Tree Languages

- **Alphabet with arity** is a finite set Σ of operators together with a function $\rho : \Sigma \to \mathbb{N}_0$, **arity**.

- $\Sigma_k = \{a \in \Sigma \mid \rho(a) = k\}$.

- The **homogeneous tree language** over Σ is the following inductively defined set $T(\Sigma)$:
  - $a \in T(\Sigma)$ for all $a \in \Sigma_0$;
  - Are $b_1, \ldots, b_k$ in $T(\Sigma)$ and is $f \in \Sigma_k$, so is $f(b_1, \ldots, b_k) \in T(\Sigma)$.

Example:
$\Sigma = \{a, cons, nil\}$,
$\rho(a) = \rho(nil) = 0$, $\rho(cons) = 2$.
Some trees over Σ:
a, $cons(nil,\ nil)$, $cons(cons(a, nil), nil)$.

Patterns, Substitutions

V infinite set of variables (arity 0).

- $p \in T(\Sigma \cup V)$ is called a **pattern** over Σ,
- p is **linear** if no variable occurs twice in p.
- A **Substitution** Θ maps variables to patterns, $\Theta : V \to T(\Sigma \cup V)$.
- Θ extended to $\Theta : T(\Sigma \cup V) \to T(\Sigma \cup V)$ by $t\Theta = x\Theta$, if $t = x \in V$ and $t\Theta = a(t_1\Theta, \ldots, t_k\Theta)$, if $t = a(t_1, \ldots, t_k)$.

Let $V = \{X\}$.
X, $cons(nil, X)$, $cons(X, nil)$ are patterns over Σ.

Regular Tree Grammars

Regular Tree Grammar (RTG)

$G = (N, \Sigma, P, S)$ consists of

- N, finite set of **non–terminals**,
- Σ, finite alphabet (with arity) of **terminals** (operators labeling nodes)
- P, finite set of **rules** $X \to s$ where $X \in N$ and $s \in T(\Sigma \cup N)$,
- $S \in N$, the start symbol.

Notions:

- $p : X \to Y$ **chain rule**,
- $p : X \to s$ has **type** $(X_1, \ldots, X_k) \to X$, if j-th occurrence of a non–terminal in s (counted from the left) is X_j.
- $\tilde{s}$ results from s by replacing non–terminal X_j by variable x_j.

Why "Regular"?

- Path words form a regular word language,
- Regular tree languages are closed under union, intersection, and complement,
- Emptiness and therefore containment are decidable.

Example: Lists

- $G_1 = (N_1, \Sigma, P_1, L)$
- $\Sigma = \{a, cons, nil\}$
  where $\rho(a) = \rho(nil) = 0, \rho(cons) = 2$
- $N_1 = \{E, L\}$ and
- $P_1 = \{ \begin{array}{lcl} L & \rightarrow & nil, \\ L & \rightarrow & cons(E, L), \\ E & \rightarrow & a\} \end{array}$

$L(TG_1)$ is the language of linear lists of a's including the empty list,
i.e. $L(G_1) = \{nil, cons(a, nil), cons(a, cons(a, nil)), \ldots\}$.

Example: Machine Grammar

- $G_m = (N_m, \Sigma, P_m, REG)$;
- $\Sigma = \{const, m, plus, REG\}$
  where $\rho(const) = 0; \rho(m) = 1, \rho(plus) = 2$,
- $N_m = \{REG\}$
- $P_m = \{$

$$
\begin{array}{lllll}
addmc: & REG & \rightarrow & plus(m(const), REG), \\
addm: & REG & \rightarrow & plus(m(REG), REG), \\
add: & REG & \rightarrow & plus(REG, REG), \\
ldmc: & REG & \rightarrow & m(const), \\
ldc: & REG & \rightarrow & const, \\
ld: & REG & \rightarrow & REG\}
\end{array}
$$

G_m describes a subset of an instruction set of a simple processor, rules are marked with names of instructions.

