e)})$. $H$ is finitary since it is a finite union of a product of finitary sets. We claim $A \times B \subseteq H \subseteq O$.

Let $(u, v)$ be a point of $A \times B$. Because of $A \subseteq G \subseteq \uparrow E$, there is $e$ in $E$ below $u$, i.e. $u \in \uparrow e$. In addition, $\{a(e)\} \times B \subseteq O_{a(e)} \times F_{a(e)}$ holds, whence $v \in B \subseteq F_{a(e)}$. Thus, we obtain $(u, v) \in \uparrow e \times F_{a(e)}$ for some $e$ in $E$.

If $p$ is a point of $H$, then $p$ is above a point $(e, v)$ where $e$ in $E$ and $v \in F_{a(e)}$. By $e \in O_{a(e)}$, we get $(e, v) \in O_{a(e)} \times F_{a(e)} \subseteq O$. Since $O$ is upper, $p$ is in $O$, too. □

# Chapter 5

# Completeness properties

In the next four chapters, we present various classifications of domains. Domains belonging to small classes usually have useful additional properties that are not satisfied by arbitrary domains. If a certain domain class is given, we are also interested in whether it is closed w.r.t. finite or even infinite products, and w.r.t. function domain formation. A class that is closed w.r.t. finite products and function domains is called *Cartesian closed* because it forms a Cartesian closed full sub-category of the Cartesian closed category of domains and continuous functions. If C and D are two classes of domains, we denote their intersection by C & D.

Throughout these chapters, we shall identify a domain with its Scott space. Thus, we shall apply topological notions directly to a domain. If we refer to closed, open, or compact sets of a domain, we always mean Scott closed, Scott open, or Scott compact sets. Consequently, the Scott closure '$\mathsf{cl}_S$' is abbreviated to '$\mathsf{cl}$'.

After having considered some trivial classes in section 5.1, we continue by introducing some classes characterized by completeness properties. Let $P$ be a possible property of sets, e.g. finitary or compact. Then a domain $\mathbf{X}$ is $P$-complete if the whole domain $\mathbf{X}$ has property $P$, and for every two points $x$ and $y$ of $\mathbf{X}$, the set of common upper bounds $\uparrow x \cap \uparrow y$ has property $P$. Some of these completeness properties have already established names which we also use. In particular, we consider the following classifications:

- (Cone) Complete (CC): $P$ is the property to be an upper cone (section 5.3).
- Bounded complete (BC): $P$ means either being empty or an upper cone (section 5.2).
- Finitarily complete (FC) = property $\mathsf{M}$: $P$ means finitary (section 5.4).
- Strongly compactly complete (SC): $P$ means strongly compact (section 5.5).
- Compactly complete (KC): $P$ means compact (section 5.5).

By their definition, these domain classes form an increasing hierarchy. The first three classifications apply to posets since no directed lubs or topological notions are involved in their definition. The latter two classifications will be applied to domains only. In the respective sections of this chapter, we present equivalent definitions of these classes and various examples.

The defining property of class FC is also known as property $\mathsf{M}$ [Jun88]. By Th. 6.4.4, we shall see that algebraic domains are in SC, iff they are in KC, iff their base has property $\mathsf{M}$.

## 5.1 Some trivial classes

The class of all domains is denoted by DOM. As we have already seen, it allows to form arbitrary products and function domains.

*Finite* posets are characterized by the finiteness of their carrier set. All finite posets are domains. Finite products of finite domains are finite, and the function domain of two finite domains is finite. Hence, the class FIN of finite domains is Cartesian closed.

*Discrete* posets are characterized by the degeneration of their order: $a \leq b$ holds iff $a$ equals $b$. They consist of a set of uncomparable points. All discrete posets are domains. Arbitrary products of discrete domains are discrete. If $\mathbf{X}$ is an arbitrary domain and $\mathbf{Y}$ is discrete, then $[\mathbf{X} \to \mathbf{Y}]$ is discrete. Thus, the class DIS of discrete domains is Cartesian closed. By Prop. 4.3.5, there is an equivalent topological characterization of discreteness: a domain is discrete iff its Scott space is (T1).

The class of domains with least element is denoted by $\mathrm{DOM}_\perp$. Similarly, we denote the class of all domains belonging to some class C and having a least element by $\mathrm{C}_\perp$. The class $\mathrm{DOM}_\perp$ is preserved by arbitrary products and function domains. If $(\mathbf{X}_i)_{i \in I}$ is a family of domains with least elements $\perp_i$, then $\prod_{i \in I} \mathbf{X}_i$ has the least element $(\perp_i)_{i \in I}$. If $\mathbf{X}$ is an arbitrary domain and $\mathbf{Y}$ is a domain with least element $\perp$, then $[\mathbf{X} \to \mathbf{Y}]$ has a least element, namely $\lambda x.\perp$. Domains $\mathbf{D}$ with least element satisfy the famous Kleene fixed point theorem: Every continuous function $f : [\mathbf{D} \to \mathbf{D}]$ has a unique least fixed point.

## 5.2 Bounded completeness

We start the presentation of the completeness classes by BC instead of the smallest class CC since some facts about CC may be proved using facts about BC.

The class BC of bounded complete domains was investigated early. In connection with algebraicity, bounded completeness delivers a Cartesian closed class of domains with particularly nice properties. Bounded complete algebraic domains may also be characterized alternatively in terms of information systems.

There is some confusion around whether bounded complete domains should contain a least element or not. (Should *every* bounded subset have a supremum, or should every *non-empty* bounded subset have a supremum?) We decided that bounded complete domains should possess a least element.

The definition of bounded completeness is presented in form of a proposition claiming several conditions to be equivalent. It defines bounded complete posets; bounded complete domains are considered later.

**Proposition 5.2.1 (Bounded complete posets)**

For a *poset* $\mathbf{P}$, the following statements are equivalent:

(1) $\mathbf{P}$ is an upper cone, and for every two points $x$ and $y$, the set $\uparrow x \cap \uparrow y$ is empty or an upper cone.

(2) $\mathbf{P}$ has a least element, and every two points with a common upper bound have a common least upper bound.

(3) The set of upper bounds of a finite set is either empty or an upper cone.

(4) Every finite bounded subset of $\mathbf{P}$ has a supremum.

Posets with these properties are called *bounded complete.*

**Proof:**

$(1) \Rightarrow (2):$ $\uparrow x \cap \uparrow y$ is empty iff $x$ and $y$ have no common upper bound. Every upper cone has a least element.

$(2) \Rightarrow (4):$ The proof is performed by induction on the cardinality of the finite set. The empty set is bounded since $\mathbf{P}$ is not empty. Thus, it has a lub, namely the least element of $\mathbf{P}$. Singletons trivially have a lub.

If $A$ is a finite set with at least two points and upper bound $u$, let $a$ be a point of $A$ and $A' = A \setminus \{a\}$. $A'$ is also bounded by $u$, whence it has a lub by induction hypothesis. Both $\bigsqcup A'$ and $a$ are below $u$, whence they have a common upper bound. By (2), $a \sqcup \bigsqcup A' = \bigsqcup\{\bigsqcup\{a\}, \bigsqcup A'\}$ exists. By Prop. 2.4.8, $\bigsqcup(\{a\} \cup A') = \bigsqcup A$ also exists.

$(4) \Rightarrow (3):$ The finite set is either bounded, then it has a lub $x$, and the set of its upper bounds is $\uparrow x$. Or, the finite set is not bounded, then the set of its upper bounds is empty.

$(3) \Rightarrow (1):$ $\mathbf{P}$ is the set of upper bounds of $\emptyset$. Hence, it is an upper cone. Two points with a common upper bound form a finite set with a non-empty set of upper bounds. Hence, this set is an upper cone. $\square$

By the definition above, every *finite* bounded set has a supremum in a bounded complete poset. The stronger property that every bounded set has a supremum comes in two equivalent statements.

**Proposition 5.2.2** For a *poset* $\mathbf{P}$, the following statements are equivalent:

(1) Every bounded subset of $\mathbf{P}$ has a supremum.

(2) Every non-empty subset of $\mathbf{P}$ has an infimum.

**Proof:**

$(1) \Rightarrow (2):$ Let $A$ be a non-empty subset of $\mathbf{X}$, and let $B$ be the set of lower bounds of $A$. $B$ is bounded by the elements of $A$, whence it possesses a supremum $x$ by (1). Every member of $A$ is an upper bound of $B$, whence it is above the least upper bound $x$. Thus, $x$ is a lower bound of $A$. Since it is above all members of $B$, it is the greatest lower bound.

$(2) \Rightarrow (1):$ Let $A$ be a bounded subset of $\mathbf{X}$, and let $B$ be the set of upper bounds of $A$. $B$ is not empty since $A$ is bounded, whence it possesses an infimum $x$ by (2). By similar arguments as above, $x$ is shown to be the supremum of $A$. $\square$

Posets satisfying the criteria of Prop. 5.2.2 are not necessarily domains. For instance, in the poset $\mathbf{N}_0 = \{0 < 1 < 2 < \cdots\}$, every non-empty subset has an infimum. It is however not a domain since $\infty$ is missing.

There are bounded complete posets that however do not satisfy the stronger condition of Prop. 5.2.2. An example is given by $\mathbf{P} = \{0 < 1 < 2 < \cdots < [a, b]\}$, i.e. an infinite ascending chain with two incomparable upper bounds. **P** is bounded complete, but the chain $\{0, 1, \ldots\}$ is bounded without having a lub, and correspondingly, the non-empty set $\{a, b\}$ has no infimum.

For domains however, the two groups of conditions are equivalent.

**Proposition 5.2.3 (Bounded complete domains)**
For a *domain* **X**, the statements of Prop. 5.2.1 and Prop. 5.2.2 are equivalent.

The class of bounded complete domains is denoted by BC.

**Proof:** Let $S$ be an arbitrary set bounded by some point $u$. Let $\mathcal{F}$ be the set of finite subsets of $S$, and let $D = \{\bigsqcup F \mid F \in \mathcal{F}\}$ where all lubs exist since all subsets of $S$ are bounded by $u$. Set $D$ is directed, since $\bigsqcup \emptyset$ is in $D$ and $\bigsqcup(F_1 \cup F_2)$ is an upper bound of $\bigsqcup F_1$ and $\bigsqcup F_2$ in $D$ by Prop. 2.4.7. Since **X** is a domain, $\bigsqcup D$ exists. By Prop. 2.4.8, $\bigsqcup(\bigcup \mathcal{F})$ exists. The union of all finite subsets of $S$ equals $S$, whence $\bigsqcup S$ exists. $\square$

Examples for bounded complete domains are provided towards the end of the next section. We now turn to the preservation of BC by products and function domains.

**Proposition 5.2.4**
Arbitrary products of bounded complete domains are bounded complete.

**Proof:** Let $A$ be a subset of $\prod_{i \in I} \mathbf{X}_i$ bounded by a point $u = (u_i)_{i \in I}$. Let $A_i = \pi_i[A]$ for all $i$ in $I$. $A_i$ is bounded by $u_i$, whence $\bigsqcup A_i$ exists for all $i$ in $I$. By Prop. 2.4.10, $\bigsqcup A$ exists. $\square$

**Proposition 5.2.5** If **X** and **Y** are domains, then bounded completeness of **Y** implies bounded completeness of $[\mathbf{X} \to \mathbf{Y}]$.

**Proof:** Let $A$ be a subset of $[\mathbf{X} \to \mathbf{Y}]$ bounded by a function $g$. For all $x$ in **X**, the set $\{f x \mid f \in A\}$ is bounded by $g\,x$. Hence, all these sets possess a supremum. By Lemma 3.4.2, $\bigsqcup A$ exists. $\square$

## 5.3 Complete domains

In analogy to the previous section, we first present complete posets and then complete domains. Because it seems inappropriate to coin posets without lubs of directed sets complete, we speak of finitely complete posets instead of complete posets.

**Proposition 5.3.1 (Finitely complete posets)**
For a *poset* **P**, the following statements are equivalent:

(1) **P** is an upper cone, and for every two points $x$ and $y$, the set of upper cones $\uparrow x \cap \uparrow y$ is an upper cone.

(2) **P** has a least element, and every two points have a common least upper bound.

(3) The set of upper bounds of a finite set is an upper cone.

(4) Every finite subset of **P** has a supremum.

Posets with these properties are called *finitely complete*.

**Proof:** The proof is a simplified version of that of Prop. 5.2.1. □

In analogy to the conditions of Prop. 5.2.2, we now consider those posets where every subset has a supremum. In contrast to the situation in the previous section, such posets are always domains.

**Proposition 5.3.2 (Complete domains)**
For a *poset* $\mathbf{X}$, the following statements are equivalent:

(1) $\mathbf{X}$ is a finitely complete domain.

(2) All finite and all directed subsets of $\mathbf{X}$ have suprema.

(3) Every subset of $\mathbf{X}$ has a supremum.

(4) Every subset of $\mathbf{X}$ has an infimum.

(5) $\mathbf{X}$ is a bounded complete domain with a greatest element.

We call these posets complete domains. The class of complete domains is denoted by CC. Complete domains are also known as complete lattices.

**Proof:**

(1) $\Rightarrow$ (5) : If every finite set has a supremum, then more than ever every bounded finite set has a supremum.

Since $\mathbf{X}$ has a least element, and every two points of $\mathbf{X}$ have an upper bound, the whole domain $\mathbf{X}$ is directed. By the definition of domains, $\mathbf{X}$ has an upper bound that represents the greatest element of $\mathbf{X}$.

(5) $\Rightarrow$ (4) : In a bounded complete domain, every non-empty set has an infimum. The empty set also possesses an infimum, namely the greatest element.

(4) $\Rightarrow$ (3) : First, the empty set has an infimum. This represents the greatest element $\top$ of $\mathbf{X}$. Second, every non-empty set has an infimum, whence by Prop. 5.2.2, every bounded set has a supremum. All sets are bounded by $\top$.

(3) $\Rightarrow$ (2) is trivial.

(2) $\Rightarrow$ (1) : Finitely complete means all finite sets have suprema, and domain means all directed sets have suprema. □

The class CC has the same preservation properties as BC.

**Proposition 5.3.3** Arbitrary products of complete domains are complete.

**Proposition 5.3.4**
If $\mathbf{X}$ and $\mathbf{Y}$ are domains, then completeness of $\mathbf{Y}$ implies completeness of $[\mathbf{X} \to \mathbf{Y}]$.

The proofs are simpler variants of those for the class BC.

In a complete domain, every two points $x$ and $y$ have a supremum $x \sqcup y$ and an infimum $x \sqcap y$. The operation '$\sqcup$' is always continuous, whereas '$\sqcap$' is generally not continuous (see example 7 below).

**Proposition 5.3.5** In a complete domain $\mathbf{X}$, the operation $\sqcup : \mathbf{X} \times \mathbf{X} \to \mathbf{X}$ is continuous.

**Proof:** By Prop. 3.3.6 and commutativity of '$\sqcup$', we only have to show continuity in the second argument. Let $x$ be a point and $D$ a directed set of $\mathbf{X}$, and let $f_x = \lambda y.\, x \sqcup y$. Then $\bigsqcup f_x[D] = \bigsqcup\{x \sqcup d \mid d \in D\} = \bigsqcup(\bigcup_{d \in D}\{x, d\}) = \bigsqcup(\{x\} \cup D) = x \sqcup \bigsqcup D = f_x(\bigsqcup D)$ holds by applying Prop. 2.4.8 twice. $\square$

**Examples:**

(1) For any set $X$, the powerset $2^X$ ordered by '$\subseteq$' is a complete domain.

(2) $\mathbf{N}_0^\infty = \{0 < 1 < 2 < \cdots < \infty\}$ is a complete domain.

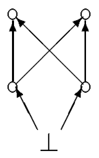
(3) The domain $\mathbf{N}'$ of lazy naturals is bounded complete, but not complete.

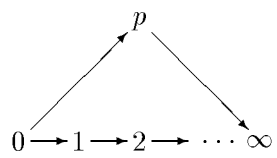
(4) $\{\mathsf{T} > \perp < \mathsf{F}\}$ is a bounded complete domain. It is not complete.

(5) The only discrete domain that is bounded complete is the one-point domain $\mathbf{1}$. $\mathbf{1}$ is even complete. The other discrete domains have no least element.

(6) Finite domains need not be bounded complete. The smallest domain with least element that is not bounded complete is depicted below to the left.

(7) Let $\mathbf{X} = \{p, 0, 1, \ldots, \infty\}$ where $0 < 1 < \cdots < \infty$ and $0 < p < \infty$ holds. $\mathbf{X}$ is depicted below to the right. It is a complete domain where the operation '$\sqcap$' is not continuous. The directed set $D = \{0, 1, \ldots\}$ has lub $\infty$. $\bigsqcup\{p \sqcap d \mid d \in D\} = \bigsqcup\{0\} = 0$ holds, whereas $p \sqcap \bigsqcup D = p \sqcap \infty = p$.




## 5.4 Finitary completeness or property M

In bounded complete posets, the set of upper bounds of two points is either empty or has a least element. As a generalization one may require the set of upper bounds to be finitary. This property is known as property M [Jun88]. We also use the term 'finitarily complete'.

As usual, there are several equivalent definitions collected in the following proposition.

**Proposition 5.4.1 (Finitarily complete)**

For a *poset* $\mathbf{P}$, the following statements are equivalent:

(1) The set of upper bounds of every finite set is finitary.

(2) $\mathbf{P}$ is finitary, and for every two points $x$ and $y$, the set $\uparrow x \cap \uparrow y$ is finitary.

(3) $\mathbf{P}$ is finitary, and the intersection of two finitary upper sets is finitary.

(4) Finite intersections of finitary sets are finitary.

A poset satisfying these conditions is said to have property M. We also say it is *finitarily complete*. The class of all *domains* with property M is called FC.

Note that finite intersections also include empty intersections. Empty intersections always result in the whole carrier $\mathbf{P}$.

**Proof:**

(1) $\Rightarrow$ (2) : $\mathbf{P}$ is the set of upper bounds of $\emptyset$, and $\uparrow x \cap \uparrow y$ is the set of upper bounds of $\{x, y\}$.

(2) $\Rightarrow$ (3) : Let $A$ and $B$ be two finitary upper sets. Then there are finite sets $E$ and $F$ such that $A = \uparrow E$ and $B = \uparrow F$. Hence, $A \cap B = \uparrow E \cap \uparrow F = \bigcup_{e\in E,\, f\in F} \uparrow e \cap \uparrow f$ is a finite union of finitary sets by (2). Thus, it is finitary.

(3) $\Rightarrow$ (4) : By induction. The empty intersection is $\mathbf{P}$.

(4) $\Rightarrow$ (1) : If $E$ is finite, then the set of upper bounds of $E$ is $\bigcap_{e\in E} \uparrow e$. Upper cones are finitary, whence this is a finite intersection of finitary sets. □

**Proposition 5.4.2**

(1) Every finite poset is finitarily complete.

(2) Every bounded complete poset is finitarily complete.

**Proof:**

(1) All finite sets are finitary.

(2) By Prop. 5.2.1 (1), the set of upper bounds of a finite set is either empty or an upper cone, whence finitary in any case. □

The preservation properties of the class FC are worse than those of CC and BC.

**Proposition 5.4.3** If $\mathbf{X}$ and $\mathbf{Y}$ are in FC, then so is $\mathbf{X} \times \mathbf{Y}$.

**Proof:** Products of finitary sets are finitary by Prop. 4.8.1. Hence, $\mathbf{X} \times \mathbf{Y}$ is finitary. Furthermore, $\uparrow(x, y) \cap \uparrow(x', y') = (\uparrow x \times \uparrow y) \cap (\uparrow x' \times \uparrow y') = (\uparrow x \cap \uparrow x') \times (\uparrow y \cap \uparrow y')$ holds. The final outcome is finitary as product of finitary sets. □

Infinite products and function domains of FC domains are generally not FC.

**Example 1:** Let $\mathbf{X} = \{a, b\}$ where $a$ and $b$ are incomparable. If $I$ is an infinite index set, then $\mathbf{Y} = \prod_{i\in I} \mathbf{X}$ is infinite and discrete, whence it cannot be finitary.

**Example 2:** Let $\mathbf{X} = \{[a_1, b_1] < [a_2, b_2] < \cdots < \infty\}$:

$$\begin{tikzcd}
a_1 \arrow[r] \arrow[dr] & a_2 \arrow[r] \arrow[dr] & a_3 & \\
 & & & \cdots \quad \infty \\
b_1 \arrow[r] \arrow[ur] & b_2 \arrow[r] \arrow[ur] & b_3 &
\end{tikzcd}$$

The domain $\mathbf{X}$ is in FC, but not in BC. For every $n$ in $\mathbf{N}$, the function $f : \mathbf{X} \to \mathbf{X}$ defined by $f(a_i) = a_i$ and $f(b_i) = b_i$ for $i < n$, and $f(a_i) = f(b_i) = a_n$ for $i \geq n$, is minimal in $[\mathbf{X} \to \mathbf{X}]$. Hence, $[\mathbf{X} \to \mathbf{X}]$ has an infinite number of minimal points and thus cannot be finitary.

