Injective Convergence Spaces and Equilogical Spaces via Pretopological Spaces

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Abstract

Sierpinski space Ω is injective in the category Top of topological spaces, but not in any of the larger cartesian closed categories Conv of convergence spaces and Equ of equilogical spaces. We show that this negative result extends to all sub-cccs of Equ and Conv that are closed under subspaces and contain Top. On the other hand, we study the category PrTop of pretopological spaces that lies in-between Top and Conv/Equ, identify its injective spaces, and show that they are also injective in Conv and Equ.

Key words: Topological Spaces, Convergence Spaces, Equilogical Spaces, Injective Objects

1 Introduction

An object Z of a category is *injective* for a morphism $e : X \to Y$ if for every morphism $f : X \to Z$, there is some morphism $F : Y \to Z$ satisfying $F \circ e = f$. If $e : X \hookrightarrow Y$ is a subspace embedding, F can be thought of as an extension of the function f defined on the subspace X to the entire space Y.

The category Top of topological spaces has a large supply of injective objects, including Sierpinski space Ω, which is injective for pre-embeddings. Yet Top is not cartesian closed, which can be remedied by embedding it into the cartesian closed categories Conv of *convergence spaces* or Equ of *equilogical spaces*. Unfortunately, most injectivity results are lost in this process: Ω is neither injective in Conv nor in Equ. This situation leaves the following question asked by Paul Taylor [20]:

- Is Ω injective in some cartesian closed subcategory of Conv or Equ still containing Top?

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A natural candidate for this subcategory is the least cartesian closed subcategory closed under pre-subspaces and containing Top, which is the same for Conv and Equ (up to equivalence), namely the category EpiTop of *epitopological* or *Antoine* spaces [12,16,15,5,6,8], called Ω-initial spaces in [10], where we already asked whether Ω is injective in that category. Now we can provide a negative answer: we show that Ω is *not* injective in EpiTop by means of an example in Section 11.

Yet the failure of injectivity of Ω in Conv and Equ leaves another question:

- Are there any other non-trivial injective spaces in Conv or Equ?

At first glance, Andrej Bauer provided a negative answer to this question in case of Equ: he showed that the only injective space in Equ is the terminal space **1** [2]. Yet he used the original definition of Equ in that note, which is based on equivalence relations in $\mathcal{T}_0$ topological spaces or on modest sets. This version of Equ does not contain all topological spaces, but only the $\mathcal{T}_0$ ones. Later many people turned to a more comprehensive definition based on equivalence relations in arbitrary topological spaces or on assemblies [19], which includes all of Top. Therefore it admits injective spaces other than **1**, namely the indiscrete topological spaces, for the trivial reason that all functions to such spaces are continuous. In this paper, we show that Equ and Conv also admit some more interesting non-trivial (i.e., not indiscrete) injective spaces. We shall even see that any equilogical space can be embedded into an injective space. Our injective spaces are intimately related with a modest generalization of Top, the category PrTop of *pretopological spaces*. Like Top, PrTop is not cartesian closed, but can be embedded into the cartesian closed categories Conv and Equ. Unlike Top, this embedding does not destroy injectivity properties: the injective objects of PrTop are still injective in Conv and Equ.

Section 2 presents the interval power set, which is useful in studying pretopological spaces. Section 3 introduces pretopological spaces by various equivalent definitions. After defining subbases for pretopologies in Section 4, we present initial constructions in PrTop (Section 5), which include products and subspaces. Section 6 introduces the specialization relation of a pretopological space, which is useful in describing finite pretopological spaces. In Section 7, we study the embedding of Top into PrTop as a reflective subcategory and show that Sierpinski space Ω is not injective in PrTop. Section 8 introduces the special space Λ, whose role for PrTop is analogous to the role of Sierpinski space Ω for Top; in particular it is injective—not only in PrTop, but also in Conv (Section 9) and Equ (Section 10). Section 10 also presents a simple characterization of injective spaces in Equ. The final section 11 contains an example implying that—in contrast to Λ—Sierpinski space is not injective in any sub-ccc of Conv and Equ that is closed under subspace and includes Top.

2 The Interval Power Set

The space of real numbers embeds quite nicely into the interval domain of the reals. This embedding has been generalized to various kinds of topological or metric spaces. Here we embed power sets into interval power sets, which turn out to be quite useful in studying pretopological spaces. The idea is quite simple and the basic properties of the construction are easily proved. Thus the main purpose of this section is to present the notations used in the following sections.

Let X be a set and $(\mathcal{P}X, \subseteq)$ its power set lattice. The *interval power set* $\mathcal{IP}X$ of X has elements $[A, B]$ where A, $B \in \mathcal{P}X$ with $A \subseteq B$. We consider the intervals $[A, B]$ not so much as sets of sets, but as special pairs (A, B) of sets subject to the side condition $A \subseteq B$. The intuition of $P = [A, B]$ is that of a set with a kind of fuzzy membership: A is the set of elements which are certainly in P, while B is the set of elements that may or may not be in P. (This idea is worked out further at the beginning of Section 8.) "Exact" intervals $[A, A]$ are abbreviated to $[A]$.

The interval power set $\mathcal{IP}X$ can be ordered by $[A, B] \leq [A', B']$ if $A \subseteq A'$ and $B \subseteq B'$. The poset $(\mathcal{IP}X, \leq)$ is a complete lattice with least element $[\emptyset]$, greatest element $[X]$, joins $\bigvee_{i\in I}[A_i, B_i] = [\bigcup_{i\in I} A_i, \bigcup_{i\in I} B_i]$, and meets $\bigwedge_{i\in I}[A_i, B_i] = [\bigcap_{i\in I} A_i, \bigcap_{i\in I} B_i]$.

Another useful order on $\mathcal{IP}X$ is the approximation order defined by $[A, B] \sqsubseteq [A', B']$ if $A \subseteq A'$ and $B' \subseteq B$. The poset $(\mathcal{IP}X, \sqsubseteq)$ is not a complete lattice, but a bounded complete dcpo. Bounded and directed joins are given by $\bigsqcup_{i\in I}[A_i, B_i] = [\bigcup_{i\in I} A_i, \bigcap_{i\in I} B_i]$, and non-empty meets by $\bigsqcap_{i\in I}[A_i, B_i] = [\bigcap_{i\in I} A_i, \bigcup_{i\in I} B_i]$. The joins and meets w.r.t. '$\leq$', which are of course monotonic w.r.t. '$\leq$', are also monotonic w.r.t. '$\sqsubseteq$': if $P_i \sqsubseteq Q_i$, then $\bigvee_{i\in I} P_i \sqsubseteq \bigvee_{i\in I} Q_i$ and $\bigwedge_{i\in I} P_i \sqsubseteq \bigwedge_{i\in I} Q_i$. The least element of $(\mathcal{IP}X, \sqsubseteq)$ is the completely undetermined interval $[\emptyset, X]$, while its maximal elements are the "exact" intervals $[A] = [A, A]$ $(A \in \mathcal{P}X)$. In fact, the lattice $(\mathcal{IP}X, \leq)$ restricted to the $\sqsubseteq$-maximal elements is isomorphic to $(\mathcal{P}X, \subseteq)$ via the correspondence $[A] \leftrightarrow A$.

Although $(\mathcal{IP}X, \leq)$ is not a Boolean lattice in general, it has an involution $\sim$ defined by $\sim[A, B] = [\neg B, \neg A]$ where $\neg$ is complement in $\mathcal{P}X$. The involution is its own inverse ($\sim\sim P = P$), turns around '$\leq$' ($P \leq Q$ iff $\sim P \geq \sim Q$), and transforms $\bigvee$ to $\bigwedge$ and vice versa. On the other hand, it leaves '$\sqsubseteq$' straight ($P \sqsubseteq Q$ iff $\sim P \sqsubseteq \sim Q$) and commutes over existing $\bigsqcup$ and $\bigsqcap$.

A function $f : X \to Y$ induces two functions on the power sets: direct image $f^+ : \mathcal{P}X \to \mathcal{P}Y$ with $f^+A = \{fa \mid a \in A\}$, and inverse image $f^- : \mathcal{P}Y \to \mathcal{P}X$ with $f^-B = \{x \in X \mid fx \in B\}$. These functions can be easily extended to the interval power sets by defining $f^\oplus[A, A'] = [f^+A, f^+A']$ and $f^\ominus[B, B'] = [f^-B, f^-B']$. The properties of f^+ and f^- for power sets immediately induce analogous properties of $f^\oplus$ and $f^\ominus$ for interval power sets: both functions are monotonic w.r.t. '$\leq$' and '$\sqsubseteq$', $f^\oplus$ commutes over $\bigvee$, $f^\ominus$

commutes over $\bigvee$, $\bigwedge$, $\sim$, and all existing $\bigsqcup$ and $\bigsqcap$. The functions are connected by $f^{\oplus}P \leq Q \Leftrightarrow P \leq f^{\ominus}Q$, and the relation $P \leq f^{\ominus}(f^{\oplus}P)$ always holds, while $P = f^{\ominus}(f^{\oplus}P)$ holds for injective functions f.