The first three instructions add

- the contents of a memory cell, whose address is given by a constant,
- the contents of a memory cell, whose address is in a register, resp.,
- the contents of a register

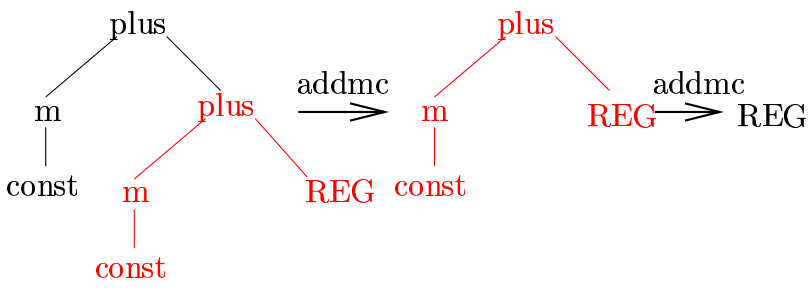
to the contents of a register and put the result into a register.

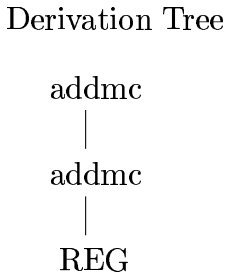
The last three rules describe load instructions, which load

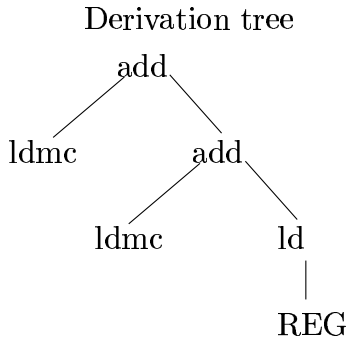
- the contents of a memory cell, whose address is given by a constant,
- a constant, resp.,
- the contents of a register

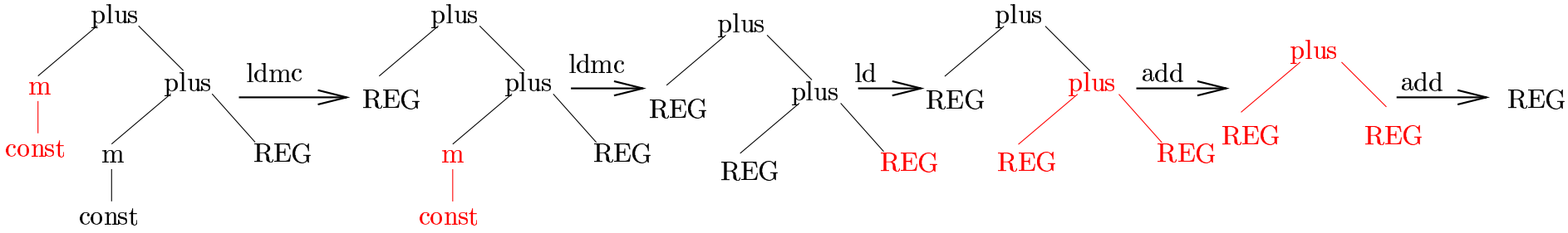
into a register.

Example Derivations





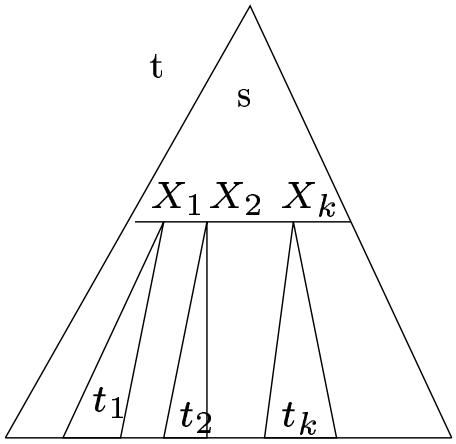


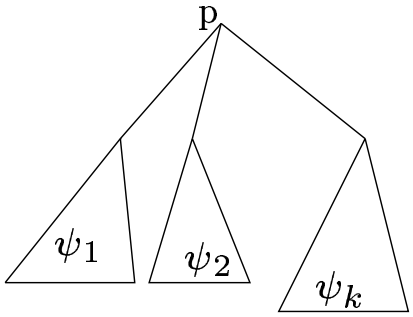


Derivation Tree

An X**–derivation tree** for tree $t \in T(\Sigma \cup N)$ according to tree grammar G is a tree $\psi \in T(P \cup N)$, such that