There are two typical examples of domains not in FC: $\{[a, b] < \cdots < 3 < 2 < 1\}$ and $\{[a, b] < [1, 2, 3, \ldots]\}$. In both cases, the set $\uparrow a \cap \uparrow b$ is not finitary.

In connection with Rudin's Lemma, we are able to prove further properties of finitarily complete domains.

**Proposition 5.4.4** Let $\mathbf{X}$ be a domain in FC. A subset $S$ of $\mathbf{X}$ is a strongly compact upper set iff it is the intersection of some $\supseteq$-directed set of finitary upper sets. One such set is the set of all finitary upper supersets of $S$.

**Proof:** One direction is provided by Prop. 4.7.8. For the opposite direction, let $S$ be a strongly compact upper set, and let $\mathcal{F}$ be the set of all finitary upper supersets of $S$. $\mathcal{F}$ is $\supseteq$-directed by the FC condition. $S \subseteq \bigcap \mathcal{F}$ obviously holds. For the opposite inclusion, we apply Lemma 4.4.4. If $S \subseteq O$ where $O$ is open, then $F \subseteq O$ holds for some $F$ in $\mathcal{F}$ by strong compactness. $\bigcap \mathcal{F} \subseteq F \subseteq O$ follows. □

## 5.5 The classes SC and KC

The class FC may be further generalized by replacing the finitary sets by strongly compact sets in its definition. Since strong compactness is a topological notion, we only consider domains, not general posets.

**Proposition 5.5.1 (SC)** For a domain $\mathbf{X}$, the following two statements are equivalent:

(1) $\mathbf{X}$ is strongly compact, and for every two points $x$ and $y$, the set $\uparrow x \cap \uparrow y$ is strongly compact.

(2) $\mathbf{X}$ is finitary, and the intersection of two finitary upper sets is strongly compact.

Domains with these properties are called *strongly compactly complete*. The class of such domains is denoted by SC.

**Proof:** Since $\mathbf{X}$ is open, $\mathbf{X}$ being strongly compact and being finitary is equivalent by Prop. 4.7.7.

Upper cones are finitary. Conversely, if $A$ and $B$ are finitary upper sets, then there are finite sets $E$ and $F$ such that $A = \uparrow E$ and $B = \uparrow F$. $A \cap B = \bigcup_{e \in E, f \in F} \uparrow e \cap \uparrow f$ is strongly compact as a finite union of strongly compact sets. □

In contrast to finitary completeness, we are neither able to derive from SC that the set of upper bounds of every finite set is strongly compact, nor that the intersection of two strongly compact upper sets is strongly compact.

The class of compactly complete domains is defined analogously to SC.

**Proposition 5.5.2 (KC)** For a domain $\mathbf{X}$, the following two statements are equivalent:

(1) $\mathbf{X}$ is compact, and for every two points $x$ and $y$, the set $\uparrow x \cap \uparrow y$ is compact.

(2) $\mathbf{X}$ is finitary, and the intersection of two finitary upper sets is compact.

Domains with these properties are called *compactly complete*. The class of such domains is denoted by KC.

**Proof:** Analogous to that of Prop. 5.5.1 using Prop. 4.6.8 instead of Prop. 4.7.7. □

The class FC is a subclass of SC, and SC is in turn a subclass of KC. The examples of section 5.4 provide domains in class FC whose infinite product and function domain are not contained in FC since their whole carrier is not finitary. By part (2) of Prop. 5.5.2, they are in KC neither. Thus, neither SC nor KC are closed w.r.t. infinite products and function domains.

Class SC is however closed w.r.t. finite products:

**Proposition 5.5.3** If $\mathbf{X}$ and $\mathbf{Y}$ are in SC, then their product $\mathbf{X} \times \mathbf{Y}$ is in SC.

**Proof:** Analogous to the proof of Prop. 5.4.3 using the fact that the product of two strongly compact sets is strongly compact (Prop. 4.9.1). $\square$

We cannot show an analogous property for KC because the analogue of Prop. 4.9.1 is missing.

**Problem 3** Is the class KC closed w.r.t. finite products?

# Chapter 6

# Algebraic and continuous domains

In the previous chapter, we presented a hierarchy of properties defined by a common scheme: from CC to KC. In this chapter, we present two well-known domain classes: algebraic and continuous domains. We start by introducing the smaller, but more well-known class of algebraic domains and then consider continuous domains.

The classical definitions of these notions will be shown to be equivalent to purely topological definitions that could be applied to all (T0)-spaces (Theorems 6.2.5 and 6.7.9). These results do not occur in the literature as far as I know it. On the other hand, they are fundamental and easy to prove such that it is unlikely that they are really new.

Sections 6.1 through 6.5 handle algebraicity. In section 6.1, isolated points are studied in arbitrary domains. Section 6.2 starts by the usual order-theoretic definition in terms of isolated points, and derives the equivalent topological definition. In section 6.3, some examples for algebraic domains are presented. In section 6.4 we show how the completeness properties of an algebraic domain may be characterized in terms of the base. Section 6.5 then treats the relations between an algebraic domain and its base: monotonic functions defined on the base can be uniquely extended to continuous functions on the whole domain (Th. 6.5.1). Also, the whole domain can be recovered from the base by ideal completion (Th. 6.5.3).

Sections 6.6 through 6.8 handle continuity of domains. Concerning the theory of continuous domains, I owe much to the paper [Law88]. In section 6.6, the way-below relation is introduced. Section 6.7 first defines continuous domains classically by means of the way-below relation and then derives the equivalent topological characterization: in every environment of every point, there is an environment that is an upper cone. In section 6.8, we present some further properties of continuous domains, and provide some examples for continuous and non-continuous domains.

## 6.1 Isolated points

Algebraic domains are characterized by a base from which all domain points can be generated by directed lubs. The points of this base are required to have a special property that is investigated in this section.

The domain $\mathbf{X} = \{p, 0, 1, \ldots, \infty\}$ where $0 < 1 < \cdots < \infty$ and $0 < p < \infty$ was introduced in section 5.3 as an example for a complete domain where the operation '$\sqcap$' is not continuous. It suffers from the fact that the point $p$ is below the limit of the directed set $D = \{0, 1, \ldots\}$, but not below any of the members of $D$.

**Definition 6.1.1** A point $a$ in a domain $\mathbf{X}$ is *isolated* (or: finite, or: compact) iff for all directed sets $D \subseteq \mathbf{X}$, the relation $a \leq \bigsqcup D$ implies $a \in \downarrow D$, i.e. there is an element $d$ in $D$ such that $a \leq d$.
The set of all isolated points of $\mathbf{X}$ is denoted by $\mathbf{X}^0$. Finite sets of isolated points are called *iso-finite*.

We adopt the name 'isolated' in order to avoid the name conflict with finite or compact sets. The name 'finite' is inspired by powersets where the isolated 'points' are just the finite sets (cf. section 6.3). The name 'compact' probably was chosen because of the characterization of compact sets by part (2) of Prop. 4.6.1. As the next proposition shows, there is also some motivation to call the isolated points 'open'.

**Proposition 6.1.2** A point $x$ is isolated iff the upper cone $\uparrow x$ is Scott open.

**Proof:** A Scott open set is a d-open upper set. The definition of $x$ being isolated coincides with statement (3) in Prop. 3.5.5, i.e. $x$ is isolated iff $\uparrow x$ is d-open. □

The upper cone being open implies a further topological property of isolated points.

**Proposition 6.1.3** A point $x$ is isolated iff for all sets $S$, $x \in \mathsf{cl}\, S$ implies $x \in \downarrow S$.

**Proof:** Let $x$ be isolated and $x \in \mathsf{cl}\, S$. By Prop. 6.1.2, $\uparrow x$ is an open environment of $x$. By Prop. 4.2.2, it meets $S$ by a point $y$. $y \in \uparrow x$ implies $x \leq y$, whence $x$ is in $\downarrow S$.

Conversely, let $x$ be a point such that $x \in \mathsf{cl}\, S$ implies $x \in \downarrow S$ for all sets $S$. Let $A = \mathsf{co} \uparrow x$. We show $A$ is closed, whence its complement $\uparrow x$ is open and $x$ is isolated by Prop. 6.1.2. Let $y$ be a point of $\mathsf{cl}\, A$. Assume $y$ is not in $A$. Then it is in $\uparrow x$, i.e. $y \geq x$ holds. Since $\mathsf{cl}\, A$ is lower, $x \in \mathsf{cl}\, A$ follows, whence $x \in \downarrow A = A$ by precondition. $x \in A$ means $x \not\geq x$ — a contradiction. Thus, $\mathsf{cl}\, A = A$ holds. □

After having considered the topological properties of isolated points, we now turn to their order-theoretic properties. The following proposition is simple, but sometimes useful.

**Proposition 6.1.4** Let $D$ be a directed set whose lub $a$ is isolated. Then $a$ is in $D$.

**Proof:** Since $a$ is isolated and below the lub of $D$ by reflexivity, there is some $d$ in $D$ such that $a \leq d$ holds. $d \leq a$ holds since $a$ is an upper bound of $D$. □

**Proposition 6.1.5** If it exists, the lub of an iso-finite set is isolated.

**Corollary:** If $\mathbf{X}$ has a least element $\perp$, then $\perp$ is isolated.

**Proof:** Let $E$ be an iso-finite set with lub $x = \bigsqcup E$. By Lemma 2.4.6, $\uparrow x = \bigcap_{e \in E} \uparrow e$ holds. The claim then follows from Prop. 6.1.2 and the fact that the intersection of a finite number of Scott open sets is Scott open. Even the intersection of zero open sets is open, since it is the whole domain.

The corollary holds, since $\perp$ is the lub of the empty set. A more direct argument is that $\uparrow\perp = \mathbf{X}$ is Scott open. □

The isolated points of a product are easily characterized.

**Proposition 6.1.6** Let $\mathbf{X}$ and $\mathbf{Y}$ be two domains. $(x, y)$ is isolated in $\mathbf{X} \times \mathbf{Y}$ iff $x$ is isolated in $\mathbf{X}$ and $y$ is isolated in $\mathbf{Y}$. Hence, $(\mathbf{X} \times \mathbf{Y})^0 = \mathbf{X}^0 \times \mathbf{Y}^0$.

**Proof:** We apply Prop. 6.1.2. By Prop. 2.2.10, $\uparrow(x, y) = \uparrow x \times \uparrow y$ holds. By Prop. 4.8.1 and 4.8.2, $\uparrow(x, y)$ is open iff both $\uparrow x$ and $\uparrow y$ are open. □

We now define a variant of the lower closure that only contains isolated points.

**Definition 6.1.7** For points $x$ in a domain $\mathbf{X}$, let $\downarrow^0 x = \{a \in \mathbf{X} \mid a \text{ isolated and } \leq x\} = \downarrow x \cap \mathbf{X}^0$ be the set of isolated points below $x$.

Obviously, $x \leq y$ implies $\downarrow^0 x \subseteq \downarrow^0 y$. The inverse implication is not true in general domains.

## 6.2 Algebraic domains

An important class of domains is characterized by the fact that every point of theirs may be reached as a directed lub of isolated points.

**Definition 6.2.1** A domain $\mathbf{X}$ is *algebraic* iff for all $x$ in $\mathbf{X}$, there is a directed set of isolated points with lub $x$. The set $\mathbf{X}^0$ of isolated points of an algebraic domain is called its *base*. The class of all algebraic domains is denoted by ALG.

An equivalent definition is given by the following proposition:

**Proposition 6.2.2** A domain $\mathbf{X}$ is algebraic iff for all $x$ in $\mathbf{X}$, the set $\downarrow^0 x$ of isolated points below $x$ is directed with lub $x$.

**Proof:** The direction from right to left is trivial. For the opposite direction, assume $\mathbf{X}$ is algebraic, and let $x \in \mathbf{X}$. Then there is a directed set $D$ of isolated points with lub $x$. $D$ obviously is a subset of $\downarrow^0 x$. We show $D$ is cofinal in $\downarrow^0 x$, whence by Prop. 3.1.8, the set $\downarrow^0 x$ is directed with lub $x$.

We have to show $\downarrow^0 x \subseteq \downarrow D$. Let $a$ be in $\downarrow^0 x$. Then $a$ is isolated and below $x$ i.e. below the lub of $D$. By definition of isolated, $a$ is below some member of $D$, i.e. $a \in \downarrow D$. □

The proposition above implies some conclusions about the '$\downarrow^0$' operator in algebraic domains.

**Proposition 6.2.3** For all points $x$ and $y$ in an algebraic domain, $\downarrow^0 x \subseteq \downarrow^0 y$ is equivalent to $x \leq y$. Hence, $\downarrow^0 x = \downarrow^0 y$ is equivalent to $x = y$.

**Proof:** The second claim follows from the first one be anti-symmetry.

Let $x \leq y$. Then every isolated point below $x$ is below $y$ by transitivity. Conversely, let $\downarrow^0 x \subseteq \downarrow^0 y$. Then $x = \bigsqcup \downarrow^0 x \leq \bigsqcup \downarrow^0 y = y$ holds by Prop. 2.4.7. □

In Prop. 6.1.2, we saw that in a domain, an upper cone is open iff its vertex is isolated. In algebraic domains, arbitrary open sets may be characterized.

**Proposition 6.2.4** In an algebraic domain, a set $O$ is open iff there is a set $S$ of isolated points such that $O = \uparrow S$.

**Proof:** Let $O$ be an open set in the algebraic domain $\mathbf{X}$, and let $S = O \cap \mathbf{X}^0$. $S$ is a set of isolated points. $S \subseteq O$ implies $\uparrow S \subseteq \uparrow O = O$. For the opposite inclusion, let $x$ be a point of $O$. $x$ is the lub of the directed set $\downarrow^0 x$. Since $O$ is open, there is $a$ in $\downarrow^0 x$ such that $a$ in $O$. $a$ is in $S$ and below $x$, whence $x$ is in $\uparrow S$.

Conversely, let $O = \uparrow S = \uparrow \bigcup_{s\in S}\{s\} = \bigcup_{s\in S} \uparrow s$. By Prop. 6.1.2, the cones $\uparrow s$ are open, whence their union is. □

The propositions above prepare a characterization of algebraicity in topological terms that might be applied to arbitrary topological spaces.

**Theorem 6.2.5** A domain $\mathbf{X}$ is algebraic iff it has the 'local open upper cone' property: for all points $x$ in all open sets $O$, there is an open upper cone $\uparrow y$ such that $x \in \uparrow y \subseteq O$.

**Proof:** First, let $\mathbf{X}$ be algebraic, and let $x$ be a point in an open set $O$. $x$ is the lub of the directed set $\downarrow^0 x$, whence some member $y$ of $\downarrow^0 x$ is in $O$. $y \leq x$ implies $x \in \uparrow y$, and $y \in O$ implies $\uparrow y \subseteq O$. $y$ being isolated implies $\uparrow y$ is open.

For the opposite direction, assume $\mathbf{X}$ satisfies the topological property. Let $x$ be a point of $\mathbf{X}$. We show $\downarrow^0 x$ is directed with lub $x$. If $a$ and $b$ are in $\downarrow^0 x$, then $x$ is contained in the open sets $\uparrow a$ and $\uparrow b$. Hence, $x$ is in the open intersection $\uparrow a \cap \uparrow b$. By the precondition, there is an open upper cone $\uparrow c$ such that $x \in \uparrow c \subseteq \uparrow a \cap \uparrow b$. Then $c$ is isolated and below $x$, whence $c$ is in $\downarrow^0 x$. By $\uparrow c \subseteq \uparrow a \cap \uparrow b$, $c \geq a, b$ holds.

Since all points of $\downarrow^0 x$ are below $x$, $\bigsqcup \downarrow^0 x \leq x$ holds. To show $x \leq \bigsqcup \downarrow^0 x$, we apply Prop. 4.3.1. Assuming $x \in O$ open, we have to show $\bigsqcup \downarrow^0 x \in O$. By the precondition, $x \in O$ implies there is an open upper cone $\uparrow y$ such that $x \in \uparrow y \subseteq O$. Then $y$ is in $\downarrow^0 x$ and $y$ is in $O$, whence $\bigsqcup \downarrow^0 x$ is in $O$ because $O$ is upper. □

Next, we present a proposition which allows to efficiently prove algebraicity and determine the base.

**Proposition 6.2.6** Let $\mathbf{X}$ be a domain and $B$ a subset of $\mathbf{X}$ such that

(1) all members of $B$ are isolated,

(2) for all $x$ in $\mathbf{X}$, there is a directed subset of $B$ with lub $x$.

Then $\mathbf{X}$ is algebraic and $B$ its base, i.e. $B$ contains *all* isolated points of $\mathbf{P}$.

**Proof:** The definition of algebraicity is trivially satisfied. By (1), $B \subseteq \mathbf{X}^0$ holds. We have to show $\mathbf{X}^0 \subseteq B$. Let $a$ be an isolated point of $\mathbf{X}$. By (2), there is a directed set $D \subseteq B$ with lub $a$. By Prop. 6.1.4, $a \in D \subseteq B$ holds. □

In section 8.4, we shall see that binary products of algebraic domains are algebraic. The function domain of two algebraic domains is however not generally algebraic again. An example is provided is section 6.8.

## 6.3 Examples for algebraic domains

We start by some classes of examples introduced by several propositions.

**Proposition 6.3.1** Every finite poset $\mathbf{P}$ is an algebraic domain whose base is $\mathbf{P}$ itself.

**Proof:** By Prop. 3.1.2, every directed set in $\mathbf{P}$ contains a greatest element which is the lub of $\mathbf{P}$. Hence, all directed sets have a lub, i.e. $\mathbf{P}$ is a domain. Furthermore, for all directed sets $D$, $\bigsqcup D \in D$ holds, whence all members of $\mathbf{P}$ are isolated. By $x = \bigsqcup\{x\}$, every point of $\mathbf{P}$ is the lub of a directed set. Applying Prop. 6.2.6 to all these facts, $\mathbf{P}$ is shown to be an algebraic domain with base $\mathbf{P}$. □

**Proposition 6.3.2** Every discrete poset $\mathbf{P}$ is an algebraic domain whose base is $\mathbf{P}$ itself.

**Proof:** Since two distinct points in $\mathbf{P}$ have no common upper bound, all directed sets of $\mathbf{P}$ are singletons. All singletons $\{x\}$ have lub $x$, whence $\mathbf{P}$ is a domain and all its points are isolated. □

**Proposition 6.3.3** If $X$ is any set, then the powerset of $X$ forms an algebraic domain when ordered by inclusion '$\subseteq$'. The isolated points are just the finite sets.

**Proof:** The powerset is a domain since every set of sets has a lub, namely their union.

Let $E$ be a finite set, and let $\mathcal{D}$ be a directed set of sets such that $E \subseteq \bigcup\mathcal{D}$. For every $e$ in $E$, there is $S_e$ in $\mathcal{D}$ such that $e \in S_e$. Let $S$ be the upper bound in $\mathcal{D}$ of the finite set $\{S_e \mid e \in E\}$. Then $E \subseteq S$ and $S \in \mathcal{D}$ holds.

Every set $S$ is the union of all its finite subsets. The set of finite subsets of $S$ is directed. Hence, every set is a lub of a directed set of isolated sets. Applying Prop. 6.2.6, we obtain that the powerset is algebraic and all isolated sets are finite. □

- The domain $\mathbf{N}_0^\infty = \{0 < 1 < 2 < \cdots < \infty\}$ is algebraic. All points except $\infty$ are isolated.
- The domain $\mathbf{N}_0^{\infty+1} = \{0 < 1 < 2 < \cdots < \infty < \infty + 1\}$ is algebraic. Again, all points except $\infty$ are isolated. Point $\infty + 1$ shows that there might be isolated points above non-isolated points. It also shows that calling the isolated points finite is inappropriate in this case.

## 6.4 Algebraicity and completeness

In this section, we investigate how the containment of an algebraic domain $\mathbf{D}$ in a completeness class can be decided by looking at the base $\mathbf{D}^0$. We start by considering bounds in algebraic domains.

In an algebraic domain, there is always an isolated point between an iso-finite set and one of its upper bounds.

**Proposition 6.4.1** Let $E$ be an iso-finite set in an algebraic domain, and let $u$ be an upper bound of $E$. Then there is an isolated upper bound of $E$ below $u$.

