Title: Similarity, Topology, and Uniformity

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Extended Abstract: Metric spaces have been generalized in many different ways: symmetry has been dropped, self-distances need not be 0, and the target domain of the distance function has been generalized from \( \mathbb{R}^+ \) to more general domains. We generalize these generalizations even further to a state without axioms and with arbitrary topological \( T_0 \) spaces \( S \) as possible target domains. For convenience, we order these target domains by their specialization relation, which corresponds to the opposite of the usual ordering in case of \( \mathbb{R}^+ \). Therefore, we speak of similarity instead of distance (if a point \( x \) moves toward a point \( y \), the distance between \( x \) and \( y \) shrinks, but their similarity increases).

**Definitions:**

A generalized similarity space or shortly gss is a tuple \( (X, S, \sigma) \) where \( X \) is a set (the set of points), \( S \) is a \( T_0 \) topological space, and \( \sigma : X \times X \rightarrow S \) is a function (the similarity function). A gss is symmetric if \( \sigma(x, x') = \sigma(x', x) \) holds for all \( x, x' \in X \). Let \( x \) be a point of \( X \) and \( u \) an open set of \( S \). The right and left pre-open balls about \( x \) with radius \( u \) are defined as \( B^R(x, u) = \{ x' \in X \mid \sigma(x, x') \in u \} \) and \( B^L(x, u) = \{ x' \in X \mid \sigma(x', x) \in u \} \).

This covers pseudo-quasi-metric spaces and partial metric spaces by taking the “middle point” of the triangle.

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We first study the properties of a single gss to find out which hypotheses are needed to prove results known from more familiar classes of generalized metrics. For instance, to show that the pre-open balls are open in the induced topology, a weak form of triangle inequality is sufficient in which the binary operation that takes the “middle point” of the triangle.

**Definition:** A gss \( (X, S, \sigma) \) is locally transitive if there is a family \((\ast_y)_{y \in X}\) of functions \( \ast_y : S \times S \rightarrow S \) continuous in each argument separately such that

\[
\sigma(x, y) \ast y \sigma(y, z) \leq \sigma(x, z), \quad \sigma(x, y) \ast y \sigma(y, y) = \sigma(x, y), \quad \sigma(y, y) \ast y \sigma(y, z) = \sigma(y, z).
\]

It is globally transitive if it is locally transitive and the operations \( \ast_y \) do not depend on \( y \).

**Proposition:** The left and right pre-open balls are open in locally transitive gss.

Note that this covers partial metrics: After turning around the role of addition is not required to be commutative or associative and may even vary dependent on the “middle point” of the triangle.

**Definition:** A gss \( (X, S, \sigma) \) is left-locally continuous (LLC) if there is a family \((\ast_y)_{y \in X}\) of functions \( \ast_y : S \times S \rightarrow S \) continuous w.r.t. the neighborhood system induced by the left pre-open balls;

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A similarity space proper is a gss \((X, S, \sigma)\) where \( S \) is a continuous lattice with its Scott topology. A generalization of the well-known injectivity of continuous lattices can be used to prove various equivalences:

- **RLC \( \Leftrightarrow \) continuous w.r.t. the neighborhood system induced by the right pre-open balls;**
- **LLC \( \Leftrightarrow \) continuous w.r.t. the neighborhood system induced by the left pre-open balls;**
- **LC \( \Leftrightarrow \) pairwise continuous w.r.t. both neighborhood systems;**
- **GC \( \Leftrightarrow \) uniformly continuous in the following sense:** for every \( x \) in \( X \) and open \( v \) of \( S \) containing \( \sigma_y(fx, fx) \), there is an open \( S \) of \( S \) containing \( \sigma_X(x, x) \) with the property \( \sigma_X(x_1, x_2) \in u \Rightarrow \sigma_y(fx_1, fx_2) \in v \).
- The category of locally transitive similarity spaces + LC functions is equivalent to \( \text{BiTop} \) (bitopological spaces + pairwise continuous functions).
- The category of symmetric locally transitive similarity spaces + LC functions is equivalent to \( \text{Top} \) (topological spaces + continuous functions).
- The category of globally transitive similarity spaces with a constant value for \( \sigma(x, x) + GC \) functions is equivalent to the category of quasi-uniform spaces + uniformly continuous functions. The equivalence restricts to the symmetric spaces on both sides. (Symmetric quasi-uniform spaces are uniform spaces.)