- Is $\psi \in N$, then $\psi = X = t$.
- Is $\psi \notin N$, then $\psi = p(\psi_1, \ldots, \psi_k)$ for a rule $p : X \to s \in P$ of type $(X_1, \ldots, X_k) \to X$, such that $t = \tilde{s}\{x_1/t_1, \ldots, x_k/t_k\}$ and ψ_j are X_j-derivation trees for the t_j.





The **generated language** $L(TG) = \{t \in T(\Sigma) \mid \exists \psi \in T(P \cup N) : \psi \text{ is } S\text{-derivation tree for } t\}$.

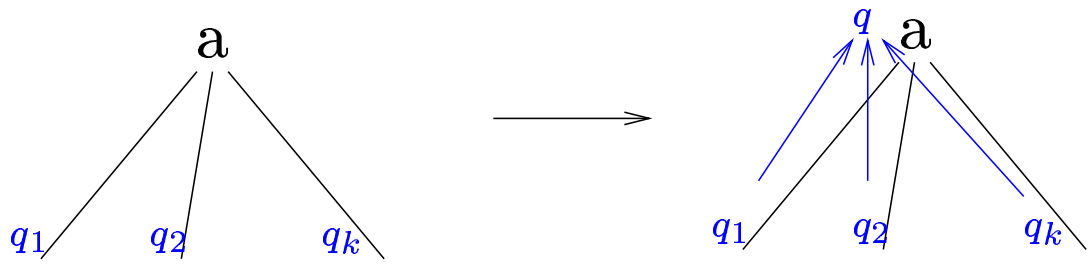
The Tree Analysis Problem

- An instance of the **tree analysis problem** consists of an RTG G and a tree t.
- A solution consists of the set of all derivation trees of t according to G,
- A **Tree Analyzer** for G solves the tree analysis problem for G and all its trees,
- A **Tree Analyzer Generator** generates a tree analyzer for each RTG.

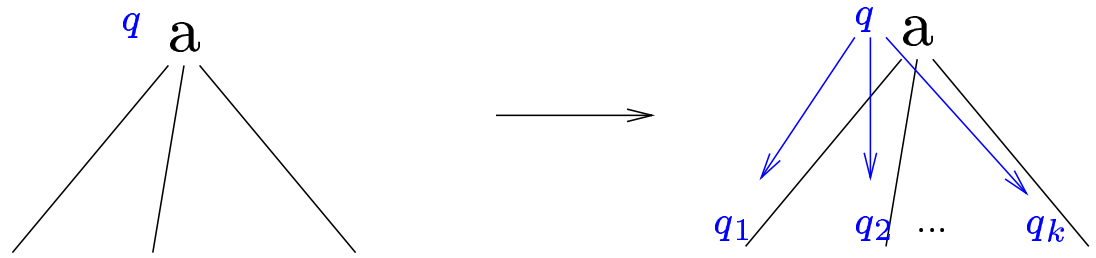
Finite Tree Automata, Intuition

- Generalization of finite word automata to trees,
- Transitions $(q, a, q_1, \ldots, q_k)$, where
  $a \in \Sigma_k$, q state at node n labeled a,
  $q_1, \ldots, q_k$ state at children of n,
- Traversal strategies,

  bottom up:



  top down:



Finite Tree Automata, Definition

Finite tree automaton (FTA)
$A = (Q, \Sigma, \delta, Q_F)$, where

- Q, finite set of **states**,
- $Q_F \subseteq Q$, **final states**,
- Σ, input alphabet (with arity),
- $\delta \subseteq \bigcup_{j \geq 0} Q \times \Sigma_j \times Q^j$, **transition relation**.
- A is **top down deterministic**, if
  - exactly one final state, and
  - at most one transition $(q, a, q_1 \ldots, q_k) \in \delta$ for all a and q.
- A is **bottom up deterministic**, if
  at most one transition $(q, a, q_1 \ldots q_k) \in \delta$ for all a and all $q_1, \ldots, q_k$.
  In this case, we write δ as partial function:
  $\delta : \bigcup_{j \geq 0} \Sigma_j \times Q^j \to Q$

Computation

- A annotates the nodes with states;
  hence new alphabet $\Sigma \times Q = \{\langle a, q\rangle \mid a \in \Sigma,\ q \in Q\}$,
  where $\rho(\langle a, q\rangle) = \rho(a)$.