3 Various Definitions of Pretopological Spaces

A topological space with point set X can be described by several different data: the topology, i.e., the collection of open subsets of X, the collection of closed subsets of X, or the closure operator, which is a function $\mathcal{P}X \to \mathcal{P}X$. Likewise, a pretopological space with point set X can be described in several different ways. From the following list, descriptions (i)–(iv) are well-known and classical (see e.g., [7] where pretopological spaces are called closure spaces, or [1, Exercise 5N] where the name "pretopological" and the abbreviation PrTop occur). Descriptions (v)–(vii) are more innovative.

(i) A preclosure operator $\mathsf{pc} : \mathcal{P}X \to \mathcal{P}X$, which is increasing ($A \subseteq \mathsf{pc}\, A$) and distributes over finite unions. Such functions are automatically monotonic w.r.t. $\subseteq$. This is a generalization of a topological closure; a preclosure operator defines a topological space if it has the additional property $\mathsf{pc}\,(\mathsf{pc}\, A) = \mathsf{pc}\, A$ (idempotence).

   A function $f : X \to Y$ is continuous if $f^{+}(\mathsf{pc}\, A) \subseteq \mathsf{pc}\,(f^{+}A)$, or equivalently $\mathsf{pc}\, A \subseteq f^{-}(\mathsf{pc}\,(f^{+}A))$, or $\mathsf{pc}\,(f^{-}B) \subseteq f^{-}(\mathsf{pc}\, B)$. These characterizations of continuity exactly correspond to the topological ones in terms of the closure operator. Hence, Top is a full subcategory of PrTop, the category of pretopological spaces and continuous functions.

(ii) A pre-interior operator $\mathsf{pi} : \mathcal{P}X \to \mathcal{P}X$, which is decreasing ($\mathsf{pi}\, A \subseteq A$) and distributes over finite intersections. The connection with pc is via $\mathsf{pi}\, B = \neg\, \mathsf{pc}\, \neg B$ and $\mathsf{pc}\, A = \neg\, \mathsf{pi}\, \neg A$. A function f is continuous iff $f^{-}(\mathsf{pi}\, B) \subseteq \mathsf{pi}\,(f^{-}B)$.

(iii) A relation $<$ between points and point sets. The connection with pi is $x < A \Leftrightarrow x \in \mathsf{pi}\, A$. The axioms for $<$ are membership $x < A \Rightarrow x \in A$, extension $x < A \subseteq A' \Rightarrow x < A'$, and intersection: $x < A_i$ for all i in a finite set I implies $x < \bigcap_{i \in I} A_i$. Continuity is characterized by $fx < B \Rightarrow x < f^{-}B$.

   In a topological space, the sets A with $x < A$ are the neighborhoods of x. Following [8], we call them *vicinities* in a pretopological space.

(iv) A vicinity filter $\mathcal{V}(x) \subseteq \mathcal{F}(x) := \{A \subseteq X \mid x \in A\}$ for every point x. The connection with $<$ is $A \in \mathcal{V}(x) \Leftrightarrow x < A$. A function f is continuous iff $\mathcal{V}(fx) \subseteq f^{*}(\mathcal{V}(x))$ where for a filter $\mathcal{A}$, $f^{*}\mathcal{A} = \{B \subseteq Y \mid f^{-}B \in \mathcal{A}\}$.

(v) A relation $<$ between point sets, which is derived from $<$ in (iii) or from pi by $A < B \Leftrightarrow \forall a \in A : a < B \Leftrightarrow A \subseteq \mathsf{pi}\, B$. Its axioms are subset $A < B \Rightarrow A \subseteq B$, extension $A' \subseteq A < B \subseteq B' \Rightarrow A' < B'$, intersection ($A < B_i$ for all i in a finite set I implies $A < \bigcap_{i \in I} B_i$), and union ($A_i < B$ for all i in an arbitrary set I implies $\bigcup_{i \in I} A_i < B$). Note that the empty cases of intersection and union state that $\emptyset < A < X$ holds for any set A. Because of the extension axiom, intersection can be reformulated as "$A_i < B_i$ for all i in a finite set I implies $\bigcap_{i\in I} A_i < \bigcap_{i\in I} B_i$", and union as "$A_i < B_i$ for all i in an arbitrary set I implies $\bigcup_{i\in I} A_i < \bigcup_{i\in I} B_i$". Continuity of f is characterized by $f^+A < B \Rightarrow A < f^-B$, or equivalently by $B < B' \Rightarrow f^-B < f^-B'$.

(vi) The subset axiom $A < B \Rightarrow A \subseteq B$ allows to interpret the elements $A < B$ of the $<$ relation as elements $[A, B]$ of the interval power set $\mathcal{IP}X$. Call an interval $[A, B]$ with $A < B$ *preopen* and the collection of all preopen intervals a *pretopology*. Then the (reformulated) union and intersection axioms of $<$ are equivalent to saying that a pretopology is closed under arbitrary join $\bigvee$ and finite meet $\bigwedge$, and the extension axiom of $<$ becomes the property that P' is preopen whenever $P' \sqsubseteq P$ for a preopen P. The second characterization of continuity in terms of $<$ translates into the property that $f^{\ominus}Q$ is preopen for every preopen Q of Y.

(vii) The involution $\sim[A, B] = [\neg B, \neg A]$ can be used to define that an interval P is preclosed iff $\sim P$ is preopen. An interval $[A, B]$ is preclosed iff $\mathsf{pc}\, A \subseteq B$. The property to be preclosed is preserved by finite joins $\bigvee$, arbitrary meets $\bigwedge$, and going down in the approximation order $\sqsubseteq$. Function f is continuous if $f^{\ominus}Q$ is preclosed for every preclosed Q of Y.

Bourdaud [5] already worked with preopen intervals as introduced in (vi), but did not consider pretopologies and their defining properties.

We finally note a criterion for membership in the preclosure $\mathsf{pc}\, S$:

Proposition 3.1 *A point x is in $\mathsf{pc}\, S$ iff every vicinity of x meets S, iff for every preopen interval $[U, V]$, $x \in U$ implies $V \cap S \neq \emptyset$.*

Proof. If x is not in $\mathsf{pc}\, S$, then $x \in \neg\, \mathsf{pc}\, S = \mathsf{pi}\,(\neg S)$. Hence $\neg S$ is a vicinity of x that does not meet S. Conversely, if V is a vicinity of x that does not meet S, then $x \in \mathsf{pi}\, V \subseteq \mathsf{pi}\,(\neg S) = \neg\, \mathsf{pc}\, S$. This proves the first equivalence. For the second, note that $[U, V]$ is preopen iff $U \subseteq \mathsf{pi}\, V$. Hence $x \in U$ implies $x \in \mathsf{pi}\, V$, i.e., V is a vicinity of x and thus meets S. Conversely, let V be a vicinity of x. Then $[\{x\}, V]$ is preopen with $x \in \{x\}$, whence V meets S. □

In the sequel, we shall write a pretopological space with point set X as $\mathbf{X}$ or as $X_{\mathcal{P}}$ where $\mathcal{P}$ is the pretopology.

4 Subbases of Pretopologies

According to Section 3 (vi), a pretopology on X is a subset of $\mathcal{IP}X$ that is closed under arbitrary join $\bigvee$ and finite meet $\bigwedge$, and is down-closed w.r.t. $\sqsubseteq$. Obviously, intersections of pretopologies are again pretopologies. Hence, any subset $\mathcal{S}$ of $\mathcal{IP}X$ is contained in a least pretopology $\langle \mathcal{S} \rangle$, the pretopology generated by $\mathcal{S}$. In analogy to the topological case, we call $\mathcal{S}$ a *subbasis* of

$\langle \mathcal{S} \rangle$. It is instructive to see how $\langle \mathcal{S} \rangle$ can be built from $\mathcal{S}$. The first two steps are familiar from topology while the third is specific for pretopologies.

Proposition 4.1 *Given $\mathcal{S} \subseteq \mathcal{IP}X$, let $\mathcal{S}_1$ be the set of finite meets $\bigwedge_{i \in F} P_i$ of elements P_i of $\mathcal{S}$, $\mathcal{S}_2$ the set of all joins $\bigvee_{j \in J} P_j$ of elements P_j of $\mathcal{S}_1$, and $\mathcal{S}_3$ the set of all P that are $\sqsubseteq P'$ for some P' in $\mathcal{S}_2$. Then $\mathcal{S}_3$ is the pretopology $\langle \mathcal{S} \rangle$ generated by $\mathcal{S}$.*

Proof. Since $(\mathcal{IP}X, \leq)$ is a frame, $\mathcal{S}_2$ is closed under all joins and finite meets. This property carries over to $\mathcal{S}_3$ since joins and meets are monotonic w.r.t. $\sqsubseteq$, i.e., $P_i \sqsubseteq P'_i$ implies $\bigvee_{i \in I} P_i \sqsubseteq \bigvee_{i \in I} P'_i$ and $\bigwedge_{i \in I} P_i \sqsubseteq \bigwedge_{i \in I} P'_i$. Thus $\mathcal{S}_3$ is a pretopology, and it is obviously contained in all pretopologies containing $\mathcal{S}$. □

As with topologies, only subbasic preopens need to be considered in checking continuity.