**Proof:** For all $e \in E$, $e \leq u = \bigsqcup {\downarrow}^0 u$ holds. Since $e$ is isolated, there is a point $d_e$ in ${\downarrow}^0 u$ with $e \leq d_e$. The set $\{d_e \mid e \in E\}$ is finite since $E$ is finite, and thus has an upper bound $d$ in ${\downarrow}^0 u$ because ${\downarrow}^0 u$ is directed. Then for all $e$ in $E$, $e \leq d_e \leq d$ holds, i.e. $d$ is an upper bound of $E$. Since $d$ is in ${\downarrow}^0 u$, it is below $u$. □

By Lemma 6.1.5, the least upper bound of an iso-finite set in a domain is isolated if it exists. In an algebraic domain, this statement may be generalized to minimal upper bounds.

**Lemma 6.4.2**
In an algebraic domain, the minimal upper bounds of an iso-finite set are isolated.

**Proof:** Let $E$ be an iso-finite set, and let $m$ be minimal in the set of upper bounds of $E$. By Prop. 6.4.1, there is an isolated upper bound $a$ of $E$ below $m$. Since $m$ is a minimal upper bound, $m = a$ is isolated. □

**Proposition 6.4.3** Let $\mathbf{D}$ be an algebraic domain. Then $\mathbf{D}$ and $\mathbf{D}^0$ have the same minimal elements, and $\mathbf{D} = \uparrow min\, \mathbf{D}$ holds in $\mathbf{D}$ iff $\mathbf{D}^0 = \uparrow min\, \mathbf{D}^0$ holds in $\mathbf{D}^0$.

The addition 'holds in ...' is supplied since the $\uparrow$-operator depends on the underlying poset.

**Proof:** Every point $x$ of $\mathbf{D}$ is the lub of a directed set of isolated points. Hence, there is an isolated point below every point of $x$. From this fact, all the statements may be proved. □

The connections between completeness of an algebraic domain and completeness of its base are given by the following theorem:

**Theorem 6.4.4** Let $\mathbf{D}$ be an algebraic domain.

CC: $\mathbf{D}$ is complete iff the poset $\mathbf{D}^0$ is finitely complete.

BC: $\mathbf{D}$ is bounded complete iff the poset $\mathbf{D}^0$ is bounded complete.

SC: $\mathbf{D}$ is in SC, iff it is in KC, iff the poset $\mathbf{D}^0$ is in FC.

**Proof:**

BC: Assume $\mathbf{D}$ is in BC. Let $E$ be a finite bounded set in $\mathbf{D}^0$. Then $E$ is bounded in $\mathbf{D}$, too, whence it has a least upper bound $x$ in $\mathbf{D}$. By Prop. 6.1.5, $x$ is in $\mathbf{D}^0$.

Conversely, assume $\mathbf{D}^0$ is in BC. Then $\mathbf{D}^0$ has a least element $\perp$, and every two bounded points of $\mathbf{D}^0$ have a lub in $\mathbf{D}^0$. Every point $x$ of $\mathbf{D}$ is the lub of a directed set of isolated points. Thus, there is at least one member of $\mathbf{D}^0$ below $x$, whence $\perp$ is below $x$. Thus, $\perp$ is also the least element of $\mathbf{D}$.

Let $x$ and $y$ be two points of $\mathbf{D}$ bounded in $\mathbf{D}$. Let $D = \downarrow^0 x \times \downarrow^0 y$. $D$ is directed as product of two directed sets. Every pair $(a, b)$ in $D$ is bounded in $\mathbf{D}$ whence it is also bounded in $\mathbf{D}^0$ by Prop. 6.4.1. Thus, $a \sqcup b$ exists. The set $D' = \{a \sqcup b \mid (a, b) \in D\}$ is directed by Prop. 2.4.7. Let $z$ be its lub in $\mathbf{D}$. $z = x \sqcup y$ may then easily be verified.

CC: The proof is analogous to a simplified version of the proof for BC.

SC: If $\mathbf{D}$ is in SC, then it is also in KC. Now assume $\mathbf{D}$ is in KC, and we want to show $\mathbf{D}^0$ in FC. First we show $\mathbf{D}^0$ is finitary. By Prop. 5.5.2, $\mathbf{D}$ is finitary. By Prop. 6.4.3, $\mathbf{D}^0$ is finitary, too. Let $a$ and $b$ be two points of $\mathbf{D}^0$. In $\mathbf{D}$, the set $\uparrow a \cap \uparrow b$ is open by Prop. 6.1.2 and compact because of KC. By Prop. 4.6.8, it is finitary in $\mathbf{D}$. Since $\{a, b\}$ forms an iso-finite set, the minimal points of $\uparrow a \cap \uparrow b$ are in $\mathbf{D}^0$. Thus, the set of upper bounds of $a$ and $b$ in $\mathbf{D}^0$ is finitary.

Now assume $\mathbf{D}^0$ is in FC, and we want to show $\mathbf{D}$ is in SC. $\mathbf{D}$ is finitary since $\mathbf{D}^0$ is. Let $x$ and $y$ be two points of $\mathbf{D}$. Let $D = \downarrow^0 x \times \downarrow^0 y$. $D$ is directed as product of two directed sets. For every pair $(a, b)$ in $D$, the set of common upper bounds of $a$ and $b$ is finitary in $\mathbf{D}^0$, whence $\uparrow a \cap \uparrow b$ is finitary in $\mathbf{D}$. Thus, using Prop. 2.4.6, one obtains $\uparrow x \cap \uparrow y = \bigcap_{a \in \downarrow^0 x} \uparrow a \;\cap\; \bigcap_{b \in \downarrow^0 y} \uparrow b = \bigcap_{(a,b) \in D} \uparrow a \cap \uparrow b$ is a $\supseteq$-directed intersection of finitary sets, whence strongly compact by Prop. 4.7.8. □

By the last statement of the theorem, the domains in KC & ALG are just the domains whose bases have property M. These domains enjoy many useful properties, whence they were used in early papers of mine such as [Hec90c, Hec90a]. The nice properties of the domains in KC & ALG now turned out to depend on KC, not on algebraicity. Examples for such properties are the Lemmas 8.10.1 and 8.10.2, and in connection with power domains, the theorems 21.5.1 and 22.6.1.

## 6.5 Continuous extension and ideal completion

In this section, we consider the relation between an algebraic domain and its base a bit closer. A monotonic function defined on the base may be uniquely extended to a continuous function on the whole domain. This also holds for embeddings. Two domains with isomorphic bases are completely isomorphic. From a given non-empty poset $\mathbf{P}$, an algebraic domain may be constructed whose base is isomorphic to $\mathbf{P}$.

**Theorem 6.5.1 (Continuous extension)**
Let $\mathbf{X}$ be an algebraic domain and $\mathbf{Y}$ a domain. For every continuous function $f : [\mathbf{X} \to \mathbf{Y}]$, the restriction of $f$ to the base $\mathbf{X}^0$ of $\mathbf{X}$ is monotonic. Conversely, for every monotonic function $f^0 : \mathbf{X}^0 \to \mathbf{Y}$, there is exactly one continuous extension $f : [\mathbf{X} \to \mathbf{Y}]$, i.e. $f$ is continuous and equals $f^0$ when restricted to $\mathbf{X}^0$.

**Proof:** The first claim concerning the restriction of $f$ to $\mathbf{X}^0$ is trivial.

First, we show the uniqueness of the continuous extension. If $f_1$ and $f_2$ are two continuous functions from $\mathbf{X}$ to $\mathbf{Y}$ which both equal $f^0$ on $\mathbf{X}^0$, then with Prop. 6.2.2, $f_1 x = f_1(\bigsqcup \downarrow^0 x) = \bigsqcup f_1[\downarrow^0 x] = \bigsqcup f^0[\downarrow^0 x] = \bigsqcup f_2[\downarrow^0 x] = f_2(\bigsqcup \downarrow^0 x) = f_2 x$ holds. This also shows how the continuous extension has to be constructed.

For $x$ in $\mathbf{X}$, we define $fx = \bigsqcup f^0[\downarrow^0 x] = \bigsqcup \{f^0 a \mid a \in \downarrow^0 x\}$. This is well defined, since $\downarrow^0 x$ is directed by Prop. 6.2.2, and the image of a directed set by a monotonic function is directed by Prop. 3.1.4.

To show the monotonicity of $f$, let $x \leq y$. Then $\downarrow^0 x \subseteq \downarrow^0 y$, whence $f^0[\downarrow^0 x] \subseteq f^0[\downarrow^0 y]$, whence $fx \leq fy$ by Prop. 2.4.7.

To show the continuity of $f$, let $D$ be a directed set in $\mathbf{X}$. Then

$$\begin{aligned}
\bigsqcup f[D] &= \bigsqcup\{fd \mid d \in D\} = \bigsqcup\{\bigsqcup f^0[\downarrow^0 d] \mid d \in D\} \\
&= \bigsqcup f^0[\textstyle\bigcup_{d \in D} \downarrow^0 d] && \text{by Prop. 2.4.8} \\
&= \bigsqcup f^0[\downarrow^0(\bigsqcup D)] && \text{by isolatedness} \\
&= f(\bigsqcup D)
\end{aligned}$$

Finally, we have to show $fa = f^0 a$ for points $a$ in $\mathbf{X}^0$. By definition of $f$, $fa = \bigsqcup f^0[\downarrow^0 a]$ holds. $a$ is an upper bound of $\downarrow^0 a$, whence $f^0 a$ is an upper bound of $f^0[\downarrow^0 a]$ by monotonicity of $f^0$. $f^0 a$ is in $f^0[\downarrow^0 a]$, since $a$ is in $\downarrow^0 a$. By Prop. 2.4.3, $f^0 a$ is the lub of $f^0[\downarrow^0 a]$. □

Next, we consider embeddings. If a poset embedding $f^0 : \mathbf{X}^0 \to \mathbf{Y}$ is given, its continuous extension $f$ may not be an embedding. The poset $\mathbf{X} = \mathbf{N}_0^{\infty+1}$ is algebraic with base $\mathbf{X}^0 = \mathbf{N}_0^{\infty+1} - \{\infty\} = \{0, 1, \ldots, \infty + 1\}$. Let $\mathbf{Y} = \mathbf{N}_0^{\infty}$ and $f^0 : \mathbf{X}^0 \to \mathbf{Y}$ be defined by $f^0 n = n$ for $n \in \mathbf{N}_0$ and $f^0(\infty + 1) = \infty$. This function is an embedding, but its continuous extension $f : \mathbf{X} \to \mathbf{Y}$ is not since $f(\infty) = f(\infty + 1) = \infty$ holds.

To ensure that the continuous extension is an embedding, the function $f^0$ has to map base points into isolated points.

**Proposition 6.5.2** Let $\mathbf{X}$ be an algebraic domain and $\mathbf{Y}$ a domain. A continuous function $f : [\mathbf{X} \to \mathbf{Y}]$ is a domain embedding if its restriction $f^0 : \mathbf{X}^0 \to \mathbf{Y}$ is a poset embedding and maps into $\mathbf{Y}^0$.

**Proof:** Let $f^0 : \mathbf{X}^0 \to \mathbf{Y}^0$ be a poset embedding, and let $x$ and $x'$ be points of $\mathbf{X}$ such that $fx \leq fx'$ holds, i.e. $\bigsqcup f^0[\downarrow^0 x] \leq \bigsqcup f^0[\downarrow^0 x']$. Let $a$ be a point of $\downarrow^0 x$. Then $f^0 a$ is in $f^0[\downarrow^0 x]$, whence $f^0 a \leq \bigsqcup f^0[\downarrow^0 x] \leq \bigsqcup f^0[\downarrow^0 x']$ follows. Since $f^0 a$ is in $\mathbf{Y}^0$, there is a point $b$ in $\downarrow^0 x'$ such that $f^0 a \leq f^0 b$. Since $f^0$ is an embedding, $a \leq b \leq x'$ holds. Because $a \leq x'$ holds for all $a \in \downarrow^0 x$, $x = \bigsqcup \downarrow^0 x \leq x'$ follows. □

Finally, we present how to construct an algebraic domain from a given base.

**Theorem 6.5.3 (Ideal completion)**
Let $\mathbf{P}$ be a given non-empty poset, and let $\mathcal{I}\mathbf{P}$ be the poset of ideals of $\mathbf{P}$ ordered by set inclusion. Then $\downarrow : \mathbf{P} \to \mathcal{I}\mathbf{P}$ is a poset embedding, and $\mathcal{I}\mathbf{P}$ is an algebraic domain whose base is the image of $\mathbf{P}$ under '$\downarrow$'. Thus, the base of $\mathcal{I}\mathbf{P}$ is isomorphic to $\mathbf{P}$.

**Proof:** For all $x$ in $\mathbf{P}$, the cone $\downarrow x$ is an ideal since it is a lower set and directed because every subset of $\downarrow x$ has the upper bound $x \in \downarrow x$. The mapping $\downarrow : \mathbf{P} \to \mathcal{I}\mathbf{P}$ is a poset embedding by Prop. 2.2.5. The poset $\mathcal{I}\mathbf{P}$ is directed complete by Lemma 3.1.5; the lub of a directed set of ideals is its union.

Let $B = \{\downarrow x \mid x \in \mathbf{P}\} \subseteq \mathcal{I}\mathbf{P}$. We first show that all members of $B$ are isolated. Let $\mathcal{D}$ be a directed set of ideals, and $\downarrow x \leq \bigsqcup \mathcal{D}$, i.e. $x \in \downarrow x \subseteq \bigsqcup \mathcal{D}$. Then, there is $I$ in $\mathcal{D}$ such that $x \in I$. Since $I$ is a lower set, $\downarrow x \subseteq I$ holds.

Next, we show that all ideals are directed lubs of members of $B$. Let $I$ be an ideal. Then $I = \bigcup_{i \in I}\{i\} \subseteq \bigcup_{i \in I} \downarrow i \subseteq I$ holds, since $I$ is a lower set. Because it is directed, $\bigcup_{i \in I} \downarrow i$ is a directed lub of members of $B$.

By Prop. 6.2.6, $\mathcal{I}\mathbf{P}$ is algebraic with base $B$. □

## 6.6 The way-below relation

The class of algebraic domains is generalized to the class of continuous domains by generalizing the concept of isolated points to the way-below relation. We say that a point $x$ is *way-below* or *essentially below* a point $y$ iff every computation converging to a point above $y$ must eventually produce partial results above $x$.

**Definition 6.6.1** A point $x$ in a domain $\mathbf{X}$ is *way-below* a point $y$, written $x \ll y$, iff for all directed sets $D$ of $\mathbf{X}$, $y \le \bigsqcup D$ implies $x \in \downarrow D$.

For points $x$ in $\mathbf{X}$, we write $\Downarrow x = \{y \mid y \ll x\}$ and $\Uparrow x = \{y \mid x \ll y\}$.

By comparing the definitions, it is obvious that a point $x$ is isolated iff $x \ll x$ holds. We start by proving some simple properties of '$\ll$'.

**Proposition 6.6.2** $x \ll y$ implies $x \le y$. Hence, $\Downarrow x \subseteq \downarrow x$ and $\Uparrow x \subseteq \uparrow x$.

**Proof:** Let $D = \{y\}$. $D$ is a directed set with lub $y$. $x \ll y$ implies there is a point $d$ in $D$ such that $x \le d$. $d = y$ implies $x \le y$. □

**Proposition 6.6.3** If $x \le y$ and $y \ll z$, then $x \ll z$ follows. If $x \ll y$ and $y \le z$, then $x \ll z$ also follows. Hence, '$\ll$' is transitive, $\Downarrow x$ is lower, $\Uparrow x$ is upper, $x \le y$ implies $\Downarrow x \subseteq \Downarrow y$ and $\Uparrow x \supseteq \Uparrow y$.

**Proof:**

(1) $z \le \bigsqcup D$ implies $y \in \downarrow D$, whence $x \in \downarrow D$.

(2) $z \le \bigsqcup D$ implies $y \le \bigsqcup D$, whence $x \in \downarrow D$.

The statements about '$\Downarrow$' and '$\Uparrow$' are direct conclusions. □

**Proposition 6.6.4** If $\mathbf{X}$ has a least element $\perp$, then $\perp \ll x$ holds for all $x$ in $\mathbf{X}$.

**Proof:** For every directed set $D$, $\perp \in \downarrow D$ holds since $D$ is not empty. □

## 6.7 Continuous domains

In a continuous domain, every point is a directed limit of points that are way-below it.

**Definition 6.7.1** A domain $\mathbf{X}$ is *continuous* iff for all points $x$ of $\mathbf{X}$, there is a directed subset $D$ of $\Downarrow x$ with lub $x$. The class of continuous domains is denoted by CONT.

Continuous domains could also have been defined by a different condition.

**Proposition 6.7.2**
A domain $\mathbf{X}$ is continuous iff for all points $x$ of $\mathbf{X}$, the set $\Downarrow x$ is directed with lub $x$.

**Proof:** The implication from right to left is trivial. For the opposite implication, let $D$ be a directed subset of $\Downarrow x$ with lub $x$. We show that $D$ is cofinal in $\Downarrow x$. Then $\Downarrow x$ is directed with the same lub $x$ by Prop. 3.1.8.

We have to show $\Downarrow x \subseteq \downarrow D$. Let $u$ be a member of $\Downarrow x$. Then $u \ll x = \bigsqcup D$, whence $u \in \downarrow D$ by definition of '$\ll$'. □

**Proposition 6.7.3** Every algebraic domain is continuous.

**Proof:** In an algebraic domain, the set $\downarrow^0 x$ is directed with lub $x$ for every point $x$. We show $\downarrow^0 x \subseteq \Downarrow x$. If $a$ is in $\downarrow^0 x$, then $a \ll a \le x$ holds, whence $a \ll x$ by Prop. 6.6.3. □

In continuous domains, the way-below relation satisfies an interpolation property.

**Proposition 6.7.4 (Interpolation)** Let $\mathbf{X}$ be a continuous domain. For every two points $x$ and $z$ in $\mathbf{X}$ with $x \ll z$, there is a point $y$ in $\mathbf{X}$ such that $x \ll y \ll z$.

**Proof:** Since $\mathbf{X}$ is continuous, $\Downarrow z$ is directed and $z = \bigsqcup \Downarrow z$ holds. By Prop. 6.6.3 and continuity of $\mathbf{X}$, the set $\{\Downarrow a \mid a \in \Downarrow z\}$ is $\subseteq$-directed. By Prop. 3.1.5, its union $D$ is directed. By Prop. 2.4.8, the lub of $D$ equals $\bigsqcup_{a \in \Downarrow z} \bigsqcup \Downarrow a = \bigsqcup \Downarrow z = z$. Because of $x \ll z = \bigsqcup D$, there is some $d$ in $D$ such that $x \leq d$. $d$ is contained in $\Downarrow y$ for some $y$ in $\Downarrow z$. Hence, $x \leq d \ll y \ll z$ follows. □

The next results will finally lead to a topological characterization of domain continuity.

**Proposition 6.7.5** Let $\mathbf{X}$ be a continuous domain, $x$ a point of $\mathbf{X}$ and $D$ a directed subset of $\mathbf{X}$. Then $x \ll \bigsqcup D$ implies $x \ll d$ for some $d$ in $D$.

**Proof:** By interpolation, there is a point $y$ in $\mathbf{X}$ such that $x \ll y \ll \bigsqcup D$. By $y \ll \bigsqcup D$, there is some $d$ in $D$ such that $y \leq d$. Then $x \ll y$ implies $x \ll d$. □

The '$\Downarrow$' operator is continuous:

**Proposition 6.7.6**
For all directed sets $D$ in a continuous domain, $\Downarrow(\bigsqcup D) = \bigcup_{d \in D} \Downarrow d$ holds.

**Proof:** By Prop. 6.6.3, $d \leq \bigsqcup D$ implies $\Downarrow d \subseteq \Downarrow(\bigsqcup D)$, whence '$\supseteq$' holds. Conversely, if $x$ is in $\Downarrow(\bigsqcup D)$, then $x \ll \bigsqcup D$, whence $x \ll d$ for some $d$ in $D$ by Prop. 6.7.5. □

**Proposition 6.7.7** Let $\mathbf{X}$ be a continuous domain. For every $x$ in $\mathbf{X}$, the set $\Uparrow x$ is open.

**Proof:** $\Uparrow x$ is upper by Prop. 6.6.3. By Prop. 6.7.5, it is d-open. □

The way-below relation may be characterized topologically.

**Proposition 6.7.8** In a continuous domain $\mathbf{X}$, $x \ll y$ holds iff there is an open set $O$ such that $\uparrow x \supseteq O \ni y$. The implication from right to left holds in every domain.