- q-**computation** ϕ of A on tree $t = a(t_1, \ldots, t_m)$:
  a tree $\langle a, q\rangle(\phi_1, \ldots, \phi_m) \in T(\Sigma \times Q)$, where
  ϕ_j are q_j-computations for the t_j, $j = 1, \ldots, m$,
  $(q, a, q_1 \ldots q_m)$ is a transition.

- Is $q \in Q_F$, then ϕ is **accepting**.

- The language $L(TA)$ consists of the trees with accepting computations.

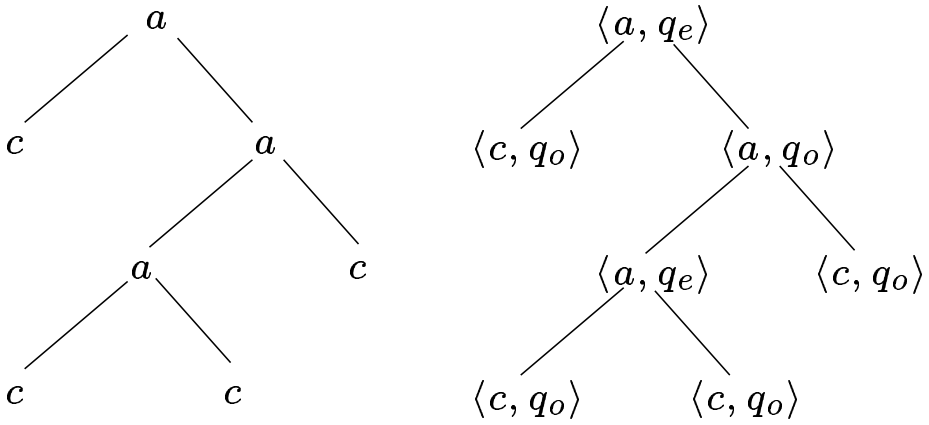
- A state resp. transition is **superfluous** if it does not occur in any accepting computation.

Example Computation

DFTA $A_b = (Q_b, \Sigma_b, \delta_b, Q_{F,b})$ with
states $Q_b = \{q_e, q_o\}$,
alphabet $\Sigma_{b,0} = \{c\}$ and $\Sigma_{b,2} = \{a\}$,
final states $Q_{F,b} = \{q_e\}$
transitions: $\delta_b = \{\ (q_o, c)$
(q_e, a, q_o, q_o)
(q_o, a, q_e, q_o)
(q_o, a, q_o, q_e)
$(q_e, a, q_e, q_e)\}$

Accepts trees with even number of c's.

Tree and q_e–computation



Determinism – Non–determinism

- Bottom up NFTAs and Top down NFTAs are equivalent,

- Bottom up DFTAs and Top down DFTAs are not equivalent;
  example language cannot be recognized by top down DFTA.

- NFTAs are equivalent to bottom up DFTAs (powerset construction).

(Bottom up) DFTA:

- At most one computation for each tree,

- At most one state at each node,

- δ extended to a partial function $\delta : T(\Sigma) \to Q$ by: $\delta(t) = \delta(a, \delta(t_1) \ldots \delta(t_k))$, if $t = a(t_1, \ldots, t_k)$.

- $\delta(t) = q$ iff there is a q–computation for t.

Generating Pattern Matchers

- Be τ a linear pattern in $B(\Sigma \cup V)$. The FTA A_τ recognizes whether τ matches a given input tree.

- Intuitively: If a pattern matches a subtree there is a region near the root of the tree where the pattern 'covers' the subtree precisely, i.e. a region where the operators of the subtree correspond precisely to the operators of the pattern. Outside this region A_τ takes an unspecific state $\perp$.

