



Relative Competitive Analysis of Cache Replacement Policies

Jan Reineke, Daniel Grund

Department of Computer Science
Saarland University

LCTES, June 2008



1 Introduction

- Motivation
- Approach

2 Relative Competitiveness

- Definition
- Automatic Computation

3 Results

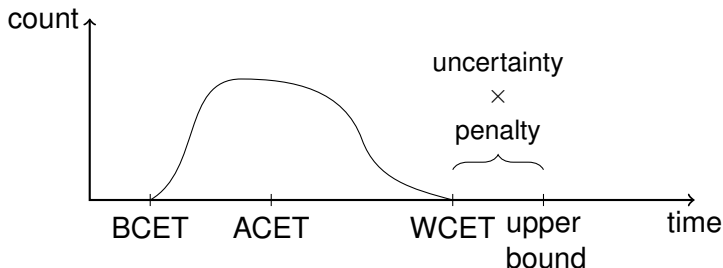
- Automatically Computed
- Generalizations

4 Summary

Motivation



- Caches used in hard real-time systems
 - Need to derive upper and lower bounds on WCET and BCET
- Need cache analysis



- In literature: almost exclusively LRU
- In practice: LRU, FIFO, PLRU, Pseudo Round-Robin, ...



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- 2 Compute performance of task T for policy Q by cache analysis.

$$m_Q(T)$$



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$$m_P \leq k \cdot m_Q + c$$

- 2 Compute performance of task T for policy Q by cache analysis.

$$m_Q(T)$$

- 3 Calculate upper bounds on the number of misses for P using the cache analysis results for Q and the competitiveness results of P relative to Q .

$$m_P \leq k \cdot m_Q + c \quad m_Q(T) = m_P(T)$$



Two types of cache analysis:

- 1 Global guarantees: bounds on cache hits/misses
[GMM98, CPHL01]
- 2 Local guarantees: classification of individual accesses
[FMW97, FW99, WHW⁺97, RM05]

→ Can provide both!



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- **Competitiveness** (Sleator and Tarjan, 1985):
worst-case performance of an online policy *relative to the optimal offline policy* (MIN, OPT, BEL)
 - ▶ used to evaluate online policies, many extensions
- **Relative competitiveness** (Reineke and Grund, 2008):
worst-case performance of an online policy *relative to another online policy*

Examples – Relative Miss-Competitiveness



P is 3-miss-competitive relative to Q with additive constant 4.
If Q incurs 7 misses, then P can incur at most $3 \cdot 7 + 4 = 25$ misses.



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$$\implies m_P \leq \frac{1}{2} \cdot m_Q \text{ on all access sequences.}$$





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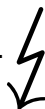
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Best: P is 1-miss-competitive relative to Q .

Worst: P is not-miss-competitive (or ∞ -miss-competitive) relative to Q .

Definition – Relative Miss-Competitiveness



Notation

$m_P(p, s)$ = *number of misses that policy P incurs on access sequence $s \in M^*$ starting in state $p \in C^P$*



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Definition (Relative miss competitiveness)

Policy P is k -miss-competitive relative to policy Q with additive constant c , if

$$m_P(p, s) \leq k \cdot m_Q(q, s) + c$$

for all access sequences $s \in M^*$ and compatible cache-set states $p \in C^P, q \in C^Q$.

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Definition (Competitive miss ratio of P relative to Q)

The smallest k , such that P is k -miss-competitive relative to Q .

Examples – Relative Hit-Competitiveness



P is $\frac{2}{3}$ -hit-competitive relative to Q with subtractive constant 3.
If Q has 27 hits, then P has at least $\frac{2}{3} \cdot 27 - 3 = 15$ hits.

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Best: P is 1-hit-competitive relative to Q .

Equivalent to 1-miss-competitiveness.

Worst: P is 0-hit-competitive relative to Q .

Analogue to ∞ -miss-competitiveness.

Definition – Relative Hit-Competitiveness



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$h_P(p, s)$ = number of hits that policy P incurs on access sequence $s \in M^*$ starting in state $p \in C^P$

Definition (Relative hit-competitiveness)

Policy P is k -hit-competitive relative to policy Q with *subtractive* constant c , if

$$h_P(p, s) \geq k \cdot h_Q(q, s) - c$$

for all access sequences $s \in M^*$ and compatible cache-set states $p \in C^P, q \in C^Q$.