Proposition 4.2 *Let $X_\mathcal{P}$ and $Y_\mathcal{Q}$ be pretopological spaces, and $\mathcal{S}$ a subbasis of $\mathcal{Q}$. Then $f : X_\mathcal{P} \to Y_\mathcal{Q}$ is continuous iff $f^\ominus Q$ is in $\mathcal{P}$ for all Q in $\mathcal{S}$.*

Proof. Let $\mathcal{Q}' = \{Q \in \mathcal{IP}Y \mid f^\ominus Q \in \mathcal{P}\}$, which is a pretopology since $f^\ominus$ preserves $\bigvee$, $\bigwedge$, and $\sqsubseteq$. Hence $\mathcal{Q} = \langle \mathcal{S} \rangle \subseteq \mathcal{Q}'$ (i.e., f is continuous) iff $\mathcal{S} \subseteq \mathcal{Q}'$ (i.e., $f^\ominus Q \in \mathcal{P}$ for all Q in $\mathcal{S}$). □

5 Initial Functions and Initial Pretopologies

A family $(g_i : \mathbf{Y} \to \mathbf{Z}_i)_{i \in I}$ of functions from a space $\mathbf{Y}$ to a family $(\mathbf{Z}_i)_{i \in I}$ of spaces is *initial* if all functions g_i are continuous, and for all spaces $\mathbf{X}$ and functions $f : \mathbf{X} \to \mathbf{Y}$, continuity of all the compositions $g_i \circ f$ implies continuity of f.

Proposition 5.1 *Given a point set Y and a family of functions $(g_i : Y \to \mathbf{Z}_i)_{i \in I}$ to a family of spaces $(\mathbf{Z}_i)_{i \in I}$, there is a unique pretopology on Y that makes the family $(g_i)_{i \in I}$ initial, namely the pretopology generated by the subbasis $\{g_i^\ominus Q \mid i \in I,\ Q$ preopen in $\mathbf{Z}_i\}$. This pretopology is called the* initial pretopology *for the family $(g_i)_{i \in I}$.*

Proof. Analogous to the topological case, using 4.2. □

Special cases of this general construction are the product space, which is initial for the projections $\pi_i : \prod_{i \in I} \mathbf{Z}_i \to \mathbf{Z}_i$, and the subspace, which is initial for the subset inclusion $Y \hookrightarrow \mathbf{Z}$ (here the index set I is a singleton). Consequently, we call initial functions $e : \mathbf{Y} \to \mathbf{Z}$ *pre-embeddings* and injective initial functions $e : \mathbf{Y} \to \mathbf{Z}$ *embeddings* (see also [1, Def. 8.6]). Since we later want to study injective spaces w.r.t. embeddings, we need to characterize (pre-)embeddings more explicitly.

Proposition 5.2 *The following are equivalent:*

(i) *$g : Y_\mathcal{Q} \to Z_\mathcal{R}$ is initial;*

(ii) *Q is in $\mathcal{Q}$ iff there is R in $\mathcal{R}$ such that $Q \sqsubseteq g^{\ominus}R$;*

(iii) *for all $B \subseteq Y$, $\mathsf{pc}_{\mathcal{Q}}\, B = g^{-}(\mathsf{pc}_{\mathcal{R}}(g^{+}B))$.*

Proof. We first show the equivalence of (i) and (ii). By 5.1, $\mathcal{Q}$ is generated by the subbasis $\{g^{\ominus}R \mid R \in \mathcal{R}\}$. This subbasis is already closed under arbitrary joins $\bigvee$ and finite meets $\bigwedge$. Hence the first two steps of 4.1 may be skipped. Only the third step is remaining, i.e., $\mathcal{Q}$ is the $\sqsubseteq$-down-closure of the subbasis. This is what (ii) says.

To prove the equivalence of (i) and (iii), we show that a function satisfying (iii) is initial and refer to the uniqueness of the initial pretopology. Recall from Section 3 (i) that a function h is continuous iff $\mathsf{pc}\, S \subseteq h^{-}(\mathsf{pc}\,(h^{+}S))$ holds. Hence (iii) implies continuity of g. Now let $f : X_{\mathcal{P}} \to Y_{\mathcal{Q}}$ be a function such that $g \circ f$ is continuous. Then $\mathsf{pc}_{\mathcal{P}}\, A \subseteq (g \circ f)^{-}(\mathsf{pc}_{\mathcal{R}}((g \circ f)^{+}A))$. Because of (iii), the right hand side equals $f^{-}(\mathsf{pc}_{\mathcal{Q}}(f^{+}A))$, whence f is continuous. □

In case of embeddings, one can get even equality in (ii).

Proposition 5.3 *If $g : Y_{\mathcal{Q}} \hookrightarrow Z_{\mathcal{R}}$ is an embedding, then Q is in $\mathcal{Q}$ iff there is R in $\mathcal{R}$ such that $Q = g^{\ominus}R$.*

Proof. If $Q = g^{\ominus}R$ for some R in $\mathcal{R}$, then Q is in $\mathcal{Q}$ by 5.2. Conversely, if Q is in $\mathcal{Q}$, then $Q \sqsubseteq g^{\ominus}R$ for some R in $\mathcal{R}$ by 5.2. Since pretopologies are down-closed w.r.t. $\sqsubseteq$, $R' = R \sqcap g^{\oplus}Q$ is also in $\mathcal{R}$. Now $g^{\ominus}R' = g^{\ominus}R \sqcap g^{\ominus}(g^{\oplus}Q) = g^{\ominus}R \sqcap Q = Q$ using that $g^{\ominus}$ distributes over $\sqcap$ and the equality $g^{\ominus}(g^{\oplus}Q) = Q$ that holds for injective functions g. □

If in particular $Y \subseteq Z$ and g is the subset inclusion, then $g^{+}B = B$ and $g^{-}C = Y \cap C$, whence $g^{\ominus}R = [Y] \wedge R$. Thus we get from 5.2 and 5.3 the following:

Proposition 5.4 *If $Y \subseteq Z$ and $Y_{\mathcal{Q}}$ is the corresponding subspace of $Z_{\mathcal{R}}$, then $\mathsf{pc}_{\mathcal{Q}}\, B = Y \cap \mathsf{pc}_{\mathcal{R}}\, B$ for all $B \subseteq Y$, and Q is in $\mathcal{Q}$ iff $Q = [Y] \wedge R$ for some R in $\mathcal{R}$.*

6 The Specialization Relation

The specialization preorder defined by $x \leq y$ if $x \in \mathsf{cl}\,\{y\}$ is an important tool in studying general (non-$\mathcal{T}_1$) topological spaces. Finite topological spaces are even completely characterized by their specialization preorder. Similar results hold for pretopological spaces, but transitivity of the specialization relation is lost.

Proposition 6.1 *For two points x and y of a pretopological space, the following are equivalent:*

(i) *x is in $\mathsf{pc}\,\{y\}$;*

(ii) *every vicinity of x contains y;*

(iii) *for every preopen interval $[U, V]$, $x \in U$ implies $y \in V$.*

We call the relation characterized by these properties the specialization relation *and denote it by $x \to y$.*

Proof. The equivalences are a special case of 3.1 ($S = \{y\}$). □

Continuous functions preserve the specialization relation ($x \to y$ implies $fx \to fy$). The specialization relation is always reflexive ($x \to x$), but not necessarily transitive (it is transitive for topological spaces). Conversely, given any reflexive relation $\to$ on a set X, the definition $\mathsf{pc}\, S = \{x \in X \mid x \to y$ for some $y \in S\}$ yields a pretopological space with specialization relation $\to$. Since pc preserves finite union, all finite pretopological spaces are of this kind. Thus the finite part of PrTop is isomorphic to the category whose objects are finite sets carrying a reflexive relation and whose morphisms are relation-preserving functions.

7 PrTop and Top

We first recall the well-known fact that Top is a reflective subcategory of PrTop, and then show that the reflection does not preserve embeddings.

A topological space $\mathbf{Y}$ with closure cl can be considered as a pretopological space $E^{\mathsf{P}}_{\mathsf{T}}\mathbf{Y}$ with preclosure $\mathsf{pc} = \mathsf{cl}$. Since the characterizations of PrTop-continuity in terms of pc and Top-continuity in terms of cl look identical, $E^{\mathsf{P}}_{\mathsf{T}}$ is a functor embedding Top into PrTop. (The indices at $E^{\mathsf{P}}_{\mathsf{T}}$ indicate the type of this functor.)

An interval $[A, B]$ is preclosed iff $\mathsf{pc}\, A \subseteq B$. In $E^{\mathsf{P}}_{\mathsf{T}}\mathbf{Y}$, this means $\mathsf{cl}\, A \subseteq B$, which is equivalent to $A \subseteq C \subseteq B$ for some closed set C. The inclusion chain may alternatively be written as $[A, B] \sqsubseteq [C]$. Applying the involution $\sim$, it follows that P is preopen in $E^{\mathsf{P}}_{\mathsf{T}}\mathbf{Y}$ iff $P \sqsubseteq [O]$ for some open O of $\mathbf{Y}$, i.e., the set of all $[O]$ with open O forms a subbasis for the pretopology of $E^{\mathsf{P}}_{\mathsf{T}}\mathbf{Y}$.