- We ignore the numbering of variables and replace them by $\perp$, i.e. suppose that $\tau \in B(\Sigma \cup \{\perp\})$.

- $A_\tau = (Q_\tau, \Sigma, \delta_\tau, Q_{\tau F})$, where $Q_\tau = \{s \mid s \text{ subtree of } \tau\} \cup \{\perp\}$, $Q_{\tau F} = \{\tau\}$ and δ_τ:
  - $(\perp, a, \perp \ldots \perp) \in \delta_\tau$
  - if $s \in Q_\tau$ and $s = a(s_1, \ldots, s_k)$, then $(s, a, s_1 \ldots s_k) \in \delta_\tau$.

- A q-computation of A corresponds to a q-derivation tree.

Generating Tree Parsers

The generation (and the explanation) process:

Input: G

1. Generate NFTA A_G,

2. Apply powerset construction to obtain DFTA $P(G)$.

Later: Consider variant with costs.

A_G, Definition

$A_G = (Q_G, \Sigma, \delta_G, \{S\})$, where

- $Q_G = N \cup \{s' \mid \exists (X \to s) \in P,$
  where s' is proper subpattern of $s\}$.

- Transition relation δ_G: transitions of the forms
  $\{(s, a, s_1 \ldots s_k) \mid s = a(s_1, \ldots, s_k) \in Q_G\}$ and
  $\{(X, a, s_1 \ldots s_k) \mid \exists (X \to s) \in P$ and
  $s = a(s_1, \ldots, s_k)\}$.

Problem with chain rules:

- A_G would have to "step on the spot" doing chain reductions.
  However, A_G has to consume at least one terminal per step,

- Chain reductions are precomputed and integrated into δ.

$$\begin{aligned}\delta_G := \quad & \{(s, a, s_1 \ldots s_k) \mid s = a(s_1, \ldots, s_k) \in Q_G\} \cup \\ & \{(X, a, s_1 \ldots s_k) \mid \exists (X' \to s) \in P : \\ & \exists X\text{–derivation tree for } X' \\ & \text{and } s = a(s_1, \ldots, s_k)\}\end{aligned}$$

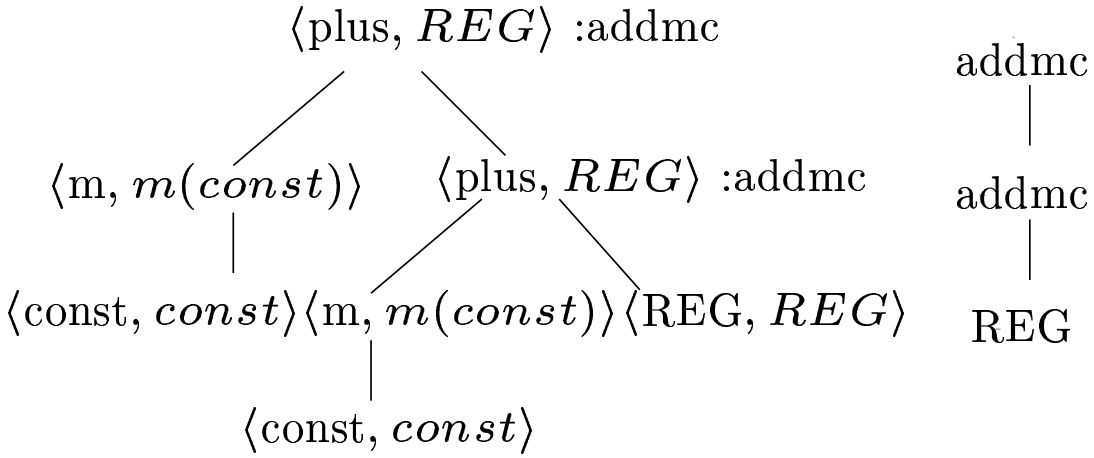
Example A_{G_m}

$A_{G_m} = (Q_{G_m}, \Sigma_{G_m}, \delta_{G_m}, Q_{F,G_m})$ for G_m has the state set
$Q_{G_m} = \{const, REG, m(const), m(REG)\}$

and the transitions

$$\begin{aligned}\delta_{G_m} = \{\ & (const, const, \epsilon)\\ & (REG, const, \epsilon)\\ & (REG, REG, \epsilon)\\ & (m(const), m, const)\\ & (REG, m, const)\\ & (m(REG), m, REG)\\ & (REG, plus, m(const)\ REG)\\ & (REG, plus, m(REG)\ REG)\\ & (REG, plus, REG\ REG)\}\end{aligned}$$