Definition (Competitive hit ratio of P relative to Q)

The greatest k , such that P is k -hit-competitive relative to Q .



Let P be 1-(miss-)competitive relative to Q with constant 0:

$$m_P(p, s) \leq 1 \cdot m_Q(q, s) + 0$$

$$\Leftrightarrow m_P(p, s) \leq m_Q(q, s)$$



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- 1 If Q “hits” so does P , and
- 2 if P “misses” so does Q .



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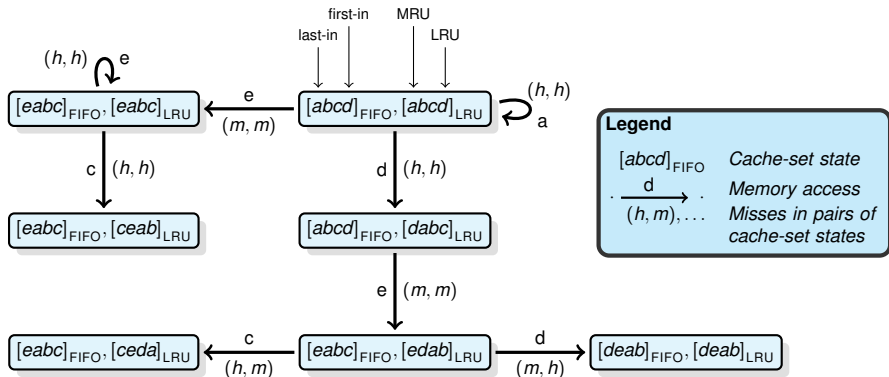
- 1 If Q “hits” so does P , and
- 2 if P “misses” so does Q .

As a consequence,

- 1 a *must*-analysis for Q is also a sound *must*-analysis for P , and
- 2 a *may*-analysis for P is also a sound *may*-analysis for Q .



P and Q induce transition system (running example):



Competitive miss ratio = maximum ratio of misses in policy P relative to the number of misses in policy Q in transition system

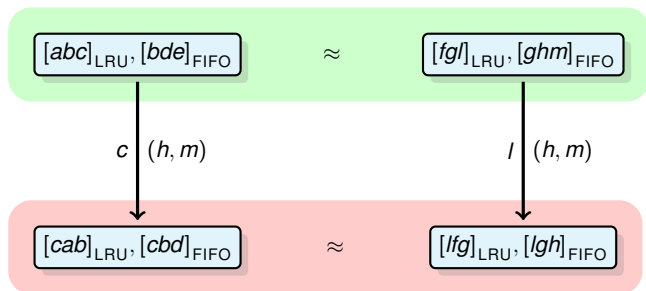
Transition System is ∞ Large



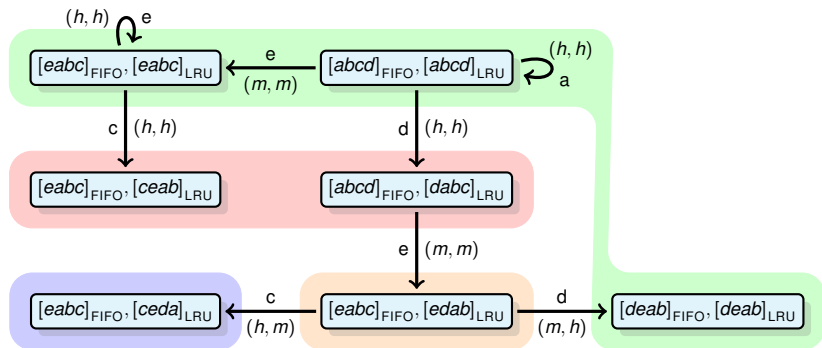
Problem: The induced transition system is ∞ large.

Goal: Construct *finite transition system* with same properties.