The topological reflection $R^{\mathsf{T}}_{\mathsf{P}}\mathbf{X}$ of a pretopological space $\mathbf{X}$ is obtained by defining a set C to be closed iff $\mathsf{pc}\, C = C$. Since $[A, B]$ is preclosed iff $\mathsf{pc}\, A \subseteq B$, this means that C is closed iff $[C]$ is preclosed, or by involution that O is open iff $[O]$ is preopen. Since $[\bigcup_{i \in I} O_i] = \bigvee_{i \in I}[O_i]$ and $[\bigcap_{i \in I} O_i] = \bigwedge_{i \in I}[O_i]$, this really defines a topology, and since $[f^- O] = f^{\ominus}[O]$ holds, $R^{\mathsf{T}}_{\mathsf{P}}$ is a functor from PrTop to Top. We have $R^{\mathsf{T}}_{\mathsf{P}}(E^{\mathsf{P}}_{\mathsf{T}}\mathbf{Y}) = \mathbf{Y}$ since C is closed in $R^{\mathsf{T}}_{\mathsf{P}}(E^{\mathsf{P}}_{\mathsf{T}}\mathbf{Y})$ iff $\mathsf{pc}\, C = C$ iff $\mathsf{cl}\, C = C$ iff C is closed in $\mathbf{Y}$.

To conclude the proof that $R^{\mathsf{T}}_{\mathsf{P}}$ is a reflection, we argue that the identity on the point set X is continuous as a function from $\mathbf{X}$ to $E^{\mathsf{P}}_{\mathsf{T}}(R^{\mathsf{T}}_{\mathsf{P}}\mathbf{X})$. To show this, let Q be preopen in $E^{\mathsf{P}}_{\mathsf{T}}(R^{\mathsf{T}}_{\mathsf{P}}\mathbf{X})$. Then $Q \sqsubseteq [O]$ for some open O of $R^{\mathsf{T}}_{\mathsf{P}}\mathbf{X}$, i.e., $Q \sqsubseteq [O]$ where $[O]$ is preopen in $\mathbf{X}$. Since pretopologies are $\sqsubseteq$-down-closed, Q is also preopen in $\mathbf{X}$.

Because of the reflection, the embedding $E^{\mathsf{P}}_{\mathsf{T}}$ of Top into PrTop preserves initiality, hence pre-embeddings and embeddings, i.e., if $e : \mathbf{X} \hookrightarrow \mathbf{Y}$ is a subspace embedding in Top, then $e : E^{\mathsf{P}}_{\mathsf{T}}\mathbf{X} \hookrightarrow E^{\mathsf{P}}_{\mathsf{T}}\mathbf{Y}$ is also a subspace embedding in PrTop. There is however no reason why the reflection $R^{\mathsf{T}}_{\mathsf{P}}$ should preserve embeddings, and in fact, it fails to do so as badly as possible:

Proposition 7.1 *Any pretopological space* $\mathbf{Y}$ *that is not topological (i.e.,* $\mathbf{Y}$*'s preclosure is not idempotent) has a subspace* $\mathbf{X}$ *such that* $R^{\mathsf{T}}_{\mathsf{P}}\mathbf{X}$ *is* not *a topological subspace of* $R^{\mathsf{T}}_{\mathsf{P}}\mathbf{Y}$*.*

Proof. Let A be a subset of $\mathbf{Y}$ such that $\mathsf{pc}(\mathsf{pc}\,A) \neq \mathsf{pc}\,A$, let y be a point in $\mathsf{pc}(\mathsf{pc}\,A)\backslash\mathsf{pc}\,A$, and let $\mathbf{X}$ be the subspace of $\mathbf{Y}$ spanned by the set $X = A \cup \{y\}$. Then $\mathsf{pc}_{\mathbf{X}}\,A = X \cap \mathsf{pc}\,A = A$ since $y \notin \mathsf{pc}\,A$. Hence A is closed in $R^{\mathsf{T}}_{\mathsf{P}}\mathbf{X}$. If A were also closed in the corresponding topological subspace of $R^{\mathsf{T}}_{\mathsf{P}}\mathbf{Y}$, then $A = X \cap C$ would hold for some closed set C of $R^{\mathsf{T}}_{\mathsf{P}}\mathbf{Y}$. But this would imply $\mathsf{pc}(\mathsf{pc}\,A) \subseteq C$ and so $y \in X \cap C = A \subseteq \mathsf{pc}\,A$—a contradiction. □

This negative result shows that Sierpinski space Ω (or rather $E^{\mathsf{P}}_{\mathsf{T}}\Omega$) fails to be injective in PrTop. For, continuous functions $p : \mathbf{X} \to E^{\mathsf{P}}_{\mathsf{T}}\Omega$ are in one-to-one correspondence with continuous functions $p : R^{\mathsf{T}}_{\mathsf{P}}\mathbf{X} \to \Omega$ and thus with open sets of $R^{\mathsf{T}}_{\mathsf{P}}\mathbf{X}$, and 7.1 shows that there are open sets U of $R^{\mathsf{T}}_{\mathsf{P}}\mathbf{X}$ that do not appear as inverse image $e^{-}V$ of an open set V of $R^{\mathsf{T}}_{\mathsf{P}}\mathbf{Y}$ under the embedding $e : \mathbf{X} \to \mathbf{Y}$. Yet PrTop has its own injective spaces presented in the next section.

8 The Space Λ

Bourdaud [5,6] considers a special pretopological space Λ with 3 points called 0, 1, 2 and a pretopological structure defined in terms of vicinities. Although the definition looks quite ad-hoc, he is able to show that continuous functions to Λ correspond to preopen intervals. In the sequel, we present a kind of rational reconstruction of Λ (with different names for the points), which allows to conclude that Λ plays the same role for PrTop as Sierpinski space plays for Top.

The elements x of a set X are in one-to-one correspondence with the functions $() \mapsto x$ from $\mathbf{1} = \{()\}$ to X. Any function $f : \mathbf{1} \to X$ induces a function $f^{\ominus} : \mathcal{IP}X \to \mathcal{IP}\mathbf{1}$ as defined in Section 2. Putting these pieces together yields a generalized membership function $\varepsilon : X \times \mathcal{IP}X \to \mathcal{IP}\mathbf{1}$ defined by $\varepsilon(x, P) = (() \mapsto x)^{\ominus}P$. The interval power set $\mathcal{IP}\mathbf{1}$ has exactly three elements $0 = [\emptyset] = [\emptyset, \emptyset]$, $1 = [\mathbf{1}] = [\mathbf{1}, \mathbf{1}]$, and $* = [\emptyset, \mathbf{1}]$.[2] So we rename $\mathcal{IP}\mathbf{1}$ into $\Lambda = \{0, 1, *\}$ and obtain

$$\varepsilon(x, [A, B]) = \begin{cases} 1 & \text{if } x \in A \\ * & \text{if } x \in B \setminus A \\ 0 & \text{if } x \notin B \end{cases}$$

This predicate in turn induces a function $\chi : \mathcal{IP}X \to (X \to \Lambda)$ where $\chi_P = (x \mapsto \varepsilon(x, P))$ is the characteristic function of the interval P. As in the case of

[2] Originally we used $\perp$ instead of $*$, but this would lead to confusion in Section 10.

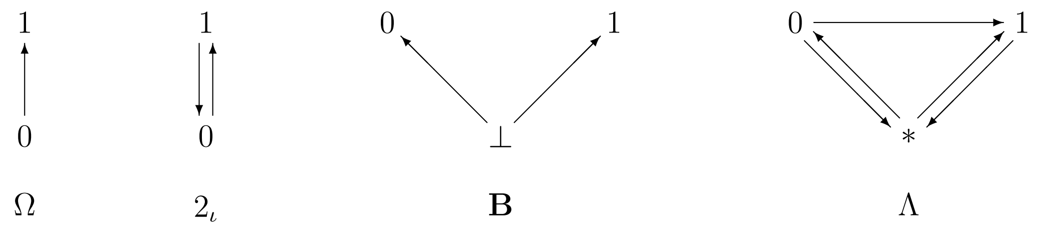

Fig. 1. Specialization relations of some pretopological spaces

the proper power set, χ is actually a bijection; its inverse $\psi : (X \to \Lambda) \to \mathcal{IP}X$ is given by $\psi\, p = p^{\ominus} I$ where I is the special interval $[\{1\}, \{*, 1\}]$ from $\mathcal{IP}\Lambda$.

Now we turn the set Λ into a pretopological space by endowing it with the pretopology generated by the subbasis $\{I\}$. The subbasis criterion 4.2 for continuity then immediately yields the following:

Proposition 8.1 *Let* $\mathbf{X}$ *be a pretopological space. A function* $p : \mathbf{X} \to \Lambda$ *is continuous iff* $p^{\ominus} I$ *is preopen in* $\mathbf{X}$*. An interval* $P \in \mathcal{IP}\mathbf{X}$ *is preopen iff its characteristic function* $\chi_P : \mathbf{X} \to \Lambda$ *is continuous.*

The entire pretopology of Λ can be constructed using 4.1: the meet closure of $\{I\}$ is $\{I, [\Lambda]\}$, whose join closure is $\{[\emptyset], I, [\Lambda]\}$, whose $\sqsubseteq$-down-closure is $\{[U, V] \mid U = \emptyset \text{ or } V = \Lambda \text{ or } [U, V] = I\}$. Hence, $[\{x\}, V]$ is preopen iff $V = \Lambda$ or $x = 1$ and $V = \{*, 1\}$. Therefore, Λ's vicinity filters are $\mathcal{V}(0) = \mathcal{V}(*) = \{\Lambda\}$ and $\mathcal{V}(1) = \{\{*, 1\}, \Lambda\}$. This corresponds to the definition in [5,6] with 0, 1, 2 renamed into 0, $*$, 1.