Example Computation of A_{G_m}



Properties

G RTG and t input tree.

- Exists X–derivation tree for t according to G iff exists an X–computation for t in A_G.

  In particular: $L(G) = L(A_G)$.

Principle of the Powerset Construction

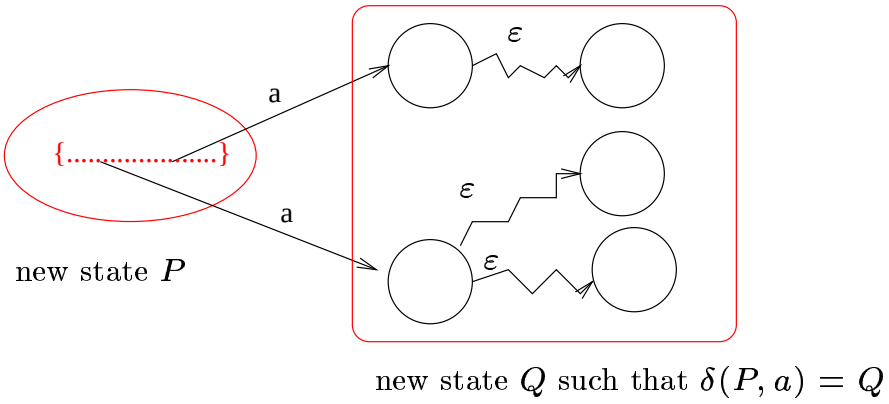
Finite Word Automata:

old states q_1, q_2 and word w such that $(q_0, w) \vdash_M^* (q_1, \varepsilon)$ and $(q_0, w) \vdash_M^* (q_2, \varepsilon)$ $\Longrightarrow$ $\exists$ new state Q such that $q_1, q_2 \in Q$ and $\delta_d(q_d, w) = Q$

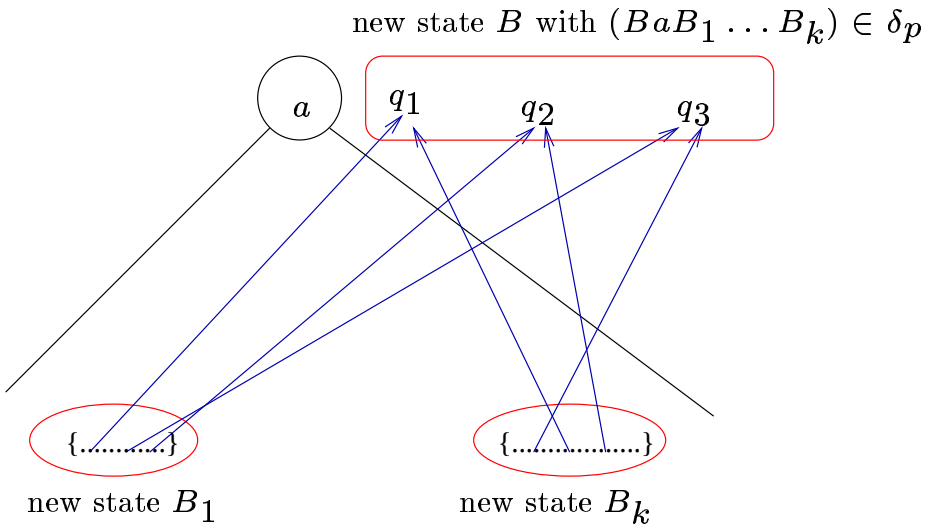
Finite Tree Automata:

old states q_1, q_2 and tree t such that $\exists q_1 -$ and $q_2 -$ computations for t $\Longrightarrow$ $\exists$ new state B such that $q_1, q_2 \in B$

Word automata:



Tree automata:



Powerset Construction

Powerset automaton $P(A)$ is built iteratively,
$Q_p^{(n)}$ and $\delta_p^{(n)}$ occur in computations on trees of height $\leq n-1$.