Observation: Only the *relative positions* of elements matter:

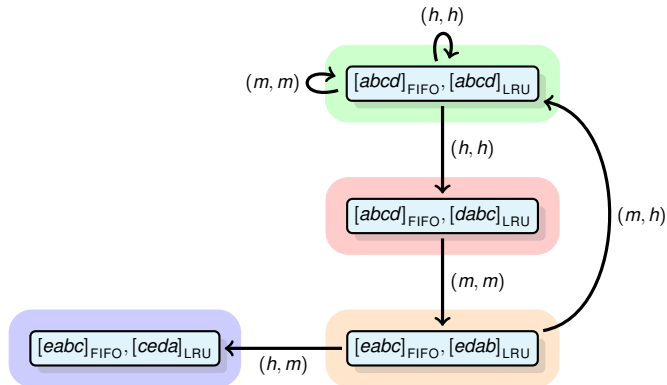


\approx -Equivalent States in Running Example





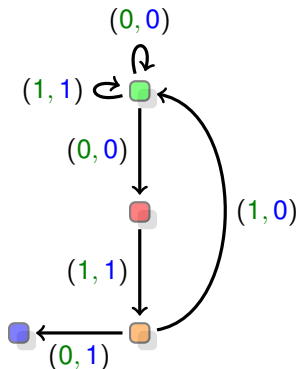
Merging \approx -equivalent states yields a finite quotient transition system:



Competitive Ratio = Maximum Cycle Ratio



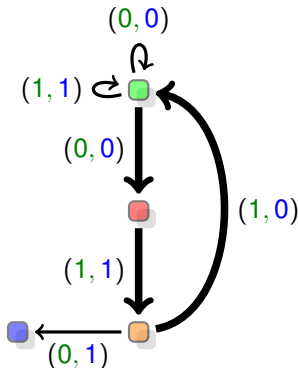
Competitive miss ratio = maximum ratio of **misses in policy P** relative to the number of **misses in policy Q** in transition system



Competitive Ratio = Maximum Cycle Ratio



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$$\text{Maximum cycle ratio} = \frac{0+1+1}{0+1+0} = 2$$



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Miss-Competitiveness Results



Miss-competitiveness *ratios, constants* relating FIFO, PLRU, and LRU at the same associativity:

Associativity:			2	3	4	5	6	7	8
LRU	vs	FIFO	2, 1	3, 2	4, 3	5, 4	6, 5	7, 6	8, 7
FIFO	vs	LRU	2, 1	3, 2	4, 3	5, 4	6, 5	7, 6	8, 7
LRU	vs	PLRU	1, 0	—	2, 1	—	—	—	5, 4
PLRU	vs	LRU	1, 0	—	∞	—	—	—	∞
FIFO	vs	PLRU	2, 1	—	4, 4	—	—	—	8, 8
PLRU	vs	FIFO	2, 1	—	∞	—	—	—	∞

Example:

LRU(4) is 2-miss-competitive relative to PLRU(4) with constant 1.

PLRU(4) is not miss-competitive relative to LRU(4) at all.

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Hit-Competitiveness Results



Hit-competitiveness *ratios, constants* relating FIFO, PLRU, and LRU at the same levels of associativity:

Associativity:			2	3	4	5	6	7	8
LRU	vs	FIFO	0,0	0,0	0,0	0,0	0,0	0,0	0,0
FIFO	vs	LRU	$\frac{1}{2}, \frac{1}{2}$	$\frac{1}{2}, 1$	$\frac{1}{2}, \frac{3}{2}$	$\frac{1}{2}, 2$	$\frac{1}{2}, \frac{5}{2}$	$\frac{1}{2}, 3$	$\frac{1}{2}, \frac{7}{2}$
LRU	vs	PLRU	1,0	—	$\frac{1}{2}, 1$	—	—	—	$\frac{1}{8}, \frac{15}{8}$
PLRU	vs	LRU	1,0	—	$\frac{1}{2}, 1$	—	—	—	$\frac{1}{4}, \frac{3}{2}$
FIFO	vs	PLRU	$\frac{1}{2}, \frac{1}{2}$	—	$\frac{1}{4}, \frac{5}{4}$	—	—	—	$\frac{1}{11}, \frac{19}{11}$
PLRU	vs	FIFO	0,0	—	0,0	—	—	—	0,0



Identified patterns and proved generalizations by hand.



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Previously unknown facts:

PLRU(k) is 1 comp. rel. to LRU($1 + \log_2 k$) with constant 0 ,
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FIFO(k) is $\frac{1}{2}$ hit-comp. rel. to LRU(k), whereas

LRU(k) is 0 hit-comp. rel. to FIFO(k), but

LRU($2k - 1$) is 1 comp. rel. to FIFO(k) with constant 0.
→ LRU-*may*-analysis can be used for FIFO



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- ... bounds performance of an online policy by that of another one,
- ... allows to derive guarantees on cache performance,
- ... can be computed automatically by building quotient system!



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Thank you for your attention!
Questions?



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