Although the points of Λ have similar names as the points of the Scott domain $\mathbf{B}$ that is normally used to model the Boolean data type, the pretopological structures of $\mathbf{B}$ and Λ are actually quite different. Figure 1 shows the specialization relations of Sierpinski space Ω, the indiscrete two-point space $\mathbf{2}_\iota$, the Boolean domain $\mathbf{B}$, and of Λ (the self-relations $x \to x$ are omitted for simplicity). While the two subspaces $\{\bot, 0\}$ and $\{\bot, 1\}$ of $\mathbf{B}$ are homeomorphic to Ω, the corresponding subspaces of Λ are indiscrete. On the other hand, the subspace $\{0, 1\}$ of $\mathbf{B}$ is discrete, while the corresponding subspace of Λ is Ω. The function that exchanges 0 and 1 is a continuous bijection of $\mathbf{B}$, while the only continuous bijection of Λ is the identity. The Boolean domain is topological, while Λ is not; this is witnessed by the non-transitivity of its specialization relation ($1 \to * \to 0$, but not $1 \to 0$). The topological reflection of Λ is indiscrete, but Λ itself is not indiscrete since the relation $1 \to 0$ is missing.

We now derive the main properties of Λ from Proposition 8.1.

Proposition 8.2 Λ *is injective for embeddings in* PrTop*. This means that for every embedding* $e : \mathbf{X} \to \mathbf{Y}$ *and continuous function* $f : \mathbf{X} \to \Lambda$*, there is a (not necessarily unique) continuous function* $F : \mathbf{Y} \to \Lambda$ *extending* f *along* e *(i.e.,* $F \circ e = f$*).*

Proof. Since $f : \mathbf{X} \to \Lambda$ is continuous, f is the characteristic function χ_P of the preopen interval $P = f^{\ominus}I \in \mathcal{IP}\mathbf{X}$. By 5.3, there is a preopen $Q \in \mathcal{IP}\mathbf{Y}$ such that $P = e^{\ominus}Q$. The characteristic function $F = \chi_Q$ of Q is a continuous function from $\mathbf{Y}$ to Λ. The extension property $F \circ e = f$ follows from $P = e^{\ominus}Q$. □

Proposition 8.3 *Every pretopological space can be embedded into a power of* Λ.

Proof. Consider a space $\mathbf{X}$ with pretopology $\mathcal{P}$. The characteristic functions $\chi_P : \mathbf{X} \to \Lambda$ of the intervals $P \in \mathcal{P}$ are continuous. Hence their tupling $E = \langle \chi_P \rangle_{P \in \mathcal{P}}$ is a continuous function from $\mathbf{X}$ to $\Lambda^{\mathcal{P}} = \prod_{P \in \mathcal{P}} \Lambda$. For each preopen P, the projection $\pi_P \circ E$ of E is χ_P. Hence $P = \chi_P^{\ominus} I$ can be obtained as inverse image $E^{\ominus}(\pi_P^{\ominus} I)$ of the preopen $\pi_P^{\ominus} I$ of $\Lambda^{\mathcal{P}}$. This proves that E is a pre-embedding. We still have to show that it is injective. For any x in $\mathbf{X}$, $P_x = [\{x\}, \mathbf{X}] \sqsubseteq [\mathbf{X}]$ is preopen. For $x \neq x'$, we have $\chi_{P_x}(x) = 1$, while $\chi_{P_x}(x') \neq 1$, whence $E(x) \neq E(x')$. □

In topology, Sierpinski space Ω is injective for pre-embeddings. This is not true for Λ: For instance, the unique function $e : \mathbf{2}_\iota \to \mathbf{1}$ is a pre-embedding, but the continuous function $f : \mathbf{2}_\iota \to \Lambda$ with $f(0) = *$ and $f(1) = 1$ has no “extension” $F : \mathbf{1} \to \Lambda$ since $F \circ e = f$ would imply $* = f(0) = F() = f(1) = 1$. Yet this weakness of Λ on the injectivity side does not cause harm since it is compensated by a strength on the embedding side: Proposition 8.3 features an *embedding* (even for non-$\mathcal{T}_0$ topological spaces!) while the analogous topological property involving Ω gives only a *pre*-embedding. Thus we can conclude in complete analogy to the corresponding topological theorem:

Proposition 8.4 *In* PrTop*, the injective spaces (w.r.t. embeddings) are exactly the retracts of the powers of* Λ.

In Top, one has the additional bonus that the injective spaces (w.r.t. pre-embeddings) can be characterized internally as the continuous lattices endowed with their Scott topology. Such a characterization is also possible in case of PrTop, but is actually much simpler.

Definition 8.5 A point $*$ of a pretopological space $\mathbf{X}$ is an *indefinite point* iff $* \leftrightarrow x$ holds for all x in $\mathbf{X}$, where $* \leftrightarrow x$ abbreviates $* \to x$ and $x \to *$.

Theorem 8.6 *A pretopological space is injective if and only if it contains at least one indefinite point.*

Proof. The point $*$ of Λ is an indefinite point (cf. Figure 1). If $*_i$ is an indefinite point of $\mathbf{X}_i$, then $(*_i)_{i \in I}$ is an indefinite point of $\prod_{i \in I} \mathbf{X}_i$. If $r : \mathbf{Y} \to \mathbf{X}$ is a continuous retraction and $*_{\mathbf{Y}}$ is an indefinite point of $\mathbf{Y}$, then $r(*_{\mathbf{Y}})$ is an indefinite point of $\mathbf{X}$. By 8.4, every injective space is a retract of a power of Λ and therefore contains an indefinite point by the above arguments.

Conversely, let $\mathbf{Z}$ be a pretopological space with an indefinite point $*$, and let $\mathbf{X}$ be a subspace of $\mathbf{Y}$. For a continuous $f : \mathbf{X} \to \mathbf{Z}$, define $F : \mathbf{Y} \to \mathbf{Z}$ by

$Fy = fy$ if $y \in \mathbf{X}$, and $Fy = *$ otherwise. To show continuity of F, let $[A, B]$ be a preopen interval of $\mathbf{Z}$. If $A = \emptyset$, then $F^{\ominus}[A, B] = [\emptyset, F^{-}B]$ is preopen in $\mathbf{Y}$. If A contains $*$, then $B = \mathbf{Z}$ because $* \to z$ for all z in $\mathbf{Z}$, and thus $F^{\ominus}[A, B] = [F^{-}A, \mathbf{Y}]$ is preopen in $\mathbf{Y}$. The remaining case is that A is not empty and does not contain $*$. Since $a \to *$ holds for any a in $\mathbf{Z}$, B must contain $*$ in this case. Then $F^{\ominus}[A, B] = [F^{-}A, F^{-}B] = [f^{-}A, f^{-}B \cup (Y \setminus X)]$. Since this interval is $\sqsubseteq [f^{-}A, f^{-}B] = f^{\ominus}[A, B]$, it is preopen as required. □

9 Convergence Spaces

The notion of convergence space is built around the notion of filter. A *filter* $\mathcal{A}$ on a set X is a subset of $\mathcal{P}X$ that is up-closed w.r.t. $\subseteq$ and closed under finite intersection. Special filters of interest are the point filters $\mathcal{F}(x) = \{A \subseteq X \mid x \in A\}$ for x in X. The set of all filters on X is denoted by ΦX.

There are several notions of convergence spaces in the literature, and worse, there are several names for the same thing: some authors prefer the name *filter spaces* [14,13], while others use the name *convergence spaces* [21,6,16]. Our definition below corresponds to the convergence spaces of [21,6,8] and the filter spaces of [14], while the convergence spaces of [16] and the filter spaces of [13] form a smaller class.

Convergence spaces are characterized by specifying which filters converge to which points. Formally, a *convergence space* is a set X together with a relation '$\downarrow$' between ΦX and X such that $\mathcal{F}(x) \downarrow x$ holds for all x in X (point filter axiom), and $\mathcal{A} \downarrow x$ and $\mathcal{B} \supseteq \mathcal{A}$ implies $\mathcal{B} \downarrow x$ (subfilter axiom). $\mathcal{A} \downarrow x$ is usually read as '$\mathcal{A}$ converges to x', or 'x is a limit of $\mathcal{A}$'. A function $f : X \to Y$ between two convergence spaces is *continuous* if $\mathcal{A} \downarrow x$ implies $f^{*}\mathcal{A} \downarrow fx$, where $f^{*}\mathcal{A} = \{B \subseteq Y \mid f^{-}B \in \mathcal{A}\}$. The category of convergence spaces and continuous functions is called Conv.

For pre-embeddings, the implication in the definition of continuity becomes an equivalence: $g : \mathbf{Y} \to \mathbf{Z}$ is a pre-embedding iff $\mathcal{B} \downarrow y \Leftrightarrow g^{*}\mathcal{B} \downarrow g\,y$. This guarantees that continuity of $g \circ f$ implies continuity of f.

We now present the well-known embedding of PrTop as a reflective subcategory into Conv. Since convergence is defined for filters, it is natural to base the definition of embedding $E^{\mathsf{C}}_{\mathsf{P}}$ and reflection $R^{\mathsf{P}}_{\mathsf{C}}$ on the description of pretopological spaces via vicinity filters $\mathcal{V}(x)$ (Section 3 (iv)).