**Proof:** Assume there is an open set $O$ such that $\uparrow x \supseteq O \ni y$. Let $D$ be directed such that $y \leq \bigsqcup D$. $y \in O$ implies $\bigsqcup D$ in $O$. Since $O$ is open, there is some $d$ in $D$ contained in $O$. $d \in O \subseteq \uparrow x$ implies $x \leq d$.

Conversely, assume $x \ll y$ holds in a continuous domain. Then $O = \Uparrow x$ is open by Prop. 6.7.7. $x \ll y$ implies $y \in \Uparrow x = O$. $O = \Uparrow x \subseteq \uparrow x$ holds by Prop. 6.6.2. □

Now, we are ready for the topological characterization of domain continuity.

**Theorem 6.7.9** A domain $\mathbf{X}$ is continuous iff it has the 'local upper cone' property: for every point $x$ in every open set $O$, there are an open set $O'$ and a point $x'$ such that $x \in O' \subseteq \uparrow x' \subseteq O$.

**Proof:** Assume first $\mathbf{X}$ is continuous. Let $x$ be a point in an open set $O$. $\bigsqcup \Downarrow x = x \in O$ implies there is some $x'$ in $\Downarrow x$ contained in $O$. $x' \in O$ implies $\uparrow x' \subseteq O$. $x' \in \Downarrow x$ implies $x' \ll x$, whence there is an open set $O'$ by Prop. 6.7.8 such that $x \in O' \subseteq \uparrow x'$.

Conversely, let $\mathbf{X}$ be a domain satisfying the topological property. Let $x$ be a point of $\mathbf{X}$. We have to provide a directed subset $D$ of $\Downarrow x$ with lub $x$.

Let $D$ be the set of all points $u$ in $\mathbf{X}$ such that there is an open set $O_u$ with $x \in O_u \subseteq \uparrow u$. By the direction of Prop. 6.7.8 that holds for all domains, $D$ is a subset of $\Downarrow x$. Since $\mathbf{X}$ itself is open, $x \in \mathbf{X}$ implies there are a point $y$ and an open set $O_y$ such that $x \in O_y \subseteq \uparrow y$. Thus, $D$ is not empty. Let $u$ and $v$ be two points of $D$. We look for a common upper bound of $u$ and $v$ in $D$. $u$ and $v$ in $D$ implies there are open sets $O_u$ and $O_v$ such that $x \in O_u \subseteq \uparrow u$ and $x \in O_v \subseteq \uparrow v$. Then $x \in O_u \cap O_v$ follows, whence by precondition there are a point $w$ and an open set $O_w$ such that $x \in O_w \subseteq \uparrow w \subseteq O_u \cap O_v$. Then $w$ is in $D$, and $\uparrow w \subseteq O_u \subseteq \uparrow u$ implies $w \geq u$. Similarly, $w \geq v$ follows.

Now, we know $D$ is directed. We have to show $\bigsqcup D = x$. For every $d$ in $D$, $x \in O_d \subseteq \uparrow d$ holds, whence $d \leq x$. $\bigsqcup D \leq x$ follows. For the opposite relation, we apply Prop. 4.3.1. Let $x$ be a point of an open set $O$. We have to show $\bigsqcup D \in O$. By the precondition, there are a point $y$ and an open set $O_y$ such that $x \in O_y \subseteq \uparrow y \subseteq O$. $y$ in $D$ and $y$ in $O$ follow. By $y \leq \bigsqcup D$, $\bigsqcup D \in O$ follows. □

## 6.8 Further properties and examples

In this section, we present some further properties of continuous domains, and provide some examples for continuous and non-continuous domains.

We start by a sample domain that is continuous without being algebraic. Let $\mathbf{X} = [0..1]$ be the unit interval of the real line ordered by the usual order such that 0 is least and 1 is greatest. $\mathbf{X}$ forms a complete domain since every subset has a supremum. In $\mathbf{X}$, $x \ll y$ holds iff $x < y$ or $x = y = 0$. $\mathbf{X}$ is continuous since $x = \bigsqcup\{y \mid y < x\}$ holds for all $x > 0$. It is however not algebraic because it contains only one isolated point, namely 0, and this single point is not capable to generate all domain points by directed lubs.

In section 5.3, we presented a complete domain whose binary infimum operation is not continuous. It was $\mathbf{X} = \{p, 0, 1, 2, \ldots, \infty\}$ where $0 < 1 < 2 < \cdots < \infty$ and $0 < p < \infty$ holds. This domain cannot be continuous because of the following proposition:

**Proposition 6.8.1** Let $\mathbf{X}$ be a continuous domain, where every two points $x$ and $y$ have a greatest lower bound $x \sqcap y$. Then $\sqcap : \mathbf{X} \times \mathbf{X} \to \mathbf{X}$ is continuous.

**Proof:** By Prop. 3.3.6 and commutativity of '$\sqcap$', we only have to show $a \sqcap \bigsqcup D = \bigsqcup_{d \in D}(a \sqcap d)$. Let the point to the left be $x$ and the point to the right be $y$. For every $d$ in $D$, $a \sqcap d \leq a \sqcap \bigsqcup D$ holds, whence $x \geq y$ follows.

For the opposite relation $x \leq y$, we apply Prop. 4.3.1. Let $x \in O$ for some open set $O$. We have to show $y \in O$.

By Th. 6.7.9, there are a point $x'$ and an open set $O'$ such that $x \in O' \subseteq \uparrow x' \subseteq O$. $x = a \sqcap \bigsqcup D$ in $O'$ implies $a \in O'$ and $\bigsqcup D \in O'$ since $O'$ is upper. Because $O'$ is open, some point $d$ of $D$ is in $O'$. Then $a, d \in O' \subseteq \uparrow x'$ implies $a, d \geq x'$, whence $a \sqcap d \geq x'$. Thus, $a \sqcap d \in \uparrow x' \subseteq O$ follows, whence $y$ is in $O$ because it is above $a \sqcap d$. □

In section 8.4, we shall prove that the classes ALG and CONT are closed w.r.t. finite products. They are however not closed w.r.t. function domain forming. The example for this is prepared by the following proposition.

**Proposition 6.8.2** All minimal points in a continuous domain are isolated.

**Proof:** Let $x$ be a minimal point in a continuous domain $\mathbf{X}$. Then $\Downarrow x$ is directed and in particular not empty. Let $y$ be a point of $\Downarrow x$. Then $y \ll x$ holds whence $y \leq x$ by Prop. 6.6.2. Since $x$ is minimal, $y = x$ follows. Thus, $x \ll x$ holds, i.e. $x$ is isolated. $\square$

Let $\mathbf{X} = \{[a_1, b_1] < [a_2, b_2] < \cdots < \infty\}$:

$$
\begin{tikzcd}
a_1 \arrow[r] \arrow[dr] & a_2 \arrow[r] \arrow[dr] & a_3 & \\
 & & & \cdots \quad \infty \\
b_1 \arrow[r] \arrow[ur] & b_2 \arrow[r] \arrow[ur] & b_3 &
\end{tikzcd}
$$

We already met this domain in section 5.4 as an example of a domain $\mathbf{X}$ in class FC where the function domain $[\mathbf{X} \to \mathbf{X}]$ is not even contained in the larger class KC. Now, we shall show that $\mathbf{X}$ is algebraic whereas $[\mathbf{X} \to \mathbf{X}]$ is not even continuous. Algebraicity of $\mathbf{X}$ is proved easily; all points except $\infty$ are isolated.

The identity function $id : [\mathbf{X} \to \mathbf{X}]$ is minimal in $[\mathbf{X} \to \mathbf{X}]$. Let $f$ be a function below $id$. Then $f(a_1) \leq a_1$, whence $f(a_1) = a_1$. Similarly, $f(b_1) = b_1$ holds. Assume as an induction hypothesis $f(a_n) = a_n$ and $f(b_n) = b_n$ hold. Then $f(a_{n+1}) \geq f(a_n) = a_n$ and $f(a_{n+1}) \geq f(b_n) = b_n$ hold by monotonicity, and $f(a_{n+1}) \leq a_{n+1}$ since $f \leq id$. These relations imply $f(a_{n+1}) = a_{n+1}$. Similarly, $f(b_{n+1}) = b_{n+1}$ is shown. By induction, we obtain that $f$ and $id$ coincide for all points $a_n$ and $b_n$. By monotonicity, $f(\infty) \geq f(a_n) = a_n$ holds for all $n$, whence $f(\infty) = \infty$.

Let $D$ be the set of all constant functions $\lambda x.\, c$ where $c < \infty$. $D$ is easily shown to be directed. Its lub is the function $\lambda x.\, \infty$. The identity function is below $\lambda x.\, \infty$, but not below any function in $D$. Thus, $id$ is not isolated.

The identity is a minimal element of $[\mathbf{X} \to \mathbf{X}]$ that is not isolated. By Prop. 6.8.2, $[\mathbf{X} \to \mathbf{X}]$ is not continuous.

# Chapter 7

# Functions and domain classes

In section 7.1, we introduce two domain classes, the finitely algebraic (also: bifinite, profinite, or strongly algebraic) and the finitely continuous domains, that are defined by properties of the function domain. The former class was studied in [Gun87], whereas I learnt about the latter class from [Law88]. These two classes are inserted in the hierarchy of the previous two chapters in sections 7.1 and 7.2. In section 7.3, their Cartesian closedness is shown.

In section 7.4, we define functors and show that locally continuous functors that map finite domains to finite domains, also preserve finite algebraicity and finite continuity. In section 7.5, we show that arbitrary functors that preserve algebraicity also preserve continuity of domains. The proof is done by using retracts that provide a close relationship between algebraic and continuous domains. I found the notion of retracts in [Law88].

## 7.1 Finite algebraicity and finite continuity

The classes of finitely algebraic and of finitely continuous domains are defined by approximations of the identity function.

**Definition 7.1.1** Let $\mathbf{X}$ be a domain. A function $f : [\mathbf{X} \to \mathbf{X}]$ is a *deflation* iff $f \leq id$ holds and $f$ has a finite image $f[\mathbf{X}]$. A function $f : [\mathbf{X} \to \mathbf{X}]$ is *idempotent* iff $f \circ f = f$.

$\mathbf{X}$ is *finitely continuous* iff the identity function $id : [\mathbf{X} \to \mathbf{X}]$ is the lub of a directed set $\mathcal{D}$ of deflations. $\mathbf{X}$ is *finitely algebraic* (or: bifinite, or: profinite) iff the identity is the lub of a directed set $\mathcal{D}$ of idempotent deflations.

The class of all finitely algebraic domains is called F-ALG, and the class of finitely continuous domains is denoted by F-CONT.

Obviously, F-ALG is a subclass of F-CONT by definition. The names finitely *algebraic* and *continuous* suggest that F-ALG is a subclass of ALG, and F-CONT of CONT. This is indeed true as we shall see soon.

**Proposition 7.1.2** Every finite poset is a finitely algebraic domain.

**Proof:** Every finite poset is a domain by Prop. 6.3.1. If $\mathbf{X}$ is finite, then the identity $id : [\mathbf{X} \to \mathbf{X}]$ has finite image. Thus, $\mathcal{D} = \{id\}$ is a directed set of idempotent deflations, whose lub is $id$. □

The next Lemma holds for arbitrary domains, not only for finitely continuous ones.

**Lemma 7.1.3** Let $\mathbf{X}$ be a domain, and $f : [\mathbf{X} \to \mathbf{X}]$ a deflation. Then $fx \ll x$ holds for all $x$ in $\mathbf{X}$. If $f$ is furthermore idempotent, then $fx$ is isolated for all $x$ in $\mathbf{X}$.

**Proof:** We first show $fx \ll x$. Let $D$ be a directed set such that $x \leq \bigsqcup D$. Then $fx \leq f(\bigsqcup D) = \bigsqcup f[D]$ follows by continuity of $f$. $f[D]$ is a *finite* directed set, whence $\bigsqcup f[D] \in f[D]$ by Prop. 3.1.2. Thus, there is $d$ in $D$ such that $\bigsqcup f[D] = fd$. Hence, $fx \leq \bigsqcup f[D] = fd \leq d$ holds for some $d$ in $D$.

If $f$ is idempotent, then for all $x$ in $\mathbf{X}$, $fx = f(fx) \ll fx$ holds as we just proved. Thus, $fx$ is isolated. □

Now we are ready to show the inclusions of F-ALG in ALG and F-CONT in CONT.

**Proposition 7.1.4** Every finitely algebraic domain is algebraic, and every finitely continuous domain is continuous.

**Proof:** Let $\mathcal{D}$ be the directed set of deflations approximating identity. For all points $x$, $x = id\,x = (\bigsqcup \mathcal{D})x = \bigsqcup\{fx \mid f \in \mathcal{D}\}$ holds by Prop. 3.4.2. The set $\{fx \mid f \in \mathcal{D}\}$ is directed and has lub $x$. Lemma 7.1.3 shows that it is a subset of $\Downarrow x$ in case of F-CONT, and of $\downarrow^0 x$ in case of F-ALG. □

## 7.2 Relations to completeness

By Prop. 7.1.4, class F-ALG is a subclass of ALG. In this section, we investigate its position inside ALG more closely. BC & ALG is a subclass of F-ALG which in turn is a subclass of SC & ALG. Analogous inclusions hold for F-CONT.

**Theorem 7.2.1** Every bounded complete algebraic domain is finitely algebraic. Every bounded complete continuous domain is finitely continuous.

**Proof:** We start by showing the statement about F-CONT. Let $\mathbf{X}$ be a bounded complete continuous domain. Then $\mathbf{X}$ has a least element $\bot$, and $[\mathbf{X} \to \mathbf{X}]$ is also bounded complete. Let $\mathcal{D}$ be the set of all deflations $f : [\mathbf{X} \to \mathbf{X}]$. In particular, $\mathcal{D}$ contains $\lambda x.\,\bot$ whose image is $\{\bot\}$. If $f$ and $g$ are two members of $\mathcal{D}$, then they are bounded by $id$, whence $h = f \sqcup g \leq id$ exists in $[\mathbf{X} \to \mathbf{X}]$. By Prop. 3.4.2, $hx = fx \sqcup gx$ holds for all $x$ in $\mathbf{X}$, whence $h[\mathbf{X}] = \{u \sqcup v \mid u \in f[\mathbf{X}],\, v \in g[\mathbf{X}]\}$ is finite. Hence, $h$ is in $\mathcal{D}$. Thus, we showed $\mathcal{D}$ is directed.

Since all members of $\mathcal{D}$ are below identity, $\bigsqcup \mathcal{D} \leq id$ obviously holds. For all points $a$ in $\mathbf{X}$, we define an auxiliary function $f_a$ mapping $\Uparrow a$ to $a$ and all other points to $\bot$. $f_a$ is continuous by Lemma 3.7.1 since $\Uparrow a$ is open by Prop. 6.7.7. $f_a$ is below identity since $a$ is below all members of $\Uparrow a$. Finally, $f_a[\mathbf{X}] \subseteq \{\bot, a\}$ holds. Thus, $f_a$ is in $\mathcal{D}$ for all $a$ in $\mathbf{X}$.

By continuity of $\mathbf{X}$, we obtain $id\,x = x = \bigsqcup \Downarrow x = \bigsqcup\{a \mid x \in \Uparrow a\} = \bigsqcup\{f_a x \mid x \in \Uparrow a\} \leq \bigsqcup\{fx \mid f \in \mathcal{D}\} = (\bigsqcup \mathcal{D})x$ whence $id \leq \bigsqcup \mathcal{D}$.

Now, we turn to the algebraic case. Here, we define $\mathcal{D}$ to contain all idempotent deflations in $[\mathbf{X} \to \mathbf{X}]$. $\lambda x.\perp$ is idempotent, whence in $\mathcal{D}$. If both $f$ and $g$ are idempotent and below identity, then $f \sqcup g$ is also idempotent: $(f \sqcup g) \circ (f \sqcup g) \leq id \circ (f \sqcup g) = f \sqcup g$ holds, and conversely $(f \sqcup g) \circ (f \sqcup g) \geq f \circ f = f$ and also $(f \sqcup g) \circ (f \sqcup g) \geq g \circ g = g$, whence $(f \sqcup g) \circ (f \sqcup g) \geq f \sqcup g$. Thus, $\mathcal{D}$ is directed.

Again, $\bigsqcup \mathcal{D} \leq id$ holds obviously, and for the opposite direction, we define some auxiliary functions. For every isolated point $b$, let $g_b$ be the function mapping the open set $\uparrow b$ to $b$ and all other points to $\perp$. In particular, this function maps $b$ to $b$ and $\perp$ to $\perp$, whence it is idempotent. Thus, it is in $\mathcal{D}$.

By algebraicity of $\mathbf{X}$, we obtain $id\,x = x = \bigsqcup \downarrow^0 x = \bigsqcup \{b \mid b \in \mathbf{X}^0,\ x \in \uparrow b\} = \bigsqcup \{g_b\,x \mid b \in \mathbf{X}^0,\ x \in \uparrow b\} \leq \bigsqcup \{fx \mid f \in \mathcal{D}\} = (\bigsqcup \mathcal{D})x$ whence $id \leq \bigsqcup \mathcal{D}$. $\square$

Since there are finite domains which are not bounded complete, the inclusion of BC & ALG in F-ALG is proper.

**Theorem 7.2.2** Every finitely continuous domain is in SC. Hence, F-ALG is a subclass of SC & ALG, and F-CONT a subclass of SC & CONT.

**Proof:** We have to show that the whole domain $\mathbf{X}$ is finitary and that for every two points $a$ and $b$, the set of upper bounds $\uparrow a \cap \uparrow b$ is strongly compact.

The directed set $\mathcal{D}$ approximating identity is not empty. Let $f$ be a member of $\mathcal{D}$. The set $E = f[\mathbf{X}]$ is finite. For all $x$ in $\mathbf{X}$, $fx \leq x$ holds, whence $\mathbf{X} \subseteq \uparrow E$. Thus, $\mathbf{X}$ is finitary.

Let $a$ and $b$ be two points and $U = \uparrow a \cap \uparrow b$. For all $f$ in $\mathcal{D}$, the image $f[U]$ is finite. For all $x$ in $U$ and $f$ in $\mathcal{D}$, $x \geq fx \in f[U]$ holds, whence $U \subseteq \uparrow f[U]$ for all $f$ in $\mathcal{D}$. Thus, we obtain $U \subseteq \bigcap \{\uparrow f[U] \mid f \in \mathcal{D}\}$.

Next, we show the opposite inclusion. Let $y$ be a point of the intersection. Then for all $f$ in $\mathcal{D}$, there is some $u$ in $U$ such that $y \geq fu$. By $u \geq a$, $y \geq fa$ follows for all $f$ in $\mathcal{D}$, whence $y \geq \bigsqcup_{f \in \mathcal{D}} fa = a$. Similarly, $y \geq b$ holds. Together, this means $y \in U$.

Let $f \leq g$. Then for $y \in g[U]$, $y \geq gu \geq fu \in f[U]$ holds for some $u$ in $U$. Hence, $f \leq g$ implies $\uparrow f[U] \supseteq \uparrow g[U]$, whence the set $\{\uparrow f[U] \mid f \in \mathcal{D}\}$ is $\supseteq$-directed as $\mathcal{D}$ is directed. Thus, $U$ is a $\supseteq$-directed intersection of finitary upper sets, whence it is strongly compact by Prop. 4.7.8. $\square$

The domain $\mathbf{X} = \{[a_1, b_1] < [a_2, b_2] < \cdots < \infty\}$, which already occurred in sections 5.4 and 6.8, is in SC & ALG — even in FC & ALG. It is however not finitely algebraic since the identity is minimal in it as shown in section 6.8. Hence, the identity cannot be the lub of a directed set of functions with finite image.

## 7.3 Cartesian closedness

In this section, we show that the classes F-ALG and F-CONT are Cartesian closed — in contrast to ALG and CONT. Let $\mathbf{X}$ and $\mathbf{Y}$ be two domains in F-CONT whose identities are approximated by $\mathcal{D}_1$ and $\mathcal{D}_2$ respectively. Then let $\mathcal{D} = \{f \times g \mid f \in \mathcal{D}_1,\ g \in \mathcal{D}_2\}$ where '$\times$' is the Cartesian combinator of Prop. 3.4.7. It is defined by $(f \times g)(x, y) = (fx, gy)$.

The members of $\mathcal{D}$ have the correct typing $[\mathbf{X} \times \mathbf{Y} \to \mathbf{X} \times \mathbf{Y}]$. By continuity of '$\times$', $\bigsqcup \mathcal{D} = \bigsqcup \mathcal{D}_1 \times \bigsqcup \mathcal{D}_2 = id_{\mathbf{X}} \times id_{\mathbf{Y}} = id_{\mathbf{X}\times\mathbf{Y}}$ holds as required.