Let $A = (Q, \Sigma, \delta, Q_F)$ be a NFTA.
Its **powerset automaton** is the DFTA
$P(A) = (Q_p, \Sigma, \delta_p, Q_{p,F})$ where

- $Q_p = 2^Q$,
- $Q_{p,F} := \{B \in Q_p \mid B \cap Q_F \neq \emptyset\}$,
- states and transitions are computed in the iteration: $Q_p := \bigcup_{n \geq 0} Q_p^{(n)}$ and $\delta_p := \bigcup_{n \geq 0} \delta_p^{(n)}$, where:
  - $Q_p^{(0)} = \emptyset$;
  - Be $n > 0$. For $a \in \Sigma_k$ and $B_1, \ldots, B_k \in Q_p^{(n-1)}$ let $B := \{q \in Q \mid \exists q_1 \in B_1, \ldots, q_k \in B_k : (q, a, q_1 \ldots q_k) \in \delta\}$.
    Is $B \neq \emptyset$,
    then $B \in Q_p^{(n)}$ and $(B, a, B_1 \ldots B_k) \in \delta_p^{(n)}$.

Example

The powerset automaton for A_{G_m} has the state set $Q_{G_m} = \{q_1, q_2, q_3, q_4\}$ where

$$\begin{array}{lcl} q_1 & = & \{REG\} \\ q_2 & = & \{const, REG\} \\ q_3 & = & \{m(REG)\} \\ q_4 & = & \{m(const), REG, m(REG)\} \end{array}$$

and the transition function δ_{G_m}:

$state$	$operator$	$children\ state(s)$	
q_1	REG	ε	
q_2	$const$	ε	
q_3	m	q_1	
q_4	m	q_2	
q_1	$plus$	q_1	q_1
q_1	$plus$	q_4	q_1
q_1	$plus$	q_3	q_1

Properties

1. For each $t \in T(\Sigma)$:
   - Is $\delta_p(t)$ defined, then $\delta_p(t) = \{q \mid \exists q\text{–computation on } t\}$.
   - Is $\delta_p(t)$ undefined, then there is no $q \in Q$ with a q–computation of A for t.
   - $\delta_p(t) \cap N = \{X' \in N \mid \exists X'\text{–derivation tree for } t\}$.

2. $L(A) = L(P_p(A))$.

3. For each state $B \in Q_r$ there exists a tree t, such that $\delta_p(t) = B$.

Tree Automata with Costs

- Machine grammars usually are ambiguous. Thus among all possible derivation trees the cheapest should be selected.

- Required: cost measure for derivation trees.

- Assume: rules of the grammar are annotated with cost functions.

- For each rule p of type $(X_1, \ldots, X_k) \to X$: k-ary function $C(p) : \mathbb{N}_0^k \to \mathbb{N}$.

- The cost measure C can be extended to a function which assigns a cost $C(\Psi)$ to every derivation tree Ψ.

- C is called *monotonic*, or *additive*, if $C(p)$ is monotonic, or of the form $C(p) = c_p + x_1 + \ldots + x_k,\ c_p \in \mathbb{N}_0$, for all $p \in P$.

Tree Automata with Costs (c'ed)

- Examples:
  - number of required clock cycles
  - number of referenced memory cells

- Translation of the cost annotation C of the grammar G into a cost annotation C^* of the associated automaton A_G. Assumption: C additive $\Rightarrow$ cost of each rule can be described by a constant, i.e. $C' : P \to \mathbb{N}$. The cost function C^* for the transitions of A_G:
  - If $\tau = (s, a, s_1 \ldots s_k)$ with $s = a(s_1, \ldots, s_k)$, then $C^*(\tau) = 0$.
  - if $\tau = (X, a, s_1 \ldots s_k)$ then $C^*(\tau)$ corresponds to the minimal cost of an X-derivation tree for $a(s_1, \ldots, s_k)$.