Proposition 9.1 *For $\mathbf{Y}$ in PrTop, define $E^{\mathsf{C}}_{\mathsf{P}}\mathbf{Y}$ in Conv by $\mathcal{B} \downarrow y \Leftrightarrow \mathcal{B} \supseteq \mathcal{V}(y)$. For $\mathbf{X}$ in Conv, define $R^{\mathsf{P}}_{\mathsf{C}}\mathbf{X}$ in PrTop by $\mathcal{V}(x) = \bigcap\{\mathcal{A} \mid \mathcal{A} \downarrow x\}$. Then $(E^{\mathsf{C}}_{\mathsf{P}}, R^{\mathsf{P}}_{\mathsf{C}})$ embeds PrTop as a concrete reflective subcategory into Conv.*

Proof. If $f : \mathbf{X} \to \mathbf{Y}$ is PrTop-continuous, i.e., $f^{*}(\mathcal{V}(x)) \supseteq \mathcal{V}(fx)$, then $\mathcal{A} \supseteq \mathcal{V}(x)$ implies $f^{*}\mathcal{A} \supseteq f^{*}(\mathcal{V}(x)) \supseteq \mathcal{V}(fx)$, and so $f : E^{\mathsf{C}}_{\mathsf{P}}\mathbf{X} \to E^{\mathsf{C}}_{\mathsf{P}}\mathbf{Y}$ is Conv-continuous.

Conversely, let $f : \mathbf{X} \to \mathbf{Y}$ be Conv-continuous. To prove that $f : R^{\mathsf{P}}_{\mathsf{C}}\mathbf{X} \to R^{\mathsf{P}}_{\mathsf{C}}\mathbf{Y}$ is PrTop-continuous, we show that $B \in \mathcal{V}(fx)$ implies $f^{-}B \in \mathcal{V}(x) =$

$\bigcap\{\mathcal{A} \mid \mathcal{A} \downarrow x\}$. To do so, let $\mathcal{A} \downarrow x$. Then $f^*\mathcal{A} \downarrow fx$, and so $B \in \mathcal{V}(fx) \subseteq f^*\mathcal{A}$, whence $f^- B \in \mathcal{A}$.

The equality $R^{\mathsf{P}}_{\mathsf{C}}(E^{\mathsf{C}}_{\mathsf{P}}\mathbf{X}) = \mathbf{X}$ holds since $\bigcap\{\mathcal{A} \mid \mathcal{A} \supseteq \mathcal{V}(x)\} = \mathcal{V}(x)$. Finally, the identity on the point set X is continuous as a function from $\mathbf{X}$ to $E^{\mathsf{C}}_{\mathsf{P}}(R^{\mathsf{P}}_{\mathsf{C}}\mathbf{X})$ since $\mathcal{A} \downarrow x$ in $\mathbf{X}$ implies $\mathcal{A} \supseteq \mathcal{V}(x)$, whence $\mathcal{A} \downarrow x$ in $E^{\mathsf{C}}_{\mathsf{P}}(R^{\mathsf{P}}_{\mathsf{C}}\mathbf{X})$. □

We also want to derive the preclosure pc of $R^{\mathsf{P}}_{\mathsf{C}}\mathbf{X}$ directly from the convergence relation of $\mathbf{X}$. Recall 3.1: x is in $\mathsf{pc}\, S$ iff every vicinity of x meets S. For a filter $\mathcal{A}$ and a set S, define $\mathcal{A} \between S$ if all A in $\mathcal{A}$ meet S. Then $\mathsf{pc}\, S$ can be characterized concisely by $x \in \mathsf{pc}\, S \Leftrightarrow \mathcal{V}(x) \between S$. (Relation $\between$ is a special case of relation $\#$ in [8].) We need a few properties of $\between$:

Proposition 9.2 *$\mathcal{A} \not\between S$ iff $\neg S \in \mathcal{A}$.*

Proof. If $\neg S \in \mathcal{A}$, then $\mathcal{A}$ contains a set $(\neg S)$ that does not meet S. Conversely, if $\mathcal{A}$ contains a set A with $A \cap S = \emptyset$, then $A \subseteq \neg S$ and so $\neg S$ is in $\mathcal{A}$. □

Proposition 9.3 *$\bigcap_{i\in I} \mathcal{A}_i \between S$ iff there is i in I such that $\mathcal{A}_i \between S$.*

Proof. The contraposition of the claimed equivalence is $\bigcap_{i\in I} \mathcal{A}_i \not\between S$ iff $\mathcal{A}_i \not\between S$ for all i in I. With 9.2, this is equivalent to $\neg S \in \bigcap_{i\in I} \mathcal{A}_i$ iff $\neg S \in \mathcal{A}_i$ for all i in I, which is true. □

Applying 9.3 to the definition $\mathcal{V}(x) = \bigcap\{\mathcal{A} \mid \mathcal{A} \downarrow x\}$ yields the following derivation of pc from the convergence relation:

Proposition 9.4 *In $R^{\mathsf{P}}_{\mathsf{C}}\mathbf{X}$, x is in $\mathsf{pc}\, S$ iff there is a filter $\mathcal{A}$ such that $\mathcal{A} \between S$ and $\mathcal{A} \downarrow x$.*

A reflection need not preserve embeddings. In Section 7, we have seen this for the reflection $R^{\mathsf{T}}_{\mathsf{P}} : \mathsf{PrTop} \to \mathsf{Top}$, and the same is true for the reflection $R^{\mathsf{T}}_{\mathsf{P}} \circ R^{\mathsf{P}}_{\mathsf{C}}$ of Conv into Top. Therefore, it comes as a surprise that the reflection $R^{\mathsf{P}}_{\mathsf{C}} : \mathsf{Conv} \to \mathsf{PrTop}$ *does* preserve embeddings.

Proposition 9.5 *If $e : \mathbf{X} \to \mathbf{Y}$ is an embedding or pre-embedding, then so is $e : R^{\mathsf{P}}_{\mathsf{C}}\mathbf{X} \to R^{\mathsf{P}}_{\mathsf{C}}\mathbf{Y}$.*

Proof. Since injectivity of e is preserved anyway, we can concentrate on initiality. We apply criterion 5.2 (iii), i.e., show $\mathsf{pc}_{\mathcal{P}}\, S = e^-(\mathsf{pc}_{\mathcal{Q}}(e^+ S))$ where $\mathcal{P}$ and $\mathcal{Q}$ are the pretopologies of $R^{\mathsf{P}}_{\mathsf{C}}\mathbf{X}$ and $R^{\mathsf{P}}_{\mathsf{C}}\mathbf{Y}$, respectively. The inclusion '$\subseteq$' follows from continuity of e; it is the opposite inclusion that matters. If x is such that ex is in $\mathsf{pc}_{\mathcal{Q}}(e^+ S)$, then there is a filter $\mathcal{B}$ on $\mathbf{Y}$ such that $\mathcal{B} \between e^+ S$ and $\mathcal{B} \downarrow ex$. Let $\mathcal{A}$ be the filter on $\mathbf{X}$ that is the up-closure of $\{e^- B \mid B \in \mathcal{B}\}$. Since $B \cap e^+ S \neq \emptyset$ implies $e^- B \cap S \neq \emptyset$, $\mathcal{A} \between S$ holds. Furthermore, $\mathcal{B}$ is a subset of $e^*\mathcal{A}$ because $B \in \mathcal{B}$ implies $e^- B \in \mathcal{A}$. By the subfilter axiom of convergence spaces, $e^*\mathcal{A} \downarrow ex$ follows from $\mathcal{B} \downarrow ex$. Since $e : \mathbf{X} \to \mathbf{Y}$ is a pre-embedding, $e^*\mathcal{A} \downarrow ex$ implies $\mathcal{A} \downarrow x$. So we have $\mathcal{A} \between S$ and $\mathcal{A} \downarrow x$, whence $x \in \mathsf{pc}_{\mathcal{P}}\, S$ by 9.4. □

The above fact is the key for deriving the following result on injective spaces.

Proposition 9.6 *If* $\mathbf{Z}$ *is injective in* PrTop*, then* $E^{\mathsf{C}}_{\mathsf{P}}\mathbf{Z}$ *is injective in* Conv*. Hence all retracts of powers of* $E^{\mathsf{C}}_{\mathsf{P}}\Lambda$ *are injective in* Conv*.*

Proof. Let $e : \mathbf{X} \hookrightarrow \mathbf{Y}$ be an embedding in Conv, and $f : \mathbf{X} \to E^{\mathsf{C}}_{\mathsf{P}}\mathbf{Z}$ a Conv-continuous function. By 9.5, $e : R^{\mathsf{P}}_{\mathsf{C}}\mathbf{X} \to R^{\mathsf{P}}_{\mathsf{C}}\mathbf{Y}$ is an embedding in PrTop, and by reflection, $f : R^{\mathsf{P}}_{\mathsf{C}}\mathbf{X} \to \mathbf{Z}$ is PrTop-continuous. Injectivity of $\mathbf{Z}$ yields a PrTop-continuous function $F : R^{\mathsf{P}}_{\mathsf{C}}\mathbf{Y} \to \mathbf{Z}$ such that $F \circ e = f$. By reflection again, $F : \mathbf{Y} \to E^{\mathsf{C}}_{\mathsf{P}}\mathbf{Z}$ is Conv-continuous. The second statement follows from 8.4 and the preservation of products by reflective embeddings. □

10 Equilogical Spaces

In December 1996, Dana Scott and his group proposed the category Equ of *equilogical spaces* [18,3,4], a cartesian closed complete category that contains Top_0, the category of $\mathcal{T}_0$ topological spaces, as a full subcategory.