All members of $\mathcal{D}$ have finite image since $(f \times g)[\mathbf{X} \times \mathbf{Y}] \subseteq f[\mathbf{X}] \times g[\mathbf{Y}]$. This shows that $\mathbf{X}\times\mathbf{Y}$ is in F-CONT again. If $f$ and $g$ are idempotent functions, then $f \times g$ is also idempotent since $(f \times g) \circ (f \times g) = (f \circ f) \times (g \circ g)$. Thus, F-ALG is also closed w.r.t. product.

Now turning to the function domain, we construct $\mathcal{D} = \{f \to g \mid f \in \mathcal{D}_1,\ g \in \mathcal{D}_2\}$ where $f \to g = \lambda h. g \circ h \circ f$. If $f : [\mathbf{X} \to \mathbf{X}]$ and $g : [\mathbf{Y} \to \mathbf{Y}]$, then $f \to g$ maps functions in $[\mathbf{X} \to \mathbf{Y}]$ into functions in $[\mathbf{X} \to \mathbf{Y}]$. Since composition is a continuous operation (Prop. 3.4.6), $\mathcal{D}$ approximates the identity of $[\mathbf{X} \to \mathbf{Y}]$: $(\bigsqcup \mathcal{D})h = (\bigsqcup \mathcal{D}_2) \circ h \circ (\bigsqcup \mathcal{D}_1) = id \circ h \circ id = h$ holds for all $h$ in $[\mathbf{X} \to \mathbf{Y}]$.

Next, we have to show that $f \to g$ has finite image. For all functions $h$, the function $(f \to g)h = g \circ h \circ f$ maps $\mathbf{X}$ first to the finite set $f[\mathbf{X}]$, then to $h[f[\mathbf{X}]] \subseteq \mathbf{Y}$, and finally into the finite set $g[\mathbf{Y}]$. Hence, only the restriction of $g \circ h$ to the finite function space $f[\mathbf{X}] \to g[\mathbf{Y}]$ matters, and $(f \to g)[\mathbf{X} \to \mathbf{Y}] \subseteq \{\varphi \circ f \mid \varphi : [f[\mathbf{X}] \to g[\mathbf{Y}]]\}$ is finite. This shows F-CONT is closed w.r.t. function domain forming.

Finally, we show that $f \to g$ is idempotent whenever $f$ and $g$ are idempotent:
$(f \to g)((f \to g)\, h) = (f \to g)(g \circ h \circ f) = g \circ (g \circ h \circ f) \circ f = g \circ h \circ f = (f \to g)h$
This proves that F-ALG is also closed w.r.t. function domains.

**Theorem 7.3.1** The classes CC & ALG, BC & ALG, F-ALG, CC & CONT, BC & CONT, and F-CONT are all Cartesian closed.

**Proof:** For F-ALG and F-CONT, this was shown above. Since BC & ALG is a subclass of F-ALG by Th. 7.2.1, it coincides with the class BC & F-ALG. This class is the intersection of the two Cartesian closed classes BC (Prop. 5.2.4 and 5.2.5) and F-ALG, and hence Cartesian closed itself. The proof for the remaining classes is analogous. □

## 7.4 Functors

In this section, we define functors for domains as an instance of a more general categorical concept. We show that functors which preserve finiteness also preserve finite algebraicity and finite continuity. In the next section, we shall show that functors preserving ALG also preserve CONT.

**Definition 7.4.1** A *functor* $\mathcal{F}$ defined on a class C of domains maps every domain $\mathbf{X}$ in C to a domain $\mathcal{F}\mathbf{X}$, and also maps every function $f : [\mathbf{X} \to \mathbf{Y}]$ where $\mathbf{X}$ and $\mathbf{Y}$ are in C into a function $\mathcal{F}f : [\mathcal{F}\mathbf{X} \to \mathcal{F}\mathbf{Y}]$ such that the two functorial properties hold:

(1) For every $\mathbf{X}$ in C, $\mathcal{F}\, id_{\mathbf{X}} = id_{\mathcal{F}\mathbf{X}}$ holds.

(2) For every domains $\mathbf{X}$, $\mathbf{Y}$, and $\mathbf{Z}$ in C and functions $f : [\mathbf{X} \to \mathbf{Y}]$ and $g : [\mathbf{Y} \to \mathbf{Z}]$, the equality $\mathcal{F}(g \circ f) = \mathcal{F}g \circ \mathcal{F}f$ holds in $[\mathcal{F}\mathbf{X} \to \mathcal{F}\mathbf{Z}]$.

The functor $\mathcal{F}$ is *locally continuous* iff the mapping $\mathcal{F} : [\mathbf{X} \to \mathbf{Y}] \to [\mathcal{F}\mathbf{X} \to \mathcal{F}\mathbf{Y}]$ is continuous for all domains $\mathbf{X}$ and $\mathbf{Y}$ in C.

We already met a functor, namely ideal completion $\mathcal{I}$ where $\mathcal{I}f(A) = \downarrow f[A]$ for all ideals $A$. $\mathcal{I}$ is not locally continuous. The main field of application of the theory developed below is however the theory of power constructions: every power construction is a locally continuous functor.

**Definition 7.4.2** A functor $\mathcal{F}$ is said to map a class $\mathrm{C}_1$ into a class $\mathrm{C}_2$ iff it is defined on a superclass of $\mathrm{C}_1$, and $\mathcal{F}\mathbf{X}$ is in $\mathrm{C}_2$ for every domain $\mathbf{X}$ in $\mathrm{C}_1$.

A functor $\mathcal{F}$ is said to *preserve* a class C iff it maps C into C.

Our goal is to provide a criterion for functors to preserve the classes F-ALG and F-CONT. The first step is to consider the behavior of a functor operating on functions with finite image.

**Proposition 7.4.3** Let $\mathcal{F}$ be a functor which preserves the class FIN of finite domains and is defined on a superclass C of FIN. Then for all domains $\mathbf{X}$ and $\mathbf{Y}$ in C holds: If $f : [\mathbf{X} \to \mathbf{Y}]$ is a morphism with finite image $f[\mathbf{X}]$, then $\mathcal{F}f : [\mathcal{F}\mathbf{X} \to \mathcal{F}\mathbf{Y}]$ has finite image again.

**Proof:** Let $\mathbf{Z} = f[\mathbf{X}]$ be the image of $f$. It is a domain in the order inherited from $\mathbf{Y}$ since it is finite, and the embedding $e : [\mathbf{Z} \to \mathbf{Y}]$ is continuous. The original morphism $f$ may be corestricted to $f' : [\mathbf{X} \to \mathbf{Z}]$ such that $f = e \circ f'$. Then $\mathcal{F}f = \mathcal{F}(e \circ f') = \mathcal{F}e \circ \mathcal{F}f'$ follows. $\mathcal{F}f'$ maps from $\mathcal{F}\mathbf{X}$ to $\mathcal{F}\mathbf{Z}$, and the latter is finite as $\mathcal{F}$ preserves FIN. Thus, the image of $\mathcal{F}f$ is finite because $(\mathcal{F}f)[\mathcal{F}\mathbf{X}] \subseteq (\mathcal{F}e)[\mathcal{F}\mathbf{Z}]$. □

The claim of the Proposition is needed to prove the following theorem:

**Theorem 7.4.4** Let $\mathcal{F}$ be a locally continuous functor defined on a superclass of F-CONT. If $\mathcal{F}$ preserves FIN, then it also preserves F-ALG and F-CONT.

**Proof:** Let $\mathbf{X}$ be a finitely continuous domain. Then, there is a directed set $\mathcal{D}$ of functions from $\mathbf{X}$ to $\mathbf{X}$ with finite image such that $\bigsqcup \mathcal{D} = id$. By local continuity of $\mathcal{F}$ and Prop. 7.4.3, $\mathcal{F}[\mathcal{D}]$ is a directed set of functions from $\mathcal{F}\mathbf{X}$ to $\mathcal{F}\mathbf{X}$ with finite image, whose lub is $\bigsqcup \mathcal{F}[\mathcal{D}] = \mathcal{F}(\bigsqcup \mathcal{D}) = \mathcal{F}\, id = id$. Thus, $\mathcal{F}\mathbf{X}$ is finitely continuous again.

If $\mathbf{X}$ is finitely algebraic, there is the additional condition $f \circ f = f$ for all $f$ in $\mathcal{D}$. It implies $\mathcal{F}f \circ \mathcal{F}f = \mathcal{F}(f \circ f) = \mathcal{F}f$, whence $\mathcal{F}\mathbf{X}$ is finitely algebraic again. □

## 7.5 Retracts

In this section, we want to prove that a functor preserving ALG also preserves CONT. The technical means for performing the proof are retracts.

**Definition 7.5.1** A domain $\mathbf{Y}$ is a *retracts* of a domain $\mathbf{X}$ if there are continuous functions $r : [\mathbf{X} \to \mathbf{Y}]$ and $e : [\mathbf{Y} \to \mathbf{X}]$ such that $r \circ e = id_{\mathbf{Y}}$.

A class C is *retract closed* if all retracts of all domains in C are in C again.

A factor domain $\mathbf{X}$ is a retract of a product $\mathbf{X} \times \mathbf{Y}$ by means of $e = \lambda x.\,(x,\, p)$ and $r = \pi_1$ where $p$ is an arbitrary point of $\mathbf{Y}$. Similarly, $\mathbf{Y}$ is a retract of $\mathbf{X} \times \mathbf{Y}$. A domain $\mathbf{Y}$ is also retract of $[\mathbf{X} \to \mathbf{Y}]$ by means of $e = \lambda y.\, \lambda x.\, y$ and $r = \lambda f.\, fp$ where $p$ is an arbitrary point of $\mathbf{X}$.

We already know several retract closed domain classes.

**Proposition 7.5.2** The class $\mathrm{DOM}_\perp$ of domains with least element, the class of finitary domains, the completeness classes CC, BC, FC, SC, and KC, the class DIS of discrete domains, the class FIN of finite domains, and the classes F-CONT and CONT are retract closed.

In connection with the remarks above, we get a host of corollaries, e.g. if $\mathbf{X} \times \mathbf{Y}$ is continuous, then $\mathbf{X}$ and $\mathbf{Y}$ are continuous, if $[\mathbf{X} \to \mathbf{Y}]$ is continuous, then so is $\mathbf{Y}$, etc.

**Proof:** Let always $\mathbf{Y}$ be a retract of $\mathbf{X}$ by means of $r : [\mathbf{X} \to \mathbf{Y}]$ and $e : [\mathbf{Y} \to \mathbf{X}]$. Assuming that $\mathbf{X}$ is in the class under consideration, we have to show that $\mathbf{Y}$ is also in it.

Least element: Let $\perp$ be the least element of $\mathbf{X}$. For all $y$ in $\mathbf{Y}$, $\perp \leq ey$ holds, whence $r(\perp) \leq r(ey) = y$ holds in $\mathbf{Y}$.

Finitary: Let $M$ be a finite subset of $\mathbf{X}$ such that $\mathbf{X} = \uparrow M$. For all $y$ in $\mathbf{Y}$, there is some $m$ in $M$ such that $m \leq ey$ holds, whence $rm \leq r(ey) = y$ holds in $\mathbf{Y}$. Thus, we obtain $\mathbf{Y} = \uparrow r[M]$ where $r[M]$ is finite.

BC: Let $a$ and $b$ be two points of $\mathbf{Y}$ bounded by $u$. Then $ea$ and $eb$ are bounded by $eu$ in $\mathbf{X}$, whence $ea \sqcup eb$ exists. Let $c = r(ea \sqcup eb)$. $c$ is an upper bound of $a$ and $b$ since $ea \sqcup eb \geq ea$ implies $c \geq r(ea) = a$. $c$ is the least upper bound of $a$ and $b$ since $a, b \leq u$ implies $ea \sqcup eb \leq eu$, whence $c \leq r(eu) = u$.

CC: Analogous.

FC / SC / KC: $\mathbf{Y}$ is finitary as shown above. For two points $a$ and $b$ of $\mathbf{Y}$, we claim $\uparrow a \cap \uparrow b = r[\uparrow ea \cap \uparrow eb]$. If $u$ is a point of the left hand set, then $u \geq a, b$ holds, whence $eu \geq ea, eb$. By $u = r(eu)$, $u$ is then in the right hand set. Conversely, if $w$ is in $r[\uparrow ea \cap \uparrow eb]$, then $w = rv$ holds for some $v$ in $\uparrow ea \cap \uparrow eb$. $v \geq ea$ implies $w = rv \geq r(ea) = a$.

The set $\uparrow a \cap \uparrow b$ is finitary / strongly compact / compact as continuous image by $r$ of the finitary / strongly compact / compact set $\uparrow ea \cap \uparrow eb$.

DIS: $y \leq y'$ in $\mathbf{Y}$ implies $ey \leq ey'$ in $\mathbf{X}$. Since $\mathbf{X}$ is discrete, $ey = ey'$ follows, whence $y = r(ey) = r(ey') = y'$.

FIN: $e[\mathbf{Y}]$ is a subdomain of $\mathbf{X}$ that is isomorphic to $\mathbf{Y}$. $\mathbf{X}$ being finite implies $e[\mathbf{Y}]$ is finite.

F-CONT: Let $\mathcal{D}$ be a directed set of deflations approximating the identity of $\mathbf{X}$. We define $\mathcal{D}' = \{r \circ f \circ e \mid f \in \mathcal{D}\}$. By continuity of composition, $\mathcal{D}'$ is a directed set with lub $\bigsqcup \mathcal{D}' = r \circ (\bigsqcup \mathcal{D}) \circ e = r \circ id_{\mathbf{X}} \circ e = id_{\mathbf{Y}}$. All members of $\mathcal{D}'$ have finite image since $r[f[e[\mathbf{Y}]]] \subseteq r[f[\mathbf{X}]]$ is finite.

CONT: We apply the topological criterion of Th. 6.7.9. Let $y$ be a point of an open set $O$ in $\mathbf{Y}$. By $r(ey) = y \in O$, point $ey$ is in the open set $r^{-1}[O]$. By continuity of $\mathbf{X}$, there are an open set $O'$ and a point $x$ such that $ey \in O' \subseteq \uparrow x \subseteq r^{-1}[O]$. First, $y$ is in $e^{-1}[O']$. Second, every point $u$ of $e^{-1}[O']$ satisfies $eu \in O' \subseteq \uparrow x$, whence $eu \geq x$, whence in turn $u = r(eu) \geq rx$. Third, every point $v$ above $rx$ is in $O$ since $x$ is in $r^{-1}[O]$. Summarizing, we obtain $y \in e^{-1}[O'] \subseteq \uparrow rx \subseteq O$. $\square$

By the proposition above, any retract of any continuous domain is continuous. Hence, every retract of any algebraic domain is continuous. The converse of this statement also holds.

**Proposition 7.5.3** Every continuous domain $\mathbf{X}$ is a retract of $\mathcal{I}\mathbf{X}$ by $r : [\mathcal{I}\mathbf{X} \to \mathbf{X}]$, $r(A) = \bigsqcup A$, and $e : [\mathbf{X} \to \mathcal{I}\mathbf{X}]$, $ex = \Downarrow x$.

Thus, every continuous domain is a retract of some algebraic domain.

**Proof:** $\mathcal{I}\mathbf{X}$ is algebraic by Th. 6.5.3.

For every $x$ in $\mathbf{X}$, the set $\Downarrow x$ is lower by Prop. 6.6.3 and directed by Prop. 6.7.2. Hence, it is an ideal. Conversely, every ideal of $\mathbf{X}$ has a lub in $\mathbf{X}$ since $\mathbf{X}$ is a domain. This defines mapping $r$. By continuity of $\mathbf{X}$, $r(ex) = \bigsqcup \Downarrow x = x$ holds for all $x$ in $\mathbf{X}$.

For continuity of $e$, we have to show $e(\bigsqcup D) = \bigsqcup e[D]$ for all directed sets $D$. The left hand side equals $\Downarrow(\bigsqcup D)$, and the right hand side is $\bigcup_{d \in D} \Downarrow d$. Both sides are equal by Prop. 6.7.6.

For continuity of $r$, we have to show $r(\bigsqcup \mathcal{D}) = \bigsqcup r[\mathcal{D}]$ for all directed sets $\mathcal{D}$ of ideals. The left hand side equals $\bigsqcup \bigcup_{A \in \mathcal{D}} A$, and the right hand side is $\bigsqcup_{A \in \mathcal{D}} (\bigsqcup A)$. Both sides are equal by Prop. 2.4.8. □

Now, we return to functors.

**Proposition 7.5.4** Let $\mathcal{F}$ be a functor defined for two domains $\mathbf{X}$ and $\mathbf{Y}$. If $\mathbf{Y}$ is a retract of $\mathbf{X}$, then $\mathcal{F}\mathbf{Y}$ is a retract of $\mathcal{F}\mathbf{X}$.

**Proof:** $\mathbf{Y}$ is a retract of $\mathbf{X}$ if there are continuous $r : [\mathbf{X} \to \mathbf{Y}]$ and $e : [\mathbf{Y} \to \mathbf{X}]$ with $r \circ e = id_{\mathbf{Y}}$. Then $\mathcal{F}r \circ \mathcal{F}e = \mathcal{F}(r \circ e) = \mathcal{F}id_{\mathbf{Y}} = id_{\mathcal{F}\mathbf{Y}}$, whence $\mathcal{F}\mathbf{Y}$ is a retract of $\mathcal{F}\mathbf{X}$. □

By now, we are able to prove the announced preservation property for ALG and CONT.

**Proposition 7.5.5** Let $\mathcal{F}$ be a functor defined on a superclass of CONT. If $\mathcal{F}$ maps ALG to CONT, then it preserves CONT.

**Proof:** If $\mathbf{X}$ is continuous, then it is a retract of some algebraic domain $\mathbf{Y}$ by Prop. 7.5.3. Then, $\mathcal{F}\mathbf{X}$ is a retract of the continuous domain $\mathcal{F}\mathbf{Y}$ by Prop. 7.5.4, whence it is continuous by Prop. 7.5.2. □

By strengthening the pre-condition, we obtain:

**Proposition 7.5.6** Let $\mathcal{F}$ be a functor defined on a superclass of CONT. If $\mathcal{F}$ preserves ALG, then it also preserves CONT.

We shall meet retracts again in the theory of power constructions in section 14.4.

# Chapter 8

# Large domain classes

In this chapter, we introduce some domain classes that are much larger than the class CONT of continuous domains. We shall see that for these classes still powerful structural properties may be shown. All these classes are defined topologically. Hence, they would primarily provide a classification of (T0)-spaces. By employing the Scott topology as the standard topology for domains, the space classes induce however domain classes.

We mostly consider domains in this chapter, not topological spaces. The reason is that we sometimes apply Rudin's Lemma in the versions Lemma 3.9.3 and 3.9.4 that do not hold in arbitrary (T0)-spaces.

In sections 8.1 through 8.4, we introduce generalizations of algebraicity and continuity, namely multi-algebraicity and multi-continuity. A long time, I believed these classifications to be new. Multi-continuity as defined topologically in this thesis is however equivalent to the notion of quasicontinuity defined order-theoretically in [GLS83].

The topological notion of sobriety is closely related with upper power domain constructions as pointed out in [Smy83]. Some authors e.g. [Law88] define sobriety via irreducible closed sets (see section 8.5), whereas others e.g. [Smy78] define it via prime or open filters (see section 8.6). The equivalence of all three definitions is shown in [HM81] by means of lattice theory. We present the definitions together with new, more direct proofs of their equivalence in section 8.7. In section 8.8, we show that all multi-continuous domains are sober. This is probably a new result since multi-continuity was never considered topologically before.

In section 8.9, we introduce the novel classes K-RD and S-RD characterized by certain generalizations of Rudin's Lemma. All sober domains are in K-RD, and K-RD is a subclass of S-RD. These classes provide the maximal classes where the upper power domains in terms of (strongly) compact sets make sense as power domains.

In section 8.10, we prove a couple of interesting Lemmas that hold for domains in KC & K-RD as well as in SC & S-RD. Section 8.11 then presents Johnstone's non-sober domain published in [Joh81, Joh82]. This domain is also not in S-RD. Thus, we do not know any domain that would separate the classes SOB of sober domains, K-RD, and S-RD.

The final section 8.12 summarizes all domain classes introduced in the last four chapters.

## 8.1 Locality classes

The topological criteria for algebraicity (Th. 6.2.5) and continuity (Th. 6.7.9) deal with upper cones. They may easily be generalized by replacing the upper cones by finitary, strongly compact, or compact sets.