- Bottom-up computation: For each node (a, B) of a computation Φ of the subset automaton A_r we tabulate the costs c_q and the transitions d_q for all $q \in B$. Let t be the subtree of the input tree for a node of Φ. Then c_q is the cost of a cheapest q-computation of t and d_q is the corresponding transition of A.

Tree Automata with Costs: Example

Be $G_m = (N_m, \Sigma, P_m, REG)$ the tree grammar of the previous examples with cost annotations:

$$
\begin{array}{lllll}
addmc: & REG & \rightarrow & plus(m(const), REG) & Cost: 3 \\
addm: & REG & \rightarrow & plus(m(REG), REG) & Cost: 3 \\
add: & REG & \rightarrow & plus(REG, REG) & Cost: 2 \\
ldmc: & REG & \rightarrow & m(const) & Cost: 2 \\
ldc: & REG & \rightarrow & const & Cost: 1 \\
ld: & REG & \rightarrow & REG & Cost: 1
\end{array}
$$

The NFTA $A = (Q, \Sigma, \delta, Q_F)$ has the states $Q = \{const, REG, m(const), m(REG)\}$ and the transitions $\delta = \{$

$$
\begin{array}{ll}
(const, const, \epsilon) & Cost: 0 \\
(REG, const, \epsilon) & Cost: 1 \\
(REG, REG, \epsilon) & Cost: 0 \\
(m(const), m, const) & Cost: 0 \\
(REG, m, const) & Cost: 2 \\
(m(REG), m, REG) & Cost: 0 \\
(REG, plus, m(const)\ REG) & Cost: 3 \\
(REG, plus, m(REG)\ REG) & Cost: 3 \\
(REG, plus, REG\ REG) & Cost: 2\}
\end{array}
$$

Powerset Construction with Costs

Usually for machine grammars $G_m = (N_m, \Sigma, P_m, S)$ the cost differences of X derivation trees with minimal costs are bounded by a constant. The finite many cost differences can be directly integrated in the states of the subset automaton $A_C = (Q_C, \Sigma, \delta_C, Q_{CF})$. A cost difference is assigned to each reachable state $q \in Q$, i.e. $B \subseteq \{(q, d) \mid q \in Q \wedge d \in \mathbb{N}_0\}$. For $(q, d) \in B$ the value d describes the cost difference between a q-computation of A to a cheapest computation.

Powerset Construction with Costs (c'ed)

Be $A = (Q, \Sigma, \delta, Q_F)$ a finite tree automaton and $C : \delta \to \mathbb{N}_0$ a cost function assigning to each transition from δ a cost in $\mathbb{N}_0$. The associated (reduced) subset automaton with integrated cost is $P_C(A) = (Q_C, \Sigma, \delta_C, Q_{CF})$ with $Q_{CF} = \{B \in Q_C \mid (q, d) \in B \wedge q \in Q_F\}$ whose states and transitions are computed iteratively by $Q_C = \bigcup_{n \geq 0} Q_C^{(n)}$, and $\delta_C = \bigcup_{n \geq 0} \delta_c^{(n)}$ where

- $Q_C^{(0)} = \emptyset$
- Be $n > 0$. For $a \in \Sigma_k$ and $B_1, \ldots, B_k \in Q_C^{(n-1)}$: $B := \{(q, d) \mid \exists (q_1, d_1) \in B_1, \ldots, (q_k, d_k) \in B_k$ and $\tau = (q, a, q_1 \ldots q_k) \in \delta$ so that $d = C(\tau) + d_1 + \ldots + d_k$ is minimal $\}$.

  If $B \neq \emptyset$, $norm(B) \in Q_C^{(n)}$ and $(norm(B), a, B_1 \ldots B_k) \in \delta_c^{(n)}$ where $norm(B) = \{(q, (d - \epsilon)) \mid (q, d) \in B\}$, $\epsilon = \min\{d \mid (q, d) \in B\}$.