In [18], the objects of Equ are equivalence relations on topological $\mathcal{T}_0$ spaces. In [3], Scott's group showed that Equ is equivalent to the category of *modest sets* (on algebraic lattices), and also considered the larger category of *assemblies* (on algebraic lattices), which is equivalent to the category of equivalence relations on arbitrary topological spaces. The latter category was also called Equ by some people [19]. Here, we do the same, basing our equilogical spaces on assemblies. This is essential since Andrej Bauer has shown that the original $\mathcal{T}_0$-based version of Equ does not have any injective objects other than the terminal object **1** [2].

An *assembly* is a tuple $(X, D, \vdash)$ consisting of a point set X, an algebraic or continuous lattice D (both choices produce equivalent categories), and a realizability relation $\vdash$ between D and X such that for all x in X there is some a in D with $a \vdash x$. Given two assemblies $(X, D, \vdash)$ and $(Y, E, \vdash)$, a function $f : X \to Y$ is *realizable* if there is a Scott-continuous function $\varphi : D \to E$ such that $a \vdash x$ implies $\varphi\, a \vdash f x$; this implication is abbreviated to $\varphi \vdash f$ and φ is called a realizer of f.

Two assemblies $\mathcal{X}_1$ and $\mathcal{X}_2$ with the same point set X are *equivalent* if the identity on X is realizable in both directions $\mathcal{X}_1 \to \mathcal{X}_2$ and $\mathcal{X}_2 \to \mathcal{X}_1$. Since it is not required that the two realizers of id_X are inverse to each other, the lattices of two equivalent assemblies may be of radically different size and shape. This is quite different from many familiar categories including Top, PrTop, and Conv where equivalent objects would be equal. To overcome this difference, we perform an *amnestic modification* [1]: we do not consider the assemblies themselves as equilogical spaces, but only as representations of the equilogical spaces; the spaces correspond to equivalence classes of assemblies. (This is slightly non-standard, but can be compared with the situation when working with bases in topology or domain theory: a topological space or a continuous dcpo can be represented using many different bases.) Since equivalent assemblies share the same point set, this common point set can be attributed to the equilogical space represented by the assemblies. Our definition of Equ is completed by saying that a function $f : \mathbf{X} \to \mathbf{Y}$ between two equilogical spaces (or rather their point sets) is continuous if $f : \mathcal{X} \to \mathcal{Y}$ is realizable for *some* assembly representations $\mathcal{X}$ and $\mathcal{Y}$ of $\mathbf{X}$ and $\mathbf{Y}$, or equivalently if $f : \mathcal{X} \to \mathcal{Y}$ is realizable for *all* assembly representations $\mathcal{X}$ and $\mathcal{Y}$ of $\mathbf{X}$ and $\mathbf{Y}$. To get a concise notation, we write $\mathbf{X} = [(X, D, \vdash)]$ if the space $\mathbf{X}$ is represented by the assembly $(X, D, \vdash)$.

Like the other categories considered here, Equ allows the formation of arbitrary subspaces: if $\mathbf{Y} = [(Y, D, \vdash)]$ and X is a subset of Y, then the subspace of $\mathbf{Y}$ induced by X is $[(X, D, \vdash_X)]$ where $\vdash_X$ is the restriction of $\vdash$ to X. Since Equ contains indiscrete spaces, these concrete subspaces coincide with the regular subobjects.

Recall that PrTop is a reflective subcategory of Conv (9.1) where the convergence relation of $E^{\mathsf{C}}_{\mathsf{P}}\mathbf{X}$ is defined by $\mathcal{A} \downarrow x \Leftrightarrow \mathcal{A} \supseteq \mathcal{V}(x)$. This convergence relation clearly satisfies the additional axiom $\mathcal{A}_1 \downarrow x,\ \mathcal{A}_2 \downarrow x \Rightarrow \mathcal{A}_1 \cap \mathcal{A}_2 \downarrow x$, which is called *prototopological* in [8] and *merge-nice* in [11]. As shown in [9] by developing ideas from [14,13], the corresponding subcategory $\mathsf{ProtoTop}$ of Conv embeds as a reflective subcategory into Equ (in [9], convergence spaces are called filter spaces, Conv and $\mathsf{ProtoTop}$ are FIL^a and FIL^c, respectively, and assemblies are used directly). The full embedding $\mathsf{ProtoTop} \hookrightarrow \mathsf{Equ}$ is performed by mapping $(X, \downarrow)$ into the equilogical space $[(X, \Phi X, \vdash)]$ where ΦX is the algebraic lattice of filters on X ordered by subset inclusion, and $\mathcal{A} \vdash x$ iff $\mathcal{A} \downarrow x$ and $\mathcal{A} \subseteq \mathcal{F}(x)$, the point filter at x. This embedding can be composed with $E^{\mathsf{C}}_{\mathsf{P}}$ from 9.1 to get a full reflective embedding $E^{\mathsf{E}}_{\mathsf{P}} : \mathsf{PrTop} \hookrightarrow \mathsf{Equ}$ where $E^{\mathsf{E}}_{\mathsf{P}}\mathbf{X} = [(X, \Phi X, \vdash_\Phi)]$ with $\mathcal{A} \vdash_\Phi x$ iff $\mathcal{V}(x) \subseteq \mathcal{A} \subseteq \mathcal{F}(x)$.

In the sequel, we derive an internal characterization of the injective equilogical spaces similar to the one for injective pretopological spaces (Theorem 8.6). To this end, we consider the equilogical space $E^{\mathsf{E}}_{\mathsf{P}}\Lambda$ more closely, which we call Λ again for simplicity. Since Λ is finite, all filters on Λ are principal, i.e., of the form $\mathcal{F}(A) = \{A' \subseteq \Lambda \mid A' \supseteq A\}$ for $A \subseteq \Lambda$. Hence, $\Phi\Lambda$ has 8 elements and is isomorphic to $\mathcal{P}\Lambda$, but note that $\mathcal{F}(A) \subseteq \mathcal{F}(B)$ iff $A \supseteq B$. The realizability relation is given by $\mathcal{A} \vdash_\Phi 0$ iff $\mathcal{F}(\Lambda) \subseteq \mathcal{A} \subseteq \mathcal{F}\{0\}$, $\mathcal{A} \vdash_\Phi *$ iff $\mathcal{F}(\Lambda) \subseteq \mathcal{A} \subseteq \mathcal{F}\{*\}$, and $\mathcal{A} \vdash_\Phi 1$ iff $\mathcal{F}\{*, 1\} \subseteq \mathcal{A} \subseteq \mathcal{F}\{1\}$. It is hard to see any useful structure in this, but fortunately, Λ has another more concise assembly representation, namely $(\Lambda, \Omega, \vdash)$ with the lattice $\Omega = \{0 < 1\}$ and $0 \vdash 0$, $0 \vdash *$, $1 \vdash *$, and $1 \vdash 1$. Here, the equivalence is established by $\varphi : \Omega \to \Phi\Lambda$ with $\varphi(0) = \mathcal{F}(\Lambda)$ and $\varphi(1) = \mathcal{F}\{*, 1\}$, and $\psi : \Phi\Lambda \to \Omega$ with $\psi(\mathcal{A}) = 0 \Leftrightarrow \mathcal{A} \subseteq \mathcal{F}\{0\}$ and $\psi(\mathcal{A}) = 1 \Leftrightarrow \mathcal{A} \supseteq \mathcal{F}\{*, 1\}$. Note that $*$ is realized by all elements of Ω. We call such a point *indefinite*.

Definition 10.1 A point $*$ of an equilogical space $\mathbf{X}$ is an *indefinite point* if $\mathbf{X}$ has an assembly representation $(X, D, \vdash)$ such that $*$ is realized by all elements of D.

Note that $\mathbf{X}$ may have other assembly representations where the indefinite point is not universally realized. This is for instance the case for the filter representation of Λ presented above.

Proposition 10.2 *All spaces $\mathbf{Z}$ containing at least one indefinite point $*$ are injective w.r.t. embeddings in* Equ.