**Definition 8.1.1** Let $\mathbf{X}$ be a domain. $\mathbf{X}$ is *multi-algebraic* iff for all points $x$ in an open set $O$, there is a finitary open set $F$ such that $x \in F \subseteq O$.

$\mathbf{X}$ is *locally finitary* or *multi-continuous* iff for all points $x$ in an open set $O$, there are an open set $O'$ and a finitary set $F$ such that $x \in O' \subseteq F \subseteq O$.

$\mathbf{X}$ is *locally compact* iff for all points $x$ in an open set $O$, there are an open set $O'$ and a compact set $K$ such that $x \in O' \subseteq K \subseteq O$.

The class of all multi-algebraic domains is called M-ALG, and the class of multi-continuous domains is denoted by M-CONT. L-COMP is the class of locally compact domains.

In the definition of multi-algebraicity, it makes no difference to replace 'finitary open' by 'compact open' because of Th. 4.6.8. We did not define locally strongly compact domains since in the situation $x \in O' \subseteq S \subseteq O$ where $S$ is strongly compact and $O$ is open, there is a finitary set $F$ such that $S \subseteq F \subseteq O$. Thus, local strong compactness coincides with local finitarity.

For multi-algebraic and multi-continuous domains, a *degree of multiplicity* may be defined. If $F$ is a finitary set, then $min\, F$ is finite. We define the size of $F$ to be the cardinality of $min\, F$. For every pair $(x, O)$ of a point $x$ in an open set $O$, we choose a finitary set $F(x, O)$ with minimal size that satisfies the conditions in Def. 8.1.1. The degree of multiplicity of a domain $\mathbf{X}$ is then the supremum of the sizes of the sets $F(x, O)$. It may be $\infty$. Since finitary sets of size 1 are upper cones, a multi-algebraic domain has degree 1 iff it is algebraic.

By means of the degree, an infinite hierarchy is obtained between ALG and M-ALG, and between CONT and M-CONT. If we refer to the class of all domains in M-ALG with degree at most $d$, then we write M$^d$-ALG. The class of all multi-algebraic domains with finite degree is denoted by M$^*$-ALG. Thus, we obtain ALG = M$^1$-ALG $\subset$ M$^2$-ALG $\subset \cdots \subset$ M$^*$-ALG $\subset$ M-ALG. An analogous hierarchy is contained in M-CONT.

**Proposition 8.1.2** M-CONT and all classes M$^k$-CONT as well as M$^*$-CONT are retract closed. L-COMP is also retract closed.

**Proof:** We first show the statement for M-CONT. Let $\mathbf{Y}$ be a retract of $\mathbf{X}$ by means of $r : [\mathbf{X} \to \mathbf{Y}]$ and $e : [\mathbf{Y} \to \mathbf{X}]$ where $r \circ e = id_{\mathbf{Y}}$. Let $y$ be a point in an open set $O$. By $r(ey) = y \in O$, the point $ey$ is in $r^{-1}[O]$. As $\mathbf{X}$ is multi-continuous, there are an open set $O'$ and a finitary upper set $F$ such that $ey \in O' \subseteq F \subseteq r^{-1}[O]$. First, $y$ is in the open set $e^{-1}[O']$. Second, if $u$ is in $e^{-1}[O']$, then $eu$ is in $O' \subseteq F$, whence $u = r(eu)$ is in $r[F]$. Set $r[F]$ is finitary by Prop. 2.7.2 (5). Third, if $v$ is in $r[F]$, then $v = rw$ where $w \in F \subseteq r^{-1}[O]$, whence $v = rw \in O$. Thus, we obtain $y \in e^{-1}[O'] \subseteq r[F] \subseteq O$.

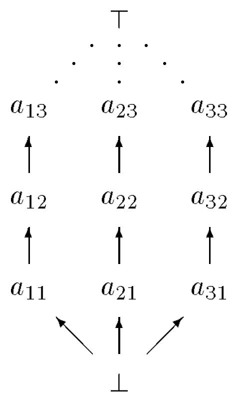
If $F$ is finitary, then $F \subseteq \uparrow(min\, F)$ holds. By Prop. 2.6.2 (7), $min\, r[F] \subseteq r[min\, F]$ follows. Hence, the size of $r[F]$ is bounded by the size of $F$. Thus, all the intermediate classes are retract closed.

The proof for local compactness is analogous to the proof for multi-continuity. $r[K]$ is compact by Prop. 4.6.5. □

## 8.2 Examples for multi-algebraic domains

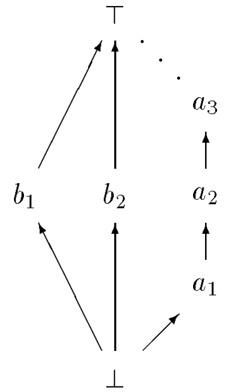
In this section, we shall provide examples for multi-algebraic domains of any degree. We shall also meet domains that are not multi-continuous. All the domains presented in this section will be complete.

For all $k$ in $\{1, 2, \ldots, \infty\}$, we define a domain $\mathbf{X}_k$ that consists of a least point $\bot$, a greatest point $\top$, and $k$ incomparable ascending chains $A_1, \ldots, A_k$ where $A_i$ is given by $\{a_{i1} < a_{i2} < a_{i3} < \cdots\}$. All the ascending chains share the same lub $\top$. The domain $\mathbf{X}_3$ is depicted below.




For every $k < \infty$, the domain $\mathbf{X}_k$ is multi-algebraic of degree $k$. This is because every non-empty open set contains $\top$ and thus also the ends of all $k$ chains. The domain $\mathbf{X}_\infty$ is not multi-continuous however since all open sets containing $\top$ must contain the ends of all the infinitely many chains and thus cannot be subset of a finitary set.

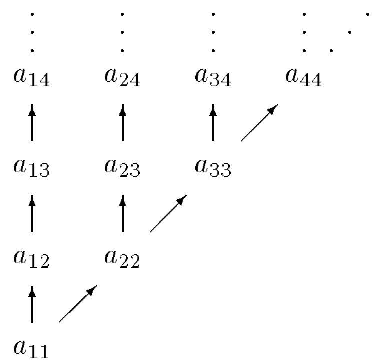
As a second series of examples, for all $n$ in $\{0, 1, 2, \ldots, \infty\}$, we define a domain $\mathbf{Y}_n = \{\bot, a_1, a_2, a_3, \ldots, b_1, b_2, \ldots, b_n, \top\}$ ordered such that $\bot$ is least, $\top$ is greatest, the $a$-points form an ascending chain $a_1 < a_2 < \cdots$ with lub $\top$, and the $b$-points are incomparable to each other and to the $a$-points. The domains $\mathbf{Y}_n$ differ in the number of $b$-points they contain. The domain $\mathbf{Y}_2$ is depicted below.




The domain $\mathbf{Y}_0$, which consists of a chain only, is algebraic. All other domains $\mathbf{Y}_n$, even $\mathbf{Y}_\infty$, are multi-algebraic of degree 2. Let a point $x$ be in an open set $O$. Then $O$ contains at least $x$, $\top$, and some end of the chain of $a$-points starting at some point $a_k$. The set $F = \uparrow\{x, a_k\}$ is open no matter what $x$ and $k$ are. $x \in F \subseteq O$ holds. Thus, the degree is at most 2. It is not 1 since $b_1$ is neither isolated nor a lub of isolated points. These examples show that the domains in M$^2$-ALG may be arbitrarily complex.

In all the domains $\mathbf{X}_k$ where $k > 1$, and $\mathbf{Y}_n$ where $n > 0$, the binary infimum '$\sqcap$' is not continuous. Prop. 6.8.1 states that '$\sqcap$' is continuous in continuous domains. The examples above show that this fact cannot be generalized to M-CONT, not even to M$^2$-ALG. One might believe that the converse of Prop. 6.8.1 holds, i.e. that a complete domain with continuous '$\sqcap$' operation be continuous. This belief is however wrong; we shall meet examples for this in the theory of upper power constructions.

We have seen examples for multi-algebraic domains of any *finite* degree and for a domain that is not multi-algebraic. An example for a multi-algebraic domain of degree $\infty$ is provided now. The domain $\mathbf{Z}$ consists of points $a_{ij}$ where $1 \leq i \leq j < \infty$ and a special point $\top$. The points $a_{ij}$ form an infinite number of ascending chains $a_{ii} < a_{i,i+1} < \cdots$ with the common lub $\top$. All the chains are incomparable except that they rise at a diagonal chain $a_{11} < a_{22} < \cdots$ whose lub is also $\top$. The domain is depicted below; the point $\top$ is omitted for simplicity.




Every non-empty open set of $\mathbf{Z}$ contains $\top$, and hence some end of the diagonal chain from say $a_{kk}$. Furthermore, it then contains some ends of the first $k-1$ vertical chains. Thus, a finitary open set of size $k$ is in it. Conversely, let $x = a_{kk}$ be a point of the finitary open set $O = \uparrow\{a_{1k}, \ldots, a_{kk}\}$. There is no finitary open set of size smaller than $k$ between $x$ and $O$. Thus, $\mathbf{Z}$ is multi-algebraic with unbounded degree of multiplicity.

## 8.3 The main properties of multi-continuous domains

From the three domain classes defined in the beginning of this chapter, we mainly consider the class of multi-continuous domains. Local compactness is too general to allow to prove analogous properties, whereas multi-algebraicity does not allow to prove significantly stronger statements.

The main theorem about multi-continuous domains provides several criteria equivalent to multi-continuity. Before we state and prove it, we define some auxiliary notions.

**Definition 8.3.1** Let $\mathbf{X}$ be a domain. For a set $A$ of $\mathbf{X}$, let $\mathcal{F}(A)$ be the set of all finitary upper environments of $A$, i.e. the set of all finitary upper sets $F$ such that there is an open set $O$ with $A \subseteq O \subseteq F$. For points $x$, $\mathcal{F}(\{x\})$ is abbreviated by $\mathcal{F}(x)$.

**Theorem 8.3.2** For a domain $\mathbf{X}$ are equivalent:

(1) $\mathbf{X}$ is multi-continuous.

(2) For all points $x$ and open sets $O$ of $\mathbf{X}$, $x \in O$ implies there is $F$ in $\mathcal{F}(x)$ such that $F \subseteq O$.

(3) For all points $x$, the set $\mathcal{F}(x)$ is $\supseteq$-directed and $\bigcap \mathcal{F}(x) = \uparrow x$ holds.

(4) For all compact sets $K$ and open sets $O$ of $\mathbf{X}$, $K \subseteq O$ implies there is $F$ in $\mathcal{F}(K)$ such that $F \subseteq O$.

(5) For all compact sets $K$, the set $\mathcal{F}(K)$ is $\supseteq$-directed and $\bigcap \mathcal{F}(K) = \uparrow K$ holds.

Statement (3) allows to prove that our notion of multi-continuity is equivalent to the quasi-continuity of [GLS83].

**Proof:** Statements (1) and (2) are equivalent by the definitions of multi-continuity and $\mathcal{F}(.)$. The remaining equivalences are shown in the order (2) $\Rightarrow$ (4) $\Rightarrow$ (5) $\Rightarrow$ (3) $\Rightarrow$ (2).

(2) $\Rightarrow$ (4) : Let $K \subseteq O$ where $K$ is compact and $O$ is open. By (2), for every $x$ in $K$, there are a finitary upper set $F_x$ and an open set $O_x$ such that $x \in O_x \subseteq F_x \subseteq O$. Thus, $K \subseteq \bigcup_{x\in K} O_x$ whence by compactness there is a finite set $E \subseteq K$ such that $K \subseteq \bigcup_{e\in E} O_e \subseteq F \subseteq O$ where $F = \bigcup_{e\in E} F_e$ is finitary upper and in $\mathcal{F}(K)$.

(4) $\Rightarrow$ (5) : $K$ is a subset of the open set $\mathbf{X}$, whence $\mathcal{F}(K)$ is not empty by (4). Let $F_1$ and $F_2$ be two members of $\mathcal{F}(K)$. Then there are open sets $O_1$ and $O_2$ such that $K \subseteq O_i \subseteq F_i$. $K \subseteq O_1 \cap O_2$ follows, whence there is a set $F$ in $\mathcal{F}(K)$ by (4) such that $K \subseteq F \subseteq O_1 \cap O_2$. $K \subseteq F \subseteq F_i$ follows, whence $\mathcal{F}(K)$ is $\supseteq$-directed.

All members of $\mathcal{F}(K)$ are upper supersets of $K$. Hence, $\uparrow K \subseteq F$ holds for all $F$ in $\mathcal{F}(K)$, whence $\uparrow K \subseteq \bigcap \mathcal{F}(K)$. For the opposite inclusion $\bigcap \mathcal{F}(K) \subseteq \uparrow K$, we apply Lemma 4.4.4. Let $\uparrow K \subseteq O$ for some open set $O$. We have to show $\bigcap \mathcal{F}(K) \subseteq O$.

$\uparrow K \subseteq O$ implies $K \subseteq O$ whence by (4) there is $F$ in $\mathcal{F}(K)$ such that $F \subseteq O$. Thus, $\bigcap \mathcal{F}(K) \subseteq F \subseteq O$ holds.

(5) $\Rightarrow$ (3) is immediate since $\{x\}$ is compact.

(3) $\Rightarrow$ (2) : Let $x$ be in an open set $O$. By (3), $\bigcap \mathcal{F}(x) = \uparrow x \subseteq O$ holds. Since $\mathcal{F}(x)$ is a $\supseteq$-directed set of finitary upper sets, Rudin's Lemma 3.9.4 applies and states $F \subseteq O$ for some $F$ in $\mathcal{F}(x)$. □

As an immediate conclusion of statement (4) of this theorem, one obtains:

**Proposition 8.3.3** In a multi-continuous domain every compact set is strongly compact.

For multi-algebraicity, an analogous theorem holds. Instead of $\mathcal{F}(.)$, we have to consider a slightly different set:

**Definition 8.3.4** Let $\mathbf{X}$ be a domain. For a set $A$ of $\mathbf{X}$, let $\mathcal{FO}(A)$ be the set of all finitary open supersets of $A$. For points $x$, $\mathcal{FO}(\{x\})$ is abbreviated by $\mathcal{FO}(x)$.

Using this notion, we may state:

**Theorem 8.3.5** For a domain $\mathbf{X}$ are equivalent:

(1) $\mathbf{X}$ is multi-algebraic.

(2) For all points $x$ and open sets $O$ of $\mathbf{X}$, $x \in O$ implies there is $F$ in $\mathcal{FO}(x)$ such that $F \subseteq O$.

(3) For all points $x$, the set $\mathcal{FO}(x)$ is $\supseteq$-directed and $\bigcap \mathcal{FO}(x) = \uparrow x$ holds.

(4) For all compact sets $K$ and open sets $O$ of $\mathbf{X}$, $K \subseteq O$ implies there is $F$ in $\mathcal{FO}(K)$ such that $F \subseteq O$.

(5) For all compact sets $K$, the set $\mathcal{FO}(K)$ is $\supseteq$-directed and $\bigcap \mathcal{FO}(K) = \uparrow K$ holds.

The proof of this theorem is very similar to that of Th. 8.3.2. One mainly has to replace 'continuous' by 'algebraic' and '$\mathcal{F}(.)$' by '$\mathcal{FO}(.)$', and to simplify some arguments involving a finitary set $F$ and an open set $O$ to arguments involving finitary open sets.

## 8.4 Multi-continuity and product

Generally, the Scott topology of the product of two domains cannot be directly obtained from the Scott topologies of the factors. For multi-continuous domains, this is however possible.

**Theorem 8.4.1** Multi-continuous domains are $\times$-conform to each other.

**Proof:** Let $\mathbf{X}_1$ and $\mathbf{X}_2$ be multi-continuous domains, and let $(x_1, x_2)$ be a point in an open set $O$ of $\mathbf{X}_1 \times \mathbf{X}_2$. We have to show that there are open sets $O_i$ in $\mathbf{X}_i$ such that $(x_1, x_2) \subseteq O_1 \times O_2 \subseteq O$.

By Th. 8.3.2, the sets $\mathcal{F}(x_i)$ are $\supseteq$-directed sets of finitary upper sets of $\mathbf{X}_i$, and $\bigcap \mathcal{F}(x_i) = \uparrow x_i$ holds. Let $\mathcal{F} = \{F_1 \times F_2 \mid F_i \in \mathcal{F}(x_i)\}$. $\mathcal{F}$ is also $\supseteq$-directed and consists of finitary upper sets by Prop. 4.8.1. The intersection of $\mathcal{F}$ is $\uparrow(x_1, x_2)$.

$\bigcap \mathcal{F} = \uparrow(x_1, x_2) \subseteq O$ implies $F \subseteq O$ for some $F$ in $\mathcal{F}$ by Rudin's Lemma 3.9.4. $F = F_1 \times F_2$ holds for some $F_i$ in $\mathcal{F}(x_i)$. Hence, there are open sets $O_i$ in $\mathbf{X}_i$ such that $x_i \in O_i \subseteq F_i$. Thus, we get $(x_1, x_2) \in O_1 \times O_2 \subseteq F_1 \times F_2 = F \subseteq O$. □

Using this theorem, we may show that some subclasses of M-CONT are closed w.r.t. products.

**Theorem 8.4.2**
The classes ALG, CONT, M-ALG, and M-CONT are closed w.r.t. finite products.

**Proof:** Let $\mathbf{X}_1$ and $\mathbf{X}_1$ be the factors, and let $x = (x_1, x_2)$ be a point of some open set $O$ in $\mathbf{X}_1 \times \mathbf{X}_2$. By Th. 8.4.1, there are open sets $O_i$ in $\mathbf{X}_i$ such that $x \in O_1 \times O_2 \subseteq O$, whence $x_i$ in $O_i$.

M-CONT: $x_i$ in $O_i$ open implies there are open sets $O'_i$ and finitary sets $F_i$ such that $x_i \in O'_i \subseteq F_i \subseteq O_i$. Hence, $x \in O'_1 \times O'_2 \subseteq F_1 \times F_2 \subseteq O_1 \times O_2 \subseteq O$ holds as required. $O'_1 \times O'_2$ is open and $F_1 \times F_2$ is finitary by Prop. 4.8.1.

M-ALG: $x_i$ in $O_i$ open implies there are open finitary sets $F_i$ such that $x_i \in F_i \subseteq O_i$. Hence, $x \in F_1 \times F_2 \subseteq O_1 \times O_2 \subseteq O$ holds as required. $F_1 \times F_2$ is finitary open by Prop. 4.8.1.

The proofs for ALG and CONT are analogous; one only has to replace the finitary sets by upper cones. □

## 8.5 Sobriety in terms of irreducible closed sets

The notion of sobriety is closely related with upper power domain constructions as pointed out in [Smy83]. Some authors e.g. [Law88] define sobriety via irreducible closed sets, whereas others e.g. [Smy78] define it via prime filters. The equivalence of both definitions is shown in [HM81] by means of lattice theory. We present both definitions together with new, more direct proofs of their equivalence.

We first present the definition by means of irreducible closed sets. As often, there are several equivalent definitions of this concept.

**Proposition 8.5.1** Let $\mathbf{X}$ be a topological space. For a closed set $C$ of $\mathbf{X}$, the following statements are equivalent.

(1) $C = A \cup B$ for closed sets $A$ and $B$ implies $C = A$ or $C = B$.

(2) $C \subseteq A \cup B$ for closed sets $A$ and $B$ implies $C \subseteq A$ or $C \subseteq B$.

(3) Whenever two open sets meet $C$, their intersection meets $C$.

Closed sets with these properties are called *irreducible closed.*

**Proof:** Let $C$ be a closed set satisfying (1) and let $O_1$ and $O_2$ be two open sets with $O_i \cap C \neq \emptyset$. Let $C_i = C \setminus O_i$. The sets $C_i$ are closed since they are intersections of $C$ and the closed complements of $O_i$. $C_i \neq C$ holds since at least one point of $C$ is in $O_i$. Since $C$ is irreducible, $C_1 \cup C_2$ cannot be $C$, i.e. there is a point $x$ in $C$ with $x \notin C_1 \cup C_2$, whence $x \in O_1 \cap O_2 \cap C$.

Let $C$ be a closed set satisfying (3). Let $C \subseteq C_1 \cup C_2$ for two closed sets $C_1$ and $C_2$. If $O_i$ is the complement of $C_i$, then $O_1 \cap O_2$ is the complement of $C_1 \cup C_2 \supseteq C$, whence $C \cap O_1 \cap O_2 = \emptyset$. By the criterion, $C \cap O_i = \emptyset$ holds for at least one $i$ in $\{1, 2\}$, whence $C \subseteq C_i$.

Let $C = C_1 \cup C_2$. Then $C \subseteq C_1 \cup C_2$ holds, whence by (2), $C \subseteq C_i$ for some $i$. $C_i \subseteq C$ also holds. □

**Proposition 8.5.2** Let $f : [\mathbf{X} \to \mathbf{Y}]$ be continuous. If $C$ is irreducible closed in $\mathbf{X}$, then $\mathsf{cl}\, f[C]$ is irreducible closed in $\mathbf{Y}$.

**Proof:** Let $\mathsf{cl}\, f[C] \subseteq C_1 \cup C_2$. Then $C \subseteq f^{-1}[C_1] \cup f^{-1}[C_2]$ follows, whence $C \subseteq f^{-1}[C_k]$ for some $k$ in $\{1, 2\}$. This inclusion implies $f[C] \subseteq C_k$, whence $\mathsf{cl}\, f[C] \subseteq C_k$. □

Closures of singleton sets are irreducible.

**Proposition 8.5.3** For all $x$ in $\mathbf{X}$, $\mathsf{cl}\, x$ is irreducible closed.

**Proof:** Let $\mathsf{cl}\, x \subseteq A \cup B$. Then $x$ is in one of $A$ and $B$, say $A$. $A$ being closed implies $\mathsf{cl}\, x \subseteq A$. □

The class of sober topological spaces is characterized by the converse of the proposition above.

**Definition 8.5.4** A topological space is *sober* iff all non-empty irreducible closed sets are the closure of a unique point.

Sobriety may be considered a separation property. As such, it is between (T0) and (T2), but incomparable with (T1).

**Proposition 8.5.5** Every sober space satisfies axiom (T0).

**Proof:** Let $\mathsf{cl}\, x = \mathsf{cl}\, y$. Set $\mathsf{cl}\, x$ is irreducible closed by Prop. 8.5.3. By sobriety, there is exactly one point $p$ with $\mathsf{cl}\, x = \mathsf{cl}\, p$. Hence, $x = y$ follows. □

**Proposition 8.5.6** Every (T2)-space is sober.

**Proof:** Let $C$ be an irreducible closed set. Assume there were two different points $x_1$ and $x_2$ in $C$. By (T2), there would be two disjoint open sets $O_1$ and $O_2$ with $x_i \in O_i$. Hence, both open sets meet $C$, but their intersection does not because it is empty. This would be a contradiction to Prop. 8.5.1. Thus, the assumption is wrong, i.e. all non-empty irreducible closed sets are singletons. Hence, they are closures of a unique point. □

Alternatively, sobriety may be considered a completeness property since it states that certain points exist.

## 8.6 Filters of open sets

The notion of filters is fundamental in topology. Usually, topological filters are sets of sets satisfying the properties of the definition below. The filters we are going to define here however contain only open sets.

**Definition 8.6.1 (Filters)** Let $\mathbf{X}$ be a topological space. A *filter* in $\mathbf{X}$ is a set $\mathcal{O}$ of open sets satisfying the following properties:

(1) $\mathbf{X} \in \mathcal{O}$

(2) If $O_1$ and $O_2$ are in $\mathcal{O}$, then so is their intersection $O_1 \cap O_2$.

(3) If $O$ is in $\mathcal{O}$ and $O \subseteq O'$ holds for some open set $O'$, then $O'$ is in $\mathcal{O}$.

In domain-theoretic view, a filter is an ideal in the poset $(\Omega\mathbf{X}, \supseteq)$. We distinguish different classes of filters:

**Definition 8.6.2**

- A filter $\mathcal{O}$ is *comprising* iff for all open sets $O$, $O \supseteq \bigcap \mathcal{O}$ implies $O \in \mathcal{O}$.
- A filter $\mathcal{O}$ is *open* iff for all $\subseteq$-directed sets $\mathcal{D}$ of open sets, $\bigcup \mathcal{D} \in \mathcal{O}$ implies $\mathcal{D} \cap \mathcal{O} \neq \emptyset$.
- A filter $\mathcal{O}$ is *prime* iff for all sets $\mathcal{S}$ of open sets, $\bigcup \mathcal{S} \in \mathcal{O}$ implies $\mathcal{S} \cap \mathcal{O} \neq \emptyset$.

The notion 'open' coincides with the notion of Scott open sets in the domain $(\Omega\mathbf{X}, \subseteq)$.

Filters naturally occur as sets of open environments of sets.

**Proposition 8.6.3** For every set $A \subseteq \mathbf{X}$, the set $\mathcal{O}(A) = \{O \text{ open} \mid O \supseteq A\}$ of open environments of $A$ is a comprising filter with intersection $\bigcap \mathcal{O}(A) = \uparrow A$. Conversely, if $\mathcal{O}$ is a comprising filter, then $\bigcap \mathcal{O}$ is an upper set, and $\mathcal{O} = \mathcal{O}(\bigcap \mathcal{O})$ holds.

**Proof:** $A \subseteq \mathbf{X}$ holds. $A \subseteq O_1$ and $A \subseteq O_2$ imply $A \subseteq O_1 \cap O_2$. $A \subseteq O$ and $O \subseteq O'$ imply $A \subseteq O'$.

$\bigcap \mathcal{O}(A) = \uparrow A$ holds by Lemma 4.4.4. If $O \supseteq \uparrow A$, then obviously $O \supseteq A$, i.e. $O \in \mathcal{O}(A)$. Thus, $\mathcal{O}(A)$ is comprising.

For the converse direction, $\bigcap \mathcal{O}$ is an upper set as intersection of upper sets. $O \in \mathcal{O}$ implies $O \supseteq \bigcap \mathcal{O}$. The opposite inclusion is the definition of 'comprising'. □

The next proposition is as trivial as the previous one.

**Proposition 8.6.4** A set $K$ is compact iff $\mathcal{O}(K)$ is a comprising open filter. Hence, if $\mathcal{O}$ is a comprising open filter, then $\bigcap \mathcal{O}$ is a compact upper set.

**Proof:** The equivalence holds since $\mathcal{O}(K)$ being open is just statement (2) of Prop. 4.6.1, the definition of compactness. The 'hence' part is due to Prop. 8.6.3. □

An analogue statement for prime filters is a bit more difficult (and also more useful).

**Lemma 8.6.5** If $a$ is a point, then $\mathcal{O}(a)$ is a comprising prime filter. Conversely, if $\mathcal{O}$ is a comprising prime filter, then there is a point $a$ such that $\bigcap \mathcal{O} = \uparrow a$ and thus $\mathcal{O} = \mathcal{O}(a)$.

**Proof:** $\mathcal{O}(a)$ is prime since $a \in \bigcup \mathcal{S}$ implies $a \in O$ for some $O \in \mathcal{S}$.

For the converse statement, let $\mathcal{S} = \{\mathsf{co}\downarrow x \mid x \in \bigcap \mathcal{O}\}$. $\mathcal{S}$ is a set of open sets. Assume $\bigcap \mathcal{O} \subseteq \bigcup \mathcal{S}$. Then $\bigcup \mathcal{S}$ is in $\mathcal{O}$ since $\mathcal{O}$ is comprising, whence there is $M$ in $\mathcal{S}$ with $M \in \mathcal{O}$ since $\mathcal{O}$ is prime. Hence, $M \supseteq \bigcap \mathcal{O}$. Remember $M = \mathsf{co}\downarrow x$ for some $x \in \bigcap \mathcal{O}$. For this $x$, $x \notin \downarrow x$ would hold.

Thus, $\bigcap \mathcal{O}$ is not a subset of $\bigcup \mathcal{S}$. This means, there is $a$ in $\bigcap \mathcal{O}$ with $a \notin \bigcup \mathcal{S}$, i.e. for all $M$ in $\mathcal{S}$, $a \notin M$. Hence, for all $x \in \bigcap \mathcal{O}$, $a \in \downarrow x$ holds, or equivalently $x \in \uparrow a$. Thus, $\bigcap \mathcal{O} \subseteq \uparrow a$. Conversely, $a \in \bigcap \mathcal{O}$ and this set being upper implies $\uparrow a \subseteq \bigcap \mathcal{O}$. □

## 8.7 Filters and sobriety

Filters induce equivalent definitions of sobriety:

**Theorem 8.7.1** For a (T0)-space $\mathbf{X}$, the following statements are equivalent:

(1) For every non-empty irreducible closed set $C$, there is a point $c$ such that $C = \mathsf{cl}\, c$. ($\mathbf{X}$ is sober.)

(2) Every open filter is comprising.

(3) Every prime filter is comprising.

**Proof:**

$(1) \Rightarrow (2)$ : Let $\mathcal{O}$ be an open filter. Let $\mathcal{O}' = \{O \text{ open} \mid O \supseteq \bigcap \mathcal{O},\ O \notin \mathcal{O}\}$ be the set of all witnesses for $\mathcal{O}$ not being comprising. We have to show $\mathcal{O}' = \emptyset$.

Assume $\mathcal{O}'$ were not empty. Let $\mathcal{D}$ be a $\subseteq$-directed subset of $\mathcal{O}'$. We claim $\bigcup \mathcal{D} \in \mathcal{O}'$. $\bigcup \mathcal{D}$ is certainly open and a superset of $\bigcap \mathcal{O}$. If $\bigcup \mathcal{D}$ were in $\mathcal{O}$, then some member of $\mathcal{D}$ would be in $\mathcal{O}$ since $\mathcal{O}$ is open. Thus, $\mathcal{O}'$ is a directed complete poset, a domain.

By Zorn's Lemma 3.2.2, there is a maximal element $M$ in $\mathcal{O}'$. Let $C$ be the complement of $M$. We claim $C$ is irreducible closed. $C$ is closed as complement of the open set $M$. Let $C = C_1 \cup C_2$ with closed sets $C_i$. By complementing, one obtains $M = M_1 \cap M_2$ with open sets $M_i$. $M_i \supseteq M \supseteq \bigcap \mathcal{O}$ holds. If both $M_1$ and $M_2$ were in $\mathcal{O}$, then so would be their intersection $M$ since $\mathcal{O}$ is a filter (condition (2)). Thus, one of $M_1$ and $M_2$ — say $M_1$ — is in $\mathcal{O}'$. By maximality of $M$, $M_1 = M$ holds, whence $C_1 = C$.

Hence, we got an irreducible closed set $C$. $C$ is not empty since otherwise $M$ would be $\mathbf{X}$ contradicting $\mathbf{X} \in \mathcal{O}$ (condition (1) of filters). By (1), there is a point $c$ with $C = \mathsf{cl}\, c$. $c$ is not in $M$ since $M$ is the complement of $C$. $\bigcap \mathcal{O} \subseteq M$ implies $c \notin \bigcap \mathcal{O}$, i.e. there is a set $O \in \mathcal{O}$ with $c \notin O$. Hence, $c$ is in $\mathsf{co}\, O$. This set is closed, whence $\mathsf{cl}\, c = C \subseteq \mathsf{co}\, O$. By complementing, $O \subseteq M$ follows. By condition (3) of filters, $M$ itself is in $\mathcal{O}$ — a contradiction.

$(2) \Rightarrow (3)$ : Every prime filter is open, and every open filter is comprising by (2).

$(3) \Rightarrow (1)$ : Let $C$ be a non-empty irreducible closed set, and let $\mathcal{O} = \{O \text{ open} \mid C \cap O \neq \emptyset\}$. We show $\mathcal{O}$ is a prime filter.

(1) $\mathbf{X} \in \mathcal{O}$ since $C \neq \emptyset$.

(2) If $O_1$ and $O_2$ are in $\mathcal{O}$, then so is their intersection $O_1 \cap O_2$ by Prop. 8.5.1.

(3) If $O$ is in $\mathcal{O}$, i.e. $C \cap O \neq \emptyset$, and $O \subseteq O'$ holds for some open set $O'$, then $C \cap O' \neq \emptyset$, whence $O'$ is in $\mathcal{O}$.

(4) If $\mathcal{S}$ is a set of open sets such that $C$ meets $\bigcup \mathcal{S}$, then $C$ meets some member of $\mathcal{S}$. Hence, $\mathcal{O}$ is prime.

By precondition (3), $\mathcal{O}$ is comprising. By Lemma 8.6.5, there is a point $c$ such that $\bigcap \mathcal{O} = \uparrow c$. Let $O$ be the complement of $\mathsf{cl}\, c$. If $O$ were in $\mathcal{O}$, then $c \in \bigcap \mathcal{O} \subseteq O$ would imply $c \notin \mathsf{cl}\, c$. Hence, $C \cap O = \emptyset$, whence $C \subseteq \mathsf{cl}\, c$.

If $U$ is any open environment of $c$, then $\bigcap \mathcal{O} = \uparrow c \subseteq U$ implies $U \in \mathcal{O}$ since $\mathcal{O}$ is comprising. Thus, every open environment of $c$ meets $C$, whence $c$ is in $\mathsf{cl}\, C = C$. $c \in C$ implies $\mathsf{cl}\, c \subseteq C$. Together, $C = \mathsf{cl}\, c$ follows. $\square$

## 8.8 Sober domains

After having established the main theorem about sobriety, we now turn our attention to sober domains. A domain is sober iff the corresponding Scott space is. The class of sober domains is called SOB. It is quite large as the following theorem shows. Nevertheless, non-sober domains exist. We present one in section 8.11.

**Theorem 8.8.1** Every multi-continuous domain is sober.

**Proof:** Let $C$ be a non-empty irreducible closed set of the multi-continuous domain $\mathbf{X}$. Let $\mathcal{F} = \bigcup_{c \in C} \mathcal{F}(c)$. From multi-continuity, we know every set $\mathcal{F}(c)$ is $\supseteq$-directed with intersection $\uparrow c$. We demonstrate the set $\mathcal{F}$ to be $\supseteq$-directed, too.

$\mathcal{F}$ is not empty since $C$ is not empty and every set $\mathcal{F}(c)$ is not empty. Let $F_1, F_2$ be in $\mathcal{F}$. By definition, there are open sets $O_1, O_2$ and points $c_1, c_2 \in C$ with $c_i \in O_i \subseteq F_i$. Hence, both open sets $O_1$ and $O_2$ meet $C$ whence their intersection meets $C$ because $C$ is irreducible closed (Prop. 8.5.1). Thus, there is a point $c$ in $C$ with $c \in O_1 \cap O_2$. By multi-continuity, there is $F'$ in $\mathcal{F}(c) \subseteq \mathcal{F}$ with $F' \subseteq O_1 \cap O_2 \subseteq F_1 \cap F_2$. $F'$ is a common lower bound of $F_1$ and $F_2$ in $\mathcal{F}$.

$\mathcal{F}$ is a $\supseteq$-directed set of upper finitary sets that all meet $C$. By Rudin's Lemma 3.9.3, the intersection $\bigcap \mathcal{F}$ meets $C$. Let $x$ be a point of $C \cap \bigcap \mathcal{F}$. Then $\mathsf{cl}\, x \subseteq C$ holds.

Assume $C$ were not a subset of $\mathsf{cl}\, x$. Then, there would be a point $c$ in $C$ lying in $\mathsf{co}\,(\mathsf{cl}\, x)$. By multi-continuity, there is $F$ in $\mathcal{F}(c) \subseteq \mathcal{F}$ with $F \subseteq \mathsf{co}\,(\mathsf{cl}\, x)$. $F$ in $\mathcal{F}$ implies $x \in \bigcap \mathcal{F} \subseteq F \subseteq \mathsf{co}\,(\mathsf{cl}\, x)$ — a contradiction.

Hence, $C$ is a subset of $\mathsf{cl}\, x$, and together, $C = \mathsf{cl}\, x$ follows. □

An analogous proof for locally compact domains is not correct since Rudin's Lemma is not generally true if finitary sets are replaced by compact sets (compare section 8.11).

In section 8.2, we presented a domain $\mathbf{X}_\infty$ consisting of a least point $\perp$, a greatest point $\top$, and an infinite number of incomparable ascending chains with lub $\top$. This domain is not multi-continuous, but still sober. If a closed set $C$ contains infinitely many points of at least one chain, then it contains $\top$ and is a lower cone. Otherwise, it contains only finitely many points of all chains. If it meets at least two chains, then it is not irreducible, and otherwise, it is a lower cone.

**Proposition 8.8.2** The class SOB is retract closed.

**Proof:** Let $\mathbf{X}$ be sober and $\mathbf{Y}$ a retract of $\mathbf{X}$ by means of $r : [\mathbf{X} \to \mathbf{Y}]$ and $e : [\mathbf{Y} \to \mathbf{X}]$. If $C$ is non-empty irreducible closed in $\mathbf{Y}$, then $\mathsf{cl}\, e[C]$ is non-empty irreducible closed in $\mathbf{X}$ by Prop. 8.5.2. Hence, there is $x$ in $\mathbf{X}$ such that $\mathsf{cl}\, e[C] = \mathsf{cl}\, x = \downarrow x$. We claim $C = \downarrow rx$.

If $c$ is a point of $C$, then $ec$ is in $e[C] \subseteq \downarrow x$, whence $ec \leq x$. Thus, $c = r(ec) \leq rx$ follows. Now, we proved $C \subseteq \downarrow rx$.

Assume $rx$ were not in $C$. Then $rx$ is in $\mathsf{co}\, C$, whence $x$ is in $r^{-1}[\mathsf{co}\, C]$. $x$ is also in $\mathsf{cl}\, e[C]$. By Prop. 4.2.2, the open environment $r^{-1}[\mathsf{co}\, C]$ meets $e[C]$. Let $u$ be a point of the intersection. Then $ru$ is both in $\mathsf{co}\, C$ and in $r[e[C]] = C$ — a contradiction. $rx \in C$ implies $\downarrow rx \subseteq C$. □

It is possible to prove that the product of two $\times$-conform sober domains is sober. We did however not succeed in proving or disproving the sobriety of the product of arbitrary sober domains, nor could we generalize Prop. 8.4.1 stating the $\times$-conformity of multi-continuous domains to sober domains.

**Problem 4** Is the product of sober domains sober?

**Problem 5** Is there any relation between $\times$-conformity and sobriety?

## 8.9 The Rudin classes S-RD and K-RD

Rudin's Lemma in the version of Lemma 3.9.3 or 3.9.4 deals with finitary upper sets. Analogous statements involving compact or strongly compact sets instead do not generally hold as we shall see in section 8.11. We now define classes of domains characterized by these conditions.

**Definition 8.9.1 (Rudin classes)** The class S-RD of domains is defined by the condition: for all $\supseteq$-directed sets $\mathcal{D}$ of *strongly compact upper sets* and open sets $O$, $\bigcap \mathcal{D} \subseteq O$ implies $S \subseteq O$ for some $S$ in $\mathcal{D}$.

The class K-RD of domains is defined analogously by means of compact sets instead of strongly compact sets.

Equivalent conditions are provided by using closed sets instead of open sets: Let $\mathcal{D}$ be a $\supseteq$-directed set of (strongly) compact upper sets. If all members of $\mathcal{D}$ meet a closed set $C$, then $\bigcap \mathcal{D}$ meets $C$. The equivalence is proved by letting $C$ be the complement of $O$ and vice versa (cf. the proof of Lemma 3.9.4 from Lemma 3.9.3).

By Lemma 3.9.4, a corresponding class F-RD defined by finitary sets contains all domains. Since strongly compact sets are compact, the class K-RD is a subclass of S-RD. The class S-RD is very large since it contains both SOB and FC (see below); section 8.11 will however provide an example of a domain not contained in S-RD.

**Theorem 8.9.2** Every sober domain satisfies property K-RD.

**Proof:** Let $\mathbf{X}$ be a sober domain and $\mathcal{D}$ a $\supseteq$-directed set of compact upper sets such that $\bigcap \mathcal{D} \subseteq O$ for some open set $O$. For every $K$ in $\mathcal{D}$, the set $\mathcal{O}(K)$ is an open filter by Prop. 8.6.4, i.e. open in $(\Omega\mathbf{X}, \subseteq)$ and directed in $(\Omega\mathbf{X}, \supseteq)$. Let $\mathcal{O} = \bigcup_{K \in \mathcal{D}} \mathcal{O}(K)$. $\mathcal{O}$ is again open in $(\Omega\mathbf{X}, \subseteq)$, and by Prop. 3.1.5, directed in $(\Omega\mathbf{X}, \supseteq)$. Thus $\mathcal{O}$ is an open filter. Since $\mathbf{X}$ is sober, $\mathcal{O}$ is comprising.

For the open set $O$, $O \supseteq \bigcap \mathcal{D} = \bigcap \{\bigcap \mathcal{O}(K) \mid K \in \mathcal{D}\} = \bigcap (\bigcup_{K \in \mathcal{D}} \mathcal{O}(K)) = \bigcap \mathcal{O}$ holds. Since $\mathcal{O}$ is comprising, $O \supseteq \bigcap \mathcal{O}$ implies $O \in \mathcal{O}$, i.e. $O \supseteq K$ for some $K$ in $\mathcal{D}$. □

**Theorem 8.9.3** Every finitarily complete domain is in S-RD.

**Proof:** For every strongly compact set $S$ let $\mathcal{F}'(S)$ be the set of finitary upper supersets of $S$.[1] By Prop. 5.4.4, $\mathcal{F}'(S)$ is $\supseteq$-directed and its intersection is $S$.