Proof. Let $\mathbf{X}$ be a subspace of $\mathbf{Y}$, i.e., $\mathbf{X} = [(X, D, \vdash_X)]$ and $\mathbf{Y} = [(Y, D, \vdash_Y)]$ with the same lattice D, $X \subseteq Y$, and $\vdash_X$ being the restriction of $\vdash_Y$ to X. Let $\mathbf{Z} = [(Z, E, \vdash_Z)]$ and $* \in Z$ such that $b \vdash_Z *$ for all b in E. Given $f : X \to Z$ realized by $\varphi : D \to E$, define $F : Y \to Z$ by $Fx = fx$ for $x \in X$ and $Fy = *$ for $y \in Y \setminus X$. Since all elements of E realize $*$, φ is also an realizer of F. □

We now consider the operation of adjoining an indefinite point to an equilogical space. Given an assembly $\mathcal{X} = (X, D, \vdash)$, let $\mathcal{X}^* = (X \cup \{*\}, D, \vdash')$ where $\vdash'$ is $\vdash$ plus the relations $d \vdash' *$ for all d in D. Thus $\mathcal{X}^*$ is $\mathcal{X}$ plus a new point $*$, which is indefinite by the definition of $\vdash'$. Any function $f : \mathcal{X} \to \mathcal{Y}$ induces a function $f^* : \mathcal{X}^* \to \mathcal{Y}^*$ with $f^*x = fx$ for x in X and $f^*(*) = *$. If f is realizable, then so is f^*, with the same realizers as f. Hence, the map $\mathcal{X} \mapsto \mathcal{X}^*$ preserves equivalence and thus induces an endofunctor of Equ. Examples are $\mathbf{1}^* = \mathbf{2}_\iota$ (the indiscrete 2-point space), and of course, $\Omega^* = \Lambda$ as witnessed by the Ω-representation of Λ. Since obviously $\mathbf{Z}$ is a subspace of $\mathbf{Z}^*$ and $\mathbf{Z}^*$ is injective by Prop. 10.2, we can conclude:

Proposition 10.3 *Every equilogical space can be embedded into an injective equilogical space.*

The embedding $\mathbf{X} \hookrightarrow \mathbf{X}^*$ and the injectivity of $\mathbf{Y}^*$ can also be used to extend continuous functions $\mathbf{X} \to \mathbf{Y}^*$ to continuous functions $\mathbf{X}^* \to \mathbf{Y}^*$, turning $(.)^*$ into a Kleisli triple and thus a monad.

The $(.)^*$ monad should not be confused with lifting $(.)_\perp$ although there is of course some similarity between the two monads. While lifting classifies partial maps defined on open subspaces, $(.)^*$ does much more:

Proposition 10.4 *There is a bijection between the partial continuous functions from $\mathbf{Y}$ to $\mathbf{Z}$ (where partial means defined on an* arbitrary *subspace) and total continuous functions from $\mathbf{Y}$ to $\mathbf{Z}^*$.*

Proof. A total continuous function $F : \mathbf{Y} \to \mathbf{Z}^*$ induces a partial continuous function defined on the subspace $\mathbf{X} = \{y \in \mathbf{Y} \mid Fy \neq *\}$. Conversely, any continuous function $f : \mathbf{X} \to \mathbf{Z}$ defined on a subspace $\mathbf{X}$ of $\mathbf{Y}$ can be composed with the embedding $\mathbf{Z} \hookrightarrow \mathbf{Z}^*$ and then extended to a continuous function $F : \mathbf{Y} \to \mathbf{Z}^*$ as in the proof of 10.2. □

We now continue with the characterization of injective equilogical spaces.

Proposition 10.5 *An equilogical space $\mathbf{X}$ is injective w.r.t. embeddings iff $\mathbf{X}$ is a retract of $\mathbf{X}^*$.*

Proof. If $\mathbf{X}$ is injective, the identity $\mathsf{id}_{\mathbf{X}} : \mathbf{X} \to \mathbf{X}$ can be extended along the embedding $e : \mathbf{X} \hookrightarrow \mathbf{X}^*$ to a continuous function $r : \mathbf{X}^* \to \mathbf{X}$ satisfying $r \circ e = \mathsf{id}_{\mathbf{X}}$, which makes $\mathbf{X}$ a retract of $\mathbf{X}^*$. Conversely, assume $\mathbf{X}$ is a retract of $\mathbf{X}^*$. Then $\mathbf{X}$ is injective since $\mathbf{X}^*$ is injective by 10.2 and injectivity carries over to retracts. □

The next step is to show that indefinite points are preserved by retractions.

Proposition 10.6 *If $r : \mathbf{Y} \to \mathbf{X}$ is a retraction and $*$ is indefinite in $\mathbf{Y}$, then $r(*)$ is indefinite in $\mathbf{X}$.*

Proof. Let $\mathcal{X} = (X, D, \vdash_X)$ be a representation of $\mathbf{X}$ and $\mathcal{Y} = (Y, E, \vdash_Y)$ a representation of $\mathbf{Y}$ such that $b \vdash_Y *$ for all b in E. Let further $\rho : E \to D$ be the realizer of the retraction $r : \mathbf{Y} \to \mathbf{X}$ and $\eta : D \to E$ the realizer of the corresponding section $e : \mathbf{X} \to \mathbf{Y}$ satisfying $r \circ e = \mathsf{id}_X$. (Note that the corresponding equation $\rho \circ \eta = \mathsf{id}_D$ is not required to hold.) We set up a new assembly $\mathcal{X}' = (X, E, \vdash'_X)$ using the lattice E of $\mathcal{Y}$ and a relation $\vdash'_X$ defined by $b \vdash'_X x \Leftrightarrow \rho\, b \vdash_X x$ for b in E and x in X. To show that this really forms an assembly, we have to show that for any x in X there is b in E such that $b \vdash'_X x$. Given x in X, $e\, x$ is in Y, whence there is b in E such that $b \vdash_Y e\, x$. Since ρ realizes r, $\rho\, b \vdash_X r(e\, x) = x$ follows, i.e., $b \vdash'_X x$. Next we show that $\mathcal{X}'$ forms an alternative representation of $\mathbf{X}$, i.e., that $\mathcal{X}$ and $\mathcal{X}'$ are equivalent. Since $b \vdash'_X x \Rightarrow \rho\, b \vdash_X x$, identity $\mathsf{id}_X : \mathcal{X}' \to \mathcal{X}$ is realized by ρ. For the opposite direction, we start with $a \vdash_X x$ for a in D. Since η realizes e, $\eta\, a \vdash_Y e\, x$ follows. Since ρ realizes r, this implies $\rho(\eta\, a) \vdash_X r(e\, x) = x$. By definition of $\vdash'_X$, $\eta\, a \vdash'_X x$ follows. These arguments show that $\eta : D \to E$ realizes $\mathsf{id}_X : \mathcal{X} \to \mathcal{X}'$. Finally, we show that $r(*)$ is universally realized in $\mathcal{X}'$: for any b in E, $b \vdash_Y *$ holds, whence $\rho\, b \vdash_X r(*)$, i.e., $b \vdash'_X r(*)$. This proves that $r(*)$ is an indefinite point of $\mathbf{X}$ (by virtue of the representation $\mathcal{X}'$). □

Putting all pieces together, we obtain:

Theorem 10.7 *An equilogical space is injective w.r.t. embeddings in* Equ *if and only if it contains some indefinite point.*

Proof. The 'if' direction is Prop. 10.2, and the 'only if' direction follows from 10.5 and 10.6. □

11 The Failure of Injectivity of Sierpinski Space

As already pointed out, Sierpinski space Ω is injective w.r.t. pre-embeddings in Top, but not in Equ or Conv. This leads to the question whether Ω is injective in any sub-ccc of Equ or Conv containing Top. Since ProtoTop is a reflective sub-ccc of both Equ and Conv containing Top (see section 10), one may restrict the search to that category. The smallest sub-ccc of ProtoTop closed under pre-subspace and containing Top is EpiTop, the category of *epitopological* or *Antoine* spaces [12,16,6,8], called Ω-initial spaces in [10]. These spaces are characterized by initiality of the canonical map $\lambda x.\, \lambda f.\, fx : \mathbf{X} \to [[\mathbf{X} \to \Omega] \to \Omega]$, or equivalently by the existence of some initial $e : \mathbf{X} \to [\mathbf{Y} \to \Omega]$ for some topological space $\mathbf{Y}$. A proof of this equivalence and the closure under exponentiation and pre-subspace (in fact all initial constructions) can be found in [10]. A proof that it is the smallest such category containing Top can be found in [8]. (A similar, but different property, namely injectivity of $\lambda x.\, \lambda f.\, fx$, is studied in Synthetic Domain Theory [17]).

Both PrTop and EpiTop contain Top, but are otherwise incomparable. The following example, inspired by Example 16.7 in [8], shows that the intersection of PrTop and EpiTop is strictly larger than Top. Because of 7.1, the existence of a pre-topological non-topological space in EpiTop shows that Sierpinski space cannot be injective in EpiTop.

The example space $\mathbf{Y}$ consists of three disjoint subsets A, B, and C. The points of A are a_{ij} for $i, j \in \mathbb{N}$, the set B consists of points b_i for i in $\mathbb{N}$, and C contains one further point c. The vicinities of a_{ij} are all sets containing this point a_{ij}, the vicinities of b_i are all sets containing b_i itself and all but a finite number of the points a_{ij}, $j \in \mathbb{N}$, and the vicinities of c are the sets containing c and all but a finite number of the points b_i, $i \in \mathbb{N}$. Hence, the sequences $(a_{ij})_{j \in \mathbb{N}}$ converge to b_i and the sequence $(b_i)_{i \in \mathbb{N}}$ converges to c, but c is not a limit point of A. Since $\mathsf{pc}\, A = A \cup B$ and $\mathsf{pc}\,(A \cup B) = A \cup B \cup C = \mathbf{Y}$, this pretopological space is not topological, and 7.1 shows that Ω is not injective for the embedding of the subspace $A \cup \{c\}$ into $\mathbf{Y}$. Since $\mathbf{Y}$ is Hausdorff in the sense that distinct points have disjoint vicinities, one can conclude that it is in EpiTop by using the characterization of epitopological spaces from [8, Theorem 11.3 or 11.4] that goes back to [5], or more directly the characterization of [15].

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