[1] $\mathcal{F}'(S)$ contains *all* finitary upper supersets, whereas $\mathcal{F}(S)$ contains the finitary upper *environments* only.

Let $\mathcal{D}$ be a $\supseteq$-directed set of strongly compact sets. Then $\{\mathcal{F}'(S) \mid S \in \mathcal{D}\}$ forms a $\subseteq$-directed set of $\supseteq$-directed sets of finitary upper sets. By Prop. 3.1.5, its union $\mathcal{F}$ is $\supseteq$-directed. By set-theoretic arguments, $\bigcap \mathcal{F}$ equals $\bigcap \mathcal{D}$. Thus, if $\bigcap \mathcal{D}$ is a subset of some open set $O$, then by Lemma 3.9.4, there is some $F$ in $\mathcal{F}$ such that $F \subseteq O$. $F$ is contained in $\mathcal{F}'(S)$ for some $S$ in $\mathcal{D}$. For this $S$, $S = \bigcap \mathcal{F}'(S) \subseteq F \subseteq O$ holds. □

The compact sets of domains in K-RD form a domain when ordered by '$\supseteq$':

**Proposition 8.9.4** Let $\mathbf{X}$ be a domain in K-RD (S-RD). Then the intersection of a $\supseteq$-directed set of (strongly) compact upper sets is (strongly) compact.

**Proof:** Let $\mathcal{D}$ be a $\supseteq$-directed set of strongly compact upper sets in a domain in S-RD. Let $O$ be an open set with $\bigcap \mathcal{D} \subseteq O$. By S-RD, there is some $S$ in $\mathcal{D}$ with $S \subseteq O$. Since $S$ is strongly compact, there is a finitary set $F$ with $\bigcap \mathcal{D} \subseteq S \subseteq F \subseteq O$.

Let $\mathcal{D}$ be a $\supseteq$-directed set of compact upper sets in a domain in K-RD. Let $\mathcal{O}$ be an open cover of $\bigcap \mathcal{D}$, i.e. $\bigcap \mathcal{D} \subseteq \bigcup \mathcal{O}$. $\bigcup \mathcal{O}$ is an open set. By K-RD, there is some $K$ in $\mathcal{D}$ with $K \subseteq \bigcup \mathcal{O}$. As $K$ is compact, there is finite subset $\mathcal{F}$ of $\mathcal{O}$ with $K \subseteq \bigcup \mathcal{F}$. By $\bigcap \mathcal{D} \subseteq K$, $\mathcal{F}$ also covers $\bigcap \mathcal{D}$. □

**Proposition 8.9.5** The classes K-RD and S-RD are retract closed.

**Proof:** Let $\mathbf{X}$ be in K-RD and $\mathbf{Y}$ a retract of $\mathbf{X}$ by means of $r : [\mathbf{X} \to \mathbf{Y}]$ and $e : [\mathbf{Y} \to \mathbf{X}]$. Let $\mathcal{K}$ be a $\supseteq$-directed set of compact upper sets of $\mathbf{Y}$ such that $\bigcap \mathcal{K} \subseteq O$ for some open set $O$. Let $\mathcal{K}' = \{\uparrow e[K] \mid K \in \mathcal{K}\}$. $\mathcal{K}'$ is a $\supseteq$-directed set of compact upper sets by Prop. 4.6.5 and 4.6.2. If $x$ is in $\bigcap \mathcal{K}'$, then for all $K$ in $\mathcal{K}$, there is some $y_K$ in $K$ such that $x \geq e y_K$. $rx \geq r(e y_K) = y_K \in K$ holds, whence $rx \in K$ since $K$ is upper. Thus, $rx$ is in $\bigcap \mathcal{K}$.

Now, we obtained $\bigcap \mathcal{K}' \subseteq r^{-1}[\bigcap \mathcal{K}] \subseteq r^{-1}[O]$. Since $\mathbf{X}$ is in K-RD, there is some $K'$ in $\mathcal{K}'$ such that $K' \subseteq r^{-1}[O]$. Hence, there is $K$ in $\mathcal{K}$ such that $e[K] \subseteq r^{-1}[O]$. This means, $y = r(ey) \in O$ holds for all $y$ in $K$, i.e. $K \subseteq O$.

The proof for S-RD is analogous. □

In the region of these large classes, we met a number of open problems:

**Problem 6** Are the classes SOB, K-RD, and S-RD different at all?

**Problem 7** Are the classes K-RD and S-RD closed w.r.t. products?

**Problem 8** Is FC (or any of the smaller classes BC or CC) a subclass of K-RD or of SOB?

## 8.10 Combinations of completeness and Rudin classes

In this section, we prove two Lemmas that hold for domains that are in SC & S-RD or in KC & K-RD. The first of these two classes contains FC as a subclass by Prop. 8.9.3, and also F-CONT by Prop. 7.2.2 and the inclusions F-CONT $\subset$ CONT $\subset$ M-CONT $\subset$ SOB $\subseteq$ K-RD $\subseteq$ S-RD. The classes KC & K-RD and SC & S-RD are not obviously comparable. SC is a subclass of KC, but conversely K-RD is a subclass of S-RD.

The first Lemma tells that certain lower sets are closed.

**Lemma 8.10.1** Let $\mathbf{X}$ be a domain in SC & S-RD or in KC & K-RD. Let $F$ be a finitary upper set and $C$ a closed set of $\mathbf{X}$. Then $\mathsf{cl}\,(C \cap F) = \downarrow(C \cap F)$ holds.

**Proof:** We present the proof for SC & S-RD. The other proof is analogous.

$\downarrow(C \cap F) \subseteq \mathsf{cl}\,(C \cap F)$ holds by Prop. 4.4.2. For the opposite inclusion, we have to show that $\downarrow(C \cap F)$ is Scott closed.

Let $D$ be a directed subset of $\downarrow(C \cap F)$. Then for every $d$ in $D$, there is a point $x_d$ in $C \cap F$ above $d$. Hence, for all $d$ in $D$, the set $\uparrow d \cap F$ meets $C$. The set $\{\uparrow d \cap F \mid d \in D\}$ is $\supseteq$-directed since $D$ is directed. By SC, it consists of strongly compact sets only. By S-RD, its intersection meets $C$ since all its members meet $C$.

Let $x$ be a point in $C \cap \bigcap_{d \in D}(\uparrow d \cap F)$. Then $x$ is in $C \cap F$ and also in $\bigcap_{d \in D} \uparrow d$. Thus, $x$ is an upper bound of $D$, whence it is above $\bigsqcup D$. $\bigsqcup D \leq x \in C \cap F$ implies $\bigsqcup D \in \downarrow(C \cap F)$. $\square$

The Lemma has some interesting corollaries. A particular special case is $C = \mathbf{X}$. Then the Lemma tells that $\downarrow F$ is closed for all finitary upper sets $F$. Every upper cone is finitary, whence $\downarrow\uparrow x$ is closed for all points $x$. In other words, if all members of a directed set $D$ have a common upper bound with $x$, then $\bigsqcup D$ has a common upper bound with $x$.

The next Lemma tells that the lower closure operator commutes over certain intersections.

**Lemma 8.10.2** Let $\mathbf{X}$ be a domain in SC & S-RD or in KC & K-RD. Let $\mathcal{F}$ be a $\supseteq$-directed set of finitary upper sets and $C$ a closed set of $\mathbf{X}$. Then $\downarrow(\bigcap_{F \in \mathcal{F}}(C \cap F)) = \bigcap_{F \in \mathcal{F}} \downarrow(C \cap F)$ holds.

**Proof:** Again, we present the proof for SC & S-RD.

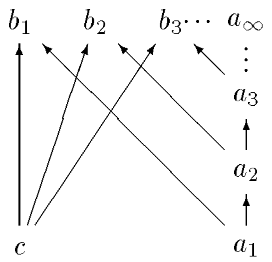
For every $F$ in $\mathcal{F}$, the intersection $\bigcap_{F \in \mathcal{F}} C \cap F$ is a subset of $C \cap F$. Applying '$\downarrow$', the left hand side is a subset of $\downarrow(C \cap F)$ for all $F$ in $\mathcal{F}$, whence the inclusion '$\subseteq$' follows.

For the opposite inclusion, let $x$ be a member of the right hand side. Then for all $F$ in $\mathcal{F}$, $x$ is in $\downarrow(C \cap F)$, whence $\uparrow x$ meets $C \cap F$. Or the other way round, for all $F$ in $\mathcal{F}$, $\uparrow x \cap F$ meets $C$. Since $\mathcal{F}$ is $\supseteq$-directed, $\{\uparrow x \cap F \mid F \in \mathcal{F}\}$ is $\supseteq$-directed, too. By SC, it consists of strongly compact sets only, and by S-RD, its intersection meets $C$.

Let $y$ be a point in $C \cap \bigcap_{F \in \mathcal{F}}(\uparrow x \cap F)$. Hence, $y$ is in both $\uparrow x$ and $\bigcap_{F \in \mathcal{F}}(C \cap F)$. Thus, $x$ is in $\downarrow(\bigcap_{F \in \mathcal{F}}(C \cap F))$. $\square$

In the special case $C = \mathbf{X}$, the Lemma tells $\downarrow \bigcap \mathcal{F} = \bigcap \downarrow[\mathcal{F}]$ for all $\supseteq$-directed sets $\mathcal{F}$ of finitary upper sets. This is a kind of continuity statement for the operation '$\downarrow$'. If the finitary upper sets are restricted to upper cones, we obtain $\downarrow\uparrow(\bigsqcup D) = \bigcap_{d \in D} \downarrow\uparrow d$ for all directed sets $D$.

To show the dependence of the two lemmas on property SC, we present some example domain. Let $\mathbf{X} = \{a_1, a_2, a_3, \ldots, a_\infty, b_1, b_2, b_3, \ldots, c\}$. There is no point $b_\infty$. The $a$-points form an ascending sequence: $a_1 < a_2 < \cdots < a_\infty$, whereas the $b$-points are incomparable. Every $a$-point is below the corresponding $b$-point: $a_n < b_n$. The remaining point $c$ is below all $b$-points, but not below any $a$-point, not even below $a_\infty$. The domain is visualized by the following picture:




This domain is algebraic. The condition SC is not satisfied since $\uparrow c \cap \uparrow a_1 = \{b_1, b_2, \ldots\}$ is an infinite discrete open set, whence it cannot be strongly compact. The set $\uparrow c$ is finitary, but $\downarrow\uparrow c$ is not closed since it contains all points $a_n$ except $a_\infty$. The sets $\uparrow a_n$ where $n < \infty$ provide a $\supseteq$-directed set of finitary sets. $c$ is in the intersection of the sets $\downarrow\uparrow a_n$, but not in $\downarrow \bigcap_{n \in \mathbf{N}} \uparrow a_n = \downarrow a_\infty$.

The defect of this domain seems to be cured by establishing the additional inequality $c \leq a_\infty$. The resulting domain is indeed in SC, but no longer algebraic. It is multi-algebraic of degree 2. Hence, it is in SC & S-RD, and the two Lemmas may be applied.

## 8.11 A non-sober domain

An example for a non-sober domain was found by Johnstone and published in [Joh81] and [Joh82]. It is also not in S-RD.

The domain $\mathbf{X}$ consists of a least element $\bot$, points $a_{ij}$ where $1 \leq i, j < \infty$, and points $b_k$ where $1 \leq k < \infty$. For every $i$, the points $a_{ij}$ form an ascending chain $A_i = \{a_{i1} < a_{i2} < \cdots\}$ with lub $b_i$. The chains are incomparable to each other. In addition, all points of level number $j$, i.e. $L_j = \{a_{1j}, a_{2j}, \ldots\}$, are below the points $b_j$, $b_{j+1}$, etc. Hence, the set of upper bounds of a point $a_{ij}$ is $\uparrow a_{ij} = \{a_{ij}, a_{i,j+1}, a_{i,j+2}, \ldots, b_i, b_j, b_{j+1}, b_{j+2}, \ldots\}$. The points $b_k$ are maximal. Their lower cone is $\downarrow b_k = \{b_k\} \cup \{a_{kj} \mid 1 \leq j < \infty\} \cup \{a_{ij} \mid 1 \leq i < \infty, 1 \leq j \leq k\} \cup \{\bot\}$.

The whole domain is closed. We show it is irreducible closed. Let $\mathbf{X} = A \cup B$ where $A$ and $B$ are closed. One of $A$ and $B$ contains an infinite number of $b$-points. Because level $L_k$ is below point $b_k$, it thus contains an infinite number of levels, whence all levels. Thus, it also contains all chains $A_i$ and so all limit points $b_i$. Summarizing, it contains all of $\mathbf{X}$. Thus, there is an irreducible closed set that obviously is not a lower cone. The corresponding prime filter is the set of all non-empty open sets.

Next, we show that $\mathbf{X}$ is not in S-RD. The sets $B_k = \{b_k, b_{k+1}, b_{k+2}, \ldots\}$ are all strongly compact. For, if $B_k \subseteq O$ for some open set $O$, then $O$ contains some member of the chain $A_k$, say $a_{kl}$ where $l > k$. The points $b_l$, $b_{l+1}$, etc. are all above $a_{kl}$. Hence, $B_k \subseteq F_{kl} = \uparrow\{a_{kl}, b_{k+1}, b_{k+2}, \ldots, b_{l-1}\} \subseteq O$. The set of all sets $B_k$ is $\supseteq$-directed and its intersection is $\emptyset$. $\emptyset$ is open, whence some $B_k$ would have to be empty if $\mathbf{X}$ were in S-RD. The sets $F_{kl}$ prove that $B_k$ may also be reached as $\supseteq$-directed intersection of finitary upper sets, namely the sets $F_{kl}$ where $l > k$.

Finally, we consider the completeness properties of $\mathbf{X}$. Since we let $\mathbf{X}$ contain a least element, the whole domain is an upper cone. Let $x$ and $y$ be two points of $\mathbf{X}$, and $U = \uparrow x \cap \uparrow y$

be the set of their common upper bounds. If $x \leq y$, then $U = \uparrow y$ is an upper cone. If $y = b_k$ for some $k$, then $U \subseteq \{b_k\}$ is empty or an upper cone since $b_k$ is maximal. The interesting case is $x = a_{ij}$ and $y = a_{i'j'}$ where $i \neq i'$. Let $k$ be the maximum of $j$ and $j'$. Then $B_k \subseteq U \subseteq \{b_i, b_{i'}\} \cup B_k$ holds where the exact value of $U$ depends on the relative sizes of $i$, $i'$, $j$, and $j'$. In any case, $U$ is strongly compact. Hence, domain $\mathbf{X}$ is in SC, but not in FC. Thus, $\mathbf{X}$ shows that class S-RD is not a superclass of SC.

## 8.12 The domain hierarchy

In this chapter and the previous ones, many domain classes were introduced. In this section, we shortly repeat their definitions and provide pictures showing their relationships. The definitions in the following list are not meant to be mathematically exact since quantifiers are missing. The domain under consideration is $\mathbf{X}$, $x$ and $y$ are points, $O$, $O_1$ etc. are open sets, $C$ closed sets, $F$ finitary upper sets, $S$ strongly compact sets, and $K$ compact sets.

| | | |
|---|---|---|
| DOM | (domains) | all domains |
| DIS | discrete | $x \leq y$ implies $x = y$ |
| FIN | finite | $\mathbf{X}$ is finite |
| CC | (cone) complete | $\mathbf{X}$ and $\uparrow x \cap \uparrow y$ are upper cones |
| BC | bounded complete | $\mathbf{X}$ and $\uparrow x \cap \uparrow y$ are empty or upper cones |
| FC | finitarily complete | $\mathbf{X}$ and $\uparrow x \cap \uparrow y$ are finitary |
| SC | strongly compactly complete | $\mathbf{X}$ and $\uparrow x \cap \uparrow y$ are strongly compact |
| KC | compactly complete | $\mathbf{X}$ and $\uparrow x \cap \uparrow y$ are compact |
| F-ALG | finitely algebraic (bifinite) | $id_{\mathbf{X}}$ is limit of idempotent deflations |
| F-CONT | finitely continuous | $id_{\mathbf{X}}$ is limit of deflations |
| ALG | algebraic | $x \in O$ implies $x \in O' = \uparrow y \subseteq O$ |
| CONT | continuous | $x \in O$ implies $x \in O' \subseteq \uparrow y \subseteq O$ |
| M-ALG | multi-algebraic | $x \in O$ implies $x \in O' = F \subseteq O$ |
| M-CONT | multi-continuous | $x \in O$ implies $x \in O' \subseteq F \subseteq O$ |
| L-COMP | locally compact | $x \in O$ implies $x \in O' \subseteq K \subseteq O$ |
| SOB | sober | $C$ non-empty irreducible closed $\Rightarrow C = \mathsf{cl}\, x$ |
| K-RD | K-Rudin | $K_i$ $\supseteq$-directed, $\bigcap K_i \subseteq O$ implies $K_k \subseteq O$ |
| S-RD | S-Rudin | $S_i$ $\supseteq$-directed, $\bigcap S_i \subseteq O$ implies $S_k \subseteq O$ |
| | ($\mathbf{X}, \mathbf{Y}$ $\times$-conform) | $(x, y) \in O$ implies $(x, y) \in O_X \times O_Y \subseteq O$ |

The following picture shows the relationships among the various classes and the one-point domain **1**. The arrows indicate inclusions. The arrow from KC to DOM is omitted for the sake of clarity.

$$\begin{tikzcd}
 & & (\times\text{-conform}) & \text{L-COMP} \arrow[rr] & & \text{DOM} \\
 & \text{M-ALG} \arrow[r] & \text{M-CONT} \arrow[u] \arrow[ur] \arrow[r] & \text{SOB} \arrow[r] & \text{K-RD} \arrow[r] & \text{S-RD} \arrow[u] \\
\text{DIS} \arrow[r] & \text{ALG} \arrow[u] \arrow[r] & \text{CONT} \arrow[u] & \text{KC} & & \\
 & \text{F-ALG} \arrow[u] \arrow[r] & \text{F-CONT} \arrow[u] \arrow[r] & \text{SC} \arrow[u] & & \\
 & \text{FIN} \arrow[u] \arrow[rr] & & \text{FC} \arrow[u] \arrow[uuurr] & & \\
 & & & \text{BC} \arrow[u] & & \\
\mathbf{1} \arrow[uuuu] \arrow[uur] \arrow[rrr] & & & \text{CC} \arrow[u] & &
\end{tikzcd}$$

As the picture shows, the classes form two almost independent hierarchies: the completeness hierarchy from CC to KC, and another hierarchy from FIN via CONT and SOB to S-RD. The next picture shows inclusions among combinations of these classes. The completeness hierarchy is spread along the $x$-axis, whereas the other hierarchy forms the $y$-axis. For simplicity, M-ALG, K-RD, BC, and KC are left off.

$$\begin{tikzcd}
 & & & \text{SC} \arrow[r] & \text{DOM} \\
\text{CC} \arrow[r] & \text{FC} \arrow[rr] & & \text{SC \& S-RD} \arrow[u] \arrow[r] & \text{S-RD} \arrow[u] \\
\text{CC \& SOB} \arrow[u] \arrow[r] & \text{FC \& SOB} \arrow[u] \arrow[rr] & & \text{SC \& SOB} \arrow[u] \arrow[r] & \text{SOB} \arrow[u] \\
\text{CC \& M-CONT} \arrow[u] \arrow[r] & \text{FC \& M-CONT} \arrow[u] \arrow[rr] & & \text{SC \& M-CONT} \arrow[u] \arrow[r] & \text{M-CONT} \arrow[u] \\
\text{CC \& CONT} \arrow[u] \arrow[r] & \text{FC \& F-CONT} \arrow[u] \arrow[r] & \text{F-CONT} \arrow[r] & \text{SC \& CONT} \arrow[u] \arrow[r] & \text{CONT} \arrow[u] \\
\text{CC \& ALG} \arrow[u] \arrow[r] & \text{FC \& F-ALG} \arrow[u] \arrow[r] & \text{F-ALG} \arrow[u] \arrow[r] & \text{SC \& ALG} \arrow[u] \arrow[r] & \text{ALG} \arrow[u] \\
\text{CC \& FIN} \arrow[u] \arrow[r] & \text{FIN} \arrow[u] & & & \text{DIS} \arrow[u]
\end{tikzcd}$$